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GRASSMANNIAN SIGMA-MODELS

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Abstract. We study solutions of Grassmannian sigma-model both in finite-dimensional and infinite-dimensional settings. Mathematically, such solutions are described by harmonic maps from the Riemann sphere $\mathbb{CP}^1$ or, more generally, compact Riemann surfaces to Grassmannians. We describe first how to construct harmonic maps from compact Riemann surfaces to the Grassmann manifold $\mathrm{G}_r(\mathbb{C}^d)$, using the twistor approach. Then we switch to the infinite-dimensional setting and consider harmonic maps from compact Riemann surfaces to the Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ of a complex Hilbert space H. Solutions of this infinite-dimensional sigma-model are, conjecturally, related to Yang–Mills fields on $\mathbb{R}^4$.

1. Introduction

In this paper we describe classical solutions of Grassmannian sigma-models in finite-dimensional and infinite-dimensional settings. The study of such solutions in the finite-dimensional case was initiated by physicists (cf. e.g., [4,8,13]). Mathematically, sigma-model solutions correspond to harmonic maps from compact Riemann surfaces to Grassmannians $\mathrm{G}_r(\mathbb{C}^d)$.

In the first part of this paper (Sections 2, 3 and 4) we explain how to construct such maps, using the twistor approach. The main idea of this approach, when applied to the construction of harmonic maps from a Riemann surface M to a given Riemannian manifold N, is to construct a certain twistor bundle $\pi : Z \to N$ over N, which has the following property. The twistor space Z is an almost complex manifold such that for any pseudoholomorphic map $\psi : M \to Z$ its projection $\varphi := \pi \circ \psi$ to N is a harmonic map $\varphi : M \to N$. In our case $N = \mathrm{G}_r(\mathbb{C}^d)$ and the role of the twistor bundle over $\mathrm{G}_r(\mathbb{C}^d)$ is played by homogeneous flag bundles $\pi : \mathcal{F}_{\mathbf{r}}(\mathbb{C}^d) \to \mathrm{G}_r(\mathbb{C}^d)$. Using the twistor approach, one can try to reduce the original "real" problem of constructing harmonic maps of compact Riemann surfaces M to $\mathrm{G}_r(\mathbb{C}^d)$ to the "complex" problem of constructing pseudoholomorphic maps $M \to \mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$. In the case of the Riemann sphere $M = \mathbb{CP}^1$ both problems are, in fact, equivalent, as shown in [5]. A complete description of harmonic maps $\mathbb{CP}^1 \to \mathrm{G}_r(\mathbb{C}^d)$, i.e., harmonic spheres in $\mathrm{G}_r(\mathbb{C}^d)$, was given by Wood in [17] and reformulated in twistor terms in [6].

In the second part of the paper (Section 5) we switch to the infinite-dimensional case and consider harmonic maps from compact Riemann surfaces to the Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ of a complex (separable) Hilbert space H, modelled on the space $L^2(S^1, \mathbb{C})$ of square integrable functions on the unit circle S^1. This Grassmannian consists of closed (infinite-dimensional) subspaces W in H, "differing not much" from the standard Hardy subspace $H_+ = H^2$ in $L^2(S^1, \mathbb{C})$. "Differing not much" means that the orthogonal projection of such a subspace W to H_+ is Fredholm, while its orthogonal projection to the orthogonal complement $H_- := H_+^{\perp}$ is Hilbert–Schmidt. All subspaces $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$ are, of course, infinite-dimensional, but they have a finite "virtual dimension", given by the Fredholm index of their projection to H_+. Using this fact, we can construct homogeneous "virtual" flag bundles $\mathcal{F}_{\mathbf{r}}(H) \to \mathrm{Gr}_{\mathrm{HS}}(H)$, playing the role of twistor bundles over the Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$. Applying again the twistor approach, we can construct harmonic maps $M \to \mathrm{Gr}_{\mathrm{HS}}(H)$ as projections of pseudoholomorphic maps $M \to \mathcal{F}_{\mathbf{r}}(H)$.

Solutions of the described infinite-dimensional Grassmann sigma-model in the case $M = \mathbb{CP}^1$ are, conjecturally, related to the Yang–Mills fields on $\mathbb{R}^4$. This conjecture is based on the Atiyah's result, asserting that for a compact Lie group G the moduli space of G-instantons on $\mathbb{R}^4$ can be identified with the space of (based) holomorphic maps from the Riemann sphere $\mathbb{CP}^1$ into the loop space ΩG. Motivated by this result, we can expect that the space of (based) harmonic maps $\mathbb{CP}^1 \to \Omega G$ can be likewise identified with the moduli space of Yang–Mills G-fields on $\mathbb{R}^4$. Since the loop space ΩG can be isometrically embedded into the Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$, we can construct harmonic spheres in ΩG, as in the finite-dimensional case, by projecting pseudoholomorphic spheres in virtual flag manifolds $\mathcal{F}_{\mathbf{r}}(H)$ to $\mathrm{Gr}_{\mathrm{HS}}(H)$.

Brief content of the paper. We start by recalling basic properties of harmonic maps of Riemannian manifolds in Section 2. In Section 3 we restrict to the case of Grassmann manifolds $\mathrm{G}_r(\mathbb{C}^d)$ and define homogeneous flag bundles over $\mathrm{G}_r(\mathbb{C}^d)$. In Section 4 a twistor construction of harmonic maps into Grassmannians, due to [6] and [5], is presented. In Section 6 we introduce the Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ and Grassmannians $\mathrm{G}_r(H)$ of virtual dimension r. The loop space ΩG can be isometrically embedded into $\mathrm{Gr}_{\mathrm{HS}}(H)$, so that harmonic maps $\varphi : M \to \Omega G$ can be considered as harmonic maps into

Grassmannians $\mathrm{G}_r(H)$. Harmonic maps $\varphi : M \to \mathrm{G}_r(H)$ may be constructed as projections of pseudoholomorphic maps $\psi : M \to \mathcal{F}_{\mathbf{r}}(H)$ to virtual flag manifolds $\mathcal{F}_{\mathbf{r}}(H)$.

2. Harmonic Maps. General Properties

Let $\varphi : (M, g) \to (N, h)$ be a smooth map of a Riemannian manifold M with a Riemannian metric g into a Riemannian manifold N with a Riemannian metric h. We define the *energy* of the map φ as the Dirichlet integral

$$E(\varphi) = \frac{1}{2} \int_M |\mathrm{d}\varphi(p)|^2 \mathrm{vol}_g \,. \tag{1}$$

The norm of the differential may be computed in local coordinates as follows. Denote by (x^i) local coordinates at $p \in M$ and by (u^α) local coordinates at $q = \varphi(p) \in N$. Then

$$|\mathrm{d}\varphi(p)|^2 = \sum_{i,j} \sum_{\alpha,\beta} g^{ij} \, \frac{\partial \varphi^\alpha}{\partial x^i} \, \frac{\partial \varphi^\beta}{\partial x^j} \, h_{\alpha\beta}$$

where $\varphi^\alpha = \varphi^\alpha(x)$ are the components of φ, (g_{ij}) and $(h_{\alpha\beta})$ are the metric tensors of M and N respectively, (g^{ij}) is the inverse matrix of (g_{ij}) and vol_g is the volume element of the metric g.

