JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 11, Number 4, Winter 1999

CONVOLUTION CALCULUS FOR A CLASS OF SINGULAR VOLTERRA INTEGRAL EQUATIONS

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Dedicated to Professor Kazuo Okamoto on the occasion of his fiftieth birthday

ABSTRACT. For a class of singular Volterra integral equations we establish a necessary and sufficient condition for unique solvability in suitable function space settings. The discussion is based on the convolution calculus associated with the one-sided Mellin transform with weight 0. This study is motivated by some inverse nonlinear Sturm-Liouville problems, whose linearizations give rise to integral equations of our class. The method developed in this paper settles them in a unified manner.

1. Statement of main theorems. This paper is concerned with the integral equation for $u(x)$:

$$\int_0^1 \Phi(t)u(xt)\,dt = f(x), \quad a \leq x \leq b, \tag{1.1}$$

where $a \leq 0 \leq b$ and the kernel Φ and the right side f are known functions. Equation (1.1) can be rewritten as a Volterra integral equation of the first kind:

$$\int_0^x \Phi(s/x)u(s)\,ds = xf(x). \tag{1.2}$$

However, in general, this can not be handled by the standard method, see, e.g., [**2**, Chapter 2], [**6**, Section 3.3], [**10**, Section 40]. Indeed, the reduction to a Volterra integral equation of the second kind cannot be applied, since $(\partial/\partial x)\Phi(s/x)$ may have a singularity at $x = 0$. Also it

Received by the editors in revised form on September 24, 1998.
1991 AMS *Mathematics Subject Classification.* 45D05, 45E10, 44A35.
Key words and phrases. Convolution calculus, singular Volterra integral equations.
This research was partially supported by Grant-in-Aid for Scientific Research No. 09640157 and No. 07640179, the Ministry of Education, Science, Culture and Sports, Japan.

would not be a good idea to use a change of variables $t = e^{-s}$, $x = e^y$; $U(y) = u(e^y)$, $F(y) = f(e^y)$, so that one gets a convolution equation. This idea is useful only when one considers (1.1) on the interval $[0, \infty)$ in the framework of L^p-spaces.

The purpose of this paper is to show that a necessary and sufficient condition for the well-posedness of (1.1) in suitable function spaces is given by

$$(\bigstar) \qquad \int_0^1 \Phi(t)\, t^z\, dt \neq 0 \quad \text{in the right half plane} \quad \operatorname{Re} z \geq 0.$$

Our work is motivated by some inverse problems of determining unknown nonlinear terms of nonlinear Sturm-Liouville problems from their spectral information, which can be reduced to solving linear integral equations of the form (1.1) through the implicit function theorem in suitable function space settings. Later we shall present two examples, see Examples 1.10 and 1.11, arising in inverse problems.

Throughout the paper we use the notation

$$J_\Phi u(x) := \int_0^1 \Phi(t) u(xt)\, dt. \tag{1.3}$$

Let $I = [a, b]$ be a bounded interval containing 0 and let $\mathcal{C}^j(I)$, $j = 0, 1, \ldots$, denote function spaces defined by

$$\begin{aligned} \mathcal{C}^j(I) := \{ f(x) \in C^j(I \setminus \{0\}) \mid\ & f(x), x f'(x), \\ & \ldots, x^j f^{(j)}(x) \in B(I \setminus \{0\}) \}, \end{aligned} \tag{1.4}$$

where $B(I \setminus \{0\})$ represents the space of bounded functions on $I \setminus \{0\}$ and $C^j(I \setminus \{0\})$ denotes the space of functions having continuous derivatives up to the order j on $I \setminus \{0\}$. The space $\mathcal{C}^j(I)$ becomes a Banach space with the norm

$$||f||_j := \sum_{i=0}^{j} \sup_{x \in I} |x^i f^{(i)}(x)|.$$

With this notation, our first main theorem is stated as follows:

Theorem A. *Let n be a nonnegative integer. Let $\Phi(t) \in C^n(0, 1] \cap C^{n+1}(0, 1)$ and assume that*

(i) $\Phi(1) = \Phi'(1) = \cdots = \Phi^{(n-1)}(1) = 0$, $\Phi^{(n)}(1) \neq 0$,

(ii) $|\Phi(t)|, |t\Phi'(t)|, \dots, |t^{n+1}\Phi^{(n+1)}(t)| \leq Mt^{\varepsilon-1}$ *with some* $M, \varepsilon > 0$.

Then:

(a) J_Φ *is a bounded linear operator from* $\mathcal{C}^0(I)$ *to* $\mathcal{C}^{n+1}(I)$.

(b) J_Φ *is an isomorphism of* $\mathcal{C}^0(I)$ *onto* $\mathcal{C}^{n+1}(I)$ *if and only if condition* ($\bigstar$) *is satisfied.*

The proof of Theorem A will be given in Section 3. However, the following remarks on the theorem may be helpful at this stage.

Remark 1.1. The *only if* part of Theorem A (b) follows directly from condition ($\bigstar$), since if $\int_0^1 \Phi(t)t^z\,dt = 0$ for some z in the right half plane $\operatorname{Re} z \geq 0$, then $u(x) = x^z$ is a nonzero solution in $\mathcal{C}^0(I)$ of the homogeneous equation

$$\int_0^1 \Phi(t)u(xt)\,dt = 0. \tag{1.5}$$

Remark 1.2. Theorem A will be proved by induction on n, namely, a reduction to the case $n = 0$. In that case, if condition ($\bigstar$) is satisfied, then the solution u of (1.1) can be written in the form

$$u(x) = \Phi(1)^{-1}\left(x^{1-\varepsilon}(x^\varepsilon f)' + x^{1-\varepsilon}\frac{d}{dx}\left\{x^\varepsilon \int_0^1 \Lambda(t)f(xt)\,dt\right\}\right) \tag{1.6}$$

with some function $\Lambda(t) \in L^1[0,1]$. The concrete form of $\Lambda(t)$ will be given in Section 3, Remark 3.5.

Theorem A, together with Remark 1.1, leads to the following alternative theorem.

Corollary 1.3. *Under assumptions* (i) *and* (ii) *of Theorem* A, *either equation* (1.1) *has a unique solution* u *in* $\mathcal{C}^0(I)$ *for each* $f \in \mathcal{C}^{n+1}(I)$ *or the homogeneous equation* (1.5) *has nonzero solutions in* $\mathcal{C}^0(I)$.

We here present an example, which illustrates an application of Theorem A.

Example 1.4. Consider the case $\Phi(t) = 1+pt+qt^2$, that is, consider

$$\int_0^1 (1+pt+qt^2)u(xt)\,dt = f(x), \quad (p,q) \in \mathbf{R}^2. \tag{1.7}$$

If $p+q+1 \neq 0$, then the assumptions in Theorem A are satisfied for $n=0$. As will be shown in Section 4, condition (★) is equivalent to

$$p+q+1>0,\ 3p+2q+6>0,\ 4p+3q+5>0 \tag{1.8}$$

or

$$p+q+1<0,\ 3p+2q+6<0.$$

Hence, when $p+q+1 \neq 0$, equation (1.7) has a unique solution $u(x)$ in $\mathcal{C}^0(I)$ for each $f \in \mathcal{C}^1(I)$ if and only if (p,q) satisfies (1.8). Moreover, if $p+q+1=0$, $(p,q) \neq (-2,1)$, then the assumptions in Theorem A are satisfied for $n=1$. In this case, condition (★) is given by $p<-4$ or $p>-2$, and (1.7) has a unique solution in $\mathcal{C}^0(I)$ for each $f \in \mathcal{C}^2(I)$ precisely when $p<-4$ or $p>-2$. Finally for $(p,q)=(-2,1)$ Theorem A applies with $n=2$. We easily see that condition (★) is satisfied and conclude that (1.7) has a unique solution in $\mathcal{C}^0(I)$ for each $f \in \mathcal{C}^3(I)$. The details will be discussed in Section 4.

In order to solve the integral equation (1.1), the first idea occurring to us would be the use of the Neumann series. A necessary and sufficient condition for it to work will be given in Section 4, Theorem 4.1. Applying Theorem 4.1 to Example 1.4, we shall see that it generally occurs that this method is not available. It is in such cases that Theorem A is of vital importance.

Condition (★) applies to the integral equation (1.1) for a kernel $\Phi(t)$ which is singular at $t=1$ as well. To explain this we need the following function space with $0<\alpha<1$:

(1.9)

$$\mathcal{C}^\alpha(I) := \left\{ u \in B(I \setminus \{0\}) \mid ||u||_\alpha := \sup_{x \neq y} \frac{|\,|y|^\alpha u(y) - |x|^\alpha u(x)|}{|y-x|^\alpha} < \infty \right\}.$$

The space $\mathcal{C}^\alpha(I)$ becomes a Banach space with the norm $||u||_\alpha$. By using this space, our second main theorem is stated as follows.

Theorem B. *Let*

$$\Phi(t) = At^{\varepsilon-1}(1-t^\beta)^{\delta-1} + R(t), \quad \beta, \varepsilon > 0,\ 0 < \delta < 1,$$

and assume that

(i) $A \neq 0$,

(ii) $R(t) \in C(0,1] \cap C^2(0,1)$, $|R(t)| \leq Mt^{\nu-1}$, $|R'(t)| \leq Mt^{\nu-2}(1-t)^{\rho-1}$, $|R''(t)| \leq Mt^{\nu-3}(1-t)^{\rho-2}$ *with some* $M, \nu, \rho > 0$.

Let $0 < \alpha < 1 - \delta$. *Then:*

(a) J_Φ *is a bounded linear operator from* $\mathcal{C}^\alpha(I)$ *to* $\mathcal{C}^{\alpha+\delta}(I)$.

(b) J_Φ *is an isomorphism of* $\mathcal{C}^\alpha(I)$ *onto* $\mathcal{C}^{\alpha+\delta}(I)$ *if and only if condition* (★) *is satisfied.*

Remark 1.5. If (★) is satisfied, then the solution u of (1.1) can be written as

$$\begin{aligned} u(x) &= x^{1-\varepsilon}\frac{d}{dx}\left\{x^\varepsilon \int_0^1 \Psi(t) f(xt)\, dt\right\} \\ &= x^{1-\varepsilon}\frac{d}{dx}\{x^\epsilon (J_\Psi f)(x)\} \end{aligned} \tag{1.10}$$

with some $\Psi(t) \in L^1[0,1]$. The concrete form of $\Psi(t)$ will be given in Section 5, Remark 5.6.

The proof of Theorem B will be given in Section 5. By using the notation

$$D_\varepsilon f(x) := x^{1-\varepsilon}\frac{d}{dx}\{x^\epsilon f(x)\}, \tag{1.11}$$

the inversion formula (1.10) can be expressed as $J_\Phi^{-1} = D_\varepsilon J_\Psi$. This situation is indicated by the following diagram:

$$\mathcal{C}^\alpha(I) \xrightarrow{\ J_\Phi\ } \mathcal{C}^{\alpha+\delta}(I) \xrightarrow{\ J_\Psi\ } \mathcal{C}^{1,\alpha}(I)$$

$$\mathcal{C}^\alpha(I) \xleftarrow{\qquad D_\varepsilon \qquad} \mathcal{C}^{1,\alpha}(I)$$

Here $\mathcal{C}^{1,\alpha}(I)$ denotes the following function space:
(1.12)
$$\mathcal{C}^{1,\alpha}(I) := \Big\{ U(x) \in C^1(I \setminus \{0\}) \mid xU'(x) \in B(I \setminus \{0\}), \\ ||U||_{1,\alpha} := \sup_{x \in I} |U(x)| + \sup_{x \neq y} \frac{|\,|y|^\alpha y U'(y) - |x|^\alpha x U'(x)|}{|y-x|^\alpha} < \infty \Big\}.$$

It should be pointed out that J_Φ, J_Ψ are "integrations" of orders δ and $1-\delta$, respectively.

