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BI-HAMILTONIAN STRUCTURES ON THE TANGENT BUNDLE TO A POISSON MANIFOLD

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Abstract. In the case when M is equipped with a bi-Hamiltonian structure (M, π_1, π_2) we show how to construct family of Poisson structures on the tangent bundle TM to a Poisson manifold. Moreover we present how to find Casimir functions for those structures and we discuss some particular examples.

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Keywords: bi-Hamiltonian structure, Casimir function, Lagrange top, Lie algebra, Lie algebroid, linear Poisson structure, tangent lift of Poisson structure

Contents

1. Introduction

The theories of Poisson and bi-Hamiltonian manifolds are one of important tools of the theory of integrable systems, see [1, 2, 10, 18, 24, 29]. The theory of Lie algebroids is another useful tool (see e.g. [3, 4, 9, 12, 13, 16, 31]) There are links between Poisson manifolds and Lie algebroids. It is well known that the total space of the dual bundle of a Lie algebroid has a canonical Poisson structure and there exists the canonical algebroid bracket of differential forms $A = T^*M$, where

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M is a Poisson manifold. In this paper we consider some modifications of this bracket. We study the connection between Poisson manifolds and bi-Hamiltonian manifolds, and some deformation of Poisson structures generated by Lie algebroid structures of differential forms.

The paper is organized as follows. In the beginning of Section 2 we recall the definitions and well known results about Lie algebroids and bi-Hamiltonian manifolds. Next sections contain the main results of the paper. In Section 3 starting from a algebroid bracket of differential forms, we describe some deformations of the Poisson structure on the dual bundle of Lie algebroid on TM. We show how the bi-Hamiltonian structure from M transfers to the tangent space TM. Moreover we discuss how to lift Casimirs functions and a family of functions in involution from M to TM. We also present some examples in Section 3, e.g. the Lagrange top. Moreover, we present how to apply Poisson and algebroid formalisms and some of their modifications to the theory of classification of real low dimensional Lie algebras, see also [7, 8].

2. Lifting of Poisson and Bi-Hamiltonian Structures

In the present section we recall some basic facts about Poisson manifolds, linear Poisson structures, Lie algebroids, bi-Hamiltonian manifolds and tangent lifts of Poisson structures and bi-Hamiltonian structures.

Let (M, π) be a N–dimensional Poisson manifold. Then the Poisson tensor $\pi \in \Gamma\left(\bigwedge^2 TM\right)$ can be written as

$$\pi(\mathbf{x}) = \sum_{i,j=1}^{N} \frac{1}{2}\pi_{ij}(\mathbf{x})\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_N)$ is a system of local coordinates on M. It leads to the Poisson bracket in the form

$$\{f, g\}(\mathbf{x}) = \sum_{i,j=1}^{N} \pi_{ij}(\mathbf{x})\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} \tag{2}$$

which is a skew-symmetric bilinear mapping satisfying the Jacobi identity

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0 \tag{3}$$

as well as the Leibniz rule

$$\{fg, h\} = f\{g, h\} + g\{f, h\}. \tag{4}$$

The bivector $\pi_{ij}(\mathbf{x}) = -\pi_{ji}(\mathbf{x}) = \{x_i, x_j\}$ satisfies the following system of equations equivalent to the Jacobi identity

$$\sum_{s=1}^{N} \left(\frac{\partial \pi_{ij}}{\partial x_s}\pi_{sk} + \frac{\partial \pi_{ki}}{\partial x_s}\pi_{sj} + \frac{\partial \pi_{jk}}{\partial x_s}\pi_{si} \right) = 0. \tag{5}$$

Given a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ of dimension N, there exists the canonical Poisson structure on $M = \mathfrak{g}^*$. This bracket is called Lie–Poisson bracket and can be defined by the following formula

$$\{f, g\}(\mathbf{x}) = \langle \mathbf{x}, [\mathrm{d}f(\mathbf{x}), \mathrm{d}g(\mathbf{x})] \rangle \tag{6}$$

where $\mathrm{d}f(\mathbf{x}), \mathrm{d}g(\mathbf{x}) \in (\mathfrak{g}^*)^* \simeq \mathfrak{g}$. There is a natural equivalence between N–dimensional linear Poisson structures and N–dimensional Lie algebras. If $[e_i, e_j] = \sum_{n=1}^{N} c_{ij}^n e_n$ then $\{x_i, x_j\} = \sum_{n=1}^{N} c_{ij}^n x_n$, where $(e_1, e_2, \ldots, e_N)$ is a basis of $\mathfrak{g}$ and c_{ij}^n are the structure constants of this Lie algebra.

We say that two Poisson tensors π_1 and π_2 are compatible if any linear combination

$$\pi_{\alpha,\beta} = \alpha\pi_1 + \beta\pi_2, \qquad \alpha, \beta \in \mathbb{R} \tag{7}$$

is also a Poisson tensor. The Poisson structures π_1 and π_2 on M are compatible if and only if their Schouten–Nijenhuis bracket vanishes $[\pi_1, \pi_2]_{S-N} = 0$, which means that

$$\begin{aligned}\sum_{s=1}^{N} \Bigg(\pi_{2,sk}\frac{\partial \pi_{1,ij}}{\partial x_s} + \pi_{1,sk}\frac{\partial \pi_{2,ij}}{\partial x_s} + \pi_{2,sj}\frac{\partial \pi_{1,ki}}{\partial x_s} \\ + \pi_{1,sj}\frac{\partial \pi_{2,ki}}{\partial x_s} + \pi_{2,si}\frac{\partial \pi_{1,jk}}{\partial x_s} + \pi_{1,si}\frac{\partial \pi_{2,jk}}{\partial x_s} \Bigg) = 0\end{aligned} \tag{8}$$

see [23,28]. The manifold M equipped with two compatible Poisson structures π_1 and π_2 is called bi-Hamiltonian manifold and we denote it as (M, π_1, π_2).

A Lie algebroid $(A, [\cdot,\cdot]_A, a)$ over a manifold M is a vector bundle $q_M \colon A \to M$ with a vector bundle map $a \colon A \to TM$, which is called the anchor, and a Lie bracket $[\cdot,\cdot]_A \colon \Gamma(A) \times \Gamma A \to \Gamma(A)$ satisfying following Leibniz rule

$$[X, fY]_A = f[X, Y] + a(X)(f)Y \tag{9}$$

for any sections $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$. The anchor a fulfill the property

$$a([X, Y]_A) = [a(X), a(Y)]_A. \tag{10}$$

It was introduced by Pradines [26], see also [5,17,21,25].