Definition 1. *A smooth map $\varphi : M \to N$ is called* harmonic *if it is extremal for the functional $E(\varphi)$ with respect to all smooth variations of φ with compact supports.*

The Euler–Lagrange equation for the energy functional $E(\varphi)$ is called otherwise the *harmonic map equation*. In the local coordinates (x^i) on M and (u^α) on N, introduced above, it has the following form

$$\Delta_M \varphi^\gamma + \sum_{i,j} g^{ij} \sum_{\alpha,\beta} {}^N\Gamma^\gamma_{\alpha\beta}(\varphi) \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_j} = 0 \tag{2}$$

where Δ_M is the standard Laplace–Beltrami operator on M, given by

$$\Delta_M \varphi^\gamma = \sum_{i,j} g^{ij} \left\{ \frac{\partial^2 \varphi^\gamma}{\partial x_i \partial x_j} - \sum_k {}^M\Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x_k} \right\} \,.$$

Here, ${}^M\Gamma^k_{ij}$ denotes the Christoffel symbols of the Levi-Civita connection ${}^M\nabla$ of M and ${}^N\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols of the Levi-Civita connection ${}^N\nabla$ of N. In the particular case $N = \mathbb{R}^n$ the equation (2) becomes linear and reduces to the Laplace–Beltrami equation

$$\Delta_M \varphi^\gamma = 0 \,, \qquad \gamma = 1, \dots, n$$

on the components of the map φ.

A non-trivial nonlinear example of harmonic maps is provided by the so called $\mathrm{SO}(3)$-*model*, arising in the theory of ferromagnets. In this example we consider smooth maps $\varphi : \mathbb{R}^2 \to S^2$ with finite energy $E(\varphi) < \infty$. The finite energy condition implies that such maps should stabilize at infinity, i.e., $\varphi(x) \to \varphi_0$ for $|x| \to \infty$. Therefore, φ extends to a map

$$\varphi : S^2 = \mathbb{R}^2 \cup \infty \longrightarrow S^2$$

which has a topological invariant, called the *degree* of the map φ

$$\deg \varphi = \int_{S^2} \varphi^* \mathrm{vol} \,.$$

Here, vol is the normalized volume form on S^2. It is useful to introduce here complex coordinates in order to have better formulas. We denote by $z = x_1 + \mathrm{i} x_2$ the complex coordinate on $\mathbb{R}^2$ and by w the complex coordinate in the image $S^2 \setminus \{\infty\}$, given by the stereographic projection.

Then the energy of the map $\varphi = w(z)$ in these coordinates will be given by the following formula

$$E(\varphi) = 2 \int\limits_{\mathbb{C}} \frac{|\partial_z w|^2 + |\partial_{\bar{z}}|^2}{(1 + |w|^2)^2} \, |\mathrm{d}z \wedge \mathrm{d}\bar{z}| \tag{3}$$

while the degree of φ is computed, according to

$$\deg \varphi = \frac{1}{2\pi} \int\limits_{\mathbb{C}} \frac{|\partial_z w|^2 - |\partial_{\bar{z}}|^2}{(1 + |w|^2)^2} \, |\mathrm{d}z \wedge \mathrm{d}\bar{z}| \,. \tag{4}$$

Comparing the last two formulas, we obtain an estimate of the energy from below

$$E(\varphi) \geq 4\pi |\deg \varphi| \,. \tag{5}$$

It follows that the minimum of the energy $E(\varphi)$ for a fixed $k = \deg\, \varphi$ is attained on holomorphic functions $w = \varphi(z)$ for $k \geq 0$, and on antiholomorphic functions $w = \varphi(z)$ for $k \leq 0$.

Fixing the asymptotic value φ_0 by the SO(3)-invariance (we set $\varphi_0 = 1$), one can write down the minima of $E(\varphi)$ for $k \geq 0$ in the form

$$w = \varphi(z) = \prod_{j=1}^{k} \frac{z - a_j}{z - b_j}$$

where a_j, b_j are arbitrary complex numbers. In particular, the space of minima for a fixed k is parameterized by $4k + 2$ real parameters.

If we compare the harmonic map equation with the Yang–Mills duality equations on $\mathbb{R}^4$, then the holomorphic (respectively, anti-holomorphic) maps $\varphi : \mathbb{R}^2 \cup \infty \to S^2$ will correspond to the instanton (respectively, anti-instanton) solutions of the duality equations. We shall see later that this correspondence can be established on a more deep level.

It may be shown that in the case of SO(3)-model the energy functional $E(\varphi)$ has no critical points, except for the described local minima. In other words, there are no other harmonic maps $\varphi : \mathbb{R}^2 \cup \infty \to S^2$, apart from the holomorphic and anti-holomorphic ones. We note that the holomorphic and anti-holomorphic maps yield the local minima of the energy $E(\varphi)$ also for smooth maps between general complex manifolds.

Namely, suppose that our Riemannian manifold (M, g) is provided with a complex (or almost complex) structure ${}^M J$, compatible with the Riemannian metric g, and, likewise, the target manifold (N, h) has a complex (or almost complex) structure ${}^N J$, compatible with the Riemannian metric h.

Definition 2. *A smooth map $\varphi : M \to N$ is called* (pseudo)holomorphic *if and only if the tangent map $\varphi_* : TM \to TN$ commutes with the (almost) complex structures on M and N, i.e.,*

$$\varphi_* \circ {}^M J = {}^N J \circ \varphi_*$$

and it is called anti-(pseudo)holomorphic *if and only if φ_* anti-commutes with the (almost) complex structures on M and N.*

The complexified tangent bundle $T^{\mathbb{C}}M = TM \otimes \mathbb{C}$ can be decomposed into the direct sum

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$$

of subbundles with fibres, given by the $(\pm \mathrm{i})$-eigenspaces of the almost complex structure operator ${}^M J$. If we extend the tangent map φ_* complex linearly to the complexified tangent bundles, then we obtain a map $\varphi_*\colon T^{\mathbb{C}}M \to T^{\mathbb{C}}N$, which, in accordance with the above decomposition, splits into the sum of four operators

$$\partial'\varphi\colon T^{1,0}M \to T^{1,0}N, \qquad \partial''\varphi\colon T^{0,1}M \to T^{1,0}N \tag{6}$$

$$\partial'\bar{\varphi} = \overline{\partial''\varphi}\colon T^{1,0}M \to T^{0,1}N, \qquad \partial''\bar{\varphi} = \overline{\partial'\varphi}\colon T^{0,1}M \to T^{0,1}N\ . \tag{7}$$

The introduced operators may be considered as sections of the bundle $(T^*M)^{\mathbb{C}} \otimes \varphi^{-1}(T^{\mathbb{C}}N)$. In these notations a map φ is pseudoholomorphic (respectively anti-pseudoholomorphic) if and only if $\partial''\varphi = 0$ (respectively $\partial'\varphi = 0$).

Generalizing the phenomena, observed for the $\mathrm{SO}(3)$-model, it may be proved (cf. [9]) that for the (almost) Kähler manifolds the holomorphic and anti-holomorphic maps $\varphi : M \to N$ always realize the local minima of the energy functional $E(\varphi)$ but, in general, there exist other critical points of $E(\varphi)$, i.e., non-minimal harmonic maps.