Since Remark 1.1 remains valid for Theorem B, we have the following:

Corollary 1.6. *Under assumptions* (i) *and* (ii) *of Theorem* B*, either equation* (1.1) *has a unique solution u in $\mathcal{C}^\alpha(I)$ for each $f \in \mathcal{C}^{\alpha+\delta}(I)$ or the homogeneous equation* (1.5) *has nonzero solutions in $\mathcal{C}^\alpha(I)$.*

The following example illustrates an application of Theorem B.

Example 1.7. Consider

$$\int_0^1 \{(1-t)^{-1/2} + c(1-t)^{3/2}\} u(xt)\, dt = f(x), \\ f(x) \in \mathcal{C}^{\alpha+1/2}(I), \tag{1.13}$$

where $c \in \mathbf{R}$ and $0 < \alpha < 1/2$. The function $\Phi(t) = (1-t)^{-1/2} + c(1-t)^{3/2}$ satisfies assumptions (i) and (ii) in Theorem B with $(\nu, \rho) = (1, 1)$. In this case we have

$$\int_0^1 \Phi(t) t^z\, dt = B(1/2, 1+z) + cB(5/2, 1+z) \\ = B(1/2, 1+z) \frac{(5+2z)(3+2z)+3c}{(5+2z)(3+2z)},$$

where $B(a,b)$ denotes the beta function. Since $B(1/2, 1+z)$ does not vanish for $z \neq -3/2, -5/2, \dots$, an elementary calculation shows that condition ($\bigstar$) can be rewritten as $c > -5$. Thus we conclude that (1.13) has a unique solution $u(x)$ in $\mathcal{C}^\alpha(I)$ for each $f \in \mathcal{C}^{\alpha+1/2}(I)$ if and only if $c > -5$.

In the case $c = 0$, equation (1.13) is Abel's integral equation and can be solved by well-known methods, see, e.g., [**2**, Chapter 2, Section 6, Chapter 5, Section 4] and [**10**, Section 41]. However, when $c \neq 0$, (1.13) cannot be solved by the standard method for a generalized Abel's equation, see, e.g., [**10**, Section 41].

We now explain our main idea for establishing Theorem B. First we shall show the following formula:

$$J_{\Omega_1} J_{\Omega_2} = J_{\Omega_1 * \Omega_2} \quad \Omega_1, \Omega_2 \in L^1[0,1], \tag{1.14}$$

where $*$ is the convolution product defined by

$$(\Omega_1 * \Omega_2)(t) := \int_t^1 \Omega_1(t/s)\Omega_2(s) \frac{ds}{s}. \tag{1.15}$$

Suppose that there exists a function $\Psi(t)$ such that

$$(\Phi * \Psi)(t) = t^{\varepsilon - 1}. \tag{1.16}$$

Then we apply J_Ψ to (1.1) and multiply both sides by x^ε to obtain

$$\int_0^x s^{\varepsilon-1} u(s)\, ds = x^\varepsilon J_\Psi f(x).$$

Differentiating this equality in x we arrive at (1.10). Thus our task becomes to find a function $\Psi(t)$ satisfying (1.16).

If we define the integral transform $\mathcal{K}$ by

$$\mathcal{K}[\Omega](\xi) := \int_0^1 \Omega(t) t^{-i\xi}\, dt, \quad \xi \in \mathbf{R}. \tag{1.17}$$

Then the following convolution formula holds:

$$\mathcal{K}[\Omega_1 * \Omega_2] = \mathcal{K}[\Omega_1]\mathcal{K}[\Omega_2]. \tag{1.18}$$

The integral transform $\mathcal{K}$ is a slight modification of the Mellin transform $\mathcal{M}$:

$$\mathcal{M}[\Omega](\xi) := \int_0^\infty \Omega(t) t^{-i\xi - 1}\, dt, \quad \xi \in \mathbf{R}.$$

The relation between $\mathcal{K}$ and $\mathcal{M}$ is given by $\mathcal{K}[\Omega(t)] = \mathcal{M}[t\Omega(t)\chi_{[0,1]}]$. Moreover note that the integral transform $\mathcal{K}$ can be expressed in terms of the Fourier transform:

(1.19)

$$\mathcal{K}[\Omega](\xi) = \int_0^1 \Omega(t)t^{-i\xi}\,dt = \int_{-\infty}^0 e^x\Omega(e^x)e^{-i\xi x}\,dx =: \mathcal{F}[e^x\tilde{\Omega}(e^x)](\xi),$$

where $\tilde{\Omega}(t)$ is the extension of $\Omega(t)$ defined by $\tilde{\Omega}(t) = 0$ for $t > 1$. We now take the integral transform $\mathcal{K}$ of (1.16) to get $\mathcal{K}[\Phi](\xi)\mathcal{K}[\Psi](\xi) = (\varepsilon - i\xi)^{-1}$ for $\xi \in \mathbf{R}$. Condition ($\bigstar$) implies $\mathcal{K}[\Phi](\xi) \neq 0$ for $\xi \in \mathbf{R}$, which yields

$$\mathcal{K}[\Psi](\xi) = \frac{1}{(\varepsilon - i\xi)\mathcal{K}[\Phi](\xi)}, \quad \xi \in \mathbf{R}. \tag{1.20}$$

Loosely speaking, condition ($\bigstar$) enables us to use the Paley-Wiener theorem, and to show the existence of a function Ψ satisfying (1.20). The procedure mentioned above is an outline of the proof of Theorem B.

If $A \neq 0$, $R(t) \equiv 0$ in Theorem B, namely $\Phi(t) = \Phi_0(t)$, where

$$\Phi_0(t) := At^{\varepsilon-1}(1 - t^\beta)^{\delta-1}, \tag{1.21}$$

then $\mathcal{K}[\Phi_0](\xi) = A\beta^{-1}B((\varepsilon - i\xi)/\beta, \delta)$. From properties of the beta function it follows that $\mathcal{K}[\Phi_0](\xi)$ is holomorphic in $\operatorname{Im}\xi > 0$, continuous and nonvanishing in $\operatorname{Im}\xi \geq 0$. Moreover, the right side of (1.20) can be computed as

$$\begin{aligned}\frac{1}{(\varepsilon - i\xi)\mathcal{K}[\Phi_0](\xi)} &= A^{-1}\left\{\frac{\varepsilon - i\xi}{\beta}B\left(\frac{\varepsilon - i\xi}{\beta}, \delta\right)\right\}^{-1} \\ &= A^{-1}\frac{\sin\pi\delta}{\pi}B\left(\frac{\varepsilon - i\xi}{\beta} + \delta, 1 - \delta\right).\end{aligned} \tag{1.22}$$

Here we have used the identity $p\,B(p,q)B(p+q, 1-q) = (\pi/\sin\pi q)$ for $\operatorname{Re} q \in (0,1)$, which follows from $\Gamma(q)\Gamma(1-q) = (\pi/\sin\pi q)$. It follows from this computation that the function

$$\Psi_0(t) := A^{-1}\beta\frac{\sin\pi\delta}{\pi}t^{\varepsilon+\beta\delta-1}(1 - t^\beta)^{-\delta} \tag{1.23}$$

satisfies (1.20). Hence the inverse $J_{\Phi_0}^{-1}$ of the operator J_{Φ_0} is given by $J_{\Phi_0}^{-1} = D_\varepsilon J_{\Psi_0}$.

Remark 1.8. Consider the special case where $A = \beta/\Gamma(\delta)$. Then the operator J_{Φ_0} is called the Eldélyi-Kober operator in the literature. Following [**5**], we use parameters (γ, δ, β) instead of $(\varepsilon, \delta, \beta)$, where they are related by $\varepsilon = \beta(\gamma + 1)$, and we denote the operator J_Φ by $I_\beta^{\gamma,\delta}$. It follows from Theorem B and (1.23) that if $\beta > 0$, $\gamma > -1$, $0 < \delta < 1$, then the operator $I_\beta^{\gamma,\delta}$ is an isomorphism of $\mathcal{C}^\alpha(I)$ onto $\mathcal{C}^{\alpha+\delta}(I)$ and its inverse $(I_\beta^{\gamma,\delta})^{-1}$ is expressed as

$$(I_\beta^{\gamma,\delta})^{-1} f(x) = \frac{1}{\Gamma(1-\delta)} D_\varepsilon \int_0^1 t^{\beta(\gamma+1+\delta)-1}(1-t^\beta)^{-\delta} f(xt)\,dt.$$

This gives a justification of the *formal* inversion formula $(I_\beta^{\gamma,\delta})^{-1} = I_\beta^{\gamma+\delta,-\delta}$, see, e.g., [**5**, p. 51].

Now we observe that formula (1.6) may be formally deduced from formula (1.10). Let $\Psi_0(t)$ be the principal part of $\Psi(t)$ defined by (1.23), and let $\Lambda(t)$ denote the remainder term of $\Psi(t)$. Then we obtain

$$\begin{aligned}(J_\Psi f)(x) &= \int_0^1 \Psi_0(t)\,dt\, f(x) \\ &\quad + \int_0^1 \Psi_0(t)(f(xt) - f(x))\,dt + \int_0^1 \Lambda(t) f(xt)\,dt \\ &= A^{-1}\frac{\sin \pi\delta}{\pi} B\left(\frac{\varepsilon}{\beta} + \delta, 1-\delta\right) f(x) \\ &\quad + \int_0^1 \Psi_0(t)(f(xt) - f(x))\,dt + \int_0^1 \Lambda(t) f(xt)\,dt \\ &= A^{-1}\frac{\Gamma((\epsilon/\beta)+\delta)}{\Gamma(\delta)\Gamma((\epsilon/\beta)+1)} f(x) \\ &\quad + \int_0^1 \Psi_0(t)(f(xt) - f(x))\,dt + \int_0^1 \Lambda(t) f(xt)\,dt.\end{aligned}$$

Since the middle term in the right side tends to zero as $\delta \to 1$, it follows from this equality that

$$J_\Psi f(x) \longrightarrow A^{-1} f(x) + J_\Lambda f(x) \quad \text{as } \delta \to 1.$$

Hence, by letting $\delta \to 1$ in (1.10), we arrive at (1.6). Thus Theorem A may be regarded as a *limit* version of Theorem B as $\delta \to 1$, in a natural fashion.

In general it is not easy to verify condition (★). The following lemma, which will be proved at the end of Section 2, gives a useful, simple sufficient condition for (★).

Lemma 1.9. *Let $\Phi(t) \in L^1[0,1] \cap C^1(0,1)$, and assume that*

$$\Phi(t),\ (t\Phi(t))' \geq 0 \quad \text{for any } t \in (0,1), \quad \Phi(t) \not\equiv 0.$$

Then condition (★) is satisfied.

Example 1.10. The function $\Phi(t) := t(1+ct)^{-3/2}$, $c > 0$, satisfies the assumptions in Lemma 1.9 and hence, in view of Theorem A, the equation

$$\int_0^1 \frac{t}{(1+ct)^{3/2}} u(xt)\, dt = f(x) \tag{1.24}$$

has a unique solution u in $\mathcal{C}^0[0,b]$ for each $f(x) \in \mathcal{C}^1[0,b]$. This equation is the linearized equation of a nonlinear integral equation which arises in a nonlinear inverse problem investigated by Lorenzi [**7**]. By solving (1.24), the second author of the present paper has improved the existence result in [**7**] and has established a uniqueness result. The reader may refer to [**4**].

