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# Classification of Canonical Bases for (*n*–2)-Dimensional Subspaces of *n*-Dimensional Vector Space

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**Abstract**

Famous K. Gauss introduced reduced row echelon forms for matrices approximately 200 years ago to solve systems of linear equations but the number of them and their structure has been unknown until 2016 when it was determined at first in the previous article given up to $(n-1)\times n$ matrices. The similar method is applied to find reduced row echelon forms for $(n-2)\times n$ matrices in this article, and all canonical bases for (*n*–2)-dimensional subspaces of *n*-dimensional vector space are found also.

**Keywords:** Vector space; Subspaces; Canonical bases

## Introduction

The canonical bases for (*n*–2)-dimensional subspaces of *n*-dimensional vector space are introduced in the article, and all nonequivalent of them are classified. Canonical bases for (*n*–1)-dimensional subspaces of *n*-dimensional vector space were classified in the previous article [2] of the same author. This new case of (*n*–2)-dimensional subspaces is interesting to be studied because some *n*-dimensional Lie algebras haven't any (*n*–1)-dimensional subalgebras. For example, in the article [3], it was proved that 6-dimensional Lie algebra of Lorentz group doesn't have any 5-dimensional subalgebra but this Lie algebra has 4-dimensional subalgebras. We start to introduce the necessary definitions.

Let $V$ be an *n*-dimensional vector space with its standard basis $\overrightarrow{e_1},\overrightarrow{e_2},....,\overrightarrow{e_n}$ . Suppose that $\overrightarrow{a_1},\overrightarrow{a_2},...,\overrightarrow{a_{n-2}}$ are (*n*–2) linearly independent vectors in the space $V$ where,

$$\overrightarrow{a_1}=a_{11}\overrightarrow{e_1}+a_{12}\overrightarrow{e_2}+...+a_{1n}\overrightarrow{e_n},\ \overrightarrow{a_2}=a_{21}\overrightarrow{e_1}+a_{22}\overrightarrow{e_2}+...+a_{2n}\overrightarrow{e_n},....,\overrightarrow{a_{n-2}}=a_{n-2,1}\overrightarrow{e_1}+...+a_{n-2,n}\overrightarrow{e_n}\ . \quad \text{(I)}$$

The vectors (I) describe all possible bases for any (*n*–2)-dimensional subspace $S$ of $V$. This description contains too many arbitrary components; their number is $(n-2)\times n$. Instead of that, we introduce canonical bases with much smaller number of arbitrary components in each of them (maximum 2(*n*–2)).

**Definition 1:** Two bases are called equivalent if they generate the same subspace of $V$, and they are called nonequivalent if they generate two different subspaces of $V$.

We will associate the following $(n-2)\times n$ matrix $M$ with a basis (I)

$$M=\begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .. & .. & .. & .. \\ a_{n-2,1} & a_{n-2,2} & .... & a_{n-2,n} \end{bmatrix} \quad \text{(II)}$$

**Definition 2:** Two matrices are called row equivalent (or just equivalent) if they have the same reduced row echelon form, and they are called nonequivalent if they have different reduced row echelon forms.

About reduced row echelon forms of matrices, see for example [1].

**Definition 3:** The basis (I) is called canonical if its vectors $\overrightarrow{a_1},\overrightarrow{a_2},...,\overrightarrow{a_{n-2}}$ are the corresponding rows in some reduced row echelon form of the matrix $M$.

Thus, there is one-to-one correspondence between nonequivalent canonical bases for (*n*–2)-dimensional subspaces of *n*-dimensional vector space and nonequivalent reduced row echelon forms for $(n-2)\times n$ matrix $M$ of the rank(*n*–2).

**Part I. Basic Examples**

Consider two examples of nonequivalent canonical bases for (*n*–2)-dimensional subspaces of *n*-dimensional vector spaces where *n*=4 and *n*=6.

Ex. 1: Let $V$ be 4-dimensional vector space with its standard basis $\overrightarrow{e_1},\overrightarrow{e_2},\overrightarrow{e_3},\overrightarrow{e_4}$ . Any 2-dimensional subspace $S$ of $V$ can be described as $S=Span\{\overrightarrow{a_1},\overrightarrow{a_2}\}$ where,

$$\overrightarrow{a_1}=a_{11}\overrightarrow{e_1}+a_{12}\overrightarrow{e_2}+a_{13}\overrightarrow{e_3},\ \overrightarrow{a_2}=a_{21}\overrightarrow{e_1}+a_{22}\overrightarrow{e_2}+a_{23}\overrightarrow{e_3}\ .$$

This arbitrary basis is equivalent to one and only one canonical basis from the next list:

(1) $\vec{a}=\overrightarrow{e_1}+a_3\overrightarrow{e_3}+a_4\overrightarrow{e_4},\ \vec{b}=\overrightarrow{e_2}+b_3\overrightarrow{e_3}+b_4\overrightarrow{e_4}$ ; (2) $\vec{a}=\overrightarrow{e_1}+a_2\overrightarrow{e_2}+a_4\overrightarrow{e_4},\ \vec{b}=\overrightarrow{e_3}+b_4\overrightarrow{e_4}$; (3) $\vec{a}=\overrightarrow{e_1}+a_2\overrightarrow{e_2}+a_3\overrightarrow{e_3},\ \vec{b}=\overrightarrow{e_4}$ ; (4) $\vec{a}=\overrightarrow{e_2}+a_4\overrightarrow{e_4},\ \vec{b}=\overrightarrow{e_3}+b_4\overrightarrow{e_4}$ ; (5) $a=e_3,\ b=e_4$ ; (6) $\vec{a}=\overrightarrow{e_3},\ \vec{b}=\overrightarrow{e_4}$ .

Details of evaluation are omitted because it is similar (but easier) to the evaluation in the example 2. The last canonical bases generate the following 6 matrices associated with them:

$$\begin{bmatrix} 1 & 0 & a_3 & a_4 \\ 0 & 1 & b_3 & b_4 \end{bmatrix},\begin{bmatrix} 1 & a_2 & 0 & a_4 \\ 0 & 0 & 1 & b_4 \end{bmatrix},\begin{bmatrix} 1 & a_2 & a_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\begin{bmatrix} 0 & 1 & 0 & a_4 \\ 0 & 0 & 1 & b_4 \end{bmatrix},\begin{bmatrix} 0 & 1 & a_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Ex. 2: Let $V$ be 6-dimensional vector space with its standard basis $\overrightarrow{e_1},\overrightarrow{e_2},\overrightarrow{e_3},\overrightarrow{e_4},\overrightarrow{e_5},\overrightarrow{e_6}$ . Any 4-dimensional subspace $S$ of $V$ can be described as $S=Span\{\vec{a},\vec{b},\vec{c},\vec{d}\}$ where,

$$\vec{a}=a_1\overrightarrow{e_1}+a_2\overrightarrow{e_2}+a_3\overrightarrow{e_3}+a_4\overrightarrow{e_4}+a_5\overrightarrow{e_5}+a_6\overrightarrow{e_6},\ \vec{b}=b_1\overrightarrow{e_1}+b_2\overrightarrow{e_2}+b_3\overrightarrow{e_3}+b_4\overrightarrow{e_4}+b_5\overrightarrow{e_5}+b_6\overrightarrow{e_6}\ ,$$

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$$\vec{c}=c_1\vec{e_1}+c_2\vec{e_2}+c_3\vec{e_3}+c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_1\vec{e_1}+d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6}\,. \quad \text{(III)}$$

Start our transforming procedure for the basis $\{\vec{a},\vec{b},\vec{c},\vec{d}\}$ to find all possible canonical nonequivalent bases.

**A.** Let at least one coefficient from $a_1$, $b_1$, $c_1$, $d_1$ in the basis (III) is not zero. Without any loss in the generality, we can suppose that $a_1 \neq 0$. Perform the linear operation $\vec{a}/a_1$ first, and operations $\vec{b}-b_1\vec{a},\ \vec{c}-c_1\vec{a},\ \vec{d}-d_1\vec{a}$ after the first one. As a result, the following basis is obtained:

$$\vec{a}=a_1\vec{e_1}+a_2\vec{e_2}+a_3\vec{e_3}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=b_1\vec{e_1}+b_2\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},$$
$$\vec{c}=c_2\vec{e_2}+c_3\vec{e_3}+c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6}\,. \quad \text{(a)}$$

**Remark 1:** The first components of vectors $\vec{a},\vec{b},\vec{c},\vec{d}$ are changed as the result of operations performed but all other components of them still have the same notations just for the common convenience. This idea will be used also in all steps of the procedure below.

1. Suppose now that at least one coefficient from $b_2$, $c_2$, $d_2$ at the basis (a) is not zero. Without any loss in generality, let $b_2 \neq 0$. Perform the linear operations: first $\vec{b}/b_2$, and then $\vec{a}-a_2\vec{b},\ \vec{c}-c_2\vec{b},\ \vec{d}-d_2\vec{b}$. As the result, the following transformed basis is obtained:

$$\vec{a}=\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},$$
$$\vec{c}=c_3\vec{e_3}+c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_3\vec{e_3}+d_4\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6}\,. \quad (1)$$

2. Suppose that at least one coefficient among $c_3$, $d_3$ at the basis (1) is not zero. Again, without any loss in the generality, let $c_3 \neq 0$. Perform the operation $\vec{c}/c_3$ first, and then operations $\vec{a}-a_3\vec{c},\ \vec{b}-b_3\vec{c},\ \vec{d}-d_3\vec{c}$. As the result, the following basis is done.

$$\vec{a}=\vec{e_1}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},$$
$$\vec{c}=\vec{e_3}+c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_4\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6}\,. \quad (2)$$

3. Suppose now that the coefficient $d_4$ at the basis (2) is not zero. Perform the operation $\vec{d}/d_4$ first, and then operations $\vec{a}-a_4\vec{d},\ \vec{b}-b_4\vec{d},\ \vec{c}-c_4\vec{d}$. As the result, the following canonical basis is obtained:

$$\vec{a}=\vec{e_1}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_5\vec{e_5}+b_6\vec{e_6},\ \vec{c}=\vec{e_3}+c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6}\,. \quad (a_1)$$

If $d_4$=0 then the basis (2) is transformed into the following one:

$$\vec{a}=\vec{e_1}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},$$
$$\vec{c}=\vec{e_3}+c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_5\vec{e_5}+d_6\vec{e_6} \quad (3)$$

Vector $\vec{d}/$ in the basis (3) has at least one nonzero coefficient $d_5$ or $d_6$. Let $d_5 \neq 0$.