We restrict now to the case, when M is a compact Riemann surface. Denote by ∇ the connection on the bundle $\varphi^{-1}(T^{\mathbb{C}}N)$ over M, induced by the Levi–Civita connection ${}^N\nabla$ on the Riemannian manifold N. If z is a local complex coordinate on M, we set

$$\delta\varphi = \varphi_*(\partial/\partial z)\ , \qquad \bar{\delta}\varphi = \varphi_*(\partial/\partial\bar{z})$$

where $\delta\varphi$ and $\bar{\delta}\varphi$ are considered as sections of the bundle $\varphi^{-1}(T^{\mathbb{C}}N)$. (More generally, we denote by $\delta = \nabla_{\partial/\partial z}$, $\bar{\delta} = \nabla_{\partial/\partial\bar{z}}$ the components of the connection ∇.) The differential $d\varphi$ is represented in the form

$$\mathrm{d}\varphi = \mathrm{d}z \otimes \delta\varphi + \mathrm{d}\bar{z} \otimes \bar{\delta}\varphi$$

and the harmonic map equation (2) may be written in the form

$$\bar{\delta}\delta\varphi = \left(\nabla_{\partial/\partial\bar{z}}\varphi_*\right)\left(\frac{\partial}{\partial z}\right) = 0 \tag{8}$$

or, equivalently, as

$$\delta\bar{\delta}\varphi = \left(\nabla_{\partial/\partial z}\varphi_*\right)\left(\frac{\partial}{\partial\bar{z}}\right) = 0\ .$$

If N is a Kähler manifold, then, according to (6)

$$\delta\varphi = \partial'\varphi + \overline{\partial''\varphi}\ , \qquad \bar{\delta}\varphi = \partial''\varphi + \overline{\partial'\varphi}$$

and the harmonic map equation for φ takes the form

$$\bar{\delta}\partial'\varphi = 0 \tag{9}$$

or equivalently

$$\delta\partial''\varphi = 0 \, .$$

According to the Koszul–Malgrange theorem (cf. [11]), any complex vector bundle E over a Riemann surface M with a connection ∇ has a unique complex structure J, such that $E \to M$ is a holomorphic vector bundle with respect to J, for which the $\bar{\partial}_J$-operator coincides with the $(0,1)$-component $\nabla^{0,1}$ of the connection ∇. This complex structure J is called the *KM-structure*.

In its terms, the first of the harmonicity conditions (8) means that $\delta\varphi$ is a holomorphic section of the bundle $\varphi^{-1}(T^{\mathbb{C}}N)$ with respect to the KM-structure on $\varphi^{-1}(T^{\mathbb{C}}N)$, induced by the connection ${}^N\nabla$. In the same way, the first of conditions (9) means that $\partial'\varphi$ is a holomorphic section of the bundle $\varphi^{-1}(T^{1,0}N)$.

3. Flag Manifolds and Flag Bundles

To define the flag manifolds in $\mathbb{C}^d$, we fix a decomposition of d into the sum of natural numbers $d = r_1 + \cdots + r_n$ and denote $\mathbf{r} := (r_1, \ldots, r_n)$.

Definition 3. *A* flag manifold $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$ *of type* $\mathbf{r}$ *in* $\mathbb{C}^d$ *consists of collections* $\mathcal{E} = (E_1, \ldots, E_n)$ *of mutually orthogonal linear subspaces* E_i *of dimension* r_i *in* $\mathbb{C}^d$ *such that* $\mathbb{C}^d = E_1 \oplus \cdots \oplus E_n$.

By this definition, a flag is a collection of mutually orthogonal subspaces, rather than a nested sequence of linear subspaces, associated with the standard image of a flag. However, one can easily produce a standard flag $(V_1, \ldots, V_n)$ in $\mathbb{C}^d$ with $V_1 \subset \cdots \subset V_n = \mathbb{C}^d$ from our collection $\mathcal{E} = (E_1, \ldots, E_n)$, setting $V_i := E_1 \oplus \cdots \oplus E_i$.

In particular, for $\mathbf{r} = (r, d-r)$ the flag manifold

$$\mathcal{F}_{(r,d-r)}(\mathbb{C}^d) = \{\mathcal{E} = (E, E^{\perp}) \, ; \, \dim E = r\} = G_r(\mathbb{C}^d)$$

coincides with the Grassmann manifold of r-dimensional subspaces in $\mathbb{C}^d$.

We have the following homogeneous representation of the flag manifold

$$\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d) = \mathrm{U}(d)/\,\mathrm{U}(r_1) \times \cdots \times \mathrm{U}(r_n) \, .$$

There is also another, complex homogeneous representation for this manifold

$$\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d) = \mathrm{GL}(d, \mathbb{C})/\,\mathcal{P}_{\mathbf{r}}$$

where $\mathcal{P}_{\mathbf{r}}$ is the parabolic subgroup of blockwise upper-triangular matrices of the form

$$\begin{pmatrix} \underset{r_1}{\boxed{\quad * \quad}}\; r_1 & * & * & \cdots & * \\ 0 & \underset{r_2}{\boxed{\quad * \quad}}\; r_2 & * & \cdots & * \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & r_n\; \overset{r_n}{\boxed{\quad * \quad}} \end{pmatrix}$$

with blocks of dimensions $r_i \times r_i$ in the boxes.

These representations imply that $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$ has a natural complex structure, which we denote by J^1. Moreover, $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$, provided with this complex structure, is a compact Kähler manifold.

In the particular case $\mathbf{r} = (r, N - r)$ we obtain well known homogeneous representations for the Grassmann manifold

$$G_r(\mathbb{C}^d) = \mathrm{U}(d)/\mathrm{U}(r) \times \mathrm{U}(d-r) = \mathrm{GL}(d,\mathbb{C})/P_{(r,d-r)} \, .$$

We construct now a series of homogeneous flag bundles over the Grassmann manifold $G_r(\mathbb{C}^c)$, which play an important role in the sequel. Let $\mathcal{F} = \mathcal{F}_{\mathbf{r}}(\mathbb{C}^N)$ be the flag manifold of type $\mathbf{r} = (r_1, \ldots, r_n)$ in $\mathbb{C}^d$ with the homogeneous representation

$$\mathcal{F} = \mathcal{F}_{\mathbf{r}}(\mathbb{C}^N) = \mathrm{U}(d)/\mathrm{U}(r_1) \times \cdots \times \mathrm{U}(r_n) \, .$$

On the Lie algebra level this representation corresponds to the decomposition of the complexified Lie algebra $\mathfrak{u}^{\mathbb{C}}(d)$ into the direct orthogonal sum

$$\begin{aligned} \mathfrak{u}^{\mathbb{C}}(d) &\cong \mathfrak{gl}(d,\mathbb{C}) \cong \overline{\mathbb{C}^d} \otimes \mathbb{C}^d \cong \left(\bar{E}_1 \oplus \cdots \oplus \bar{E}_n\right) \otimes (E_1 \oplus \cdots \oplus E_n) \\ &\cong \left[\mathfrak{u}^{\mathbb{C}}(r_1) \oplus \cdots \oplus \mathfrak{u}^{\mathbb{C}}(r_n)\right] \oplus \left[\bigoplus_{i<j} \left(\bar{E}_i E_j \oplus \bar{E}_j E_i\right)\right]. \end{aligned} \tag{10}$$

In the latter formula we have omitted the symbol of the tensor product in the expression $\bar{E}_i E_j$ and its conjugate in order to make the formulas more visible. The same rule will be applied in the sequel.