Example 1.11. Let $v(s)$ be the first eigenfunction, normalized by $v(0) = 1$, of the operator

$$Lv(s) := -\frac{d^2 v}{ds^2} + q(s)v, \quad 0 \leq s \leq \pi/2,$$

with the boundary condition $v'(0) = v(\pi/2) = 0$, and consider the integral equation

$$\int_0^{\pi/2} v(s) u(xv(s))\, ds = f(x). \tag{1.25}$$

If $v'(x) < 0$ for any $x \in (0, \pi/2]$ then this equation can be written in the form (1.1) by setting

$$\Phi(t) := -\frac{t}{v'(v^{-1}(t))},$$

where $v^{-1}(t)$ denotes the inverse function of $v(s)$. An elementary calculation shows that the function $\Phi(t)$ can be expressed as $\Phi(t) = A(1-t^2)^{-1/2} + R(t)$, where $A, R(t)$ satisfies assumptions (i) and (ii) of Theorem B, provided that $q(s)$ is sufficiently smooth and $v''(0) < 0$. Furthermore, if $v(s)$ satisfies the condition

$$v''(s)v(s) \leqq 2v'(s)^2 \quad \text{for } 0 \leqq s < \pi/2,$$

then $\Phi(t)$ satisfies the assumptions in Lemma 1.9. Therefore the integral equation (1.25) can be solved in the form (1.10). This is the core of the analysis in [**3**], where the authors established the existence of the nonlinear term g of the nonlinear Sturm-Liouville equation $Lv = \lambda v + g(v)$ realizing a given first bifurcating branch, under some conditions on $q(s)$. It should be mentioned that, in the case $q(s) \equiv 0$, equation (1.25) is obtained by differentiating Schlömilch's integral equation, see [**9**, Section 11.81].

We conclude this section with the remark that more general integral equations

$$\int_0^1 \Phi(x,t)u(xt)\,dt = f(x) \tag{1.26}$$

can be treated as perturbations of (1.1), because (1.26) can be rewritten as

$$\int_0^1 \Phi(0,t)u(xt)\,dt + \int_0^1 (\Phi(x,t) - \Phi(0,t))u(xt)\,dt = f(x),$$

where the second term of the left side may be regarded as a *residual*. For instance, we can establish that: *if $\Phi(x,t)$ is a C^1-function on $I \times [0,1]$ with $\Phi(0,t)$ satisfying ($\bigstar$), then $u(x) \mapsto \int_0^1 \Phi(x,t)u(xt)\,dt$ is an isomorphism of $\mathcal{C}^0(I)$ onto $\mathcal{C}^1(I)$.*

2. Preliminaries. In this section we shall prove some lemmas which will be used later. We first pick out basic properties of the convolution product (1.15):

Lemma 2.1. *Let* $\Omega_1, \Omega_2 \in L^1[0,1]$. *Then*:
(a) $\Omega_1 * \Omega_2 \in L^1[0,1]$ *and* $\mathcal{K}[\Omega_1 * \Omega_2] = \mathcal{K}[\Omega_1]\mathcal{K}[\Omega_2]$.
(b) $J_{\Omega_1} J_{\Omega_2} u = J_{\Omega_2} J_{\Omega_1} u = J_{\Omega_1 * \Omega_2} u$ *for any* $u \in \mathcal{C}^0(I)$.

Proof. Changing the order of integration, we have for any $u \in \mathcal{C}^0(I)$,

$$\begin{aligned} J_{\Omega_1} J_{\Omega_2} u(x) &= x^{-1} \int_0^x \Omega_1(\tau/x) \frac{d\tau}{\tau} \int_0^\tau \Omega_2(\sigma/\tau) u(\sigma)\, d\sigma \\ &= x^{-1} \int_0^x u(\sigma)\, d\sigma \int_\sigma^x \Omega_1(\tau/x)\Omega_2(\sigma/\tau) \frac{d\tau}{\tau} \\ &= \int_0^1 u(xt)\, dt \int_{xt}^x \Omega_1(\tau/x)\Omega_2(xt/\tau) \frac{d\tau}{\tau} \\ &= \int_0^1 u(xt)\, dt \int_t^1 \Omega_1(t/x)\Omega_2(s) \frac{ds}{s} \\ &= \int_0^1 (\Omega_1 * \Omega_2)(t) u(xt)\, dt. \end{aligned}$$

This proves (b). Assertion (a) follows from an inspection of the above calculation and Fubini's theorem. The proof is complete. □

In Section 5 we shall need the following estimate:

Lemma 2.2. *Assume that* $\Omega_i(t) \in C^1(0,1)$, $i = 1, 2$, *satisfy*

$$\begin{aligned} |\Omega_i(t)| &\le M_i t^{\sigma_i - 1}(1-t)^{\tau_i - 1}, \\ |\Omega_i'(t)| &\le M_i t^{\sigma_i - 2}(1-t)^{\tau_i - 2}, \end{aligned}$$

where $M_i, \sigma_i, \tau_i > 0$. *Then* $(\Omega_1 * \Omega_2)(t) \in C^1(0,1)$, *and for any* $\sigma < \min\{\sigma_1, \sigma_2\}$,

$$\begin{aligned} |(\Omega_1 * \Omega_2)(t)| &\le M t^{\sigma - 1}(1-t)^{\tau_1 + \tau_2 - 1}, \\ |(\Omega_1 * \Omega_2)'(t)| &\le M t^{\sigma - 2}(1-t)^{\tau_1 + \tau_2 - 2}, \end{aligned}$$

with some constant $M > 0$. *If* $\sigma_1 \neq \sigma_2$, *then we can take* $\sigma = \min\{\sigma_1, \sigma_2\}$.

Proof. By definition (1.15) and the substitution $s = 1-(1-t)\eta$, we have

$$(2.1)\quad (\Omega_1 * \Omega_2)(t) = \int_0^1 \Omega_1\left(\frac{t}{1-(1-t)\eta}\right)\Omega_2(1-(1-t)\eta)\frac{1-t}{1-(1-t)\eta}\,d\eta.$$

Since $|\Omega_1(t)| \leq M_1 t^{\sigma-1}(1-t)^{\tau_1-1}$, $|\Omega_2(t)| \leq M_2 t^{\sigma_2-1}(1-t)^{\tau_2-1}$, $1-(1-t)\eta \geq 1-\eta$, it follows that

$$\begin{aligned}|(\Omega_1 * \Omega_2)(t)| &\leq M_1M_2 t^{\sigma-1}(1-t)^{\tau_1+\tau_2-1}\int_0^1 \frac{\eta^{\tau_2-1}(1-\eta)^{\tau_1-1}}{(1-(1-t)\eta)^{\tau_1+\sigma-\sigma_2}}\,d\eta\\ &\leq M_1M_2 t^{\sigma-1}(1-t)^{\tau_1+\tau_2-1}\int_0^1 \eta^{\tau_2-1}(1-\eta)^{\sigma_2-\sigma-1}\,d\eta.\end{aligned}$$

Differentiating (2.1) leads to

$$\begin{aligned}&(\Omega_1 * \Omega_2)'(t)\\ &\quad= \int_0^1 \Omega_1'\left(\frac{t}{1-(1-t)\eta}\right)\Omega_2(1-(1-t)\eta)\frac{(1-\eta)(1-t)}{(1-(1-t)\eta)^3}\,d\eta\\ &\qquad+\int_0^1 \Omega_1\left(\frac{t}{1-(1-t)\eta}\right)\Omega_2'(1-(1-t)\eta)\frac{(1-t)\eta}{1-(1-t)\eta}\,d\eta\\ &\qquad-\int_0^1 \Omega_1\left(\frac{t}{1-(1-t)\eta}\right)\Omega_2(1-(1-t)\eta)\frac{1}{(1-(1-t)\eta)^2}\,d\eta.\end{aligned}$$

This, together with the assumption on Ω_1, Ω_2, yields

$$\begin{aligned}&|(\Omega_1 * \Omega_2)'(t)|\\ &\quad\leq M_1M_2 t^{\sigma-1}(1-t)^{\tau_1+\tau_2-1}\int_0^1 \frac{\eta^{\tau_2-1}(1-\eta)^{\tau_1-1}}{(1-(1-t)\eta)^{\tau_1+\sigma-\sigma_2}}\,d\eta\\ &\qquad+2M_1M_2 t^{\sigma-1}(1-t)^{\tau_1+\tau_2-2}\int_0^1 \frac{\eta^{\tau_2-1}(1-\eta)^{\tau_1-1}}{(1-(1-t)\eta)^{1+\tau_1+\sigma-\sigma_2}}\,d\eta\\ &\quad\leq 3M_1M_2 t^{\sigma-2}(1-t)^{\tau_1+\tau_2-2}\int_0^1 \eta^{\tau_2-1}(1-\eta)^{\sigma_2-\sigma-1}\,d\eta,\end{aligned}$$

where we have used $t \leq 1-(1-t)\eta$. In the case $\sigma_1 < \sigma_2$, the above estimate remains true if σ is replaced by σ_1. The proof is complete. $\square$

The following lemma is direct from the rewriting:

$$J_{t^{\varepsilon-1}} u(x) = x^{-\varepsilon} \int_0^x s^{\varepsilon-1} u(s)\, ds.$$

Lemma 2.3. *Let $\varepsilon > 0$, $0 < \alpha < 1$. Then the operator $J_{t^{\varepsilon-1}}$ is an isomorphism of $\mathcal{C}^\alpha(I)$ onto $\mathcal{C}^{1,\alpha}(I)$. Moreover the operator $J_{t^{\varepsilon-1}}$ is an isomorphism of $\mathcal{C}^0(I)$ onto $\mathcal{C}^1(I)$. In either case the inverse $J_{t^{\varepsilon-1}}^{-1}$ is given as $J_{t^{\varepsilon-1}}^{-1} = D_\varepsilon$.*

We shall use the Paley-Wiener theorem of the following form, which is verified by the argument in, e.g., [**1**, Chapter 4], [**8**].

Lemma 2.4. *Let $D \subset \mathbf{C}$ be a domain containing 0, and let $\phi(z)$ be a holomorphic function in D with $\phi(0) = 0$. If $\Omega \in L^1(-\infty, 0]$ satisfies $\mathcal{F}[\Omega](\xi) \in D$ for any ξ in the upper half plane $\operatorname{Im} \xi \geq 0$, then there exists a function $\Theta \in L^1(-\infty, 0]$ such that $\phi(\mathcal{F}[\Omega](\xi)) = \mathcal{F}[\Theta](\xi)$ for any $\xi \in \mathbf{R}$.*

Since the integral transform $\mathcal{K}$ defined in (1.17) is connected with the Fourier transform $\mathcal{F}$ by (1.19), Lemma 2.4 can be rewritten in terms of $\mathcal{K}$. In particular, in the case $\phi(z) = \pm(z/(1+z))$, we have the following:

Lemma 2.5. *If $\Omega \in L^1[0,1]$ satisfies $1 + \mathcal{K}[\Omega](\xi) \neq 0$ for any ξ in the upper half plane $\operatorname{Im} \xi \geq 0$, then there exist functions $\Theta_\pm \in L^1[0,1]$ such that*

$$\pm \frac{\mathcal{K}[\Omega](\xi)}{1 + \mathcal{K}[\Omega](\xi)} = \mathcal{K}[\Theta_\pm](\xi), \quad \xi \in \mathbf{R}.$$

We conclude this section with the

Proof of Lemma 1.9. Let $z = a + bi$, $a \geq 0$. In the case $b = 0$ the assertion follows directly from the assumption $\Phi(t) \geqq 0$, $\Phi(t) \not\equiv 0$. Hence, in what follows, we assume $b \neq 0$. An elementary calculation shows that

$$\begin{aligned}\operatorname{Im} \int_0^1 \Phi(t) t^z \, dt &= \int_0^1 t^a \Phi(t) \sin(b \log t) \, dt \\ &= -\int_0^\infty e^{-as} e^{-s} \Phi(e^{-s}) \sin(bs) \, ds.\end{aligned}$$

Since $t\Phi(t)$ is nondecreasing, $e^{-s}\Phi(e^{-s})$ is nonincreasing and hence the function $e^{-as}e^{-s}\Phi(e^{-s})$, which is not a constant function, is nonincreasing as well. Hence, by means of the alternating series test, $\operatorname{Im} \int_0^1 \Phi(t) t^z \, dt \lesseqgtr 0$ for $b \gtreqless 0$. The proof is complete. □

3. Proof of Theorem A. In this section we shall prove Theorem A. Throughout this section let $I = [a, b]$ be a bounded interval containing 0, and let $\mathcal{C}^j(I)$, $j = 0, 1, \dots$, denote the function space defined in (1.4).