Let us consider a certain Lie algebroid structure on T^*M

$$\begin{tikzcd} A = T^*M \arrow[r, "a"] \arrow[d, "q_M^*"] & TM \arrow[d, "q_M"] \\ M \arrow[r, "\mathrm{id}"] & M \end{tikzcd} \tag{11}$$

A vector bundle map a is called the anchor of the Lie algebroid $A = T^*M$ and in this case it is defined as

$$a(\mathrm{d}f)(\cdot) = \{f, \cdot\}. \tag{12}$$

Sections ΓA form a Lie algebra with a Lie bracket

$$[\mathrm{d}f, \mathrm{d}g] = \mathrm{d}\{f, g\} \tag{13}$$

where $f, g \in C^\infty(M)$. This bracket must satisfy the following conditions

$$\begin{aligned} [\mathrm{d}f, h\,\mathrm{d}g] &= h[\mathrm{d}f, \mathrm{d}g] + a(\mathrm{d}f)(h)\mathrm{d}g \\ a\left([\mathrm{d}f, \mathrm{d}g]\right) &= [a(\mathrm{d}f), a(\mathrm{d}g)] \end{aligned} \tag{14}$$

for all $\mathrm{d}f, \mathrm{d}g \in \Gamma A$, $h \in C^\infty(M)$, see [15]. On the dual space (TM, q_M, M) to the Lie algebroid (T^*M, q_M^*, M) we have the tangent Poisson structure. The Poisson bracket on $C^\infty(TM)$ is given by relations

$$\begin{aligned} \{f \circ q_M, g \circ q_M\}_{TM} &= 0 \\ \{l_{\mathrm{d}f}, l_{\mathrm{d}g}\}_{TM} &= l_{[\mathrm{d}f, \mathrm{d}g]} \\ \{f \circ q_M, l_{\mathrm{d}g}\}_{TM} &= -a(\mathrm{d}g)(f) \circ q_M \end{aligned} \tag{15}$$

where $f, g \in C^\infty(M)$. In the above formulas $l_{\mathrm{d}f} \in C^\infty(TM)$ is defined by pairing

$$l_{\mathrm{d}f}(\xi) = \big\langle \xi, \mathrm{d}f(q_M(\xi)) \big\rangle, \qquad \xi \in TM. \tag{16}$$

In this situation the tangent Poisson tensor can be expressed by formula

$$\pi_{TM}(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{c|c} 0 & \pi(\mathbf{x}) \\ \hline \pi(\mathbf{x}) & \sum_{s=1}^N \dfrac{\partial \pi}{\partial x_s}(\mathbf{x}) y_s \end{array} \right) \tag{17}$$

where $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_N, y_1 = l_{\mathrm{d}x_1}, \ldots, y_N = l_{\mathrm{d}x_N})$ is a system of local coordinates on TM.

Some of the properties of such Poisson structure are well known, see [6, 11]. If $c_1, \dots, c_r$ are Casimir functions for the Poisson structure π, i.e. $\{c, f\} = 0$ for all $f \in C^\infty(M)$, then the functions

$$c_i \circ q_M \qquad \text{and} \qquad l_{\mathrm{d}c_i} = \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s} y_s, \qquad i = 1, \dots r \tag{18}$$

are Casimir functions for the Poisson tensor π_{TM}. Subsequently if the functions $\{H_i\}_{i=1}^k$ are in involution with respect to the Poisson bracket generated by π, then the functions

$$H_i \circ q_M^* \qquad \text{and} \qquad l_{\mathrm{d}H_i} = \sum_{s=1}^{N} \frac{\partial H_i}{\partial x_s}(\mathbf{x}) y_s, \qquad i = 1, \dots k \tag{19}$$

are in involution with respect to the Poisson tensor π_{TM} given by (17).

As it was shown in [6] the algebroid structure (17) on TM can be deformed as follows

$$\pi_{TM,\lambda}(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{c|c} 0 & \pi_2(\mathbf{x}) \\ \hline \pi_2(\mathbf{x}) & \sum_{s=1}^{N} \frac{\partial \pi_2}{\partial x_s}(\mathbf{x}) y_s + \lambda \pi_1(\mathbf{x}) \end{array} \right) \tag{20}$$

using bi-Hamiltonian structure (M, π_1, π_2). This structure can also be presented globally

$$\begin{aligned} \{f \circ q_M, g \circ q_M\}_{TM} &= 0 \\ \{l_{\mathrm{d}f}, l_{\mathrm{d}g}\}_{TM} &= l_{[\mathrm{d}f, \mathrm{d}g]_2} + \lambda \{f, g\}_1 \circ q_M \\ \{f \circ q_M, l_{\mathrm{d}g}\}_{TM} &= -a(\mathrm{d}g)(f) \circ q_M = \{f, g\}_2 \circ q_M \end{aligned} \tag{21}$$

where $f, g \in C^\infty(M)$, $\lambda \in \mathbb{R}$. Some of the properties of the Poisson structure above are known, see [6]. If functions $\{H_i\}_{i=1}^k$ are in involution with respect to the both Poisson brackets given by π_1 and π_2, then the functions (19) are in involution with respect to the Poisson tensor (20). Moreover, if $c_1, \dots, c_r$, where $r = \dim M - \operatorname{rank} \pi_2$, are Casimir functions for the Poisson structure π_2 and functions f_i^λ, $i = 1, \dots, r$, satisfy the conditions

$$\{f_i^\lambda, x_j\}_1 = \{x_j, c_i\}_2, \qquad \text{for} \quad j = 1, \cdots, n \tag{22}$$

then the functions

$$c_i \circ q_M^* \qquad \text{and} \qquad \tilde{c}_i = \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s}(\mathbf{x}) y_s + \lambda f_i^\lambda(\mathbf{x}), \qquad i = 1, \dots r \tag{23}$$

are the Casimir functions for the Poisson tensor $\pi_{TM,\lambda}$ given by (20).

3. Deformations of Tangent Poisson Structures

In this section, we deform Poisson structure (17) on TM using an additional Poisson structure on the base manifold M. We are discussing also how to transfer Casimir functions and functions in involution from M to the space TM.

At the beginning we take a trivial lift of the Poisson structure π_1 from M to the tangent bundle TM in the form $\left(\begin{array}{c|c} \pi_1(\mathbf{x}) & 0 \\ \hline 0 & 0 \end{array}\right)$. In the next step, we ask about the compatibility of this structure with the algebroid structure (17) on TM.