The above decomposition of the Lie algebra $\mathfrak{u}^{\mathbb{C}}(d)$ implies that the complexified tangent space $T_o^{\mathbb{C}}\mathcal{F}$ at the origin $o \in \mathcal{F}$ coincides with

$$T_o^{\mathbb{C}}\mathcal{F} = \bigoplus_{i<j} D_{ij}^{\mathbb{C}} := \bigoplus_{i<j} \left(\bar{E}_i E_j \oplus \bar{E}_j E_i\right) .$$

Every component D_{ij} may be provided with two different complex structures: for one of them its $(1,0)$-subspace coincides with $\bar{E}_i E_j$, for another with $\bar{E}_j E_i$. By the Borel–Hirzebruch theorem [2], any $U(d)$-invariant almost complex structure J on $\mathcal{F}$ is determined by the choice of one of these two complex structures on every D_{ij}. The almost complex structure J^1, for which

$$T_o^{1,0}\mathcal{F} = \bigoplus_{i<j} \bar{E}_i E_j$$

is called *canonical*.

Fix an ordered subset $\sigma \subset \{1,\dots,n\}$. Denote by σ^c the complement of σ in $\{1,\dots,n\}$ and set $r := \sum_{i\in\sigma} r_i$. We can associate with any of such subsets σ a homogeneous bundle

$$\pi_\sigma\colon\ \mathcal{F}_{\mathbf{r}}(\mathbb{C}^N) = \frac{U(d)}{U(r_1)\times\ldots\times U(r_n)} \longrightarrow \frac{U(d)}{U(r)\times U(d-r)} = G_r(\mathbb{C}^d) \quad (11)$$

by assigning: $(E_1,\dots,E_n) \longmapsto E = \bigoplus_{i\in\sigma} E_i$.

The complexified tangent bundle $T^{\mathbb{C}}\mathcal{F}_{\mathbf{r}}(\mathbb{C}^N)$ is decomposed into the direct sum of vertical and horizontal subbundles. Namely, the vertical subspace at the origin coincides with $\bigoplus_{i,j} D_{ij}^{\mathbb{C}}$, where $i<j$ and either $i,j\in\sigma$, or $i,j\in\sigma^c$. Respectively, the horizontal subspace at the origin is equal to $\bigoplus_{i,j} D_{ij}^{\mathbb{C}}$, where $i<j$ and either $i\in\sigma, j\in\sigma^c$, or $i\in\sigma^c, j\in\sigma$.

We introduce, along with the canonical complex structure J^1, a new $\mathrm{U}(d)$-invariant almost complex structure J^2 on $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^N)$, by setting it equal to $J^2 = J^1$ on horizontal tangent vectors and $J^2 = -J^1$ on vertical tangent vectors. Note that the constructed homogeneous bundle π_σ is not, generally speaking, holomorphic with respect to both almost complex structures. Moreover, the almost complex structure J^2 is never integrable. However, it turns out that precisely this complex structure is related to harmonic maps.

4. Twistor Construction of Harmonic Maps into the Grassmannian $\mathrm{G}_r(\mathbb{C}^d)$

Recall a general definition of the twistor bundle.

Definition 4. *Let N be a Riemannian manifold and Z is an almost complex manifold. A smooth bundle $\pi : Z \to N$ is called the twistor bundle, if for any pseudoholomorphic map $\psi : M \to Z$ of any Riemann surface M into the manifold Z its projection $\varphi = \pi \circ \psi : M \to N$ is a harmonic map.*

Using the twistor bundle $\pi : Z \to N$, one can effectively construct harmonic maps $M \to N$ by projecting pseudoholomorphic maps $M \to Z$ to N. A general theory of twistor bundles is presented in [7], here we restrict to the case of Grassmann manifolds. We'll show that the homogeneous flag bundles π_σ, constructed in the previous Section, are, in fact, twistor bundles in the sense of the above definition.

Let M be a Riemann surface. Denote by $M \times \mathbb{C}^d$ the trivial bundle $M \times \mathbb{C}^d \to M$, provided with the standard Hermitian metric on $\mathbb{C}^d$. Any subbundle $E \subset M \times \mathbb{C}^d$ of rank r defines a map $\varphi_E : M \longrightarrow G_r(\mathbb{C}^d)$ by setting: $\varphi_E(p) :=$ the fibre E_p at $p \in M$. Conversely, any map $\varphi : M \to G_r(\mathbb{C}^d)$ defines a subbundle $E \subset M \times \mathbb{C}^d$ of rank r.

Consider a smooth map of a Riemann surface M into the Grassmannian $G_r(\mathbb{C}^d)$. Denote by π and $\pi^\perp$ the orthogonal projections of $M \times \mathbb{C}^d$ onto the subbundle E and its orthogonal complement $E^\perp$. The bundle E is provided with the complex KM-structure, which is determined in a local chart on M with a local coordinate z by the $\bar\partial$-operator

$$\partial''_E = \pi \circ \frac{\partial}{\partial z} \circ \pi \ .$$

The inverse image $\varphi_E^{-1}(T^{\mathbb{C}} G_r(\mathbb{C}^d))$ of the complexified tangent bundle of the Grassmannian under the map φ_E admits a decomposition

$$\varphi_E^{-1}(T^{\mathbb{C}} G_r(\mathbb{C}^d)) \cong \bar{E} E^\perp \oplus \overline{E^\perp} E \ .$$

In terms of this decomposition the differential of φ_E has local components

$$A'_E := \pi^\perp \circ \frac{\partial}{\partial z} \circ \pi \ , \qquad A''_E := \pi^\perp \circ \frac{\partial}{\partial \bar z} \circ \pi \ .$$

(In the sequel we sometimes omit the symbol $\circ$ to simplify the formulas.) In particular, a bundle $E \subset M \times \mathbb{C}^d$ is holomorphic $\iff$ $A''_E = 0$, and in this case the complex KM-structure on E coincides with the complex structure, induced from $M \times \mathbb{C}^d$. Then

$$\begin{aligned} 0 &= \pi^\perp \left[\frac{\partial}{\partial z}(\pi + \pi^\perp)\frac{\partial}{\partial \bar z} - \frac{\partial}{\partial \bar z}(\pi + \pi^\perp)\frac{\partial}{\partial z} \right] \pi \\ &= A'_E \partial''_E + \partial'_{E^\perp} A''_E - A''_E \partial'_E - \partial''_{E^\perp} A'_E = A'_E \partial''_E - \partial''_{E^\perp} A'_E \ . \end{aligned} \tag{12}$$

Otherwise speaking, the bundle $A'_E \in \mathrm{Hom}(E, E^\perp)$ is holomorphic with respect to the KM-structures on E and $E^\perp$.