Perform operation $\vec{d}/d_5$ first, and the operations $\vec{a}-a_5\vec{d},\ \vec{b}-b_5\vec{d},\ \vec{c}-c_5\vec{d}$ after the first one. As the result, the following canonical basis is obtained:

$$\vec{a}=\vec{e_1}+a_4\vec{e_4}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_4\vec{e_4}+b_6\vec{e_6},\ \vec{c}=\vec{e_3}+c_4\vec{e_4}+c_6\vec{e_6},\ \vec{d}=\vec{e_5}+d_6\vec{e_6}\,. \quad (a_2)$$

If $d_6 \neq 0$ in the basis (3), then perform operation $\vec{d}/d_6$ first, and operations $\vec{a}-a_6\vec{d},\ \vec{b}-b_6\vec{d},\ \vec{c}-c_6\vec{d}$ after the first one. We obtain the new canonical basis:

$$\vec{a}=\vec{e_1}+a_4\vec{e_4}+a_5\vec{e_5},\ \vec{b}=\vec{e_2}+b_4\vec{e_4}+b_5\vec{e_5},\ \vec{c}=\vec{e_3}+c_4\vec{e_4}+c_5\vec{e_5},\ \vec{d}=d_5\vec{e_5}+\vec{e_6}\,. \quad (a_3)$$

4. Suppose now that both coefficients $c_3$, $d_3$ at the basis (1) are zero. We have:

$$\vec{a}=\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},$$
$$\vec{c}=c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_4\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6} \quad (4)$$

Consider coefficients $c_4$, $d_4$ in the basis (4). Suppose that at least one of them is not zero. Let $c_4 \neq 0$. Perform operation $\vec{c}/c_4$ first, and perform operations $\vec{a}-a_4\vec{c},\ \vec{b}-b_4\vec{c},\ \vec{d}-d_4\vec{c}$ after the first one. The following basis is obtained,

$$\vec{a}=\vec{e_1}+a_3\vec{e_3}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_5\vec{e_5}+b_6\vec{e_6},$$
$$\vec{c}=\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_5\vec{e_5}+d_6\vec{e_6}$$

In the last basis, at least one coefficient $d_5$ or $d_6$ is not zero. Let $d_5 \neq 0$. Performing operation $\vec{d}/d_5$ first, and operations $\vec{a}-a_5\vec{d},\ \vec{b}-b_5\vec{d},\ \vec{c}-c_5\vec{d}$ after the first one, we obtain the new canonical basis:

$$\vec{a}=\vec{e_1}+a_3\vec{e_3}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_6\vec{e_6},\ \vec{c}=\vec{e_4}+c_6\vec{e_6},\ \vec{d}=\vec{e_5}+d_6\vec{e_6}\,. \quad (a_4)$$

If $d_6$ 0 then doing similarly we obtain the following canonical basis:

$$\vec{a}=\vec{e_1}+a_3\vec{e_3}+a_5\vec{e_5},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_5\vec{e_5},\ \vec{c}=\vec{e_4}+c_5\vec{e_5},\ \vec{d}=d_5\vec{e_5}+\vec{e_6}\,. \quad (a_5)$$

If $d_4 \neq 0$ in the basis (4), then there will be obtained the bases that are equivalent to $(a_4)$ and $(a_5)$.

5. Suppose now that both coefficients $c_4$, $d_4$ in the basis (4) are zero. We obtain:

$$\vec{a}=\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},$$
$$\vec{c}=c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_5\vec{e_5}+d_6\vec{e_6} \quad (5)$$

In the last basis (5), at least one coefficient among $c_5$, $d_5$ is not zero. If both coefficients $d_5, f_5$ are zero, then $\vec{c}=c_6\vec{e_6},\ \vec{d}=d_6\vec{e_6}$, and vectors $\vec{d},\vec{f}$ are linearly dependent but it's impossible for any basis. Let $c_5 \neq 0$. Perform the operation $\vec{c}/c_5$ first, and operations $\vec{a}-a_5\vec{c},\ \vec{b}-b_5\vec{c},\ \vec{c}-c_5\vec{d}$ after the first one. We obtain the following basis:

$$\vec{a}=\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}+b_6\vec{e_6},$$
$$\vec{c}=\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_6\vec{e_6}$$

In the last basis, $d_6 \neq 0$. Perform the operation $\vec{a}-a_6\vec{d},\vec{b}-b_6\vec{d},\vec{c}-c_6\vec{d}$ first, and the operations $\vec{a}-a_6\vec{d},\ \vec{b}-b_6\vec{d},\vec{c}-c_6\vec{d}$ after the first one. We obtain the new canonical basis:

$$\vec{a}=\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4},\ \vec{c}=\vec{e_5},\ \vec{d}=\vec{e_6}\,. \quad (a_6)$$

At the case when $d_5 \neq 0$ in the basis (5), we obtain the same basis $(a_6)$.

6. Suppose, in opposition to step 1, that all coefficients $b_2$, $c_2$, $d_2$ in the basis (a) are zero. We obtain:

$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_3\vec{e_3}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=b_3\vec{e_3}+b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},$$
$$\vec{c}=c_3\vec{e_3}+c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_3\vec{e_3}+d_4\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6}$$

Consider coefficients $b_3$, $c_3$, $d_3$ in the last basis. Suppose that at least one of them is not zero. Let $b_3 \neq 0$ (without any loss in generality). Perform the operation $\vec{b}/b_3$ first, and the operations $\vec{a}-a_3\vec{b},\ \vec{c}-c_3\vec{b},\ \vec{d}-d_3\mathrm{b}$ after the first one. We obtain the following result:

$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=\vec{e_3}+b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},$$
$$\vec{c}=c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_4\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6} \quad (6)$$

7. Suppose now that at least one coefficient from $c_4$, $d_4$ in the basis (6) is not zero. Let $c_4 \neq 0$. Perform the operation $\vec{c}/c_4$ first, and the operations $\vec{a}-a_4\vec{c},\ \vec{b}-b_4\vec{c},\ \vec{b}-b_4\vec{c}$ after the first one. The following basis is obtained:

$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=\vec{e_3}+b_5\vec{e_5}+b_6\vec{e_6},$$
$$\vec{c}=\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_5\vec{e_5}+d_6\vec{e_6} \quad (7)$$

In the basis (7), at least one from the coefficients $d_5$, $d_6$ is not zero.

Let $d_5 \neq 0$. Perform the operation $\vec{d}/d_5$ first, and the operations $\vec{a}-a_5\vec{d}, \vec{b}-b_5\vec{d}, \vec{c}-c_5\vec{d}$ after the first one. We obtain the new canonical basis:

$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_6\vec{e_6}, \vec{b}=\vec{e_3}+b_6\vec{e_6}, \vec{c}=\vec{e_4}+c_6\vec{e_6}, \vec{d}=\vec{e_5}+d_6\vec{e_6}. \quad (a_7)$$

If $d_6 \neq 0$ in the basis (7), we obtain the following canonical basis performing similar steps:

$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_5\vec{e_5}, \vec{b}=\vec{e_3}+b_5\vec{e_5}, \vec{c}=\vec{e_4}+c_5\vec{e_5}, \vec{d}=d_5\vec{e_5}+\vec{e_6}. \quad (a_8)$$

If $d_4$ 0 in the basis (6), we obtain the canonical bases that are equivalent to $(a_7)$ and $(a_8)$.

8. Suppose now that both coefficients $c_4$, $d_4$ in the basis (6) are zero. We obtain:

$$\begin{aligned}&\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6}, \vec{b}=\vec{e_3}+b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},\\ &\vec{c}=c_5\vec{e_5}+c_6\vec{e_6}, \vec{d}=d_5\vec{e_5}+d_6\vec{e_6}\end{aligned}. \quad (8)$$

In the basis (8), at least one coefficient from $c_5$, $d_5$ is not zero. Otherwise, vectors $\vec{c},\vec{d}$ are linearly dependent. Let $c_5 \neq 0$. Perform the operation $\vec{c}/c_5$ first, and the operations $\vec{a}-a_5\vec{c}$, $\vec{b}-b_5\vec{c}$, $\vec{d}-d_5\vec{c}$ after the first one. We obtain the following basis:

$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_4\vec{e_4}+a_6\vec{e_6}, \vec{b}=\vec{e_3}+b_4\vec{e_4}+b_6\vec{e_6}, \vec{c}=\vec{e_5}+c_6\vec{e_6}, \vec{d}=d_6\vec{e_6}.$$

Performing one more obvious step, we obtain the following canonical basis:

$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_4\vec{e_4}, \vec{b}=\vec{e_3}+b_4\vec{e_4}, \vec{c}=\vec{e_5}, \vec{d}=\vec{e_6}. \quad (a_9)$$

If $d_5$ 0 in the basis (8), then we obtain the canonical basis that is equivalent to $(a_9)$.