Theorem 1. *Let (M,π_1) and (M,π_2) be Poisson manifolds. If the Poisson tensors π_1 and π_2 satisfy the conditions*

$$\begin{gathered}\sum_{s=1}^{N}\left(\pi_{2,sk}(\mathbf{x})\frac{\partial\pi_{1,ij}}{\partial x_s}(\mathbf{x})+\pi_{1,si}(\mathbf{x})\frac{\partial\pi_{2,jk}}{\partial x_s}(\mathbf{x})+\pi_{1,sj}(\mathbf{x})\frac{\partial\pi_{2,ki}}{\partial x_s}(\mathbf{x})\right)=0\\ \sum_{s,m=1}^{N} y_s\pi_{1,mk}(\mathbf{x})\frac{\partial^2\pi_{2,ij}}{\partial x_m\partial x_s}(\mathbf{x})=0\end{gathered}\tag{24}$$

then there exists a Poisson structure on TM associated with π_1 and π_2 of the form

$$\left(\pi_1\ltimes_2\pi_2\right)(\mathbf{x},\mathbf{y})=\left(\begin{array}{c|c}\pi_1(\mathbf{x}) & \pi_2(\mathbf{x})\\ \hline \pi_2(\mathbf{x}) & \sum_{s=1}^{N}\frac{\partial\pi_2}{\partial x_s}(\mathbf{x})y_s\end{array}\right).\tag{25}$$

Proof: By the Jacobi identity we have

$$\begin{aligned}&\{\{x_i,x_j\}_{\ltimes_2},x_k\}_{\ltimes_2}+\{\{x_k,x_i\}_{\ltimes_2},x_j\}_{\ltimes_2}+\{\{x_j,x_k\}_{\ltimes_2},x_i\}_{\ltimes_2}\\ &\quad=\{\{y_i,y_j\}_{\ltimes_2},y_k\}_{\ltimes_2}+\{\{y_k,y_i\}_{\ltimes_2},y_j\}_{\ltimes_2}+\{\{y_j,y_k\}_{\ltimes_2},y_i\}_{\ltimes_2}=0\end{aligned}\tag{26}$$

$$\begin{aligned}&\{\{x_i,x_j\}_{\ltimes_2},y_k\}_{\ltimes_2}+\{\{y_k,x_i\}_{\ltimes_2},x_j\}_{\ltimes_2}+\{\{x_j,y_k\}_{\ltimes_2},x_i\}_{\ltimes_2}\\ &=\sum_{s=1}^{N}\left(\frac{\partial\pi_{1,ij}}{\partial x_s}(\mathbf{x})\pi_{2,sk}(\mathbf{x})+\frac{\partial\pi_{2,ki}}{\partial x_s}(\mathbf{x})\pi_{1,sj}(\mathbf{x})+\frac{\partial\pi_{2,jk}}{\partial x_s}(\mathbf{x})\pi_{1,si}(\mathbf{x})\right)\end{aligned}\tag{27}$$

$$\begin{aligned}&\{\{y_i,y_j\}_{\ltimes_2},x_k\}_{\ltimes_2}+\{\{x_k,y_i\}_{\ltimes_2},y_j\}_{\ltimes_2}+\{\{y_j,x_k\}_{\ltimes_2},y_i\}_{\ltimes_2}\\ &\qquad=\sum_{s,m=1}^{N}y_s\pi_{1,mk}(\mathbf{x})\frac{\partial^2\pi_{2,ij}}{\partial x_m\partial x_s}(\mathbf{x}).\end{aligned}\tag{28}$$

We see that if the conditions (24) are fulfilled then $\pi_1 \ltimes_2 \pi_2$ is a Poisson tensor. ■

This structure can be presented globally

$$\begin{aligned} \{f \circ q_M, g \circ q_M\}_{TM} &= \{f, g\}_1 \circ q_M \\ \{l_{\mathrm{d}f}, l_{\mathrm{d}g}\}_{TM} &= l_{[\mathrm{d}f, \mathrm{d}g]_2} \\ \{f \circ q_M, l_{\mathrm{d}g}\}_{TM} &= -a_2(\mathrm{d}g)(f) \circ q_M = \{f, g\}_2 \circ q_M. \end{aligned} \tag{29}$$

If the manifold M is a linear space then we can take the next simple Poisson structure $\left(\begin{array}{c|c} \pi_1(\mathbf{y}) & 0 \\ \hline 0 & 0 \end{array}\right)$ on manifold TM. This is an interesting structure as shown in the following example.

Example 2. *If we consider the linear Poisson structure given by the Poisson tensor* $\pi_1(x_1, x_2, x_3) = \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$ *associated with take Lie algebra* $\mathfrak{a}_{3,9}$, *then in the above construction, we obtain the following structure* $\pi(\mathbf{x}, \mathbf{y}) = y_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} + y_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}$ *on* $T\mathfrak{a}^*_{3,9}$. *We recognize the Lie-Poisson structure related to the Lie algebra* $\mathfrak{a}_{6,3}$ (*we use the classification given in* [27]).

In addition, the following theorem provides the conditions for compatibility of this structure with the algebroid structure (17) on TM.

Theorem 3. *Let* (M, π_1) *and* (M, π_2) *be a Poisson manifolds. If the Poisson tensors* π_1 *and* π_2 *satisfy the conditions*

$$\begin{gathered} \textstyle\sum_{s=1}^N \left(\pi_{2,sk}(\mathbf{x}) \frac{\partial \pi_{1,ij}}{\partial y_s}(\mathbf{y}) + \pi_{2,sj}(\mathbf{x}) \frac{\partial \pi_{1,ki}}{\partial y_s}(\mathbf{y}) + \pi_{2,si}(\mathbf{x}) \frac{\partial \pi_{1,jk}}{\partial y_s}(\mathbf{y})\right) = 0 \\ \textstyle\sum_{s=1}^N \Big(\sum_{m=1}^N y_m \frac{\partial \pi_{2,sk}}{\partial x_m}(\mathbf{x}) \frac{\partial \pi_{1,ij}}{\partial y_s}(\mathbf{y}) + \pi_{1,si}(\mathbf{y}) \frac{\partial \pi_{2,jk}}{\partial x_s}(\mathbf{x}) + \pi_{1,sj}(\mathbf{y}) \frac{\partial \pi_{2,ki}}{\partial x_s}(\mathbf{x})\Big) = 0 \\ \textstyle\sum_{s,m=1}^N y_s \pi_{1,mk}(\mathbf{y}) \frac{\partial^2 \pi_{2,ij}}{\partial x_m \partial x_s}(\mathbf{x}) = 0 \end{gathered} \tag{30}$$

then there exists a Poisson structure on TM *associated with* π_1 *and* π_2 *of the form*

$$(\pi_1 \ltimes_1 \pi_2)(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{c|c} \pi_1(\mathbf{y}) & \pi_2(\mathbf{x}) \\ \hline \pi_2(\mathbf{x}) & \sum_{s=1}^N \dfrac{\partial \pi_2}{\partial x_s}(\mathbf{x}) y_s \end{array}\right). \tag{31}$$