In general, we call a bundle $E \subset M \times \mathbb{C}^d$ *harmonic* if

$$A'_E \circ \partial''_E = \partial''_{E^\perp} \circ A'_E \ .$$

The harmonicity of E is equivalent to the harmonicity of the map $\varphi_E : M \to G_r(\mathbb{C}^d)$ (cf. [6]). Note also that a bundle E is harmonic $\iff$ its orthogonal complement $E^\perp$ is harmonic.

In a more general way, consider an arbitrary collection $\mathcal{E} = (E_1, \ldots, E_n)$ of mutually orthogonal subbundles E_i in $M \times \mathbb{C}^d$ of rank r_i with $r_1 + \ldots + r_n = d$, which generates a decomposition of $M \times \mathbb{C}^d$ into the direct orthogonal sum

$$M \times \mathbb{C}^d = \bigoplus_{i=1}^{n} E_i \ .$$

We call such a collection of subbundles $\mathcal{E} = (E_1, \ldots, E_n)$ the *moving flag* on M. It determines, in the same way as before, a map $\psi_{\mathcal{E}} : M \to \mathcal{F}_{r_1 \ldots r_n} = \mathcal{F}$ by assigning to a point $p \in M$ the flag, defined by the subspaces $(E_{1,p}, \ldots, E_{n,p})$. Conversely, any smooth map $\psi\colon M \to \mathcal{F}$ determines a moving flag $\mathcal{E} = (E_1, \ldots, E_n)$, where $E_i = \psi^{-1} T_i$ is the pull-back of a natural tautological bundle $T_i \to \mathcal{F}_{\mathbf{r}}$: the fibre of T_i at $\mathcal{E} \in \mathcal{F}$ coincides, by definition, with the subspace E_i for $1 \le i \le n$.

As in the Grassmann case, the differential $\psi_{\mathcal{E}}$ is determined locally by the components

$$A'_{ij} = \pi_i \circ \frac{\partial}{\partial z} \circ \pi_j \ , \qquad A''_{ij} = \pi_i \circ \frac{\partial}{\partial \bar z} \circ \pi_j$$

where $\pi_i\colon\ M \times \mathbb{C}^d \to E_i$ is the orthogonal projection.

Theorem 5. (Burstall–Salamon [5]) *The homogeneous flag bundle*

$$\pi_\sigma : (\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d), J^2) \longrightarrow G_r(\mathbb{C}^d)$$

defined by (11) (*cf. Section 3*)*, is a twistor bundle, i.e., for any J^2-holomorphic map $\psi : M \to \mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$ its projection $\varphi = \pi_\sigma \circ \psi : M \to G_r(\mathbb{C}^d)$ is harmonic.*

To prove the Theorem, it is sufficient to show that for any moving flag $\mathcal{E} = (E_1, \dots, E_n)$, corresponding to a J^2-holomorphic map $\psi_{\mathcal{E}} : M \to \mathcal{F}$, the bundle $E := \bigoplus_{i\in\sigma} E_i$ is harmonic. The holomorphicity of the map $\psi_{\mathcal{E}}$ means that

$$A'_{ij} = 0 = A''_{ji}\ , \quad \text{if} \quad \begin{cases} i > j & \text{and} \quad i, j \in \sigma \text{ or } i, j \in \sigma^c \\ i < j & \text{and} \quad i \in \sigma, j \in \sigma^c \text{ or } i \in \sigma^c, j \in \sigma\ . \end{cases}$$

If $k < l$ and $k \in \sigma$, $l \in \sigma^c$ then, as in the Grassmann case, we will have

$$\begin{aligned} 0 =& \pi_l \sum_i \left[\frac{\partial}{\partial \bar z}\pi_i \frac{\partial}{\partial z} - \frac{\partial}{\partial z}\pi_i\frac{\partial}{\partial \bar z}\right]\pi_k = \sum_i (A''_{li}A'_{ik} - A'_{li}A''_{ik}) \\ =& \sum_{i\in\sigma^c} A''_{li}A'_{ik} - \sum_{i\in\sigma} A'_{li}A''_{ik} = \Big(\sum_{i\in\sigma^c} A''_{li}\Big)\Big(\sum_{i\in\sigma^c} A'_{ik}\Big) \\ & - \Big(\sum_{i\in\sigma} A'_{li}\Big)\Big(\sum_{i\in\sigma} A''_{ik}\Big) = \pi_l \left(\partial''_{E^\perp} \circ A'_E - A'_E \circ \partial''_E\right)\pi_k\ . \end{aligned} \tag{13}$$

Analogous relations are satisfied for $k > l$, which implies that $A'_E \circ \partial''_E = \partial''_{E^\perp} \circ A'_E$, i.e., the bundle E is harmonic.

In the case when M is the Riemann sphere $\mathbb{CP}^1$, it's possible to prove a converse of Theorem 5, which is based on the Harder–Narasimhan filtration theorem for holomorphic vector bundles over $\mathbb{CP}^1$.

Suppose that E is a holomorphic vector bundle of rank r over $\mathbb{CP}^1$, identified with a subbundle of the trivial bundle $\mathbb{CP}^1 \times \mathbb{C}^d \to \mathbb{CP}^1$. Then the Harder–Narasimhan theorem ([10]) asserts that there exists a filtration of E by holomorphic subbundles

$$0 = \mathcal{B}_0 \subset \mathcal{B}_1 \subset \ldots \subset \mathcal{B}_k = E$$

having quotients of the form

$$\mathcal{B}_i / \mathcal{B}_{i-1} \cong \underbrace{L^{\beta_i} \oplus \ldots \oplus L^{\beta_i}}_{b_i \text{ times}}$$

where L^{β_i} is the β_i-th power of the standard Hopf line bundle L over $\mathbb{CP}^1$ and $\beta_1 > \cdots > \beta_k$. The subbundle $\mathcal{B}_i$ can be defined as the smallest holomorphic subbundle of E, containing the images of all meromorphic sections of E with divisors of degree, greater or equal to β_i. Using the Hermitian metric on $\mathbb{C}^d$, we can identify the quotient $\mathcal{B}_i/\mathcal{B}_{i-1}$ with the orthogonal complement B_i of $\mathcal{B}_{i-1}$ in $\mathcal{B}_i$.

We can construct an analogous filtration for the orthogonal complement $E^\perp$ of E in $\mathbb{CP}^1 \times \mathbb{C}^d \to \mathbb{CP}^1$

$$0 = \mathcal{C}_0 \subset \mathcal{C}_1 \subset \ldots \subset \mathcal{C}_l = E^\perp$$

with quotients of the form

$$\mathcal{C}_i / \mathcal{C}_{i-1} \cong \underbrace{L^{\gamma_i} \oplus \ldots \oplus L^{\gamma_i}}_{c_i \text{ times}}$$

and $\gamma_1 > \cdots > \gamma_l$. We identify again the quotient $\mathcal{C}_i/\mathcal{C}_{i-1}$ with the orthogonal complement C_i of $\mathcal{C}_{i-1}$ in $\mathcal{C}_i$.