We first prove assertion (a). Let $u \in \mathcal{C}^0(I)$. Then

$$xJ_\Phi u(x) = \int_0^x \Phi(s/x) u(s) \, ds \tag{3.1}$$

is a differentiable function with the derivative

$$\begin{aligned}\frac{d}{dx}(xJ_\Phi u(x)) &= \Phi(1)u(x) - \int_0^x \Phi'(s/x)(s/x^2) u(s) \, ds \\ &= \Phi(1)u(x) - \int_0^1 t\Phi'(t) u(xt) \, dt,\end{aligned} \tag{3.2}$$

provided that $\Phi(t) \in C(0,1] \cap C^1(0,1)$, $t\Phi'(t) \in L^1[0,1]$. Hence we have

$$x(J_\Phi u)'(x) = -J_\Phi u(x) + (xJ_\Phi u(x))' \in \mathcal{C}^0(I).$$

This proves assertion (a) in the case $n = 0$. If $\Phi(1) = 0$ then, from (3.2), we obtain

$$\frac{d}{dx}(xJ_\Phi u(x)) = \int_0^1 \Phi_1(t) u(xt) \, dt,$$

where we set $\Phi_1(t) := -t\Phi'(t)$. Hence

$$x\frac{d}{dx}(xJ_\Phi u(x)) = \int_0^x \Phi_1(s/x)u(s)\,ds$$

is a differentiable function with the derivative

$$\left(\frac{d}{dx}x\right)^2 J_\Phi u(x) = \Phi_1(1)u(x) - \int_0^1 t\Phi_1'(t)u(xt)\,dt \in \mathcal{C}^0(I),$$

provided that $\Phi_1(t) \in C(0,1] \cap C^1(0,1), t\Phi_1'(t) \in L^1[0,1]$. Similarly, by setting

$$\Phi_k(t) := \left(-\,t\frac{d}{dt}\right)^k \Phi(t), \tag{3.3}$$

it follows that for $k = 0, 1, 2, \dots,$

$$\left(\frac{d}{dx}x\right)^{k+1} J_\Phi u(x) = \Phi_k(1)u(x) + \int_0^1 \Phi_{k+1}(t)u(xt)\,dt \in \mathcal{C}^0(I), \tag{3.4}$$

provided that $\Phi_k(t) \in C(0,1] \cap C^1(0,1), \Phi_{k+1}(t) \in L^1[0,1],\ \Phi(1) = \Phi'(1) = \cdots = \Phi_{k-1}(1) = 0$. Therefore, under the assumptions in Theorem A, we have

$$\left(\frac{d}{dx}x\right) J_\Phi u(x), \dots, \left(\frac{d}{dx}x\right)^{n+1} J_\Phi u(x) \in \mathcal{C}^0(I).$$

This proves assertion (a).

We turn to the proof of assertion (b). We first treat the case $n = 0$. The following is direct from an inspection of (3.2).

Lemma 3.1. *Let $\Phi(t) \in C(0,1]$ and assume that $\Phi(t), t\Phi'(t) \in L^1[0,1]$. Then J_Φ is a bounded linear operator from $\mathcal{C}^0(I)$ to $\mathcal{C}^1(I)$, and also from $\mathcal{C}^1(I)$ to $\mathcal{C}^2(I)$.*

Let I^j denote the identity operator on $\mathcal{C}^j(I)$. The following lemma asserts that the right side of (1.6) gives an operator from $\mathcal{C}^1(I)$ to $\mathcal{C}^0(I)$.

Lemma 3.2. *Let $\varepsilon > 0$ and assume that $\Lambda(t) \in L^1[0,1]$. Then $D_\varepsilon(I^1 + J_\Lambda)$ is an operator from $\mathcal{C}^1(I)$ to $\mathcal{C}^0(I)$, where D_ε is defined by* (1.11).

Proof. It is easy to see that if $\Lambda(t) \in L^1[0,1]$ then J_Λ is an operator on $\mathcal{C}^1(I)$. On the other hand, D_ε is an operator from $\mathcal{C}^1(I)$ to $\mathcal{C}^0(I)$. This proves the lemma. □

We need the following:

Lemma 3.3. *Let $\Phi(t) \in C(0,1]$, and assume that*

$$\Phi(t), t\Phi'(t) \in L^1[0,1], \quad \lim_{t\to 0} t\Phi(t) = 0.$$

Then, as operators on $\mathcal{C}^j(I)$, $j = 0, 1, \ldots,$

$$J_{t^\varepsilon(t^{1-\varepsilon}\Phi(t))'} = \Phi(1)I^j - D_\varepsilon J_\Phi.$$

Proof. For any $u \in \mathcal{C}^j(I)$, (3.1) yields

$$\begin{aligned}(\Phi(1)I^j - D_\varepsilon J_\Phi)u &= \Phi(1)u(x) - x^{1-\varepsilon}\frac{d}{dx}x^{\varepsilon-1}\left\{\int_0^x \Phi(s/x)u(s)\,ds\right\} \\ &= (1-\varepsilon)x^{-1}\int_0^x \Phi(s/x)u(s)\,ds + \int_0^x \Phi'(s/x)(s/x^2)u(s)\,ds \\ &= \int_0^1 t^\varepsilon(t^{1-\varepsilon}\Phi(t))'u(xt)\,dt.\end{aligned}$$

This proves the lemma. □

The following result is the core of the proof of (b) for the case $n = 0$:

Proposition 3.4. *Let $\Phi(t) \in C(0,1]$, and assume that*

$$\Phi(1) \neq 0, \quad \Phi(t), t\Phi'(t) \in L^1[0,1], \quad \lim_{t\to 0} t\Phi(t) = 0.$$

Suppose that there exists a function $\Lambda(t) \in L^1[0,1]$ satisfying

$$(\Phi * \Lambda)(t) = t^{\varepsilon-1} * (t^{\varepsilon}(t^{1-\varepsilon}\Phi(t))') \quad \text{for almost every } t \in [0,1]. \tag{3.5}$$

Then J_Φ is an isomorphism of $\mathcal{C}^0(I)$ onto $\mathcal{C}^1(I)$. The inverse of J_Φ is given by $J_\Phi^{-1} = \Phi(1)^{-1} D_\varepsilon (I^1 + J_\Lambda)$.

Proof. Because $\Lambda(t) \in L^1[0,1]$, J_Λ is an operator on $\mathcal{C}^1(I)$. Hence it follows from Lemmas 2.1, 3.3, and 2.3 that

$$J_\Phi J_\Lambda = J_{t^{\varepsilon-1}} J_{t^{\varepsilon}(t^{1-\varepsilon}\Phi(t))'} = J_{t^{\varepsilon-1}}(\Phi(1) I^1 - D_\varepsilon J_\Phi) = \Phi(1) J_{t^{\varepsilon-1}} I^1 - J_\Phi,$$

as operators from $\mathcal{C}^1(I)$ to $\mathcal{C}^2(I)$. By Lemmas 2.1 and 2.3 we have

$$J_\Phi D_\varepsilon = D_\varepsilon J_{t^{\varepsilon-1}} J_\Phi D_\varepsilon = D_\varepsilon J_\Phi J_{t^{\varepsilon-1}} D_\varepsilon = D_\varepsilon J_\Phi$$

as operators on $\mathcal{C}^1(I)$. These equalities yield

$$\begin{aligned} J_\Phi[\Phi(1)^{-1} D_\varepsilon (I^1 + J_\Lambda)] &= \Phi(1)^{-1} D_\varepsilon (J_\Phi + J_\Phi J_\Lambda) \\ &= \Phi(1)^{-1} D_\varepsilon \Phi(1) J_{t^{\varepsilon-1}} I^1 = I^1. \end{aligned}$$

Similarly we have $J_\Lambda J_\Phi = \Phi(1) J_{t^{\varepsilon-1}} I^0 - J_\Phi$, which leads to

$$\begin{aligned} [\Phi(1)^{-1} D_\varepsilon (I^1 + J_\Lambda)] J_\Phi &= \Phi(1)^{-1} D_\varepsilon (J_\Phi + J_\Lambda J_\Phi) \\ &= \Phi(1)^{-1} D_\varepsilon \Phi(1) J_{t^{\varepsilon-1}} I^0 = I^0. \end{aligned}$$

The proof is complete. □

In view of Proposition 3.4, the proof of (b) in the case of $n = 0$ is reduced to finding a function $\Lambda(t) \in L^1[0,1]$ satisfying (3.5). For this we set

$$\rho(t) := -\Phi(1)^{-1} t^{\varepsilon} (t^{1-\varepsilon}\Phi(t))'. \tag{3.6}$$

Integrating by parts we have for $\operatorname{Im}\xi \geq 0$,

$$\begin{aligned} \mathcal{K}[\rho](\xi) &= -\Phi(1)^{-1} \int_0^1 t^{\varepsilon - i\xi} (t^{1-\varepsilon}\Phi(t))' \, dt \\ &= -1 + \Phi(1)^{-1} (\varepsilon - i\xi) \mathcal{K}[\Phi](\xi), \end{aligned}$$

which yields

$$(\varepsilon - i\xi)\mathcal{K}[\Phi](\xi) = \Phi(1)\{1 + \mathcal{K}[\rho](\xi)\}, \quad \operatorname{Im}\xi \geq 0. \tag{3.7}$$

Assumption (★) implies that this function does not vanish in the upper half plane $\operatorname{Im}\xi \geq 0$. Hence, by Lemma 2.5, there exists a function $\Lambda(t) \in L^1[0,1]$ such that

$$-\frac{\mathcal{K}[\rho](\xi)}{1 + \mathcal{K}[\rho](\xi)} = \mathcal{K}[\Lambda](\xi), \quad \xi \in \mathbf{R}. \tag{3.8}$$

By Lemma 2.1 (a), (3.7), (3.8) and (3.6), we have

$$\begin{aligned} \mathcal{K}[\Phi * \Lambda](\xi) &= \mathcal{K}[\Phi](\xi)\mathcal{K}[\Lambda](\xi) \\ &= -\frac{\Phi(1)}{\varepsilon - i\xi}\mathcal{K}[\rho](\xi) \\ &= \frac{1}{\varepsilon - i\xi}\mathcal{K}[t^{\varepsilon}(t^{1-\varepsilon}\Phi(t))'](\xi) \\ &= \mathcal{K}[t^{\varepsilon-1} * (t^{\varepsilon}(t^{1-\varepsilon}\Phi(t))'](\xi), \end{aligned} \tag{3.9}$$

where $\mathcal{K}[t^{\varepsilon-1}](\xi) = (\varepsilon - i\xi)^{-1}$ is used in the last equality. This shows that Λ satisfies (3.5). In view of Proposition 3.4, in the case $n = 0$, the proof of the assertion (b) is complete.