Proof: By direct calculation we obtain

$$\begin{aligned} &\{\{x_i, x_j\}_{\ltimes_1}, x_k\}_{\ltimes_1} + \{\{x_k, x_i\}_{\ltimes_1}, x_j\}_{\ltimes_1} + \{\{x_j, x_k\}_{\ltimes_1}, x_i\}_{\ltimes_1} \\ &= \sum_{s=1}^N \left(\pi_{2,sk}(\mathbf{x}) \frac{\partial \pi_{1,ij}}{\partial y_s}(\mathbf{y}) + \pi_{2,sj}(\mathbf{x}) \frac{\partial \pi_{1,ki}}{\partial y_s}(\mathbf{y}) + \pi_{2,si}(\mathbf{x}) \frac{\partial \pi_{1,jk}}{\partial y_s}(\mathbf{y})\right) \end{aligned} \tag{32}$$

$$\begin{aligned}&\{\{x_i,x_j\}_{\ltimes_1},y_k\}_{\ltimes_1}+\{\{y_k,x_i\}_{\ltimes_1},x_j\}_{\ltimes_1}+\{\{x_j,y_k\}_{\ltimes_1},x_i\}_{\ltimes_1}\\ &=\textstyle\sum_{s=1}^N\big(\sum_{m=1}^N \frac{\partial \pi_{2,sk}}{\partial x_m}(\mathbf{x})y_m\frac{\partial \pi_{1,ij}}{\partial y_s}(\mathbf{y})+\pi_{1,si}(\mathbf{y})\frac{\partial \pi_{2,jk}}{\partial x_s}(\mathbf{x})+\pi_{1,sj}(\mathbf{y})\frac{\partial \pi_{2,ki}}{\partial x_s}(\mathbf{x})\big)\end{aligned} \tag{33}$$

$$\begin{aligned}&\{\{y_i,y_j\}_{\ltimes_1},x_k\}_{\ltimes_1}+\{\{y_j,x_k\}_{\ltimes_1},y_i\}_{\ltimes_1}+\{\{x_k,y_i\}_{\ltimes_1},y_j\}_{\ltimes_1}\\ &\qquad=\sum_{s,m=1}^N \pi_{1,mk}(\mathbf{y})y_s\frac{\partial^2\pi_{2,ij}}{\partial x_m\partial x_s}(\mathbf{x})\end{aligned} \tag{34}$$

$$\{\{y_i,y_j\}_{\ltimes_1},y_k\}_{\ltimes_1}+\{\{y_j,y_k\}_{\ltimes_1},y_i\}_{\ltimes_1}+\{\{y_k,y_i\}_{\ltimes_1},y_j\}_{\ltimes_1}=0. \tag{35}$$

■

Let us observe that if we put $\pi_1 = 0$ then we reduce (31) or (25) to (17), i.e. we obtain the classical tangent Poisson structure on TM. Note that the last condition in (30) or (24) is very restrictive but it is automatically fulfilled for the linear or constant Poisson tensor π_2. In addition, if $\pi_1 = \pi_2$, then the conditions (30) or (24) are also realized. In this case if $c_1, \dots, c_r$ are Casimir functions for the Poisson structure π_2, then the functions $c_i(\mathbf{x}-\mathbf{y}) + c_i(\mathbf{x}+\mathbf{y})$, $c_i(\mathbf{x}-\mathbf{y}) - c_i(\mathbf{x}+\mathbf{y})$, $i = 1, \dots r$, are the Casimir functions for the Poisson tensor $\pi_2 \ltimes_1 \pi_2$. Similarly the functions $c_i(\mathbf{x}-\mathbf{y}) + c_i(\mathbf{x})$, $c_i(\mathbf{x}-\mathbf{y}) - c_i(\mathbf{x})$, $i = 1, \dots r$, are the Casimir functions for the Poisson tensor $\pi_2 \ltimes_2 \pi_2$. There is a natural bijection between linear Poisson structures π_2 and Lie algebras $\mathfrak{g}$. Moreover, there is a relationship between linear Poisson structure $\pi_2 \ltimes_1 \pi_2$ and Cartan decomposition of Lie algebra $T\mathfrak{g} = \mathfrak{g} + V$ satisfying the relations

$$[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}, \qquad [\mathfrak{g},V] \subset V, \qquad [V,V] \subset \mathfrak{g}. \tag{36}$$

Corollary 4. *If linear Poisson tensors π_1 and π_2 are compatible and conditions* (30) *are satisfied then conditions* (24) *are also satisfied.*

Corollary 5. *If linear or constant Poisson tensors π_1 and π_2 are compatible and conditions* (30) *are satisfied then*

$$(\pi_2 \ltimes_1 \pi_1)(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} \pi_2(\mathbf{y}) & \pi_1(\mathbf{x}) \\ \hline \pi_1(\mathbf{x}) & \sum_{s=1}^N \dfrac{\partial \pi_1}{\partial x_s}(\mathbf{x})y_s \end{array}\right) \tag{37}$$

is a Poisson tensor.

Corollary 6. *If linear or constant Poisson tensors π_1 and π_2 are compatible and conditions* (24) *are satisfied then*

$$(\pi_2 \ltimes_2 \pi_1)(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} \pi_2(\mathbf{x}) & \pi_1(\mathbf{x}) \\ \hline \pi_1(\mathbf{x}) & \sum_{s=1}^N \dfrac{\partial \pi_1}{\partial x_s}(\mathbf{x})y_s \end{array}\right) \tag{38}$$

is a Poisson tensor.

Note that the conditions (30) or (24) are met for certain classes of Poisson tensors. One of these classes is described by the following restriction.

Corollary 7. *If the Poisson tensor π_2 is constant and (M, π_1, π_2) is a bi-Hamiltonian manifold then the conditions* (30) *are satisfied.*

Let us observe that this class is rich, because it contains as π_1 the Lie-Poisson structure

$$\{f, g\}_{LP}(\mathbf{x}) = \Big\langle \mathbf{x}, [\mathrm{d}f(\mathbf{x}), \mathrm{d}g(\mathbf{x})] \Big\rangle, \qquad \mathbf{x} \in \mathfrak{g}^* \tag{39}$$

on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$, and as π_2 the frozen Poisson structure

$$\{f, g\}_{\mathbf{x}_0}(\mathbf{x}) = \Big\langle \mathbf{x}_0, [\mathrm{d}f(\mathbf{x}), \mathrm{d}g(\mathbf{x})] \Big\rangle \tag{40}$$

where $\mathbf{x}_0$ is a fixed element of $\mathfrak{g}^*$. These structures are compatible, i.e. form a pencil of Poisson structures for every freezing point $\mathbf{x}_0$, see [20]. Moreover, it is clear from Corollary 7 that they also build the Poisson tensor $(\pi_{LP} \ltimes_1 \pi_{x_0})(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{c|c} \pi_{LP}(\mathbf{y}) & \pi_{x_0}(\mathbf{x}) \\ \hline \pi_{x_0}(\mathbf{x}) & 0 \end{array}\right)$ on $T\mathfrak{g}^*$.

In general, the Casimir function c for the structure $\pi_1 \ltimes_1 \pi_2$ has to satisfy the following conditions

$$\begin{aligned} \sum_{s=1}^{N} \left(\pi_{1,is}(\mathbf{y}) \frac{\partial c}{\partial x_s}(\mathbf{x}, \mathbf{y}) + \pi_{2,is}(\mathbf{x}) \frac{\partial c}{\partial y_s}(\mathbf{x}, \mathbf{y}) \right) &= 0 \\ \sum_{s=1}^{N} \pi_{2,is}(\mathbf{x}) \frac{\partial c}{\partial x_s}(\mathbf{x}, \mathbf{y}) + \sum_{s=1}^{N} y_s \sum_{m=1}^{N} \frac{\partial \pi_{2,im}}{\partial x_s}(\mathbf{x}) \frac{\partial c}{\partial y_m}(\mathbf{x}, \mathbf{y}) &= 0. \end{aligned} \tag{41}$$

In particular, in the case described by the Corollary 7, the following theorems can be proved.