9. Suppose now that all coefficients $b_2$, $c_2$, $d_2$ and $b_3$, $c_3$, $d_3$ in the basis (a) are zero. This is the new case that opposites to the case considered in the step 6. We have:

$$\begin{aligned}&\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_3\vec{e_3}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6}, \vec{b}=b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},\\ &\vec{c}=c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6}, \vec{d}=d_4\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6}\end{aligned}$$

In the last basis, at least one coefficient from $b_4$, $c_4$, $d_4$ is not zero. Otherwise, vectors are linearly dependent but it's impossible for any basis. Let $b_4 \neq 0$ (without any loss in the generality). Perform the operation $\vec{b}/b_4$ first, and the operations $\vec{a}-a_4\vec{b}, \vec{c}-c_4\vec{b}, \vec{d}-d_4\vec{b}$ after the first one. We obtain the following basis:

$$\begin{aligned}&\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_3\vec{e_3}+a_5\vec{e_5}+a_6\vec{e_6}, \vec{b}=\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},\\ &\vec{c}=c_5\vec{e_5}+c_6\vec{e_6}, \vec{d}=d_5\vec{e_5}+d_6\vec{e_6}\end{aligned}. \quad (9)$$

In the basis (9), at least one coefficient among $c_5$, $d_5$ is not zero. Let $c_5 \neq 0$. Perform the operation $\vec{c}/c_5$ first, and the operations $\vec{a}-a_4\vec{b}$, $\vec{c}-c_4\vec{b}, \vec{d}-d_4\vec{b}$ after the first one. The following basis is obtained:

$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_3\vec{e_3}+a_6\vec{e_6}, \vec{b}=\vec{e_4}+b_6\vec{e_6}, \vec{c}=\vec{e_5}+c_6\vec{e_6}, \vec{d}=d_6\vec{e_6}.$$

The obvious linear operations transform the last basis into the new canonical basis:

$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_3\vec{e_3}+a_6\vec{e_6}, \vec{b}=\vec{e_4}+b_6\vec{e_6}, \vec{c}=\vec{e_5}+c_6\vec{e_6}, \vec{d}=d_6\vec{e_6}. \quad (a_{10})$$

If $d_5$ 0 in the basis (9), then the basis that is equivalent to $(a_{10})$ will be obtained. We have analyzed all possibilities in the situation A.

**B**. Suppose now that all coefficients $a_1$, $b_1$, $c_1$, $d_1$ are zero in (III). The following basis is obtained:

$$\begin{aligned}&\vec{a}=a_2\vec{e_2}+a_3\vec{e_3}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6}, \vec{b}=b_2\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},\\ &\vec{c}=c_2\vec{e_2}+c_3\vec{e_3}+c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6}, \vec{d}=d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6}\end{aligned}. \quad (b)$$

1. Consider coefficients $a_2$, $b_2$, $c_2$, $d_2$ in the basis (b). Suppose now that at least one coefficient among $a_2$, $b_2$, $c_2$, $d_2$ is not zero. Without any loss in generality, let $a_2 \neq 0$. Perform next linear operations: $\vec{a}/a_2$ first, and $\vec{b}-b_2\vec{a}$, $\vec{c}-c_2\vec{a}$, $\vec{d}-d_2\vec{a}$ then. As the result, the next transformed basis appears:

$$\begin{aligned}&\vec{a}=a_2\vec{e_2}+a_3\vec{e_3}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6}, \vec{b}=b_2\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},\\ &\vec{c}=c_2\vec{e_2}+c_3\vec{e_3}+c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6}, \vec{d}=d_2\vec{e_2}+d_3\vec{e_3}+d_4\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6}\end{aligned} \quad (1)$$

2. Suppose now that at least one coefficient from $b_3$, $c_3$, $d_3$ in the basis (1) is not zero. Without any loss in generality, let $b_3$ 0. Perform next linear operations: $\vec{b}/b_3$ first, and $\vec{b}-b_2\vec{a}, \vec{c}-c_2\vec{a}, \vec{d}-d_2\vec{a}$ then. As the result, the next transformed basis appears:

$$\begin{aligned}&\vec{a}=\vec{e_2}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6}, \vec{b}=\vec{e_3}+b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},\\ &\vec{c}=c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6}, \vec{d}=d_4\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6}\end{aligned} \quad (2)$$

3. Consider coefficients $c_4$, $d_4$ in the basis (2). Suppose that at least one of them is not zero. Let $c_4 \neq 0$. Perform operation $\vec{c}/c_4$ first, and operations $\vec{b}-b_2\vec{a}, \vec{c}-c_2\vec{a}, \vec{d}-d_2\vec{a}$ after the first one. We obtain the following basis:

$$\vec{a}=\vec{e_2}+a_5\vec{e_5}+a_6\vec{e_6}, \vec{b}=\vec{e_3}+b_5\vec{e_5}+b_6\vec{e_6}, \vec{c}=\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6}, \vec{d}=d_5\vec{e_5}+d_6\vec{e_6} \quad (3)$$

The vector $\vec{d}$ in the basis (3) has at least one non zero coefficient in the last basis. If $d_5 \neq 0$ then performing operations $\vec{d}/d_5, \vec{a}-a_5\vec{d}, \vec{b}-b_5\vec{d}, \vec{c}-c_5\vec{d}$, we obtain the new canonical basis:

$$\vec{a}=\vec{e_2}+a_6\vec{e_6}, \vec{b}=\vec{e_3}+b_6\vec{e_6}, \vec{c}=\vec{e_4}+c_6\vec{e_6}, \vec{d}=\vec{e_5}+d_6\vec{e_6}. \quad (b_1)$$

If $d_6 \neq 0$ then we obtain one more new canonical basis:

$$\vec{a}=\vec{e_2}+a_5\vec{e_5}, \vec{b}=\vec{e_3}+b_5\vec{e_5}, \vec{c}=\vec{e_4}+c_5\vec{e_5}, \vec{d}=d_5\vec{e_5}+\vec{e_6}. \quad (b_2)$$

The assumption $d_4$ 0 brings the same canonical bases $(b_1)$ and $(b_2)$.

4. Suppose now that both coefficients $c_4$, $d_4$ are zero in the basis (2). We have:

$$\begin{aligned}&\vec{a}=\vec{e_2}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6}, \vec{b}=\vec{e_3}+b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},\\ &\vec{c}=c_5\vec{e_5}+c_6\vec{e_6}, \vec{d}=d_5\vec{e_5}+d_6\vec{e_6}\end{aligned} \quad (4)$$

In the basis (4), at least one coefficient from $c_5$, $d_5$ is not zero. Let $c_5 \neq 0$. Perform operation $\vec{c}/c_5$ first, and operations $\vec{a}-a_5\vec{c}, \vec{b}-b_5\vec{c}, \vec{d}-d_5\vec{c}$ after the first one. We obtain the next basis:

$$\vec{a}=\vec{e_2}+a_4\vec{e_4}+a_6\vec{e_6}, \vec{b}=\vec{e_3}+b_4\vec{e_4}+b_6\vec{e_6}, \vec{c}=\vec{e_5}+c_6\vec{e_6}, \vec{d}=d_6\vec{e_6}.$$

It is easy to transform the last basis into the next canonical basis:

$$\vec{a}=\vec{e_2}+a_4\vec{e_4}, \vec{b}=\vec{e_3}+b_4\vec{e_4}, \vec{c}=\vec{e_5}, \vec{d}=\vec{e_6}. \quad (b_3)$$

If we suppose that $d_5 \neq 0$ in the basis (4), the same canonical basis $(b_3)$ will be done.

5. Suppose now that all coefficients $b_3$, $c_3$, $d_3$ in the basis (1) are zero. We obtain:

$$\begin{aligned}&\vec{a}=\vec{e_2}+a_3\vec{e_3}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6}, \vec{b}=b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},\\ &\vec{c}=c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6}, \vec{d}=d_4\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6}\end{aligned} \quad (5)$$

In the basis (5), at least one coefficient among $b_4$, $c_4$, $d_4$ is not zero. Otherwise, vectors $\vec{b},\vec{c},\vec{d}$ are linearly dependent but it's impossible. Let $b_4 \neq 0$. Perform next linear operations: $\vec{b}/b_4$ first, and $\vec{a}-a_4\vec{b}, \vec{c}-c_4\vec{b}, \vec{d}-d_4\vec{b}$ then. As the result, the next transformed basis appears:

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$$\vec{a}=\vec{e_2}+a_3\vec{e_3}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},$$
$$\vec{c}=c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_5\vec{e_5}+d_6\vec{e_6}$$

If $c_4 \neq 0$ or $d_4$ 0, then we obtain bases that are equivalent to the last basis. At least one coefficient among $c_5$, $d_5$ at the last basis is not zero. Let $c_5 \neq 0$. Perform operation $\vec{c}/c_5$ first, and operations $\vec{a}-a_5\vec{c},\ \vec{b}-b_5\vec{c},\ \vec{d}-d_5\vec{c}$ after the first one. We obtain:

$$\vec{a}=\vec{e_2}+a_3\vec{e_3}+a_6\vec{e_6},\ \vec{b}=\vec{e_4}+b_6\vec{e_6},\ \vec{c}=\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_6\vec{e_6}\ .$$

The last basis can be transformed immediately into the next canonical basis:

$$\vec{a}-a_5\vec{c},\ \vec{b}-b_5\vec{c},\ \vec{d}-d_5\vec{c} \qquad (b_4)$$

If we suppose that $d_5$ 0 instead $c_5 \neq 0$, we'll obtain the same canonical basis $(b_4)$.