We shall treat the case $n = 1, 2, \ldots$. To show the injectivity of J_Φ, we assume that $J_\Phi u = 0$ for $u \in \mathcal{C}^0(I)$. Then assumption (i) and (3.4) with $k = n - 1$ yield

$$\int_0^1 \Phi_n(t)u(xt)\,dt = 0.$$

But it follows from (3.3), repeated integration by parts, and assumption (ii) that

$$\begin{aligned} \int_0^1 \Phi_n(t)t^z\,dt &= -\int_0^1 \Phi'_{n-1}(t)t^{z+1}\,dt = (z+1)\int_0^1 \Phi_{n-1}(t)t^z\,dt \\ &= \cdots = (z+1)^n \int_0^1 \Phi(t)t^z\,dt \quad \operatorname{Re} z \geq 0. \end{aligned} \tag{3.10}$$

So condition (★) remains valid if $\Phi(t)$ is replaced by $\Phi_n(t)$. Hence we can apply the assertion (b) for $n = 0$, which has been established already, to conclude that $u = 0$.

To show the surjectivity of J_Φ, let us consider the equation

$$\int_0^1 \Phi_n(t)u(xt)\,dt = \left(\frac{d}{dx}x\right)^n f(x), \quad f \in \mathcal{C}^{n+1}(I). \tag{3.11}$$

It is easy to see from assumption (ii) that $\Phi_n(t)$ satisfies

$$|\Phi_n(t)|, |t\Phi_n'(t)| \leq Mt^{\varepsilon-1}.$$

Moreover $\Phi_n(t)$ satisfies condition (★). Hence, in view of assertion (b) for $n = 0$, equation (3.11) admits a solution $u \in \mathcal{C}^0(I)$. Put

$$U(x) := \int_0^1 \Phi_{n-1}(t)u(xt)\,dt - \left(\frac{d}{dx}x\right)^{n-1} f(x).$$

Then (3.4) yields

$$\frac{d}{dx}\{xU(x)\} = \int_0^1 \Phi_n(t)u(xt)\,dt - \left(\frac{d}{dx}x\right)^n f(x) = 0.$$

Therefore $xU(x) = C$, where C is a constant. But assumption (ii) and $f \in \mathcal{C}^{n+1}(I)$ imply $U(x) \in \mathcal{C}^0(I)$, forcing $C = 0$. Hence the solution u of (3.11) satisfies the equation

$$\int_0^1 \Phi_{n-1}(t)u(xt)dt = \left(\frac{d}{dx}x\right)^{n-1} f(x).$$

By repeating this procedure we conclude that u satisfies the equation $J_\Phi u = f$. The proof of Theorem A is complete. □

Remark 3.5. An inspection of the proof shows that

$$\Lambda(t) = \mathcal{K}^{-1}\left(\frac{\Phi(1) - (\varepsilon - i\xi)\mathcal{K}[\Phi](\xi)}{(\varepsilon - i\xi)\mathcal{K}[\Phi](\xi)}\right).$$

Remark 3.6. By the argument in the proof of Theorem A, see (3.4), solving the integral equation (1.1) under assumptions (i) and (ii) is reduced to that of

$$\Phi_n(1)u(x) + \int_0^1 \Phi_{n+1}(t)u(xt)\,dt = \left(\frac{d}{dx}x\right)^{n+1} f(x).$$

This equation can be rewritten as

$$(I + A_n)u = g, \tag{3.12}$$

where

$$\begin{aligned} A_n u(x) &:= \Phi_n(1)^{-1} \int_0^1 \Phi_{n+1}(t)u(xt)\,dt, \\ g(x) &:= \Phi_n(1)^{-1} \left(\frac{d}{dx}x\right)^{n+1} f(x). \end{aligned} \tag{3.13}$$

Since g varies in $\mathcal{C}^0(I)$ as f varies in $\mathcal{C}^{n+1}(I)$, we conclude that solving (1.1) is equivalent to showing that (3.12) has a unique solution u in $\mathcal{C}^0(I)$ for each $g \in \mathcal{C}^0(I)$.

4. Use of the Neumann series. By Remark 3.6 the operator J_Φ is an isomorphism of $\mathcal{C}^0(I)$ onto $\mathcal{C}^{n+1}(I)$ if and only if the operator $I + A_n$ defined by (3.12) is an isomorphism in $\mathcal{C}^0(I)$. In view of Theorem A (b) the former condition is equivalent to (★), while a sufficient condition for the latter is given by

$$\|A_n\| < 1, \tag{4.1}$$

where $\|A_n\|$ is the operator norm of A_n in $\mathcal{C}^0(I)$. Indeed, if (4.1) is satisfied then the Neumann series $I - A_n + A_n^2 - \cdots$ converges and gives the inverse of $I + A_n$, Hence (4.1) gives a sufficient condition for (★). This observation naturally leads us to the question: to what extent does (4.1) cover the necessary and sufficient condition (★) for the unique solvability of (1.1) ? If (4.1) were completely equivalent to (★), then the equation (1.1) could always be solved by means of the Neumann series. However, there actually exists a gap between these two conditions. To make this point transparent, we shall first obtain a necessary and sufficient condition for (4.1) in terms of $\Phi(t)$, see Theorem 4.1, and then apply it to Example 1.4 to illustrate this gap clearly.

Theorem 4.1. *Let* $\Phi_k(t)$ *be defined by* (3.3). *Then a necessary and sufficient condition for* (4.1) *is given by*

$$\int_0^1 |\Phi_{n+1}(t)|\,dt < |\Phi_n(1)|. \tag{4.2$_n$}$$

Proof. Let $\Omega(t) \in C[0,1]$ and consider the operator $B : \mathcal{C}^0(I) \to \mathcal{C}^0(I)$ defined by

$$(Bu)(x) = \int_0^1 \Omega(t) u(xt)\,dt.$$

To establish the theorem it suffices to prove that the operator norm $||B||$ of B in $\mathcal{C}^0(I)$ is expressed as

$$||B|| = \int_0^1 |\Omega(t)|\,dt, \tag{4.3}$$

since the theorem follows immediately by applying (4.3) to $\Omega(t) = \Phi_n(1)^{-1}\Phi_{n+1}(t)$. Since $\mathcal{C}^0(I)$ is dense in $L^\infty(I)$, the operator norm of $B : \mathcal{C}^0(I) \to \mathcal{C}^0(I)$ is equal to that of $B : L^\infty(I) \to L^\infty(I)$. So we shall calculate the latter norm instead of the former.

First we have for $u(x) \in L^\infty(I)$,

$$|(Bu)(x)| \le \int_0^1 |\Omega(t)||u(xt)|\,dt \le ||u||_{L^\infty} \int_0^1 |\Omega(t)|\,dt, \quad x \in I,$$

which yields

$$||Bu||_{L^\infty} \le ||u||_{L^\infty} \int_0^1 |\Omega(t)|\,dt.$$

Hence,

$$||B|| \le \int_0^1 |\Omega(t)|\,dt.$$

It is therefore enough to establish the reverse inequality. For any $u(x) \in L^\infty(I)$, the function Bu is rewritten as

$$(Bu)(x) = x^{-1} \int_0^x \Omega(t/x) u(t)\, dt.$$

It follows from the assumption $\Omega(t) \in C[0,1]$ and Lebesgue's convergence theorem that the function Bu is continuous at each point $y \in I \setminus \{0\}$. Hence we have

$$|(Bu)(y)| \le ||Bu||_{L^\infty} \le ||B|| ||u||_{L^\infty}, \quad y \in I \setminus \{0\}. \tag{4.4}$$

Let $y \in I \setminus \{0\}$ be fixed and let $u(x)$ be a function in $L^\infty(I)$ defined by

$$u(x) := \begin{cases} \operatorname{sgn} \Omega(x/y) & 0 \le (\operatorname{sgn} y)x \le |y|, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\operatorname{sgn} x := \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1, & x < 0. \end{cases}$$

Then clearly $||u||_{L^\infty} = 1$ and $u(yt) = \operatorname{sgn} \Omega(t)$ for $0 \le t \le 1$. Therefore,

$$(Bu)(y) = \int_0^1 \Omega(t) u(yt)\, dt = \int_0^1 \Omega(t) \operatorname{sgn} \Omega(t)\, dt = \int_0^1 |\Omega(t)|\, dt.$$

Substituting this into (4.4), we obtain $\int_0^1 |\Omega(t)|\, dt \le ||B||$, as required. The proof is complete. □

The rest of this section is devoted to a thorough investigation of Example 1.4, where $\Phi(t)$ is given by

$$\Phi(t) = 1 + pt + qt^2, \quad (p, q) \in \mathbf{R}^2. \tag{4.5}$$

We shall examine how Theorems A and 4.1 can be applied to the equation (1.7) in the example. The discussion will clearly illustrate the gap between the conditions (★) and $(4.2)_n$ in these theorems.

We begin with Theorem A. For the function $\Phi(t)$ in (4.5), assumption (ii) in Theorem A is satisfied for each nonnegative integer n. Moreover $\Phi_k(t)$ in (3.3) is computed as

$$\Phi_k(t) = \begin{cases} 1 + pt + qt^2 & k = 0, \\ (-1)^k t(p + 2^k qt) & k = 1, 2, 3, \dots, \end{cases} \tag{4.6}$$

which in particular leads to

$$\Phi_k(1) = \begin{cases} 1 + p + q & k = 0, \\ (-1)^k (p + 2^k q), & k = 1, 2, 3, \dots. \end{cases} \tag{4.7}$$

Therefore we easily observe that assumption (i) in Theorem A is satisfied if and only if

$$(4.8)_0 \qquad p + q + 1 \neq 0 \qquad \text{for} \quad n = 0,$$

$$(4.8)_1 \qquad p + q + 1 = 0,\ (p, q) \neq (-2, 1) \qquad \text{for} \quad n = 1,$$

$$(4.8)_2 \qquad (p, q) = (-2, 1) \qquad \text{for} \quad n = 2,$$

respectively. For $n \geq 3$ the assumption is not satisfied for any $(p, q) \in \mathbf{R}^2$. As for condition (★), a calculation shows that

$$\int_0^1 \Phi(t) t^z \, dt = \frac{az^2 + bz + c}{(z + 1)(z + 2)(z + 3)},$$

where we set

$$a = p + q + 1, \quad b = 4p + 3q + 5, \quad c = 3p + 2q + 6.$$

Accordingly (★) is equivalent to the condition that the quadratic equation $az^2+bz+c = 0$ has no roots with nonnegative real parts. This condition is satisfied if and only if one of the following three conditions holds:

(0) $a, b, c > 0$ or $a, b, c < 0$;
(1) $a = 0,\ bc > 0$;
(2) $a = b = 0,\ c \neq 0$,

which are rewritten as

$$(4.9)_0 \qquad \begin{gathered} p + q + 1 > 0,\ 3p + 2q + 6 > 0,\ 4p + 3q + 5 > 0 \\ \text{or} \quad p + q + 1 < 0,\ 3p + 2q + 6 < 0; \end{gathered}$$

$$(4.9)_1 \qquad p+q+1=0,\ p<-4 \quad \text{or} \quad p+q+1=0,\ p>-2;$$

$$(4.9)_2 \qquad (p,q)=(-2,1),$$

respectively. For each $n = 0, 1, 2$, $(4.9)_n$ gives a necessary and sufficient condition for (★) in the case $(4.8)_n$; thereby the conclusion in Example 1.4 follows.