Theorem 8. *Let $\pi_2 = \mathrm{const}$, (M, π_1, π_2) be bi-Hamiltonian and $c_1, \ldots, c_r$, where $r = \dim M - \operatorname{rank} \pi_2$, be Casimir functions for the constant Poisson structure π_2. Then the functions*

$$c_i(\mathbf{y}), \qquad \widetilde{c}_i(\mathbf{x}, \mathbf{y}) = c_i(\mathbf{x}) + \widetilde{\widetilde{c}}_i(\mathbf{y}) \tag{42}$$

where $\widetilde{\widetilde{c}}_i$ satisfies the conditions

$$\sum_{s=1}^{N} \left(\pi_{1,js}(\mathbf{y}) \frac{\partial c_i}{\partial x_s}(\mathbf{x}) + \pi_{2,js} \frac{\partial \widetilde{\widetilde{c}}_i}{\partial y_s}(\mathbf{y}) \right) = 0 \tag{43}$$

are the Casimir functions for the Poisson tensor $(\pi_1 \ltimes_1 \pi_2)$ *given by* (31).

Proof: Proof is obtained by direct calculation from formulas (41). ■

Theorem 9. *Let* $\pi_2 = const$, (M, π_1, π_2) *be bi-Hamiltonian and functions* $\{H_i\}_{i=1}^{k}$ *be in involution with respect to the both Poisson brackets given by* π_1 *and* π_2, *then the functions*

$$H_i(\mathbf{y}) \qquad \textit{and} \qquad \widetilde{H}_i(\mathbf{x},\mathbf{y}) = \sum_{s=1}^{N} \frac{\partial H_i}{\partial y_s}(\mathbf{y}) x_s \tag{44}$$

are in involution with respect to the Poisson tensor $(\pi_1 \ltimes_1 \pi_2)$ *given by* (31).

Proof: The functions H_i and H_j are in involution with respect to the Poisson structure given by (31) when they satisfy the condition

$$\begin{aligned} \{H_i(\mathbf{x},\mathbf{y}), H_j(\mathbf{x},\mathbf{y})\}_{\ltimes_1} = & \sum_{s,m=1}^{N} \Big(\pi_{1,sm}(\mathbf{y}) \frac{\partial H_i}{\partial x_s}(\mathbf{x},\mathbf{y}) \frac{\partial H_j}{\partial x_m}(\mathbf{x},\mathbf{y}) \\ & + \pi_{2,sm} \frac{\partial H_i}{\partial x_s}(\mathbf{x},\mathbf{y}) \frac{\partial H_j}{\partial y_m}(\mathbf{x},\mathbf{y}) + \pi_{2,sm} \frac{\partial H_i}{\partial y_s}(\mathbf{x},\mathbf{y}) \frac{\partial H_j}{\partial x_m}(\mathbf{x},\mathbf{y}) \Big) = 0. \end{aligned} \tag{45}$$

From the above we get $\{H_i(\mathbf{y}), H_j(\mathbf{y})\}_{\ltimes_1} = 0$. It is also easy to see that

$$\begin{aligned} \{H_i(\mathbf{y}), \widetilde{H}_j(\mathbf{x},\mathbf{y})\}_{\ltimes_1} &= \sum_{s,m=1}^{N} \pi_{2,sm} \frac{\partial H_i}{\partial y_s}(\mathbf{y}) \frac{\partial H_j}{\partial y_m}(\mathbf{y}) = \{H_i, H_j\}_2(\mathbf{y}) = 0 \\ \{\widetilde{H}_i(\mathbf{x},\mathbf{y}), \widetilde{H}_j(\mathbf{x},\mathbf{y})\}_{\ltimes_1} &= \{H_i, H_j\}_1(\mathbf{y}) + \sum_{p=1}^{N} x_p \frac{\partial}{\partial y_p} \big(\{H_i, H_j\}_2(\mathbf{y})\big) = 0 \end{aligned} \tag{46}$$

from involution with respect to the Poisson tensors π_1 and π_2. ■

In low-dimensional cases, there is sometimes another possibility to build a family of functions in involution.

Theorem 10. *Let* H_i *be Casimirs functions for the Poisson tensor* π_1 *quadratic homogeneous in* $\mathbf{x}$ *or linear homogeneous in* $\mathbf{x}$. *Then the family of functions*

$$\widehat{H}_i(\mathbf{x},\mathbf{y}) = \sum_{s=1}^{N} \frac{\partial H_i}{\partial y_s}(\mathbf{y}) x_s \tag{47}$$

and the family $\widehat{\widehat{H}}_j(x)$ *defined by the following conditions*

$$\{H_i, \widehat{\widehat{H}}_j\}_2 = 0, \qquad \sum_{s,m=1}^{N} \left(\pi_{1,sm}(\mathbf{y}) \frac{\partial \widehat{\widehat{H}}_i}{\partial x_s}(\mathbf{x}) \frac{\partial \widehat{\widehat{H}}_j}{\partial x_m}(\mathbf{x}) \right) = 0 \tag{48}$$

are in involution with respect to the Poisson tensor $(\pi_1 \ltimes_1 \pi_2)$ *given by* (31).

Proof: From the previous results and the equality $\dfrac{\partial}{\partial y_m}\left(\sum_{s=1}^N \dfrac{\partial H_i}{\partial y_s}(\mathbf{y}) x_s\right) = \dfrac{\partial}{\partial y_m}\left(\sum_{s=1}^N \dfrac{\partial H_i}{\partial x_s}(\mathbf{x}) y_s\right) = \dfrac{\partial H_i}{\partial x_m}(\mathbf{x})$ one can deduce the above theorem. ■

4. Examples

Example 11 (Lagrange top) . *Let us consider two Poisson structures*

$$\pi_1(\mathbf{x}) = \begin{pmatrix} 0 & \omega x_3 & -x_2 \\ -\omega x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}, \qquad \pi_2(\mathbf{x}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{49}$$

where $\mathbf{x} = (x_1, x_2, x_3), \quad \mathbf{y} = (y_1, y_2, y_3), \quad \omega = \mathrm{const} \quad (\omega \neq 0)$. *In this case, the Casimir function for the structure* π_1 *is*

$$F(\mathbf{x}) = x_1^2 + x_2^2 + \omega x_3^2 \tag{50}$$

and for π_2

$$c_1(\mathbf{x}) = x_3. \tag{51}$$

Let us take as a Hamiltonian

$$H(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2. \tag{52}$$

The equations of motion for this Hamiltonian computed for first Poisson structure π_1 *assume the form*

$$\dot{x}_1 = 2(\omega - 1)x_2 x_3, \qquad \dot{x}_2 = -2(\omega - 1)x_1 x_3, \qquad \dot{x}_3 = 0. \tag{53}$$