6. Consider coefficients $a_2$, $b_2$, $c_2$, $d_2$ in the basis (b). Suppose now (in opposition to the step 1) that all coefficients $a_2$, $b_2$, $c_2$, $d_2$ are zero. We have the basis:

$$\begin{aligned}&\vec{a}=a_3\vec{e_3}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=b_3\vec{e_3}+b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},\\&\vec{c}=c_3\vec{e_3}+c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_3\vec{e_3}+d_4\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6}\end{aligned} \qquad (6)$$

Consider coefficients $a_3$, $b_3$, $c_3$, $d_3$. At least one of them is not zero. Otherwise, vectors $\vec{a},\vec{b},\vec{c},\vec{d}$ are linearly dependent but it's impossible for any basis. Let $a_3 \neq 0$ (without any loss in generality). Perform the operation $\vec{a}/a_3$ first, and the operations $\vec{b}-b_3\vec{a},\ \vec{c}-c_3\vec{a},\ \vec{d}-d_3\vec{a}$ after the first one. We obtain the following basis:

$$\vec{a}=\vec{e_3}+a_4\vec{e_4}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=b_4\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},$$
$$\vec{c}=c_4\vec{e_4}+c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_4\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6}$$

Consider coefficients $b_4$, $c_4$, $d_4$ in the last basis. At least one of them is not zero. Otherwise, vectors $\vec{b},\vec{c},\vec{d}$ are linearly dependent but it's impossible. Let $b_4 \neq 0$. Perform the operation $\vec{a}-a_4\vec{b},\ \vec{c}-c_4\vec{b},\ \vec{d}-d_4\vec{b}$ first, and the operations $\vec{a}-a_4\vec{b},\ \vec{c}-c_4\vec{b},\ \vec{d}-d_4\vec{b}$ after the first one. We obtain:

$$\vec{a}=\vec{e_3}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=\vec{e_4}+b_5\vec{e_5}+b_6\vec{e_6},\ \vec{c}=c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=d_5\vec{e_5}+d_6\vec{e_6}$$

Continue this procedure; we will obtain the following canonical basis at the end:

$$\vec{a}=\vec{e_3},\ \vec{b}=\vec{e_4},\ \vec{c}=\vec{e_5},\ \vec{b}=\vec{e_6}\ .\ (b_5)$$

All other subcases in the step 6 give the same basis $(b_5)$.

The total list of all canonical bases that are found at the situations A and B is done here:

$$\vec{a}=\vec{e_1}+a_4\vec{e_4}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_4\vec{e_4}+b_6\vec{e_6},\ \vec{c}=\vec{e_3}+c_4\vec{e_4}+c_6\vec{e_6},\ \vec{d}=\vec{e_5}+d_6\vec{e_6}\ . \quad (a_1)$$
$$\vec{a}=\vec{e_1}+a_4\vec{e_4}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_4\vec{e_4}+b_6\vec{e_6},\ \vec{c}=\vec{e_3}+c_4\vec{e_4}+c_6\vec{e_6},\ \vec{d}=\vec{e_5}+d_6\vec{e_6} \quad (a_2)$$
$$\vec{a}=\vec{e_1}+a_4\vec{e_4}+a_5\vec{e_5},\ \vec{b}=\vec{e_2}+b_4\vec{e_4}+b_5\vec{e_5},\ \vec{c}=\vec{e_3}+c_4\vec{e_4}+c_5\vec{e_5},\ \vec{d}=d_5\vec{e_5}+\vec{e_6} \quad (a_3)$$
$$\vec{a}=\vec{e_1}+a_3\vec{e_3}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_6\vec{e_6},\ \vec{c}=\vec{e_4}+c_6\vec{e_6},\ \vec{d}=\vec{e_5}+d_6\vec{e_6} \quad (a_4)$$
$$\vec{a}=\vec{e_1}+a_3\vec{e_3}+a_5\vec{e_5},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_5\vec{e_5},\ \vec{c}=\vec{e_4}+c_5\vec{e_5},\ \vec{d}=d_5\vec{e_5}+\vec{e_6} \quad (a_5)$$
$$\vec{a}=\vec{e_1}+a_3\vec{e_3}+a_4\vec{e_4},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_4\vec{e_4},\ \vec{c}=\vec{e_5},\ \vec{d}=\vec{e_6} \quad (a_6)$$
$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_6\vec{e_6},\ \vec{b}=\vec{e_3}+b_6\vec{e_6},\ \vec{c}=\vec{e_4}+c_6\vec{e_6},\ \vec{d}=\vec{e_5}+d_6\vec{e_6} \quad (a_7)$$
$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_5\vec{e_5},\ \vec{b}=\vec{e_3}+b_5\vec{e_5},\ \vec{c}=\vec{e_4}+c_5\vec{e_5},\ \vec{d}=d_5\vec{e_5}+\vec{e_6} \quad (a_8)$$
$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_4\vec{e_4},\ \vec{b}=\vec{e_3}+b_4\vec{e_4},\ \vec{c}=\vec{e_5},\ \vec{d}=\vec{e_6} \quad (a_9)$$
$$\vec{a}=\vec{e_2}+a_6\vec{e_6},\ \vec{b}=\vec{e_3}+b_6\vec{e_6},\ \vec{c}=\vec{e_4}+c_6\vec{e_6},\ \vec{d}=\vec{e_5}+d_6\vec{e_6} \quad (a_{10})$$
$$\vec{a}=\vec{e_2}+a_6\vec{e_6},\ \vec{b}=\vec{e_3}+b_6\vec{e_6},\ \vec{c}=\vec{e_4}+c_6\vec{e_6},\ \vec{d}=\vec{e_5}+d_6\vec{e_6} \quad (b_1)$$
$$\vec{a}=\vec{e_2}+a_5\vec{e_5},\ \vec{b}=\vec{e_3}+b_5\vec{e_5},\ \vec{c}=\vec{e_4}+c_5\vec{e_5},\ \vec{d}=d_5\vec{e_5}+\vec{e_6} \quad (b_2)$$
$$\vec{a}=\vec{e_2}+a_4\vec{e_4},\ \vec{b}=\vec{e_3}+b_4\vec{e_4},\ \vec{c}=\vec{e_5},\ \vec{d}=\vec{e_6} \quad (b_3)$$
$$\vec{a}=\vec{e_2}+a_3\vec{e_3},\ \vec{b}=\vec{e_4},\ \vec{c}=\vec{e_5},\ \vec{d}=\vec{e_6} \quad (b_4)$$
$$\vec{a}=\vec{e_3},\ \vec{b}=\vec{e_4},\ \vec{c}=\vec{e_5},\ \vec{d}=\vec{e_6} \quad (b_5)$$

Compare these canonical bases to determine nonequivalent among them. If $d_5 \neq 0$ in the basis $(a_3)$ then this basis is equivalent to the basis $(a_2)$, so $d_5$=0 in $(a_3)$. Similarly, if $d_5 \neq 0$ in the basis $(a_5)$ then this basis is equivalent to the basis $(a_4)$, so $d_5$=0 in $(a_5)$. Again, if $d_5 \neq 0$ in the basis $(a_8)$ then this basis is equivalent to the basis $(a_7)$, so d5=0 in $(a_8)$. If $d_5 \neq$ 0 in the basis $(b_2)$ then this basis is equivalent to the basis $(b_1)$, so $d_5$=0 in $(b_2)$. The final list of nonequivalent canonical bases is:

$$\vec{a}=\vec{e_1}+a_5\vec{e_5}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_5\vec{e_5}+b_6\vec{e_6},\ \vec{c}=\vec{e_3}+c_5\vec{e_5}+c_6\vec{e_6},\ \vec{d}=\vec{e_4}+d_5\vec{e_5}+d_6\vec{e_6} \quad (a_1)$$
$$\vec{a}=\vec{e_1}+a_4\vec{e_4}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_4\vec{e_4}+b_6\vec{e_6},\ \vec{c}=\vec{e_3}+c_4\vec{e_4}+c_6\vec{e_6},\ \vec{d}=\vec{e_5}+d_6\vec{e_6} \quad (a_2)$$
$$\vec{a}=\vec{e_1}+a_4\vec{e_4}+a_5\vec{e_5},\ \vec{b}=\vec{e_2}+b_4\vec{e_4}+b_5\vec{e_5},\ \vec{c}=\vec{e_3}+c_4\vec{e_4}+c_5\vec{e_5},\ \vec{d}=\vec{e_6} \quad (a_3)$$
$$\vec{a}=\vec{e_1}+a_3\vec{e_3}+a_6\vec{e_6},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_6\vec{e_6},\ \vec{c}=\vec{e_4}+c_6\vec{e_6},\ \vec{d}=\vec{e_5}+d_6\vec{e_6} \quad (a_4)$$
$$\vec{a}=\vec{e_1}+a_3\vec{e_3}+a_5\vec{e_5},\ \vec{b}=\vec{e_2}+b_3\vec{e_3}+b_5\vec{e_5},\ \vec{c}=\vec{e_4}+c_5\vec{e_5},\ \vec{d}=\vec{e_6} \quad (a_5)$$
$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_6\vec{e_6},\ \vec{b}=\vec{e_3}+b_6\vec{e_6},\ \vec{c}=\vec{e_4}+c_6\vec{e_6},\ \vec{d}=\vec{e_5}+d_6\vec{e_6} \quad (a_6)$$
$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_6\vec{e_6},\ \vec{b}=\vec{e_3}+b_6\vec{e_6},\ \vec{c}=\vec{e_4}+c_6\vec{e_6},\ \vec{d}=\vec{e_5}+d_6\vec{e_6} \quad (a_7)$$
$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_5\vec{e_5},\ \vec{b}=\vec{e_3}+b_5\vec{e_5},\ \vec{c}=\vec{e_4}+c_5\vec{e_5},\ \vec{d}=\vec{e_6} \quad (a_8)$$
$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_4\vec{e_4},\ \vec{b}=\vec{e_3}+b_4\vec{e_4},\ \vec{c}=\vec{e_5},\ \vec{d}=\vec{e_6} \quad (a_9)$$
$$\vec{a}=\vec{e_1}+a_2\vec{e_2}+a_3\vec{e_3},\ \vec{b}=\vec{e_4},\ \vec{c}=\vec{e_5},\ \vec{d}=\vec{e_6} \quad (a_{10})$$
$$\vec{a}=\vec{e_2}+a_6\vec{e_6},\ \vec{b}=\vec{e_3}+b_6\vec{e_6},\ \vec{c}=\vec{e_4}+c_6\vec{e_6},\ \vec{d}=\vec{e_5}+d_6\vec{e_6} \quad (b_1)$$
$$\vec{a}=\vec{e_2}+a_5\vec{e_5},\ \vec{b}=\vec{e_3}+b_5\vec{e_5},\ \vec{c}=\vec{e_4}+c_5\vec{e_5},\ \vec{d}=\vec{e_6} \quad (b_2)$$
$$\vec{a}=\vec{e_2}+a_4\vec{e_4},\ \vec{b}=\vec{e_3}+b_4\vec{e_4},\ \vec{c}=\vec{e_5},\ \vec{d}=\vec{e_6} \quad (b_3)$$
$$\vec{a}=\vec{e_2}+a_3\vec{e_3},\ \vec{b}=\vec{e_4},\ \vec{c}=\vec{e_5},\ \vec{d}=\vec{e_6} \quad (b_4)$$
$$\vec{a}=\vec{e_3},\ \vec{b}=\vec{e_4},\ \vec{c}=\vec{e_5},\ \vec{d}=\vec{e_6} \quad (b_5)$$