We collect now the subbundles $B_1, \ldots, B_k, C_1, \ldots, C_l$ into a single collection of $n = k + l$ subbundles, denoted by $E_1, \ldots, E_n$, so that each of E_i is isomorphic to the direct sum of c_i copies of L^{δ_i} and $\delta_1 \leq \cdots \leq \delta_n$. (If for some j we have $\delta_j = \delta_{j+1}$, we arrange the associated subbundles E_j, E_{j+1} in such a way that E_j corresponds to some B_p and E_{j+1} to some C_q.) We introduce a subset $\sigma \subset \{1, 2, \ldots, n\}$, uniquely defined by the equalities

$$E = \bigoplus_{i \in \sigma} E_i , \qquad E^\perp = \bigoplus_{i \in \sigma^c} E_i .$$

We are ready to prove now the converse of Theorem 5.

Theorem 6. (Burstall [3]) *Any harmonic map $\varphi : \mathbb{CP}^1 \to \mathrm{G}_r(\mathbb{C}^d)$ can be obtained as the projection of a $\mathcal{J}^2$-holomorphic map $\psi : \mathbb{CP}^1 \to \mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$ with respect to some twistor bundle $\pi_\sigma : \mathcal{F}_{\mathbf{r}}(\mathbb{C}^d) \to \mathrm{G}_r(\mathbb{C}^d)$.*

To prove the Theorem, we associate, as above, with a harmonic map $\varphi : \mathbb{CP}^1 \to \mathrm{G}_r(\mathbb{C}^d)$ a harmonic subbundle E of rank r in the trivial bundle $\mathbb{CP}^1 \times \mathbb{C}^d \to \mathbb{CP}^1$. Using the Harder–Narasimhan filtration theorem, we construct, as above, a moving flag $\mathcal{E} := (E_1, \ldots, E_n)$ and fix a subset $\sigma \subset \{1, 2, \ldots, n\}$ such that

$$E = \bigoplus_{i \in \sigma} E_i , \qquad E^\perp = \bigoplus_{i \in \sigma^c} E_i .$$

Denote by $\psi_{\mathcal{E}} : M \to \mathcal{F}$ the map, associated with the moving flag $\mathcal{E}$. We have to prove that this map is J^2-holomorphic. In other words, we should prove that

$$A'_{ij} = 0 = A''_{ji} , \quad \text{if} \quad \begin{cases} i > j & \text{and} \quad i, j \in \sigma \text{ or } i, j \in \sigma^c \\ i < j & \text{and} \quad i \in \sigma, j \in \sigma^c \text{ or } i \in \sigma^c, j \in \sigma \ . \end{cases}$$

Suppose first that $i > j$ and $i, j \in \sigma$. Then $\delta_i > \delta_j$ and the subbundle E_i is contained in some holomorphic subbundle $\mathcal{B}_p$ of E, orthogonal to E_j. It follows that $A''_{ji} = 0$, which implies also that $A'_{ij} = 0$. The case $i, j \in \sigma^c$ is treated in a similar way.

Suppose next that $i < j$ and $i \in \sigma^c, j \in \sigma$. Then $E_j = B_p$ for some $B_p \subset \mathcal{B}_p$. Since E is harmonic, it follows that the differential $\mathrm{d}z \otimes A'_E$ is holomorphic (cf. (8), (9) in Section 2). Here, A'_E is considered as a section of the holomorphic bundle $\mathrm{Hom}(E, E^\perp)$. Since the image $A'_E(\mathcal{B}_p)$ is spanned by meromorphic sections of $E^\perp$ with divisors of degree, greater than $\delta_j + 1$, we have

$$A'_E(E_j) \subset \bigoplus_{q \in \sigma^c,\, q > j} E_q \,.$$

Hence, $A'_{ij} = 0$ for $i < j$, implying also that $A''_{j\,i} = 0$.The case $i \in \sigma, j \in \sigma^c$ is treated in a similar way, using the fact that the subbundle $E^\perp$ is harmonic along with E.

By the above Theorem 6 the problem of description of harmonic spheres in the Grassmann manifold $\mathrm{G}_r(\mathbb{C}^d)$ reduces to the problem of description of J^2-holomorphic spheres in flag manifolds $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$. The latter problem was solved by Wood in [17] (cf. also [5]). The Wood's method can be roughly described as follows. Consider a moving flag $\mathcal{E} = (E_1, \ldots, E_n)$, corresponding to a smooth map $\psi : M \to \mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$. If the original map ψ was J^1-holomorphic, i.e., holomorphic with respect to the canonical complex structure on $\mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$, then the subbundles $E_1, \ldots, E_n$ will be holomorphic with respect to the pulled-back complex structure $J_\psi := \psi^*(J^1)$ on M. Suppose that we know already how to construct J^1-holomorphic maps $\psi : M \to \mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$. Then one can convert J^1-holomorphic maps $\psi : M \to \mathcal{F}_{\mathbf{r}}(\mathbb{C}^d)$ into J^2-holomorphic maps by replacing some of the holomorphic subbundles E_i by anti-holomorphic subbundles $\bar{E}_i$ (and vice versa for the orthogonal complements $E_i^\perp$ of E_i).

5. Harmonic Maps into the Hilbert–Schmidt Grassmannian

We switch now to the case of infinite-dimensional Grassmann σ-models and try to extend to this case the methods, developed for finite-dimensional Grassmanians in the previous sections.

We start from the definition of the Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ of a complex (separable) Hilbert space H. We take for a model of this Hilbert space the space $L^2_0(S^1, \mathbb{C})$ of square integrable complex-valued functions on the circle S^1 with the zero average over S^1.

Suppose that H has a *polarization*, i.e., a decomposition

$$H = H_+ \oplus H_- \tag{14}$$

into the direct orthogonal sum of infinite-dimensional closed subspaces. In the case of $H = L^2_0(S^1, \mathbb{C})$ one can take for such subspaces

$$H_\pm = \{\gamma \in H\, ;\, \gamma(z) = \sum_{\pm k > 0} \gamma_k z^k\}\ .$$

Any bounded linear operator $A \in L(H)$ with respect to the polarization (14) can be written in the block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a : H_+ \to H_+\ , & b : H_- \to H_+ \\ c : H_+ \to H_-\ , & d : H_- \to H_- \end{pmatrix}\ .$$

Denote by $\mathrm{GL}(H)$ the group of linear bounded operators on H, having a bounded inverse, and introduce the *Hilbert–Schmidt group* $\mathrm{GL}_{\mathrm{HS}}(H)$, consisting of operators $A \in \mathrm{GL}(H)$, for which the "off-diagonal" terms b and c are Hilbert–Schmidt operators (briefly: HS-operators). In other words, the group $\mathrm{GL}_{\mathrm{HS}}(H)$ consists of operators $A \in \mathrm{GL}(H)$, for which the "off-diagonal" terms b and c are "small" with respect to the "diagonal" terms a and d. We denote also by $\mathrm{U}_{\mathrm{HS}}(H)$ the intersection of $\mathrm{GL}_{\mathrm{HS}}(H)$ with the group $\mathrm{U}(H)$ of unitary operators in H.

As in the finite-dimensional situation, there is a Grassmann manifold $\mathrm{Gr}_{\mathrm{HS}}(H)$, called the Hilbert–Schmidt Grassmannian, related to the group $\mathrm{GL}_{\mathrm{HS}}(H)$.