It is easy to see that the conditions (30) *from* Theorem 3 *are satisfied. Then from* (31)*, the tangent Poisson structure is given by*

$$(\pi_1 \ltimes_1 \pi_2)(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{ccc|ccc} 0 & \omega y_3 & -y_2 & 0 & -1 & 0 \\ -\omega y_3 & 0 & y_1 & 1 & 0 & 0 \\ y_2 & -y_1 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{54}$$

The Casimirs from Theorem 8 *for the structure given above assume following form*

$$c_1(\mathbf{y}) = y_3, \qquad \widetilde{c}_1(\mathbf{x},\mathbf{y}) = x_3 - \tfrac{1}{2}\left(y_1^2 + y_2^2 + \omega y_3^2\right). \tag{55}$$

The functions H and F are in involution with respect to the Poisson structure π_2, then they satisfy the assumption of Theorem 9. *In this case we obtain*

$$\begin{aligned} H_1(\mathbf{y}) &= H(\mathbf{y}) = y_1^2+y_2^2+y_3^2, & \widetilde{H}_1(\mathbf{x},\mathbf{y}) &= 2x_1y_1+2x_2y_2+2x_3y_3 \\ H_2(\mathbf{y}) &= F(\mathbf{y}) = y_1^2+y_2^2+\omega y_3^2, & \widetilde{H}_2(\mathbf{x},\mathbf{y}) &= 2x_1y_1+2x_2y_2+2\omega x_3y_3. \end{aligned} \tag{56}$$

Of course, it is also easy to see that two of the four functions can be expressed by the Casimir functions c_1 and $\widetilde{c}_1$. Let us take as a Hamiltonian

$$h = \alpha H_1 + \beta H_2 + \gamma \widetilde{H}_1 + \delta \widetilde{H}_2. \tag{57}$$

The Hamilton's equations, in this case, are given by

$$\begin{aligned} \dot{x}_1 &= 2\gamma(\omega-1)y_2y_3 - 2(\alpha+\beta)y_2 - 2(\gamma+\delta)x_2 \\ \dot{x}_2 &= -2\gamma(\omega-1)y_1y_3 + 2(\alpha+\beta)y_1 + 2(\gamma+\delta)x_1 \\ \dot{x}_3 &= 0 \\ \dot{y}_1 &= -2(\gamma+\delta)y_2 \\ \dot{y}_2 &= 2(\gamma+\delta)y_1 \\ \dot{y}_3 &= 0. \end{aligned} \tag{58}$$

Moreover, since the Casimir function F for the Poisson structure π_1 is quadratic homogeneous in $\mathbf{x}$, *we obtain that F and H satisfy also the conditions of* Theorem 10. *In this case we have only two functions*

$$\widehat{H}_2(\mathbf{x},\mathbf{y}) = 2x_1y_1+2x_2y_2+2\omega x_3y_3, \qquad \widehat{\widehat{H}}_1(\mathbf{x}) = H(\mathbf{x}) = x_1^2+x_2^2+x_3^2. \tag{59}$$

These functions and Casimir functions (55) *are the four constants of motion for the Lagrange top. The Hamiltonian of this top is given by*

$$\begin{aligned} h(\mathbf{x},\mathbf{y}) &= \frac{1}{2}\left(\widehat{\widehat{H}}_1(\mathbf{x}) + (\omega-1)c_1(\mathbf{y})\widehat{H}_2(\mathbf{x},\mathbf{y})\right) \\ &= \frac{1}{2}\left(x_1^2+x_2^2+x_3^2\right) + (\omega-1)y_3\left(x_1y_1+x_2y_2+\omega x_3y_3\right). \end{aligned} \tag{60}$$

The Hamilton's equations in this case assume the form

$$\begin{aligned} \dot{x}_1 &= x_2y_3 - x_3y_2 \\ \dot{x}_2 &= x_3y_1 - x_1y_3 \\ \dot{x}_3 &= x_1y_2 - x_2y_1 \\ \dot{y}_1 &= -x_2 - (\omega-1)y_2y_3 \\ \dot{y}_2 &= x_1 + (\omega-1)y_1y_3 \\ \dot{y}_3 &= 0 \end{aligned} \tag{61}$$

see [19, 22, 30]. *The above equations describe the motion Lagrange top.*

Table 1. Lie algebras of dimensions three.

Name	Nonzero commutation relations	Invariants
$\mathfrak{a}_{3,1}$	$[e_2, e_3] = e_1$	e_1
$\mathfrak{a}_{3,2}$	$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2$	$e_1 \exp(-e_2/e_1)$
$\mathfrak{a}_{3,3}$	$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_2$	e_2/e_1
$\mathfrak{a}_{3,4}$	$[e_1, e_3] = e_1, \quad [e_2, e_3] = -e_2$	$e_1 e_2$
$\mathfrak{a}^a_{3,5}$	$[e_1, e_3] = e_1, \quad [e_2, e_3] = ae_2 \quad 0 < \|a\| < 1)$	$e_2 e_1^{-a}$
$\mathfrak{a}_{3,6}$	$[e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1$	$e_1^2 + e_2^2$
$\mathfrak{a}^a_{3,7}$	$[e_1, e_3] = ae_1 - e_2, [e_2, e_3] = e_1 + ae_2 (a > 0)$	$(e_1^2 + e_2^2)(\frac{e_1 + ie_2}{e_1 - ie_2})^{ia}$
$\mathfrak{a}_{3,8}$	$[e_1, e_3] = -2e_2, \quad [e_1, e_2] = e_1, \quad [e_2, e_3] = e_3$	$2e_2^2 + e_1 e_3 + e_3 e_1$
$\mathfrak{a}_{3,9}$	$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2$	$e_1^2 + e_2^2 + e_3^2$

Example 12. *Let us consider all real Lie algebras of dimension equal to three. There are nine real algebras of dimension three, two of which depend on a parameter. Our list* (Table 1) *is based on the article* [27].

On the dual space $\mathfrak{a}^*_{3,i}$, $i = 1, \ldots, 9$, *of a Lie algebra* $\mathfrak{a}_{3,i}$ *is a Poisson structure defined by Poisson tensor* $\pi_{3,i}$. Table 2 *below presents the corresponding tensors for the various Lie algebras.*

The next table describes if the above structures are compatible in sense (7), *i.e.,* $\mathbb{R}^3$ *equipped with these is a bi-Hamiltonian manifold* (Table 3).