The following 15 matrices are associated with the canonical bases above:

$$\begin{bmatrix}1&0&0&0&a_5&a_6\\0&1&0&0&b_5&b_6\\0&0&1&0&c_5&c_6\\0&0&0&1&d_5&d_6\end{bmatrix},\begin{bmatrix}1&0&0&a_4&0&a_6\\0&1&0&b_4&0&b_6\\0&0&1&c_4&0&c_6\\0&0&0&0&1&d_6\end{bmatrix},\begin{bmatrix}1&0&0&a_4&a_5&0\\0&1&0&b_4&b_5&0\\0&0&1&c_4&c_5&0\\0&0&0&0&0&1\end{bmatrix},$$

$$\begin{bmatrix}1&0&a_3&0&0&a_6\\0&1&b_3&0&0&b_6\\0&0&0&1&0&c_6\\0&0&0&0&1&d_6\end{bmatrix},\begin{bmatrix}1&0&a_3&0&a_5&0\\0&1&b_3&0&b_5&0\\0&0&0&1&c_5&0\\0&0&0&0&0&1\end{bmatrix},\begin{bmatrix}1&0&a_3&a_4&0&0\\0&1&b_3&b_4&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},$$

$$\begin{bmatrix}1&0&a_3&0&0&a_6\\0&1&b_3&0&0&b_6\\0&0&0&1&0&c_6\\0&0&0&0&1&d_6\end{bmatrix},\begin{bmatrix}1&0&a_3&0&a_5&0\\0&1&b_3&0&b_5&0\\0&0&0&1&c_5&0\\0&0&0&0&0&1\end{bmatrix},\begin{bmatrix}1&0&a_3&a_4&0&0\\0&1&b_3&b_4&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},$$

$$\begin{bmatrix}1&a_2&a_3&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},\begin{bmatrix}0&1&0&0&0&a_6\\0&0&1&0&0&b_6\\0&0&0&1&0&c_6\\0&0&0&0&1&d_6\end{bmatrix},\begin{bmatrix}0&1&0&0&a_5&0\\0&0&1&0&b_5&0\\0&0&0&1&c_5&0\\0&0&0&0&0&1\end{bmatrix},$$

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$$\begin{bmatrix}0&1&0&a_4&0&0\\0&0&1&b_4&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},\begin{bmatrix}0&1&a_3&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},\begin{bmatrix}0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}.$$

All these matrices are not equivalent.

Part II. General Case:

The following statement concerning $(n-2)\times n$ matrices $M$ of the type (II) is true.

**Theorem 1:** Let $M$ be a $(n-2)\times n$ matrix (II) of the rank $(n-2)$ where $n \geq 4$. This matrix is row equivalent to one and only one of the $\frac{n(n-1)}{2}$ following matrices:

$$[1]\begin{bmatrix}1&0&0&\dots&0&a_{1,n-1}&a_{1,n}\\0&1&0&\dots&0&a_{2,n-1}&a_{2,n}\\0&0&1&\dots&0&a_{3,n-1}&a_{3,n}\\.&.&.&\dots&.&.&.\\0&0&0&\dots&0&a_{n-4,n-1}&a_{n-4,n}\\0&0&0&\dots&0&a_{n-3,n-1}&a_{n-3,n}\\0&0&0&\dots&1&a_{n-2,n-1}&a_{n-2,n}\end{bmatrix},[2]\begin{bmatrix}1&0&0&\dots&a_{1,n-2}&0&a_{1,n}\\0&1&0&\dots&a_{2,n-2}&0&a_{2,n}\\0&0&1&\dots&a_{3,n-2}&0&a_{3,n}\\.&.&.&\dots&.&.&.\\0&0&0&\dots&a_{n-4,n-2}&0&a_{n-4,n}\\0&0&0&\dots&a_{n-3,n-2}&0&a_{n-3,n}\\0&0&0&\dots&0&1&a_{n-2,n}\end{bmatrix},\dots\dots,$$

$$[n-3]\begin{bmatrix}1&0&a_{1,3}&\dots&0&0&a_{1,n}\\0&1&a_{2,3}&\dots&0&0&a_{2,n}\\0&0&0&\dots&0&0&a_{3,n}\\.&.&.&\dots&.&.&.\\0&0&0&\dots&0&0&a_{n-4,n}\\0&0&0&\dots&1&0&a_{n-3,n}\\0&0&0&\dots&0&1&a_{n-2,n}\end{bmatrix},[n-2]\begin{bmatrix}1&a_{1,2}&0&\dots&0&0&a_{1,n}\\0&0&1&\dots&0&0&a_{2,n}\\0&0&0&\dots&0&0&a_{3,n}\\.&.&.&\dots&.&.&.\\0&0&0&\dots&0&0&a_{n-4,n}\\0&0&0&\dots&1&0&a_{n-3,n}\\0&0&0&\dots&0&1&a_{n-2,n}\end{bmatrix};$$

$$[n-1]\begin{bmatrix}1&0&0&\dots&0&a_{1,n-2}&a_{1,n-1}&0\\0&1&0&\dots&0&a_{2,n-2}&a_{2,n-1}&0\\0&0&1&\dots&0&a_{3,n-2}&a_{3,n-1}&0\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&0&a_{n-4,n-2}&a_{n-4,n-1}&0\\0&0&0&\dots&1&a_{n-3,n-2}&a_{n-3,n-1}&0\\0&0&0&\dots&0&0&0&1\end{bmatrix},[n]\begin{bmatrix}1&0&0&\dots&a_{1,n-3}&0&a_{1,n-1}&0\\0&1&0&\dots&a_{2,n-3}&0&a_{2,n-1}&0\\0&0&1&\dots&a_{3,n-3}&0&a_{3,n-1}&0\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&a_{n-4,n-3}&0&a_{n-4,n-1}&0\\0&0&0&\dots&0&1&a_{n-3,n-1}&0\\0&0&0&\dots&0&0&0&1\end{bmatrix},\dots\dots,$$

$$[2n-5]\begin{bmatrix}1&a_{1,2}&0&\dots&0&0&a_{1,n-1}&0\\0&0&1&\dots&0&0&a_{2,n-1}&0\\0&0&0&\dots&0&0&a_{3,n-1}&0\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&0&0&a_{n-4,n-1}&0\\0&0&0&\dots&0&1&a_{n-3,n-1}&0\\0&0&0&\dots&0&0&0&1\end{bmatrix};\ [2n-4]\begin{bmatrix}1&0&0&\dots&a_{1,n-3}&a_{1,n-2}&0&0\\0&1&0&\dots&a_{2,n-3}&a_{2,n-2}&0&0\\0&0&1&\dots&a_{3,n-3}&a_{3,n-2}&0&0\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&a_{n-4,n-3}&a_{n-4,n-2}&0&0\\0&0&0&\dots&0&0&1&0\\0&0&0&\dots&0&0&0&1\end{bmatrix},\dots\dots,$$

$$[3n-9]\begin{bmatrix}1&a_{1,2}&0&\dots&0&a_{1,n-2}&0&0\\0&0&1&\dots&0&a_{2,n-2}&0&0\\0&0&0&\dots&0&a_{3,n-2}&0&0\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&1&a_{n-4,n-2}&0&0\\0&0&0&\dots&0&0&1&0\\0&0&0&\dots&0&0&0&1\end{bmatrix};\dots,[\frac{n(n-3)}{2}+1]\begin{bmatrix}1&a_{1,2}&a_{1,3}&\dots&0&0&0&0\\0&0&0&\dots&0&0&0&0\\0&0&0&\dots&0&0&0&0\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&0&1&0&0\\0&0&0&\dots&0&0&1&0\\0&0&0&\dots&0&0&0&1\end{bmatrix};$$

$$[\frac{n(n-3)}{2}+2]\begin{bmatrix}0&1&0&\dots&0&0&a_{1,n}\\0&0&1&\dots&0&0&a_{2,n}\\0&0&0&\dots&0&0&a_{3,n}\\.&.&.&\dots&.&.&.\\0&0&0&\dots&1&0&a_{n-3,n}\\0&0&0&\dots&0&1&a_{n-2,n}\end{bmatrix},[\frac{n(n-3)}{2}+3]\begin{bmatrix}0&1&0&\dots&0&a_{1,n-1}&0\\0&0&1&\dots&0&a_{2,n-1}&0\\0&0&0&\dots&0&a_{3,n-1}&0\\.&.&.&\dots&.&.&.\\0&0&0&\dots&1&a_{n-3,n-1}&0\\0&0&0&\dots&0&0&1\end{bmatrix},\dots\dots,$$

$$[\frac{n(n-1)}{2}-1]\begin{bmatrix}0&1&a_{1,3}&\dots&0&0&0\\0&0&0&\dots&0&0&0\\0&0&0&\dots&0&0&0\\.&.&.&\dots&.&.&.\\0&0&0&\dots&0&1&0\\0&0&0&\dots&0&0&1\end{bmatrix},[\frac{n(n-1)}{2}]\begin{bmatrix}0&0&1&\dots&0&0&0\\0&0&0&\dots&0&0&0\\0&0&0&\dots&0&0&0\\.&.&.&\dots&.&.&.\\0&0&0&\dots&0&1&0\\0&0&0&\dots&0&0&1\end{bmatrix}.$$

All matrices above are not equivalent between them.