We proceed to Theorem 4.1. For each $n = 0, 1, 2$, we shall determine those parameters (p, q) which satisfy condition $(4.2)_n$ in the case $(4.8)_n$. For $n = 0$, by using (4.6) and (4.7), condition $(4.2)_0$ is expressed as

$$(4.10)_0 \qquad \int_0^1 t|p+2qt|\,dt < |p+q+1|.$$

If the function $p+2qt$ does not change its sign in the interval $0 \le t \le 1$, that is, if either $p \ge 0, p+2q \ge 0$ or $p \le 0, p+2q \le 0$ holds, then $(4.10)_0$ is equivalent to

$$(4.10)_{01} \qquad |3p+4q| < 6|p+q+1|.$$

If $p+2qt$ changes its sign in $0 \le t \le 1$, that is, if either $p > 0, p+2q < 0$ or $p < 0, p+2q > 0$ holds, then $(4.10)_0$ is equivalent to

$$(4.10)_{02} \qquad |p|^3 + (p+2q)^2|p-4q| < 24q^2|p+q+1|.$$

By checking conditions $(4.10)_{01}$ and $(4.10)_{02}$ in more detail separately and then by summarizing the results, it follows that $(4.2)_0$ holds if and only if (p, q) lies in the following region:

$$(4.11)_0 \qquad q \ge \begin{cases} g_+(p) & (p \le -(3/2)), \\ -(9/10)p-(3/5) & (-(3/2) \le p \le 0), \\ f_-(p) & (p \ge 0), \end{cases}$$

or

$$q \le \begin{cases} f_+(p) & (p \le -3), \\ -(3/2)p-3 & (-3 \le p \le 0), \\ g_-(p) & (p \ge 0), \end{cases}$$

where $f_+(p)$ and $f_-(p)$ are defined to be the maximal and minimal real roots of the cubic equation:

$$F(p,q) := 20q^3 + 18pq^2 + 12q^2 - p^3 = 0,$$

$g_+(p)$ and $g_-(p)$ being defined in the same manner from the equation:

$$G(p,q) := 4q^3 + 6pq^2 + 12q^2 + p^3 = 0.$$

Here we wish to point out that the polynomials $F(p,q)$ and $G(p,q)$ are obtained by choosing suitable signs in $(4.10)_{02}$.

Secondly, for $n = 1$, condition $(4.2)_1$ is rewritten as

$$(4.10)_1 \qquad \int_0^1 t|p + 4qt|\, dt < |p + 2q|.$$

If the function $p+4qt$ does not change its sign in the interval $0 \le t \le 1$, that is, if either $p \ge 0, p+4q \ge 0$ or $p \le 0, p+4q \le 0$ holds, then $(4.10)_1$ is equivalent to

$$(4.10)_{11} \qquad |3p + 8q| < 6|p + 2q|.$$

If $p+4qt$ changes its sign in $0 \le t \le 1$, that is, if either $p > 0, p+4q < 0$ or $p < 0, p + 4q > 0$ holds, then $(4.10)_1$ is equivalent to

$$(4.10)_{12} \qquad |p|^3 + (p + 4q)^2|p - 8q| < 96q^2|p + 2q|.$$

A thorough check of these conditions, upon taking $(4.8)_1$ into account, shows that $(4.2)_1$ holds if and only if (p,q) lies on the half lines:

$$(4.11)_1 \qquad p + q + 1 = 0,\ p < p_1 \quad \text{or} \quad p + q + 1 = 0,\ p > p_2,$$

where $p_1 = -4.97403\ldots$ is the (unique) real root of the cubic equation

$$F(p) := 7p^3 + 48p^2 + 72p + 32 = 0,$$

and $p_2 = -1.70484\ldots$ is the minimal real root of

$$G(p) := 89p^3 + 336p^2 + 408p + 160 = 0.$$

Here the polynomials $F(p)$ and $G(p)$ are obtained by choosing suitable signs in $(4.10)_{12}$.

Finally, for $n = 2$, condition $(4.2)_2$ is expressed as

$$(4.10)_2 \qquad \int_0^1 t|p + 8qt|\,dt < |p + 4q|.$$

Since we are considering the case $(4.8)_2$, we have only to verify this condition for $(p, q) = (-2, 1)$, which can easily be done by a direct computation. It follows that condition $(4.2)_2$ becomes

$$(4.11)_2 \qquad (p, q) = (-2, 1).$$

In conclusion, for each $n = 0, 1, 2$, we have shown that in the case $(4.8)_n$ conditions ($\bigstar$) and $(4.2)_n$ become $(4.9)_n$ and $(4.11)_n$, respectively. Let X_n and Y_n be the set of parameters (p, q) satisfying conditions $(4.9)_n$ and $(4.11)_n$, respectively. Then $X_2 = Y_2$ for $n = 2$, while Y_n is a *proper* subset of X_n for $n = 0, 1$. For $n = 0$, the gap between X_0 and Y_0 is clearly indicated in Figure 1, where Y_0 and $X_0 \setminus Y_0$ are given by the thinly and thickly dotted regions, respectively.

5. Proof of Theorem B. In this section we shall prove Theorem B. Let $I = [a, b]$ be a bounded interval containing 0 and let $\mathcal{C}^\alpha(I)$, $0 \le \alpha \le 1$, denote the function space defined in (1.9).

Assertion (a) of Theorem B follows immediately from the following:

Lemma 5.1. *Let* $0 < \delta < 1$, $0 \le \alpha < 1 - \delta$, *let* $\Phi(t) \in C^1(0, 1)$ *and assume that*

$$|\Phi(t)|,\ |t(1-t)\Phi'(t)| \le M t^{\lambda-1}(1-t)^{\delta-1}$$

with some constants $M, \lambda > 0$. *Then* J_Φ *is a bounded linear operator from* $\mathcal{C}^\alpha(I)$ *to* $\mathcal{C}^{\alpha+\delta}(I)$.

Proof. Let $u \in \mathcal{C}^\alpha(I)$, and set $f := J_\Phi u$. For the proof it is enough to show that the absolute value $|y^{\alpha+\delta} f(y) - x^{\alpha+\delta} f(x)|$ is bounded by $M||u||_\alpha |y - x|^{\alpha+\delta}$ with some constant M. We only consider the case

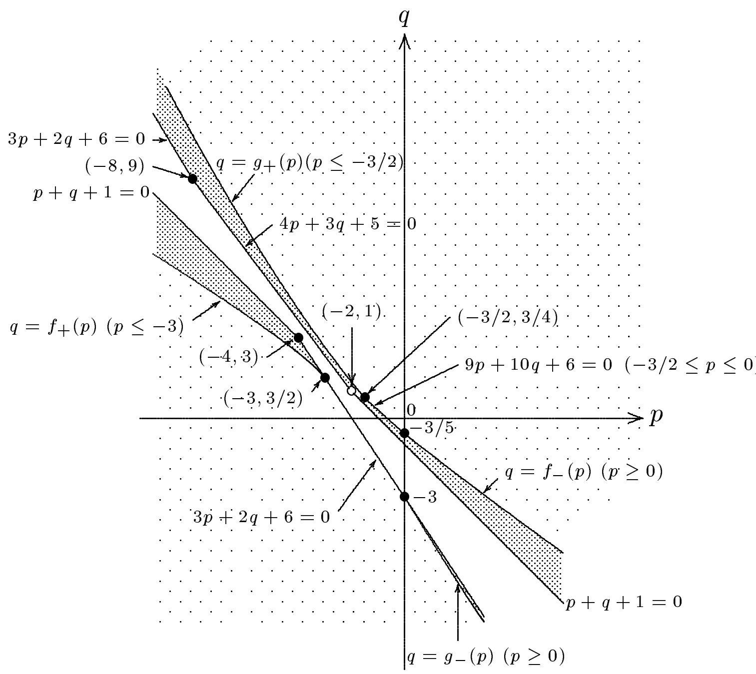

FIGURE 1.

$0 < x \leq y$, because the remaining cases can be treated similarly. A calculation shows that

$$\begin{aligned} y^{\alpha+\delta} f(y) - x^{\alpha+\delta} f(x) &= y^{\alpha+\delta-1} \int_0^y \Phi(s/y) u(s)\, ds \\ &\quad - x^{\alpha+\delta-1} \int_0^x \Phi(s/x) u(s)\, ds \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 := y^{\alpha+\delta-1} \int_x^y \Phi(s/y)(u(s) - u(x))\, ds;$$

$$I_2 := \int_0^x \{y^{\alpha+\delta-1}\Phi(s/y) - x^{\alpha+\delta-1}\Phi(s/x)\}(u(s) - u(x))\, ds;$$

$$I_3 := (y^{\alpha+\delta} - x^{\alpha+\delta})\, u(x) \int_0^1 \Phi(t)\, dt.$$

By definition (1.9) we easily deduce $|u(x)| \leq ||u||_\alpha$ for any $x \in [0,1]$. It is easily seen from this estimate that the term I_3 is bounded by $M||u||_\alpha |y-x|^{\alpha+\delta}$ with some constant M. Moreover for $0 \leq x < y$, we have from (1.9)

$$\begin{aligned}
|u(y) - u(x)| &= \left| u(y) - \frac{x^\alpha}{y^\alpha} u(x) - u(x)\left(1 - \frac{x^\alpha}{y^\alpha}\right)\right| \\
&\leq \frac{1}{y^\alpha}|y^\alpha u(y) - x^\alpha u(x)| + |u(x)|\left|1 - \frac{x^\alpha}{y^\alpha}\right| \\
&\leq \frac{||u||_\alpha}{y^\alpha}|y-x|^\alpha + C'||u||_\alpha \left|1 - \frac{x}{y}\right|^\alpha \\
&\leq C\left|1 - \frac{x}{y}\right|^\alpha,
\end{aligned}$$

where C is a constant independent of x, y. Hence, by the substitution $s = x + \eta(y-x)$, we have

$$\begin{aligned}
|I_1| &\leq C||u||_\alpha \int_0^1 \{\eta + (x/y)(1-\eta)\}^{\lambda-1-\alpha}\eta^\alpha(1-\eta)^{\delta-1}\, d\eta (y-x)^{\alpha+\delta} \\
&\leq M||u||_\alpha |y-x|^{\alpha+\delta},
\end{aligned}$$

with some constant $M > 0$. Changing the variables via the substitutions $r = x/y$, $s = xt$, and $t = \sigma/(1-\eta(1-\sigma))$, we obtain

$$\begin{aligned}
|I_2| &\leq M||u||_\alpha \int_0^x |y^{\alpha+\delta-1}\Phi(s/y) - x^{\alpha+\delta-1}\Phi(s/x)|(1-(s/x))^\alpha \, ds \\
&= M||u||_\alpha y^{\alpha+\delta} r \int_0^1 |\Phi(rt) - r^{\alpha+\delta-1}\Phi(t)|(1-t)^\alpha \, dt \\
&= M||u||_\alpha y^{\alpha+\delta} r \int_0^1 \left| \int_r^1 \frac{d}{d\eta}\left\{\Phi(\eta t) - \eta^{\alpha+\delta-1}\Phi(t)\right\} d\eta \right| (1-t)^\alpha \, dt \\
&\leq M||u||_\alpha y^{\alpha+\delta} r \int_0^1 (1-t)^\alpha \, dt \\
&\quad \cdot \int_r^1 \{t|\Phi'(\eta t)| + (1-\alpha-\delta)\eta^{\alpha+\delta-2}|\Phi(t)|\}\, d\eta \\
&\leq M||u||_\alpha y^{\alpha+\delta} r \int_0^1 t^{\lambda-1}(1-t)^\alpha \, dt
\end{aligned}$$

$$\begin{aligned}
&\cdot \int_r^1 \eta^{\lambda-2}(1-\eta t)^{\delta-2} d\eta + M||u||_\alpha y^{\alpha+\delta} r \int_0^1 t^{\lambda-1}(1-t)^{\alpha+\delta-1}\, dt \\
&\cdot \int_r^1 \eta^{\alpha+\delta-2}\, d\eta \le M||u||_\alpha y^{\alpha+\delta} r \int_r^1 \eta^{\lambda-2}\, d\eta \\
&\cdot \int_0^1 t^{\lambda-1}(1-t)^{\alpha}(1-\eta t)^{\delta-2}\, dt + M||u||_\alpha y^{\alpha+\delta}(r^{\alpha+\delta}-r) \\
&= M||u||_\alpha y^{\alpha+\delta} r \int_r^1 \eta^{\lambda-2}(1-\eta)^{\alpha+\delta-1}\, d\eta \\
&\cdot \int_0^1 \frac{\sigma^{\lambda-1}(1-\sigma)^{\alpha}}{(1-\eta(1-\sigma))^{\alpha+\delta+\lambda-1}}\, d\sigma \\
&+ M||u||_\alpha (y-x)^{\alpha+\delta} r^{\alpha} \frac{r^{\alpha+\delta}-r}{(1-r)^{\alpha+\delta}} \\
&\le M||u||_\alpha y^{\alpha+\delta} r \int_r^1 \eta^{\lambda-2}(1-\eta)^{\alpha+\delta-1}\, d\eta \\
&\cdot \int_0^1 \sigma^{-\alpha-\delta}(1-\sigma)^{\alpha} d\sigma + M||u||_\alpha (y-x)^{\alpha+\delta},
\end{aligned}$$

where we have used the assumption $\alpha + \delta < 1$. Since there exists a constant C independent of $r \in [0,1]$ such that

$$r \int_r^1 \eta^{\lambda-2}(1-\eta)^{\alpha+\delta-1}\, d\eta \le C(1-r)^{\alpha+\delta},$$

the term I_2 can be estimated as $|I_2| \le M||u||_\alpha |y-x|^{\alpha+\delta}$. The proof is complete. □

Let $\mathcal{C}^{1,\alpha}(I)$ be the function space defined in (1.12). Then we have:

Lemma 5.2. *Under the same assumption as in Lemma* 5.1, $J_\Phi D_\varepsilon = D_\varepsilon J_\Phi$ *as operators from* $\mathcal{C}^{1,\alpha}(I)$ *to* $\mathcal{C}^{\alpha+\delta}(I)$.