We can apply the analogous procedure as in Example 1 *to other three dimensional Lie algebras. We will take the linear Poisson tensor on the manifold* $\mathfrak{a}^*_{3,8}$ *and frozen Poisson tensor compatible with it*

$$\pi_1(\mathbf{x}) = \begin{pmatrix} 0 & x_1 & -2x_2 \\ -x_1 & 0 & x_3 \\ 2x_2 & -x_3 & 0 \end{pmatrix}, \qquad \pi_2(\mathbf{x}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{62}$$

The Casimir function for the structure π_1 *is*

$$H(\mathbf{x}) = x_2^2 + x_1 x_3 \tag{63}$$

and for π_2

$$c_1(\mathbf{x}) = x_3. \tag{64}$$

Table 2. Poisson tensors for Lie algebras of dimensions three.

Name	Poisson tensors	Name	Poisson tensors
$\mathfrak{a}_{3,1}^*$	$\pi_{3,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x_1 \\ 0 & -x_1 & 0 \end{pmatrix}$	$\mathfrak{a}_{3,6}^*$	$\pi_{3,6} = \begin{pmatrix} 0 & 0 & -x_2 \\ 0 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$
$\mathfrak{a}_{3,2}^*$	$\pi_{3,2} = \begin{pmatrix} 0 & 0 & x_1 \\ 0 & 0 & x_1 + x_2 \\ -x_1 & -x_1 - x_2 & 0 \end{pmatrix}$	$\left(\mathfrak{a}_{3,7}^a\right)^*$	$\pi_{3,7}^a = \begin{pmatrix} 0 & 0 & ax_1 - x_2 \\ 0 & 0 & x_1 + ax_2 \\ -ax_1 + x_2 & -x_1 - ax_2 & 0 \end{pmatrix}$
$\mathfrak{a}_{3,3}^*$	$\pi_{3,3} = \begin{pmatrix} 0 & 0 & x_1 \\ 0 & 0 & x_2 \\ -x_1 & -x_2 & 0 \end{pmatrix}$	$\mathfrak{a}_{3,8}^*$	$\pi_{3,8} = \begin{pmatrix} 0 & x_1 & -2x_2 \\ -x_1 & 0 & x_3 \\ 2x_2 & -x_3 & 0 \end{pmatrix}$
$\mathfrak{a}_{3,4}^*$	$\pi_{3,4} = \begin{pmatrix} 0 & 0 & x_1 \\ 0 & 0 & -x_2 \\ -x_1 & x_2 & 0 \end{pmatrix}$	$\mathfrak{a}_{3,9}^*$	$\pi_{3,9} = \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$
$\left(\mathfrak{a}_{3,5}^a\right)^*$	$\pi_{3,5}^a = \begin{pmatrix} 0 & 0 & x_1 \\ 0 & 0 & ax_2 \\ -x_1 & -ax_2 & 0 \end{pmatrix}$		

Table 3. Compatibility of Poisson structures for Lie algebras of dimensions three.

$\mathfrak{a}_{3,1}^*$	$\mathfrak{a}_{3,2}^*$	$\mathfrak{a}_{3,3}^*$	$\mathfrak{a}_{3,4}^*$	$\left(\mathfrak{a}_{3,5}^a\right)^*$	$\mathfrak{a}_{3,6}^*$	$\left(\mathfrak{a}_{3,7}^a\right)^*$	$\mathfrak{a}_{3,8}^*$	$\mathfrak{a}_{3,9}^*$	Name
YES	YES	YES	YES	YES	YES	YES	YES	YES	$\mathfrak{a}_{3,1}^*$
	YES	YES	YES	YES	YES	YES	NO	NO	$\mathfrak{a}_{3,2}^*$
		YES	YES	YES	YES	YES	NO	NO	$\mathfrak{a}_{3,3}^*$
			YES	YES	YES	YES	YES	YES	$\mathfrak{a}_{3,4}^*$
				YES	YES	YES	NO	NO	$\left(\mathfrak{a}_{3,5}^a\right)^*$
					YES	YES	YES	YES	$\mathfrak{a}_{3,6}^*$
						YES	NO	NO	$\left(\mathfrak{a}_{3,7}^a\right)^*$
							YES	YES	$\mathfrak{a}_{3,8}^*$
								YES	$\mathfrak{a}_{3,9}^*$

The Casimirs from Theorem 8 *for the structure*

$$(\pi_1 \ltimes_1 \pi_2)(\mathbf{x},\mathbf{y}) = \left(\begin{array}{ccc|ccc} 0 & y_1 & -2y_2 & 0 & 1 & 0 \\ -y_1 & 0 & y_3 & -1 & 0 & 0 \\ 2y_2 & -y_3 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{65}$$

assume following form

$$c_1(\mathbf{y}) = y_3, \qquad \widetilde{c}_1(\mathbf{x},\mathbf{y}) = x_3 + y_2^2 + y_1 y_3. \tag{66}$$

The Casimir function H for the Poisson structure π_1 is quadratic homogeneous in $\mathbf{x}$. *Moreover, this function and*

$$\widehat{\widehat{H}}_1(\mathbf{x}) = x_2^2 + x_1 x_3 + \lambda x_3^2 \tag{67}$$

are in involution with respect to the Poisson tensor π_2, where λ is any constant. So from Theorem 10 *we obtain that* (67) *and*

$$\widehat{H}_2(\mathbf{x},\mathbf{y}) = 2x_2 y_2 + x_1 y_3 + x_3 y_1 \tag{68}$$

are in involution with respect to the Poisson structure (65)*. If we take as the Hamiltonian*

$$h(\mathbf{x},\mathbf{y}) = x_2^2 + x_1 x_3 + \lambda x_3^2 + \alpha y_3 \left(2x_2 y_2 + x_1 y_3 + x_3 y_1\right) \tag{69}$$

we obtain the following Hamilton's equations

$$\begin{aligned} \dot{x}_1 &= 2\alpha x_2 y_3 - 4\lambda x_3 y_2 + 2y_1 x_2 - 2x_1 y_2 \\ \dot{x}_2 &= x_1 y_3 - x_3 y_1 + (2\lambda - \alpha) x_3 y_3 \\ \dot{x}_3 &= 2(x_3 y_2 - x_2 y_3) \\ \dot{y}_1 &= 2x_2 + 2\alpha y_2 y_3 \\ \dot{y}_2 &= -x_3 - \alpha y_3^3 \\ \dot{y}_3 &= 0 \end{aligned} \tag{70}$$

where α is an arbitrary constant.

Example 13. *Table 4 shows which Poisson structures related to the Lie algebras of dimension three, see* Example 2*, can be used to build a Poisson structure of dimension six. A table row $\mathfrak{a}_{3,i}^*$ and table column $\mathfrak{a}_{3,j}^*$ means that, we are building a Poisson tensor* (Theorem 3)

$$(\pi_{3,i} \ltimes_1 \pi_{3,j}) = \left(\begin{array}{c|c} \pi_{3,i}(\mathbf{y}) & \pi_{3,j}(\mathbf{x}) \\ \hline \pi_{3,j}(\mathbf{x}) & \pi_{3,j}(\mathbf{y}) \end{array}\right). \tag{71}$$

Table 4. Compatibility in the sense of Theorem 3 of Poisson structures for Lie algebras of dimensions three.