**Proof:** We will use the mathematical induction method with respect to the dimension $n$. This statement is correct in the cases $n$=4 and $n$=6 according Examples 1 and 2. Suppose that the statement is true for arbitrary $n\geq 4$, and prove it for the dimension $(n+1)$. Let $M$ be a matrix of the size $(n-1)\times(n+1)$:

$$M=\begin{bmatrix}a_{1,1}&a_{1,2}&a_{1,3}&\dots&a_{1,n-2}&a_{1,n-1}&a_{1,n}&a_{1,n+1}\\a_{2,1}&a_{2,2}&a_{2,3}&\dots&a_{2,n-2}&a_{2,n-1}&a_{2,n}&a_{2,n+1}\\a_{3,1}&a_{3,2}&a_{3,3}&\dots&a_{3,n-2}&a_{3,n-1}&a_{3,n}&a_{3,n+1}\\.&.&.&\dots&.&.&.&.\\a_{n-3,1}&a_{n-3,2}&a_{n-3,3}&\dots&a_{n-3,n-2}&a_{n-3,n-1}&a_{n-3,n}&a_{n-3,n+1}\\a_{n-2,1}&a_{n-2,2}&a_{n-2,3}&\dots&a_{n-2,n-2}&a_{n-2,n-1}&a_{n-2,n}&a_{n-2,n+1}\\a_{n-1,1}&a_{n-1,2}&a_{n-1,3}&\dots&a_{n-1,n-2}&a_{n-1,n-1}&a_{n-1,n}&a_{n-1,n+1}\end{bmatrix}.$$

Consider the $(n-2)\times n$ submatrix $M'$ located in the upper left corner of the matrix $M$. According the assumption, this submatrix can be transformed into one of the matrices listed in this statement. We will substitute submatrix $M'$ by the corresponding matrix, and then transform the special matrix $M$ into reduced rom echelon form. The standard linear operations with rows (vectors) will be utilized: (a) interchange any two rows, (b) multiply any row by a nonzero constant, (c) add a multiple of some row to another row.

1. At the first case, we have:

$$M=\begin{bmatrix}1&0&0&\dots&0&a_{1,n-1}&a_{1,n}&a_{1,n+1}\\0&1&0&\dots&0&a_{2,n-1}&a_{2,n}&a_{2,n+1}\\0&0&1&\dots&0&a_{3,n-1}&a_{3,n}&a_{3,n+1}\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&0&a_{n-3,n-1}&a_{n-3,n}&a_{n-3,n+1}\\0&0&0&\dots&1&a_{n-2,n-1}&a_{n-2,n}&a_{n-2,n+1}\\a_{n-1,1}&a_{n-1,2}&a_{n-1,3}&\dots&a_{n-1,n-2}&a_{n-1,n-1}&a_{n-1,n}&a_{n-1,n+1}\end{bmatrix}.$$

Perform linear transformations $\overrightarrow{a_{n-1}}-a_{n-1,1}\overrightarrow{a_1},\ \overrightarrow{a_{n-1}}-a_{n-1,2}\overrightarrow{a_2},\ \dots,\ \overrightarrow{a_{n-1}}-a_{n-1,n-2}\overrightarrow{a_{n-2}}$. The result of the operations is the following matrix:

$$M=\begin{bmatrix}1&0&0&\dots&0&a_{1,n-1}&a_{1,n}&a_{1,n+1}\\0&1&0&\dots&0&a_{2,n-1}&a_{2,n}&a_{2,n+1}\\0&0&1&\dots&0&a_{3,n-1}&a_{3,n}&a_{3,n+1}\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&0&a_{n-3,n-1}&a_{n-3,n}&a_{n-3,n+1}\\0&0&0&\dots&1&a_{n-2,n-1}&a_{n-2,n}&a_{n-2,n+1}\\0&0&0&\dots&0&a_{n-1,n-1}&a_{n-1,n}&a_{n-1,n+1}\end{bmatrix}.$$

At least one components among $a_{n-1,n-1}$, $a_{n-1,n}$, $a_{n-1,n+1}$ is not zero but all other components of the $(n-1)$ row are zero. Let $a_{n-1,n-1}\neq 0$. Perform the operation $\overrightarrow{a_{n-1}}/a_{n-1,n-1}$ first, and the operations $\overrightarrow{a_1}-a_{1,n-1}\overrightarrow{a_{n-1}},\ \overrightarrow{a_2}-a_{2,n-1}\overrightarrow{a_{n-1}},\ \dots,\overrightarrow{a_{n-2}}-a_{n-2,n-1}\overrightarrow{a_{n-1}}$ after the first one. We obtain:

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$$M=\begin{bmatrix}1&0&0&\dots&0&0&a_{1,n}&a_{1,n+1}\\0&1&0&\dots&0&0&a_{2,n}&a_{2,n+1}\\0&0&1&\dots&0&0&a_{3,n}&a_{3,n+1}\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&0&0&a_{n-3,n}&a_{n-3,n+1}\\0&0&0&\dots&1&0&a_{n-2,n}&a_{n-2,n+1}\\0&0&0&\dots&0&1&a_{n-1,n}&a_{n-1,n+1}\end{bmatrix}.$$

It is the matrix of the first type from the list above as we need. Let $a_{n-1,\,n-1}=0$, and $a_{n-1,\,n}\neq 0$. Perform the operation $\overrightarrow{a_{n-1}}/a_{n-1,n}$ first, and the operations $\overrightarrow{a_1}-a_{1,n}\overrightarrow{a_{n-1}}$, $\overrightarrow{a_2}-a_{2,n}\overrightarrow{a_{n-1}}$, …, $\overrightarrow{a_{n-2}}-a_{n-2,n}\overrightarrow{a_{n-1}}$ after the first one. We obtain:

$$M=\begin{bmatrix}1&0&0&\dots&0&a_{1,n-1}&0&a_{1,n+1}\\0&1&0&\dots&0&a_{2,n-1}&0&a_{2,n+1}\\0&0&1&\dots&0&a_{3,n-1}&0&a_{3,n+1}\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&0&a_{n-3,n-1}&0&a_{n-3,n+1}\\0&0&0&\dots&1&a_{n-2,n-1}&0&a_{n-2,n+1}\\0&0&0&\dots&0&0&1&a_{n-1,n+1}\end{bmatrix}.$$

The last matrix is of the second type matrix from the list as we need. Let $a_{n-1,\,n-1}=0$, $a_{n-1,\,n}=0$, and $a_{n-1,\,n+1}\neq 0$. Perform the operation $\overrightarrow{a_{n-1}}/a_{n-1,n+1}$ first, and the operations $\overrightarrow{a_1}-a_{1,n+1}\overrightarrow{a_{n-1}}$, $\overrightarrow{a_2}-a_{2,n+1}\overrightarrow{a_{n-1}}$, …, $\overrightarrow{a_{n-2}}-a_{n-2,n+1}\overrightarrow{a_{n-1}}$ after the first one. We obtain:

$$M=\begin{bmatrix}1&0&0&\dots&0&a_{1,n-1}&a_{1,n}&0\\0&1&0&\dots&0&a_{2,n-1}&a_{2,n}&0\\0&0&1&\dots&0&a_{3,n-1}&a_{3,n}&0\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&0&a_{n-3,n-1}&a_{n-3,n}&0\\0&0&0&\dots&1&a_{n-2,n-1}&a_{n-2,n}&0\\0&0&0&\dots&0&0&0&1\end{bmatrix}.$$

The last matrix is of the $(n-1)$ type matrix from the list above. The statement is proved for the 1st case.