Definition 7. *The* Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{HS}(H)$ *is the set of all closed subspaces $W \subset H$ such that the orthogonal projection $pr_+ : W \to H_+$ is a Fredholm operator, and the orthogonal projection $pr_- : W \to H_-$ is a Hilbert–Schmidt operator. Equivalently a subspace $W \in \mathrm{Gr}_{HS}(H)$ if and only if it coincides with the image of a linear operator $w : H_+ \to H$ such that $w_+ := pr_+ \circ w$ is a Fredholm operator, and $w_- := pr_- \circ w$ is a Hilbert–Schmidt operator.*

In other words, the Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ consists of the subspaces $W \subset H$, which differ "little" from the subspace H_+ in the sense that the projection $\mathrm{pr}_+ : W \to H_+$ is "close" to an isomorphism and the projection $\mathrm{pr}_- : W \to H_-$ is "small".

We have the following homogeneous space representation of $\mathrm{Gr}_{\mathrm{HS}}(H)$

$$\mathrm{Gr}_{\mathrm{HS}}(H) = \mathrm{U}_{\mathrm{HS}}(H)/\,\mathrm{U}(H_+) \times \mathrm{U}(H_-)\ .$$

Since $\mathrm{U_{HS}}(H)$ acts transitively on the Grassmannian $\mathrm{Gr_{HS}}(H)$, we can construct an $\mathrm{U_{HS}}(H)$-invariant Kähler metric on $\mathrm{Gr_{HS}}(H)$ from an inner product on the tangent space $T_{H_+}\,\mathrm{Gr_{HS}}(H)$ at the origin $H_+ \in \mathrm{Gr_{HS}}(H)$, invariant under the action of the isotropy subgroup $\mathrm{U}(H_+)\times\mathrm{U}(H_-)$. The tangent space $T_{H_+}\,\mathrm{Gr_{HS}}(H)$ coincides with the space of Hilbert–Schmidt operators $\mathrm{HS}(H_+, H_-)$, and the invariant inner product on it is given by the formula

$$(A, B) \longmapsto \mathrm{Re}\left\{\mathrm{tr}(AB^\dagger)\right\}, \qquad A, B \in \mathrm{HS}(H_+, H_-)\ .$$

Note that the imaginary part of the complex inner product $\mathrm{tr}(AB^\dagger)$ determines a non-degenerate invariant two-form on $T_{H_+}\,\mathrm{Gr_{HS}}(H)$, which extends to an $\mathrm{U_{HS}}(H)$-invariant symplectic structure on the manifold $\mathrm{Gr_{HS}}(H)$. Hence, we have a Kähler structure on $\mathrm{Gr_{HS}}(H)$, which makes it a *Kähler Hilbert manifold.*

The evident difficulty, encountered when trying to extend the techniques, developed for the finite-dimensional Grassmanians, to the case of $\mathrm{Gr_{HS}}(H)$, is that the subspaces $W \in \mathrm{Gr_{HS}}(H)$ are infinite-dimensional. In this sense, they all have the same infinite "dimension", which does not allow to compare them. However, there is a substitution of the notion of dimension, which is more helpful for the study of such subspaces, namely, we can compare them by their "virtual dimension".

In more detail, the manifold $\mathrm{Gr_{HS}}(H)$ has a countable number of connected components, numerated by the index of the Fredholm operator w_+ for a subspace $W \in \mathrm{Gr_{HS}}(H)$, coinciding with the image of a linear operator $w : H_+ \to H$. We say that a subspace W has the *virtual dimension* d, if the index of w_+ is equal to d. Denote by $\mathrm{G}_d(H)$ the component of $\mathrm{Gr_{HS}}(H)$, consisting of subspaces W of virtual dimension d. Then we have the following decomposition of $\mathrm{Gr_{HS}}(H)$ into the disjoint union of its connected components $\mathrm{G}_d(H)$

$$\mathrm{Gr_{HS}}(H) = \bigcup_d \mathrm{G}_d(H)\ . \tag{15}$$

Due to this decomposition, the study of harmonic maps of Riemann surfaces into $\mathrm{Gr_{HS}}(H)$ is reduced to the study of harmonic maps into the Grassmannians $\mathrm{G}_d(H)$ of virtual dimension d, which may be carried on along the same lines, as in the case of the Grassmann manifold $G_r(\mathbb{C}^d)$.

As in the latter case, for any decomposition $d = r_1 + \cdots + r_n$ of d into the sum of integers we define the corresponding *virtual flag manifold* $\mathcal{F} = \mathcal{F}_{\mathbf{r}}(H)$ of type $\mathbf{r} = (r_1, \ldots, r_n)$, consisting of collections $\mathcal{W} = (W_1, \ldots, W_n)$ of mutually orthogonal subspaces $W_i \subset H$ of virtual dimension r_i. Next, for any ordered subset $\sigma \subset \{1, \ldots, n\}$ we set $r := \sum_{i\in\sigma} r_i$ and construct a homogeneous flag bundle

$$\pi\colon\ \mathcal{F}_{\mathbf{r}}(H) \longrightarrow \mathrm{G}_r(H)$$

by assigning

$$(W_1,\dots,W_n) \longmapsto W = \bigoplus W_i\ .$$

We introduce the almost complex structures J^1 and J^2 on the flag manifold $\mathcal{F}_{\mathbf{r}}(H)$, as in the finite-dimensional situation. We have the following assertion, analogous to the finite-dimensional case.

Theorem 8. *The homogeneous bundle $\pi : (\mathcal{F}_{\mathbf{r}}(H), J^2) \to \mathrm{G}_r(H)$ is a twistor bundle, i.e., for any J^2-holomorphic map $\psi : M \to \mathcal{F}_{\mathbf{r}}(H)$ its projection $\varphi = \pi \circ \psi : M \to \mathrm{G}_r(H)$ is a harmonic map.*

The proof of this Theorem is similar to the proof of Theorem 5. Due to Theorem 8, one can produce harmonic maps $M \to \mathrm{G}_r(H)$ by projecting J^2-holomorphic maps $M \to \mathcal{F}_{\mathbf{r}}(H)$ to $\mathrm{G}_r(H)$.

There is also an analogue of Theorem 6, valid for harmonic maps $\varphi : \mathbb{CP}^1 \to \mathrm{G}_r(H)$.

Theorem 9. *Any harmonic map $\varphi : \mathbb{CP}^1 \to \mathrm{G}_r(H)$ can be obtained as the projection of a J^2-holomorphic map $\psi : \mathbb{CP}^1 \to \mathcal{F}_{\mathbf{r}}(H)$ with respect to some twistor bundle $\pi_\sigma : \mathcal{F}_{\mathbf{r}}(H) \to \mathrm{G}_r(H)$.*

Theorem 9 can be proved in the same way, as Theorem 6, if one uses, instead of the Birkhoff–Grothendieck classification theorem for holomorphic vector bundles over $\mathbb{CP}^1$ (implying the Harder–Narasimhan filtration theorem) its infinite-dimensional analogue. This analogue, namely, the classification theorem for holomorphic vector bundles over $\mathbb{CP}^1$ with the structure group, consisting of invertible operators of the form $I +$ compact, is proved in [15, 16].

6. Harmonic Maps into Loop Spaces

We can apply the above results to the study of harmonic maps into the loop spaces ΩG of compact Lie groups G by embedding these loop spaces into the Hilbert–Schmidt Grassmannian. At the end of this Section we explain, why this case is particularly interesting for us.