Name	$\mathfrak{a}_{3,1}^*$	$\mathfrak{a}_{3,2}^*$	$\mathfrak{a}_{3,3}^*$	$\mathfrak{a}_{3,4}^*$	$\left(\mathfrak{a}_{3,5}^a\right)^*$	$\mathfrak{a}_{3,6}^*$	$\left(\mathfrak{a}_{3,7}^a\right)^*$	$\mathfrak{a}_{3,8}^*$	$\mathfrak{a}_{3,9}^*$
$\mathfrak{a}_{3,1}^*$	YES	YES	YES	NO	NO	NO	NO	NO	NO
$\mathfrak{a}_{3,2}^*$	YES	YES	YES	NO	NO	NO	NO	NO	NO
$\mathfrak{a}_{3,3}^*$	YES	YES	YES	YES	YES	YES	YES	NO	NO
$\mathfrak{a}_{3,4}^*$	NO	NO	YES	YES	YES	NO	NO	NO	NO
$\left(\mathfrak{a}_{3,5}^a\right)^*$	NO	NO	YES	YES	YES	NO	NO	NO	NO
$\mathfrak{a}_{3,6}^*$	NO	NO	YES	NO	NO	YES	YES	NO	NO
$\left(\mathfrak{a}_{3,7}^a\right)^*$	NO	NO	YES	NO	NO	YES	YES	NO	NO
$\mathfrak{a}_{3,8}^*$	NO	NO	NO	NO	NO	NO	NO	YES	NO
$\mathfrak{a}_{3,9}^*$	NO	NO	NO	NO	NO	NO	NO	NO	YES

Let us consider, in table given above, the row $\mathfrak{a}_{3,3}^*$ and the column $\mathfrak{a}_{3,1}^*$. It means we are building a Poisson tensor

$$\left(\pi_{3,3} \ltimes_1 \pi_{3,1}\right)(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} \pi_{3,3}(\mathbf{y}) & \pi_{3,1}(\mathbf{x}) \\ \hline \pi_{3,1}(\mathbf{x}) & \pi_{3,1}(\mathbf{y}) \end{array}\right). \tag{72}$$

In local coordinates $\mathbf{x}=(x_1,x_2,x_3), \mathbf{y}=(y_1,y_2,y_3)$ we get

$$\left(\pi_{3,3} \ltimes_1 \pi_{3,1}\right)(\mathbf{x},\mathbf{y}) = \left(\begin{array}{ccc|ccc} 0 & 0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & y_2 & 0 & 0 & x_1 \\ -y_1 & -y_2 & 0 & 0 & -x_1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 & y_1 \\ 0 & -x_1 & 0 & 0 & -y_1 & 0 \end{array}\right). \tag{73}$$

We can prove by direct calculation and changing the variables that this is a tensor for a Lie-Poisson structure related to the Lie algebra $\mathfrak{a}_{6,16}$ from the classification given in [27]. *The commutation relations for $\mathfrak{a}_{6,16}$ are $[e_1,e_3]=e_4$, $[e_1,e_4]=e_5$, $[e_1,e_5]=e_6$, $[e_2,e_3]=e_5$ and $[e_2,e_4]=e_6$, where $(x_1,x_2,x_3,y_1,y_2,y_3)\mapsto(-e_5,-e_3,e_1,e_6,e_4,-e_2)$. In this case, the Casimirs assume the following form $c_1(\mathbf{x})=\dfrac{x_2}{x_1}$ for π_1 and $c_2(\mathbf{x})=x_1$ for π_2, respectively. Moreover, the Casimirs for the $\pi_1 \ltimes_1 \pi_2$ are given by the formulas $c_1(\mathbf{x},\mathbf{y})=y_1$ and $c_2(\mathbf{x},\mathbf{y})=\frac{1}{3}x_1^3 - x_1y_1y_2+x_2y_1^2$.*

Let us take now the row $\mathfrak{a}_{3,2}^*$ and the column $\mathfrak{a}_{3,1}^*$. It means we are building a Poisson tensor

$$\left(\pi_{3,2} \ltimes_1 \pi_{3,1}\right)(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} \pi_{3,2}(\mathbf{y}) & \pi_{3,1}(\mathbf{x}) \\ \hline \pi_{3,1}(\mathbf{x}) & \pi_{3,1}(\mathbf{y}) \end{array}\right). \tag{74}$$

In local coordinates $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3)$ *we get*

$$(\pi_{3,2} \ltimes_1 \pi_{3,1})(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{ccc|ccc} 0 & 0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & y_1 + y_2 & 0 & 0 & x_1 \\ -y_1 & -y_1 - y_2 & 0 & 0 & -x_1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 & y_1 \\ 0 & -x_1 & 0 & 0 & -y_1 & 0 \end{array}\right). \tag{75}$$

We can prove by direct calculation and changing the variables that this is also a tensor for a Lie-Poisson structure related also to the Lie algebra $\mathfrak{a}_{6,16}$*, where* $(x_1, x_2, x_3, y_1, y_2, y_3) \mapsto (x_1, x_2, x_3, y_1, y_1{+}y_2, y_3) = (-e_5, -e_3, e_1, e_6, e_4, -e_2)$*. In this case, the Casimirs assume the following form* $c_1(\mathbf{x}) = x_1$ *for* π_1 *and* $c_2(\mathbf{x}) = \dfrac{x_2}{x_1}$ *for* π_2*, respectively. Moreover, the Casimirs for the* $\pi_1 \ltimes_1 \pi_2$ *are given by the formulas* $c_1(\mathbf{x}, \mathbf{y}) = y_1$ *and* $c_2(\mathbf{x}, \mathbf{y}) = \frac{1}{3}x_1^3 - x_1 y_1 (y_1 + y_2) + x_2 y_1^2$*. We will get very important cases, the Poisson structure on* $\mathfrak{so}^*(4)$*, if we consider the following case*

$$(\pi_{3,9} \ltimes_1 \pi_{3,9})(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{c|c} \pi_{3,9}(\mathbf{y}) & \pi_{3,9}(\mathbf{x}) \\ \hline \pi_{3,9}(\mathbf{x}) & \pi_{3,9}(\mathbf{y}) \end{array}\right) \cong \pi_{3,9}(\mathbf{x} + \mathbf{y}) \oplus \pi_{3,9}(\mathbf{y} - \mathbf{x}) \tag{76}$$

and $\mathfrak{e}^*(3)$ *if we take*

$$\pi(\mathbf{x}, \mathbf{y}) = \left(\begin{array}{c|c} 0 & \pi_{3,9}(\mathbf{x}) \\ \hline \pi_{3,9}(\mathbf{x}) & \pi_{3,9}(\mathbf{y}) \end{array}\right). \tag{77}$$

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