2. At the second case, we have:

$$M=\begin{bmatrix}1&0&0&\dots&a_{1,n-2}&0&a_{1,n}&a_{1,n+1}\\0&1&0&\dots&a_{2,n-2}&0&a_{2,n}&a_{2,n+1}\\0&0&1&\dots&a_{3,n-2}&0&a_{3,n}&a_{3,n+1}\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&a_{n-3,n-2}&0&a_{n-3,n}&a_{n-3,n+1}\\0&0&0&\dots&0&1&a_{n-2,n}&a_{n-2,n+1}\\a_{n-1,1}&a_{n-1,2}&a_{n-1,3}&\dots&a_{n-1,n-2}&a_{n-1,n-1}&a_{n-1,n}&a_{n-1,n+1}\end{bmatrix}.$$

Perform linear transformations $\overrightarrow{a_{n-1}}-a_{n-1,1}\overrightarrow{a_1}$, $\overrightarrow{a_{n-1}}-a_{n-1,2}\overrightarrow{a_2}$, …, $\overrightarrow{a_{n-1}}-a_{n-3,n-3}\overrightarrow{a_{n-3}}$, and $\overrightarrow{a_{n-1}}-a_{n-1,n-1}\overrightarrow{a_{n-2}}$. The result of the operations is the following matrix:

$$M=\begin{bmatrix}1&0&0&\dots&a_{1,n-2}&0&a_{1,n}&a_{1,n+1}\\0&1&0&\dots&a_{2,n-2}&0&a_{2,n}&a_{2,n+1}\\0&0&1&\dots&a_{3,n-2}&0&a_{3,n}&a_{3,n+1}\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&a_{n-3,n-2}&0&a_{n-3,n}&a_{n-3,n+1}\\0&0&0&\dots&0&1&a_{n-2,n}&a_{n-2,n+1}\\0&0&0&\dots&a_{n-1,n-2}&0&a_{n-1,n}&a_{n-1,n+1}\end{bmatrix}.$$

At least one components among $a_{n-1,\,n-2}$, $a_{n-1,\,n}$, $a_{n-1,\,n+1}$ is not zero but all other components of the $(n-1)$ row are zero. Let $a_{n-1,n-2}\neq 0$. Perform the operation $\overrightarrow{a_{n-1}}/a_{n-1,n-2}$ first, and the operations $\overrightarrow{a_1}-a_{1,n-2}\overrightarrow{a_{n-1}}$, $\overrightarrow{a_2}-a_{2,n-2}\overrightarrow{a_{n-1}}$, …, $\overrightarrow{a_{n-2}}-a_{n-3,n-2}\overrightarrow{a_{n-1}}$ after the first one. We obtain:

$$M=\begin{bmatrix}1&0&0&\dots&0&0&a_{1,n}&a_{1,n+1}\\0&1&0&\dots&0&0&a_{2,n}&a_{2,n+1}\\0&0&1&\dots&0&0&a_{3,n}&a_{3,n+1}\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&0&0&a_{n-3,n}&a_{n-3,n+1}\\0&0&0&\dots&0&1&a_{n-2,n}&a_{n-2,n+1}\\0&0&0&\dots&1&0&a_{n-1,n}&a_{n-1,n+1}\end{bmatrix}.$$

If interchange rows $\overrightarrow{a_{n-2}}$ and $\overrightarrow{a_{n-1}}$ in the last matrix, we obtain the matrix of the first type as we need. Let $a_{n-1,n-2}=0$, and $a_{n-1,\,n}\neq 0$. Perform the operation $\overrightarrow{a_{n-1}}/a_{n-1,n}$ first, and the operations $\overrightarrow{a_1}-a_{1,n}\overrightarrow{a_{n-1}}$, $\overrightarrow{a_2}-a_{2,n}\overrightarrow{a_{n-1}}$, …, $\overrightarrow{a_{n-2}}-a_{n-2,n}\overrightarrow{a_{n-1}}$ after the first one. We obtain:

$$M=\begin{bmatrix}1&0&0&\dots&a_{1,n-2}&0&0&a_{1,n+1}\\0&1&0&\dots&a_{2,n-2}&0&0&a_{2,n+1}\\0&0&1&\dots&a_{3,n-2}&0&0&a_{3,n+1}\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&a_{n-3,n-2}&0&0&a_{n-3,n+1}\\0&0&0&\dots&0&1&0&a_{n-2,n+1}\\0&0&0&\dots&0&0&1&a_{n-1,n+1}\end{bmatrix}.$$

It is the matrix of the 3rd type that follows the matrix of the second type in the list. Let $a_{n-1,n-2}=0$, $a_{n-1,\,n}=0$, and $a_{n-1,\,n+1}\neq 0$. Perform the operation $\overrightarrow{a_{n-1}}/a_{n-1,n+1}$ first, and the operations $\overrightarrow{a_1}-a_{1,n+1}\overrightarrow{a_{n-1}}$, $\overrightarrow{a_2}-a_{2,n+1}\overrightarrow{a_{n-1}}$, …, $\overrightarrow{a_{n-2}}-a_{n-2,n+1}\overrightarrow{a_{n-1}}$ after the first one. We obtain:

$$M=\begin{bmatrix}1&0&0&\dots&a_{1,n-2}&0&a_{1,n}&0\\0&1&0&\dots&a_{2,n-2}&0&a_{2,n}&0\\0&0&1&\dots&a_{3,n-2}&0&a_{3,n}&0\\.&.&.&\dots&.&.&.&.\\0&0&0&\dots&a_{n-3,n-2}&0&a_{n-3,n}&0\\0&0&0&\dots&0&1&a_{n-2,n}&0\\0&0&0&\dots&0&0&0&1\end{bmatrix}.$$

It is the matrix of the type $(n)$ in the list above. The statement is proved for the 2nd case. All the next cases are similar to the cases (1) and (2) but we consider some of them.

Case $(n)$. At this case we have:

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$$M=\begin{bmatrix} 1 & 0 & 0 & \dots & a_{1,n-3} & 0 & a_{1,n-1} & 0 & a_{1,n+1} \\ 0 & 1 & 0 & \dots & a_{2,n-3} & 0 & a_{2,n-1} & 0 & a_{2,n+1} \\ 0 & 0 & 1 & \dots & a_{3,n-3} & 0 & a_{3,n-1} & 0 & a_{3,n+1} \\ . & . & . & \dots & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 & a_{n-3,n-1} & 0 & a_{n-3,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & a_{n-2,n+1} \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \dots & a_{n-1,n-3} & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} & a_{n-1,n+1} \end{bmatrix}.$$

Perform linear transformations $\overrightarrow{a_{n-1}}-a_{n-1,1}\overrightarrow{a_1},\ \overrightarrow{a_{n-1}}-a_{n-1,2}\overrightarrow{a_2},\ \dots,\ \overrightarrow{a_{n-1}}-a_{n-1,n-2}\overrightarrow{a_{n-3}}$, and $\overrightarrow{a_{n-1}}-a_{n-1,n}\overrightarrow{a_{n-2}}$. The result of the operations is the following matrix:

$$M=\begin{bmatrix} 1 & 0 & 0 & \dots & a_{1,n-3} & 0 & a_{1,n-1} & 0 & a_{1,n+1} \\ 0 & 1 & 0 & \dots & a_{2,n-3} & 0 & a_{2,n-1} & 0 & a_{2,n+1} \\ 0 & 0 & 1 & \dots & a_{3,n-3} & 0 & a_{3,n-1} & 0 & a_{3,n+1} \\ . & . & . & \dots & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 & a_{n-3,n-1} & 0 & a_{n-3,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & a_{n-2,n+1} \\ 0 & 0 & 0 & \dots & a_{n-1,n-3} & 0 & a_{n-1,n-1} & 0 & a_{n-1,n+1} \end{bmatrix}.$$

At least one components among $a_{n-1,n-3}$, $a_{n-1,n}$, $a_{n-1,n+1}$ is not zero but all other components of the $(n-1)$ row are zero. Let $a_{n-1,n-3}\neq 0$. Perform the operation $\overrightarrow{a_{n-1}}/a_{n-1,n-3}$ first, and the operations $\overrightarrow{a_1}-a_{1,n-3}\overrightarrow{a_{n-1}},\ \overrightarrow{a_2}-a_{2,n-3}\overrightarrow{a_{n-1}},\ \dots,\ \overrightarrow{a_{n-4}}-a_{n-4,n-3}\overrightarrow{a_{n-1}}$ after the first one. We obtain:

$$M=\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & a_{1,n-1} & 0 & a_{1,n+1} \\ 0 & 1 & 0 & \dots & 0 & 0 & a_{2,n-1} & 0 & a_{2,n+1} \\ 0 & 0 & 1 & \dots & 0 & 0 & a_{3,n-1} & 0 & a_{3,n+1} \\ . & . & . & \dots & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 & a_{n-3,n-1} & 0 & a_{n-3,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & a_{n-2,n+1} \\ 0 & 0 & 0 & \dots & 1 & 0 & a_{n-1,n-1} & 0 & a_{n-1,n+1} \end{bmatrix}.$$

If we interchange the last 3 rows of this matrix, we obtain the matrix of the 2nd type from the list as we need. Let $a_{n-1,n-3}=0$, and $a_{n-1,n-1}\neq 0$. Perform the operation $\overrightarrow{a_{n-1}}/a_{n-1,n-1}$ first, and the operations $\overrightarrow{a_1}-a_{1,n-1}\overrightarrow{a_{n-1}},\ \overrightarrow{a_2}-a_{2,n-1}\overrightarrow{a_{n-1}},\ \dots,\ \overrightarrow{a_{n-3}}-a_{n-3,n-1}\overrightarrow{a_{n-1}}$ after the first one. We obtain:

$$M=\begin{bmatrix} 1 & 0 & 0 & \dots & a_{1,n-3} & 0 & 0 & 0 & a_{1,n+1} \\ 0 & 1 & 0 & \dots & a_{2,n-3} & 0 & 0 & 0 & a_{2,n+1} \\ 0 & 0 & 1 & \dots & a_{3,n-3} & 0 & 0 & 0 & a_{3,n+1} \\ . & . & . & \dots & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & a_{n-3,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & a_{n-2,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & a_{n-1,n+1} \end{bmatrix}.$$

If we interchange the rows $(n-1)$ and $(n-2)$ in this matrix, we obtain the matrix of the 3rd type from the list as we need. Let $a_{n-1,n-3}=0$, $a_{n-1,n-1}=0$, and $a_{n-1,n+1}\neq 0$. Perform the operation $\overrightarrow{a_{n-1}}/a_{n-1,n+1}$ first, and the operations $\overrightarrow{a_1}-a_{1,n+1}\overrightarrow{a_{n-1}},\ \overrightarrow{a_2}-a_{2,n+1}\overrightarrow{a_{n-1}},\ \dots,\ \overrightarrow{a_{n-2}}-a_{n-2,n+1}\overrightarrow{a_{n-1}}$ after the first one. We obtain:

$$M=\begin{bmatrix} 1 & 0 & 0 & \dots & a_{1,n-3} & 0 & a_{1,n-1} & 0 & 0 \\ 0 & 1 & 0 & \dots & a_{2,n-3} & 0 & a_{2,n-1} & 0 & 0 \\ 0 & 0 & 1 & \dots & a_{3,n-3} & 0 & a_{3,n-1} & 0 & 0 \\ . & . & . & \dots & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 & a_{n-3,n-1} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is the matrix of the $(2n-3)$ type from the list. The case $(n)$ is proved.