Proof. The same argument as in the proof of Lemma 5.1 implies that J_Φ is a bounded linear operator from $\mathcal{C}^{1,\alpha}(I)$ to $\mathcal{C}^{1,\alpha+\delta}(I)$. This fact, together with Lemmas 5.1, 2.3, and 2.1 (b), shows that

$$J_\Phi D_\varepsilon = D_\varepsilon J_{t^{\varepsilon-1}} J_\Phi D_\varepsilon = D_\varepsilon J_\Phi J_{t^{\varepsilon-1}} D_\varepsilon = D_\varepsilon J_\Phi.$$

The proof is complete. □

The following result gives sufficient conditions for the correspondence $f \mapsto u$ in (1.10) to be a bounded linear operator from $\mathcal{C}^{\alpha+\delta}(I)$ to $\mathcal{C}^{\alpha}(I)$.

Lemma 5.3. *Let* $0 < \delta < 1$ *and* $0 < \alpha \le 1 - \delta$.

(a) *If* $\Psi(t)$ *belongs to* $C^1(0,1)$ *and satisfies*

$$|\Psi(t)|,\, |t(1-t)\Psi'(t)| \le M t^{\mu-1}(1-t)^{-\delta} \tag{5.1}$$

with some constants $M > 0$, $\mu \in (0,1)$, *then* $D_\varepsilon J_\Psi$ *is a bounded linear operator from* $\mathcal{C}^{\alpha+\delta}(I)$ *to* $\mathcal{C}^{\alpha}(I)$.

(b) *If* $\Psi(t)$ *belongs to* $C^1(0,1]$ *and satisfies*

$$\Psi(t),\, t\Psi'(t) \in L^1[0,1], \tag{5.2}$$

then $D_\varepsilon J_\Psi$ *is a bounded linear operator from* $\mathcal{C}^{\alpha+\delta}(I)$ *to* $\mathcal{C}^{\alpha}(I)$.

Proof. We shall prove only assertion (a) because the proof of (b) is easier. It easily follows from definition (1.11) that D_ε is a bounded linear operator from $\mathcal{C}^{1,\alpha}(I)$ to $\mathcal{C}^{\alpha}(I)$. Hence it suffices to show that J_Ψ is an operator from $\mathcal{C}^{\alpha+\delta}(I)$ to $\mathcal{C}^{1,\alpha}(I)$. Let $f \in \mathcal{C}^{\alpha+\delta}(I)$ and set $U = J_\Psi f$. By the definition of $\mathcal{C}^{\alpha+\delta}(I)$ we have

$$|f(\eta) - f(\eta t)| \le C||f||_{\alpha+\delta}(1-t)^{\alpha+\delta}, \quad \eta \in I,\ 0 < t \le 1, \tag{5.3}$$

with some constant $C > 0$. By (5.1) and (5.3) we can use integration by parts to obtain for a fixed $x_0 \ne 0$,

$$\begin{aligned} U(x) &= \int_0^1 t\Psi(t)\left(\frac{f(xt)}{t} - \frac{f(x_0t)}{t}\right) dt + U(x_0) \\ &= \int_0^1 t\Psi(t)\frac{d}{dt}\left\{\int_{x_0}^x \eta^{-1}(f(\eta t) - f(\eta))d\eta\right\} dt + U(x_0) \\ &= \int_0^1 (t\Psi(t))'\, dt \int_{x_0}^x \eta^{-1}(f(\eta) - f(\eta t))\, d\eta + U(x_0). \end{aligned}$$

From this equality it follows that $U(x)$ is a differentiable function with the derivative

$$U'(x) = x^{-1}\int_0^1 (t\Psi(t))'(f(x) - f(xt))\, dt.$$

Hence Lebesgue's convergence theorem, together with (5.1), (5.3) and $\alpha > 0$, implies that $xU'(x) \in B(I \setminus \{0\})$. Moreover we have for $0 < x \le y$,

$$
\begin{aligned}
y^{\alpha+1}U'(y) - x^{\alpha+1}U'(x) &= y^{\alpha}\int_0^1 (t\Psi(t))'(f(y)-f(yt))\,dt \\
&\quad - x^{\alpha}\int_0^1 (t\Psi(t))'(f(x)-f(xt))\,dt \\
&= y^{\alpha}\int_{x/y}^1 (t\Psi(t))'(f(y)-f(yt))\,dt \\
&\quad - x^{\alpha}\int_{x/y}^1 (t\Psi(t))'(f(x)-f(xt))\,dt \\
&\quad + \int_0^{x/y} (t\Psi(t))'(y^{\alpha}f(y)-x^{\alpha}f(x))\,dt \\
&\quad - \int_0^{x/y} (t\Psi(t))'(y^{\alpha}f(yt)-x^{\alpha}f(xt))\,dt. \qquad (5.4)
\end{aligned}
$$

Using (5.1) and (5.3) we obtain

$$
\begin{aligned}
&\left| y^{\alpha}\int_{x/y}^1 (t\Psi(t))'(f(y)-f(yt))dt \right| \\
&\qquad \le M||f||_{\alpha+\delta}\, y^{\alpha}\int_{x/y}^1 t^{\mu-1}(1-t)^{\alpha-1}\,dt \\
&\qquad = M||f||_{\alpha+\delta}(y-x)^{\alpha}\int_0^1 \left\{1-\tau\left(1-\frac{x}{y}\right)\right\}^{\mu-1}\tau^{\alpha-1}\,d\tau \\
&\qquad \le M||f||_{\alpha+\delta}(y-x)^{\alpha}\int_0^1 \tau^{\alpha-1}(1-\tau)^{\mu-1}\,d\tau \\
&\qquad \le M||f||_{\alpha+\delta}(y-x)^{\alpha},
\end{aligned}
$$

with some constant M. The second term in the right side of (5.4) can be estimated in a similar manner. It follows from (1.9) that

$$\begin{aligned}|y^\alpha f(y) - x^\alpha f(x)| &= \left|\frac{1}{y^\delta}(y^{\alpha+\delta}f(y) - x^{\alpha+\delta}f(x)) - x^\alpha(1-(x/y)^\delta)f(x)\right| \\ &\le ||f||_{\alpha+\delta}\frac{1}{y^\delta}(y-x)^{\alpha+\delta} + C'x^\alpha||f||_{\alpha+\delta}\left(1-\frac{x}{y}\right) \\ &\le C||f||_{\alpha+\delta}(y-x)^\alpha\left(1-\frac{x}{y}\right)^\delta,\end{aligned}$$

with some constant C, which leads to

$$\begin{aligned}&\left|\int_0^{x/y}(t\Psi(t))'(y^\alpha f(y) - x^\alpha f(x))\,dt\right| \\ &\qquad \le M||f||_{\alpha+\delta}(y-x)^\alpha\left(1-\frac{x}{y}\right)^\delta\int_0^{x/y}t^{\mu-1}(1-t)^{-\delta-1}\,dt \\ &\qquad \le M||f||_{\alpha+\delta}(y-x)^\alpha,\end{aligned}$$

where we have used the fact that $\sup_{0\le r<1}(1-r)^\delta\int_0^r t^{\mu-1}(1-t)^{-\delta-1}\,dt < \infty$. The last term in the right side of (5.4) can be estimated in a similar manner. The proof is complete. □

The following is a core of the proof of Theorem B.

Proposition 5.4. *Let $0<\delta<1$ and $0<\alpha<1-\delta$. Let $\Phi(t)\in C^1(0,1)$ and assume that*

$$|\Phi(t)|,\ |t(1-t)\Phi'(t)| \le Mt^{\lambda-1}(1-t)^{\delta-1}$$

with some constants $M,\lambda>0$. Suppose that there exists a function $\Psi(t)$ satisfying the following conditions:

(i) *$\Psi(t)$ can be decomposed as*

$$\Psi(t) = \Psi_1(t) + \Psi_2(t),$$

where $\Psi_1(t)\in C^1(0,1)$ and $\Psi_2(t)\in C^1(0,1]$ satisfy (5.1) and (5.2), respectively.

(ii) *There exists a constant* $\varepsilon > 0$ *such that for almost every* $t \in [0,1]$,

$$(\Phi * \Psi)(t) = t^{\varepsilon - 1}.$$

Then J_Φ *is an isomorphism of* $\mathcal{C}^\alpha(I)$ *onto* $\mathcal{C}^{\alpha+\delta}(I)$. *The inverse of* J_Φ *is given by* $J_\Phi^{-1} = D_\varepsilon J_\Psi$.