Denote by $LG = C^\infty(S^1, G)$ the loop group of G, i.e., the space of C^∞-smooth maps $S^1 \to G$, where S^1 is identified with the unit circle in $\mathbb{C}$. It is a Lie–Frechet group with respect to the pointwise multiplication (cf. [14]), modelled on the *loop algebra* $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of the group G. The *loop space* ΩG of the group G (or the basic loop space) is the homogeneous space of the group LG of the form

$$\Omega G = LG/G \tag{16}$$

where the group G in the denominator is identified with the subgroup of constant maps $S^1 \to g_0 \in G$. Note that the loop space ΩG may be identified with the space of based maps in LG, sending $1 \in S^1$ to the unit e of the group G, and so inherits a Frechet manifold structure from the loop group LG.

The loop group LG acts on ΩG by left translations. Denote by o the origin in ΩG, represented by the class of constant maps: $o = [G]$. The tangent space of ΩG at the origin o is identified with the space $\Omega\mathfrak{g} = L\mathfrak{g}/\mathfrak{g}$. We represent vectors of the tangent space $T_o(\Omega G)$ by their Fourier series: an arbitrary vector ξ of the complexified tangent space $T_o^{\mathbb{C}}(\Omega G) = T_o(\Omega G) \otimes \mathbb{C}$ has a Fourier decomposition of the form

$$\xi = \sum_{k \neq 0} \xi_k \mathrm{e}^{\mathrm{i}k\theta}$$

where the coefficients ξ_k belong to the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$. A vector $\xi \in T_o(\Omega G) \iff \xi_{-k} = \bar{\xi}_k$.

The loop space ΩG has a natural symplectic structure, invariant under the action of the loop group LG on ΩG. Due to the invariance, it's sufficient to define its restriction to $T_o(\Omega G) = \Omega\mathfrak{g}$. For that we fix an invariant inner product $< \cdot, \cdot >$ on the Lie algebra $\mathfrak{g}$ and consider a two-form ω on $L\mathfrak{g}$ of the form

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle \, \mathrm{d}\theta, \qquad \xi, \eta \in L\mathfrak{g} \ .$$

This formula defines a left-invariant closed two-form on LG, subject to the condition: $\omega(\xi, \eta) = 0$ if and only if at least one of the maps ξ, η is constant. Hence it can be pushed down to a left-invariant two-form on $\Omega\mathfrak{g}$, which is non-degenerate and closed, and so generates a symplectic structure on ΩG.

An invariant complex structure on ΩG is provided by the "complex" representation of $\Omega G = LG/G$ as a homogeneous space of the complex Lie–Frechet group $LG^{\mathbb{C}} = C^\infty(S^1, G^{\mathbb{C}})$, where $G^{\mathbb{C}}$ is the complexification of the Lie group G. This representation has the form (cf. [12] and also [14])

$$\Omega G = LG^{\mathbb{C}}/L^+G^{\mathbb{C}} \tag{17}$$

where $L^+G^{\mathbb{C}} = \mathrm{Hol}(\Delta, G^{\mathbb{C}})$ is a subgroup of $LG^{\mathbb{C}}$, consisting of the maps $S^1 \to G^{\mathbb{C}}$, which can be extended smoothly to holomorphic maps of the disc $\Delta := \{|z| < 1\} \to G^{\mathbb{C}}$.

The invariant complex structure J^1 on ΩG, induced by the complex representation (17), has a simple meaning in terms of Fourier series. Namely, the restriction of J^1 to the complexified tangent space $T_o^{\mathbb{C}}(\Omega G) = \Omega\mathfrak{g}^{\mathbb{C}}$ at the origin is given by the following formula:

$$\xi = \sum_{k\neq 0} \xi_k \mathrm{e}^{\mathrm{i}k\theta} \quad \longmapsto \quad J^1\xi = -\mathrm{i}\sum_{k>0} \xi_k \mathrm{e}^{\mathrm{i}k\theta} + \mathrm{i}\sum_{k<0} \xi_k \mathrm{e}^{\mathrm{i}k\theta}.$$

The introduced symplectic and complex structures on ΩG are compatible in the sense that $\omega(J^1\xi, J^1\eta) = \omega(\xi, \eta)$ for all $\xi, \eta \in T_o(\Omega G)$ and the symmetric form

$$g^1(\xi, \eta) := \omega(\xi, J^1\eta) \quad \text{on} \quad T_o(\Omega G) \times T_o(\Omega G)$$

is positive definite. So this form extends to an invariant Riemannian metric g^1 on ΩG (due to the invariance of ω and J^1). In other words, the loop space ΩG is a *Kähler Frechet manifold*, provided with the Kähler metric g^1.

We shall study harmonic maps from Riemann surfaces M to the loop spaces ΩG, by embedding isometrically ΩG into the Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$.

Assume that G is a matrix group, i.e., G is represented as a subgroup of $\mathrm{U}(n)$ for some n. Then we have an isometric embedding

$$LG \longrightarrow \mathrm{U}_{\mathrm{HS}}(H)$$

given by the map

$$\gamma \in LG = C^\infty(S^1, G) \longmapsto M_\gamma \in \mathrm{U}_{\mathrm{HS}}(H)$$

where the multiplication operator M_γ is defined by:

$$f \in H = L_0^2(S^1, \mathbb{C}^n) \longmapsto (M_\gamma f)(z) := \gamma(z) f(z) \quad \text{for } z \in S^1 \ .$$

It is easy to check that $M_\gamma \in \mathrm{U}_{\mathrm{HS}}(H)$ if γ is smooth ([12]).

The constructed embedding of the loop group LG into $\mathrm{U}_{\mathrm{HS}}(H)$ induces an isometric embedding

$$\Omega G \longrightarrow \mathrm{Gr}_{\mathrm{HS}}(H) \ .$$

So we can consider harmonic maps $M \to \Omega G$ as taking values in $\mathrm{Gr}_{\mathrm{HS}}(H)$, thus reducing their study to the study of harmonic maps $M \to \mathrm{Gr}_{\mathrm{HS}}(H)$, considered above.

The motivation for the study of harmonic maps $M \to \Omega G$ comes from a result of Atiyah [1], relating G-instantons on $\mathbb{R}^4$ with holomorphic maps $\mathbb{CP}^1 \to \Omega G$, i.e., holomorphic spheres in ΩG. More precisely, it is proved in [1] that there is the following one to one correspondence

$$\left\{\begin{matrix}\text{moduli space of } G\text{-} \\ \text{instantons on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{based holomorphic maps} \\ f : \mathbb{CP}^1 \to \Omega G\end{matrix}\right\}$$

where a holomorphic map $f : \mathbb{CP}^1 \to \Omega G$ is called *based* if it sends the infinity $\infty \in \mathbb{CP}^1$ to the origin $o \in \Omega G$.

Motivated by this result, we can conjecture that there is also a one to one correspondence

$$\left\{\begin{matrix}\text{based harmonic maps} \\ h : \mathbb{CP}^1 \to \Omega G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{moduli space of solutions of} \\ \text{Yang–Mills } G\text{-equations on } \mathbb{R}^4\end{matrix}\right\}.$$

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