Case $[\frac{n(n-1)}{2}]$. At this case, we have the following matrix of $(n-1)\times(n+1)$ size.

$$M=\begin{bmatrix} 0 & 0 & 1 & \dots & 0 & 0 & 0 & a_{1,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & a_{2,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & a_{3,n+1} \\ . & . & . & \dots & . & . & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & a_{n-3,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & a_{n-2,n+1} \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \dots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} & a_{n-1,n+1} \end{bmatrix}.$$

Perform the operations $\overrightarrow{a_{n-1}}-a_{n-1,3}\overrightarrow{a_1},\ \overrightarrow{a_{n-1}}-a_{n-1,4}\overrightarrow{a_2},\ \dots,\ \overrightarrow{a_{n-1}}-a_{n-1,n}\overrightarrow{a_{n-2}}$. We obtain:

$$M=\begin{bmatrix} 0 & 0 & 1 & \dots & 0 & 0 & 0 & a_{1,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & a_{2,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & a_{3,n+1} \\ . & . & . & \dots & . & . & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & a_{n-3,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & a_{n-2,n+1} \\ a_{n-1,1} & a_{n-1,2} & 0 & \dots & 0 & 0 & 0 & a_{n-1,n+1} \end{bmatrix}.$$

At least one component among $a_{n-1,1}$, $a_{n-1,2}$, $a_{n-1,n+1}$ is not zero. Let $a_{n-1,1}\neq 0$. Perform the operation $\overrightarrow{a_{n-1}}/a_{n-1,1}$, and remove the new last row into the first position. We obtain:

$$M=\begin{bmatrix} 1 & a_{n-1,2} & 0 & \dots & 0 & 0 & 0 & a_{n-1,n+1} \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & a_{1,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & a_{2,n+1} \\ . & . & . & \dots & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & a_{n-4,n+1} \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & a_{n-3,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & a_{n-2,n+1} \end{bmatrix}.$$

It is the matrix of the 2nd type from the list as we need. Let $a_{n-1,1}=0$, and $a_{n-1,2}\neq 0$ in the previous matrix. Perform the operation $\overrightarrow{a_{n-1}}/a_{n-1,2}$ first, and then remove the new last row into the first position. We obtain:

$$M=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 & a_{n-1,n+1} \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & a_{1,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & a_{2,n+1} \\ . & . & . & \dots & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & a_{n-4,n+1} \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & a_{n-3,n+1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & a_{n-2,n+1} \end{bmatrix}.$$

It is the matrix of the type $[\frac{n(n-3)}{2}+2]$ from the list as we need. Let $a_{n-1,1}=0$, $a_{n-1,2}=0$, and $a_{n-1,n+1}\neq 0$ in the previous matrix. Perform the operation $\overrightarrow{a_{n-1}}/a_{n-1,n+1}$ first, and the operations $\overrightarrow{a_1}-a_{1,n+1}\overrightarrow{a_{n-1}},\ \overrightarrow{a_2}-a_{2,n+1}\overrightarrow{a_{n-1}},\ \dots,\ \overrightarrow{a_{n-2}}-a_{n-2,n+1}\overrightarrow{a_{n-1}}$ after the first one. We have:

$$M = \begin{bmatrix} 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ . & . & . & \dots & . & . & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is the matrix of the $[\frac{n(n-1)}{2}]$ type from the list as we need. This case is proved, and the total proof is done.

**Remark 2:** Of cause, the list of matrices in Theorem 1 doesn't contain all of them. But any missed matrix can be restored using Ladder Principle. For each subsequence of matrices (between (;) sings) in the list, imagine the ladder from the lower right corner to the upper left corner. Take the left most columns with arbitrary components, and make 1 step up along the ladder bringing this column up and to the left of the previous position. Fix elements 0 and only one element 1 at the corresponding positions in the released column. The next matrix from the list will be done.

As an obvious consequence of Theorem 1, we obtain the following statement.

**Theorem 2**: Each basis for ($n$–2)-dimensional subspaces of a $n$-dimensional vector space ($n$4) is equivalent to one and only one canonical basis from the following list.

(1) $\overrightarrow{a_1} = \overrightarrow{e_1} + a_{1,n-1}\overrightarrow{e_{n-1}} + a_{1,n}\overrightarrow{e_n},\ \overrightarrow{a_2} = \overrightarrow{e_2} + a_{2,n-1}\overrightarrow{e_{n-1}} + a_{2,n}\overrightarrow{e_n}, \dots,\ \overrightarrow{a_{n-2}} = \overrightarrow{e_{n-2}} + a_{n-2,n-1}\overrightarrow{e_{n-1}} + a_{n-2,n}\overrightarrow{e_n}$.

(2) $\overrightarrow{a_1} = \overrightarrow{e_1} + a_{1,n-2}\overrightarrow{e_{n-2}} + a_{1,n}\overrightarrow{e_n},\ \overrightarrow{a_2} = \overrightarrow{e_2} + a_{2,n-2}\overrightarrow{e_{n-2}} + a_{2,n}\overrightarrow{e_n}, \dots,\ \overrightarrow{a_{n-2}} = \overrightarrow{e_{n-1}} + a_{n-2,n}\overrightarrow{e_n}$.

$[n-2]\overrightarrow{a_1} = \overrightarrow{e_1} + a_{1,2}\overrightarrow{e_2} + a_{1,n}\overrightarrow{e_n},\ \overrightarrow{a_2} = \overrightarrow{e_3} + a_{2,n}\overrightarrow{e_n}, \dots\dots,\ \overrightarrow{a_{n-2}} = \overrightarrow{e_{n-1}} + a_{n-2,n}\overrightarrow{e_n}$.

$[n-1]\ \overrightarrow{a_1} = \overrightarrow{e_1} + a_{1,n-2}\overrightarrow{e_{n-2}} + a_{1,n-1}\overrightarrow{e_{n-1}},\ \overrightarrow{a_2} = \overrightarrow{e_2} + a_{2,n-2}\overrightarrow{e_{n-2}} + a_{2,n-1}\overrightarrow{e_{n-1}}, \dots,\ \overrightarrow{a_{n-2}} = \overrightarrow{e_n}$.

$[2n-5]\ \overrightarrow{a_1} = \overrightarrow{e_1} + a_{1,2}\overrightarrow{e_2} + a_{1,n-1}\overrightarrow{e_{n-1}},\ \overrightarrow{a_2} = \overrightarrow{e_3} + a_{2,n-1}\overrightarrow{e_{n-1}}, \dots,\ \overrightarrow{a_{n-3}} = \overrightarrow{e_{n-2}} + a_{n-3,n-1}\overrightarrow{e_{n-1}},\ \overrightarrow{a_{n-2}} = \overrightarrow{e_n}$.

$[2n-4]\ \overrightarrow{a_1} = \overrightarrow{e_1} + a_{1,n-3}\overrightarrow{e_{n-3}} + a_{1,n-2}\overrightarrow{e_{n-2}},\ \overrightarrow{a_2} = \overrightarrow{e_2} + a_{2,n-3}\overrightarrow{e_{n-3}} + a_{2,n-2}\overrightarrow{e_{n-2}}, \dots,\ \overrightarrow{a_{n-2}} = \overrightarrow{e_n}$.

$[3n-9]\ \overrightarrow{a_1} = \overrightarrow{e_1} + a_{1,2}\overrightarrow{e_2} + a_{1,n-2}\overrightarrow{e_{n-2}},\ \overrightarrow{a_2} = \overrightarrow{e_3} + a_{2,n-2}\overrightarrow{e_{n-2}},\ \dots,\ \overrightarrow{a_{n-3}} = \overrightarrow{e_{n-1}},\ \overrightarrow{a_{n-2}} = \overrightarrow{e_n}$.

$[\frac{n(n-3)}{2}+2]\ \overrightarrow{a_1} = \overrightarrow{e_2} + a_{1,n}\overrightarrow{e_n},\ \overrightarrow{a_2} = \overrightarrow{e_3} + a_{2,n}\overrightarrow{e_n}, \dots,\ \overrightarrow{a_{n-2}} = \overrightarrow{e_{n-1}} + a_{n-2,n}\overrightarrow{e_n}$.

$[\frac{n(n-3)}{2}+3]\ \overrightarrow{a_1} = \overrightarrow{e_2} + a_{1,n-1}\overrightarrow{e_{n-1}},\ \overrightarrow{a_2} = \overrightarrow{e_3} + a_{2,n-1}\overrightarrow{e_{n-1}}, \dots,\ \overrightarrow{a_{n-3}} = \overrightarrow{e_{n-2}} + a_{n-3,n-1}\overrightarrow{e_{n-1}},\ \overrightarrow{a_{n-2}} = \overrightarrow{e_n}$.

$[\frac{n(n-1)}{2}]\ \overrightarrow{a_1} = \overrightarrow{e_3},\ \overrightarrow{a_2} = \overrightarrow{e_4}, \dots,\ \overrightarrow{a_{n-4}} = \overrightarrow{e_{n-2}},\ \overrightarrow{a_{n-3}} = \overrightarrow{e_{n-1}},\ \overrightarrow{a_{n-2}} = \overrightarrow{e_n}$.

## Conclusion

Results of this article are ready to be used at any research concerning subalgebras and ideals of noncommutative algebras. Classification of canonical bases for ($n$–2)-dimensional subspaces is very effective to study reductive subalgebras and reductive pairs of any $n$-dimensional Lie algebra.

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