Proof. It follows from Lemmas 5.1 and 5.3 that J_Φ and J_Ψ are bounded linear operators from $\mathcal{C}^\alpha(I)$ to $\mathcal{C}^{\alpha+\delta}(I)$ and from $\mathcal{C}^{\alpha+\delta}(I)$ to $\mathcal{C}^{1,\alpha}(I)$, respectively. By assumption (ii) and Lemma 2.1 (b), we obtain $J_\Phi J_\Psi = J_{t^{\varepsilon-1}}$. This, together with Lemmas 5.2 and 2.3, shows that

$$J_\Phi D_\varepsilon J_\Psi = D_\varepsilon J_\Phi J_\Psi = D_\varepsilon J_{t^{\varepsilon-1}} = I^{\alpha+\delta},$$

where $I^{\alpha+\delta}$ denotes the identity operator on $\mathcal{C}^{\alpha+\delta}(I)$. On the other hand, by Lemma 2.3, we obtain

$$D_\varepsilon J_\Psi J_\Phi = D_\varepsilon J_{t^{\varepsilon-1}} = I^\alpha,$$

where I^α denotes the identity operator on $\mathcal{C}^\alpha(I)$. The proof is complete. □

In view of Proposition 5.4, the proof of Theorem B is reduced to finding a function $\Psi(t) \in L^1[0,1]$ satisfying the conditions (i) and (ii) of the proposition. Let Φ_0 be the function defined by (1.21). Then, as was computed in (1.22),

$$\frac{1}{(\varepsilon - i\xi)\mathcal{K}[\Phi_0](\xi)} = \mathcal{K}[\Psi_0](\xi), \tag{5.5}$$

where $\Psi_0(t)$ is given by (1.23). Observing that the condition (ii) of the proposition can be rewritten as

$$\begin{aligned} \mathcal{K}[\Psi](\xi) &= \frac{1}{(\varepsilon - i\xi)\mathcal{K}[\Phi](\xi)} \\ &= \frac{1}{(\varepsilon - i\xi)(\mathcal{K}[\Phi_0](\xi) + \mathcal{K}[R](\xi))} \\ &= \mathcal{K}[\Psi_0](\xi)\frac{1}{1 + (\mathcal{K}[R](\xi))/(\mathcal{K}[\Phi_0](\xi))}, \end{aligned} \tag{5.6}$$

we first seek a function $S(t) \in L^1[0,1]$ such that

$$\frac{\mathcal{K}[R](\xi)}{\mathcal{K}[\Phi_0](\xi)} = \mathcal{K}[S](\xi), \quad \operatorname{Im} z \geq 0. \tag{5.7}$$

This can be carried out by defining

$$S(t) := -t^\varepsilon \frac{d}{dt}(t^{1-\varepsilon}(\Psi_0 * R)(t)). \tag{5.8}$$

In fact, integrating by parts, noting that $t(\Psi_0 * R)(t)|_{t=0} = (\Psi_0 * R)(1) = 0$ follows from Lemma 2.2, and by (5.5), we have for $\operatorname{Im} z \geq 0$,

$$\begin{aligned} \mathcal{K}[S](\xi) &= -\int_0^1 \frac{d}{dt}(t^{1-\varepsilon}(\Psi_0 * R)(t))t^{\varepsilon - i\xi}\, dt \\ &= (\varepsilon - i\xi)\mathcal{K}[\Psi_0 * R](\xi) \\ &= \frac{\mathcal{K}[R](\xi)}{\mathcal{K}[\Phi_0](\xi)}. \end{aligned} \tag{5.9}$$

Since $\Psi_0(t)$, $t^\varepsilon(t^{1-\varepsilon}R(t))' \in L^1[0,1]$, Lemma 2.1 (a) implies that $S(t) \in L^1[0,1]$.

In what follows we assume $0 < \rho \leq 1$ because if $\rho > 1$ we may change ρ so that $0 < \rho \leq 1$. Then we have:

Lemma 5.5. *There exist $M, \mu > 0$ such that for $j = 2, 3, \ldots$,*

$$|(\overbrace{S * \cdots * S}^{j \text{ times}})(t)| \leq M t^{\mu - 1}(1-t)^{-\delta + (j-1)(1-\delta)},$$
$$|(\overbrace{S * \cdots * S}^{j \text{ times}})'(t)| \leq M t^{\mu - 2}(1-t)^{-\delta + (j-1)(1-\delta) - 1}.$$

Proof. We have

$$\begin{aligned} S(t) &= -t^\varepsilon \frac{d}{dt}\left\{t^{1-\varepsilon}\int_t^1 R(t/s)\Psi_0(s)\frac{ds}{s}\right\} \\ &= (\varepsilon - 1)(\Psi_0 * R)(t) + R(1)\Psi_0(t) - \int_t^1 R'(t/s)(t/s)\Psi_0(s)\frac{ds}{s} \\ &= R(1)\Psi_0(t) - \Psi_0 * (t^\varepsilon(t^{1-\varepsilon}R(t))'). \end{aligned}$$

By assumption (ii) in Theorem B, there exists a constant $M > 0$ such that

$$|t^{\varepsilon}(t^{1-\varepsilon}R(t))'| \leq Mt^{\nu-1}(1-t)^{\rho-1},$$
$$|\{t^{\varepsilon}(t^{1-\varepsilon}R(t))'\}'| \leq Mt^{\nu-2}(1-t)^{\rho-2}.$$

Let ν_1 be a number such that $0 < \nu_1 < \min\{\varepsilon + \beta\delta, \nu\}$. Then Lemma 2.2 yields

$$|\Psi_0 * (t^{\varepsilon}(t^{1-\varepsilon}R(t))')| \leq Mt^{\nu_1-1}(1-t)^{\rho-\delta},$$
$$|(\Psi_0 * (t^{\varepsilon}(t^{1-\varepsilon}R(t))'))'| \leq Mt^{\nu_1-2}(1-t)^{\rho-\delta-1},$$

which lead to

$$|S(t)| \leq Mt^{\nu_1-1}(1-t)^{-\delta}, \quad |S'(t)| \leq Mt^{\nu_1-2}(1-t)^{-\delta-1}.$$

Letting μ be a number such that $0 < \mu < \nu_1$ and using Lemma 2.2 repeatedly, we complete the proof of Lemma 5.5. □

By Lemma 5.5 we have:

Lemma 5.6. *Let N be an integer such that $N-2 > \delta(1-\delta)^{-1}$. Then there exists a function $\Delta \in C^1(0,1] \cap L^1[0,1]$ such that $t\Delta'(t) \in L^1[0,1]$, $\Delta(1) = 0$, and*

$$\frac{\mathcal{K}[S](\xi)^N}{1 + \mathcal{K}[S](\xi)} = \mathcal{K}[\Delta](\xi), \quad \xi \in \mathbf{R}. \tag{5.10}$$

Proof. Let $A(t)$ be the function defined by

$$A(t) := -t^{\varepsilon}\frac{d}{dt}\left(t^{1-\varepsilon}(\overbrace{S * \cdots * S}^{N-1 \text{ times}})(t)\right).$$

By the condition for N and Lemma 5.5 it follows that $A(t) \in L^1[0,1]$ and that $(\overbrace{S * \cdots * S}^{N-1 \text{ times}})(1) = 0$. Hence, in the same manner as in (5.9), we have for $\operatorname{Im} z \geq 0$,

$$\mathcal{K}[A](\xi) = (\varepsilon - i\xi)\mathcal{K}[\overbrace{S * \cdots * S}^{N-1 \text{ times}}] = (\varepsilon - i\xi)\mathcal{K}[S](\xi)^{N-1}. \tag{5.11}$$

In view of $\mathcal{K}[\Phi_0](1+\mathcal{K}[S]) = \mathcal{K}[\Phi]$ and condition ($\bigstar$), $1+\mathcal{K}[S](\xi) \neq 0$ for $\operatorname{Im} z \geq 0$. Therefore, in view of Lemma 2.5, there exists a function $B(t) \in L^1[0,1]$ such that

$$\frac{\mathcal{K}[S](\xi)}{1+\mathcal{K}[S](\xi)} = \mathcal{K}[B](\xi), \quad \xi \in \mathbf{R}. \tag{5.12}$$

By Lemma 2.1 (a) we have $A * B \in L^1[0,1]$. So we can define, for $0 < t \leq 1$,

$$\Delta(t) := t^{\varepsilon-1} \int_t^1 (A*B)(s) \frac{ds}{s^\varepsilon}. \tag{5.13}$$

Since

$$\begin{aligned}\int_0^1 |\Delta(t)| dt &\leq \int_0^1 t^{\varepsilon-1}\, dt \int_s^1 |(A*B)(r)| \frac{dr}{r^\varepsilon} \\ &= \int_0^1 |(A*B)(r)| \frac{dr}{r^\varepsilon} \int_0^r t^{\varepsilon-1}\, dt < \infty,\end{aligned}$$

the function $\Delta(t)$ belongs to $L^1[0,1]$. Moreover, from (5.11) and (5.12), we obtain for $\xi \in \mathbf{R}$,

$$\begin{aligned}\mathcal{K}[\Delta](\xi) &= \int_0^1 t^{\varepsilon-1-i\xi}\, dt \int_t^1 (A*B)(s) \frac{ds}{s^\varepsilon} \\ &= \int_0^1 (A*B)(s) \frac{ds}{s^\varepsilon} \int_0^s t^{\varepsilon-1-i\xi}\, dt \\ &= \frac{1}{\varepsilon - i\xi} \mathcal{K}[A*B](\xi) \\ &= \frac{1}{\varepsilon - i\xi} \mathcal{K}[A](\xi) * \mathcal{K}[B](\xi) \\ &= \frac{\mathcal{K}[S](\xi)^N}{1+\mathcal{K}[S](\xi)}.\end{aligned}$$

By definition (5.13) it is evident that $t\Delta'(t) \in L^1[0,1]$, $\Delta(1) = 0$. The proof of Lemma 5.6 is complete. $\square$

Observing that (5.6) can be rewritten as

$$\begin{aligned}\mathcal{K}[\Psi] &= \mathcal{K}[\Psi_0]\frac{1}{1+\mathcal{K}[S]} \\ &= \mathcal{K}[\Psi_0]\left\{1-\mathcal{K}[S]+\cdots+(-1)^{N-1}\mathcal{K}[S]^{N-1}+(-1)^N\frac{\mathcal{K}[S]^N}{1+\mathcal{K}[S]}\right\} \\ &= \mathcal{K}[\Psi_0]\{1-\mathcal{K}[S]+\cdots+(-1)^{N-1}\mathcal{K}[S]^{N-1}+(-1)^N\mathcal{K}[\Delta]\},\end{aligned}$$

we now define

(5.14)

$$\begin{aligned}\Psi_1 &:= \Psi_0-\Psi_0*S+\Psi_0*S*S-\cdots+(-1)^{N-1}\Psi_0*\overbrace{S*\cdots*S}^{N-1\text{ times}}, \\ \Psi_2 &:= (-1)^N\Psi_0*\Delta; \\ \Psi &:= \Psi_1+\Psi_2.\end{aligned}$$

From Lemmas 5.5 and 2.2 it follows that

$$|\Psi_1(t)| \le Mt^{\mu-1}(1-t)^{-\delta}, \quad |\Psi_1'(t)| \le Mt^{\mu-2}(1-t)^{-\delta-1},$$

with some positive constant M, in other words, that $\Psi_1(t)$ satisfies condition (5.1). Moreover, in view of $\Delta(1)=0$, we obtain

$$t\Psi_2'(t) = (-1)^N t\frac{d}{dt}\left(\int_t^1 \Delta(t/s)\Psi_0(s)\frac{ds}{s}\right) = (-1)^N(t\Delta'(t))*\Psi_0.$$

This shows that $\Psi_2(t)$ satisfies (5.2). Thus $\Psi(t)$ satisfies the assumption (i) of Proposition 5.4.

Finally, from (5.14), (5.10), (5.7), (5.5), we get

$$\begin{aligned}\mathcal{K}[\Psi] &= \mathcal{K}[\Psi_0]\{1-\mathcal{K}[S]+\cdots+(-1)^{N-1}\mathcal{K}[S]^{N-1}+(-1)^N\mathcal{K}[\Delta]\}, \\ &= \mathcal{K}[\Psi_0]\left\{\frac{1-(-1)^N\mathcal{K}[S]^N}{1+\mathcal{K}[S]}+(-1)^N\frac{\mathcal{K}[S]^N}{1+\mathcal{K}[S]}\right\} \\ &= \mathcal{K}[\Psi_0]\frac{1}{1+(\mathcal{K}[R]/\mathcal{K}[\Phi_0])} \\ &= \frac{1}{(\varepsilon-i\xi)\mathcal{K}[\Phi]}.\end{aligned}$$

This shows that $\Psi(t)$ satisfies assumption (ii) of Proposition 5.4. In view of Proposition 5.4 the proof of Theorem B is complete. □

Remark 5.6. From the above proof, we obtain

$$\Psi = \mathcal{K}^{-1}\left[\frac{1}{(\varepsilon - i\xi)\mathcal{K}[\Phi](\xi)}\right].$$

In practical applications this formula often enables us to compute $\Psi(t)$ explicitly. For instance we refer to [**3**, Section 5].

Acknowledgment. The authors are very grateful to the referees for their valuable suggestions which improved the original manuscript substantially.

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