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Research Article

Generalized Lie Algebroids and Connections over Pair of Diffeomorphic Base Manifolds

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Abstract Extending the definition of Lie algebroid from one base manifold to a pair of diffeomorphic base manifolds, we obtain the generalized Lie algebroid. When the diffeomorphisms used are identities, then we obtain the definition of Lie algebroid. We extend the concept of tangent bundle, and the Lie algebroid generalized tangent bundle is obtained. In the particular case of Lie algebroids, a similar Lie algebroid with the prolongation Lie algebroid is obtained. A new point of view over (linear) connections theory of Ehresmann type on a fiber bundle is presented. These connections are characterized by a horizontal distribution of the Lie algebroid generalized tangent bundle. Some basic properties of these generalized connections are investigated. Special attention to the class of linear connections is paid. The recently studied Lie algebroids connections can be recovered as special cases within this more general framework. In particular, all results are similar with the classical results. Formulas of Ricci and Bianchi type and linear connections of Levi-Civita type are presented.

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1 Introduction

In general, if $\mathcal{C}$ is a category, then we denote $|\mathcal{C}|$ the class of objects. For any $A, B \in |\mathcal{C}|$, we denote $\mathcal{C}(A,B)$ the set of morphisms of A source and B target, and $\mathrm{Iso}_{\mathcal{C}}(A,B)$ the set of $\mathcal{C}$-isomorphisms of A source and B target. Let **Liealg**, **Mod**, **Man**, **B**, and $\mathbf{B}^{\mathbf{v}}$ be the category of Lie algebras, modules, manifolds, fiber bundles, and vector bundles, respectively.

We know that if $(E,\pi,M) \in |\mathbf{B}^{\mathbf{v}}|$, $\Gamma(E,\pi,M) = \{u \in \mathbf{Man}(M,E) : u \circ \pi = \mathrm{Id}_M\}$ and $\mathcal{F}(M) = \mathbf{Man}(M,\mathbb{R})$, then $(\Gamma(E,\pi,M),+,\cdot)$ is a $\mathcal{F}(M)$-module. If $(\varphi,\varphi_0) \in \mathbf{B}^{\mathbf{v}}((E,\pi,M),(E',\pi',M'))$ such that $\varphi_0 \in \mathrm{Iso}_{\mathbf{Man}}(M,M')$, then, using the operation,

$$\begin{array}{rcl}\mathcal{F}(M) \times \Gamma(E',\pi',M') & \xrightarrow{\cdot} & \Gamma(E',\pi',M') \\ (f,u') & \longmapsto & f \circ \varphi_0^{-1} \cdot u'\end{array}$$

it results that $(\Gamma(E',\pi',M'),+,\cdot)$ is a $\mathcal{F}(M)$-module and we obtain the **Mod**-morphism:

$$\begin{array}{rcl}\Gamma(E,\pi,M) & \xrightarrow{\Gamma(\varphi,\varphi_0)} & \Gamma(E',\pi',M') \\ u & \longmapsto & \Gamma(\varphi,\varphi_0)u\end{array}$$

defined by:

$$\Gamma(\varphi,\varphi_0)u(y) = \varphi\big(u_{\varphi_0^{-1}(y)}\big) = \big(\varphi \circ u \circ \varphi_0^{-1}\big)(y),$$

for any $y \in M'$.

The theory of connections constitutes one of the most important chapter of differential geometry, which has been explored in the literature (see [2,3,4,5,6,9,12,13,20,21]). Connections theory has become an indispensable tool in various branches of theoretical and mathematical physics.

If (E,π,M) is a fiber bundle with paracompact base, then, using the pull-back action, we obtain the $\mathbf{B^v}$-morphisms:

$$\begin{tikzcd} \pi^*TM \arrow[r, hook] \arrow[d, "\pi^*\tau_M"'] & TM \arrow[d, "\tau_M"] \\ E \arrow[r, "\pi"] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \pi^*\left(T^*M\right) \arrow[r, hook] \arrow[d, "\pi^*(\overset{*}{\tau}_M)"'] & T^*M \arrow[d, "\overset{*}{\tau}_M"] \\ E \arrow[r, "\pi"] & M \end{tikzcd}$$

If (VTE,τ_E,E) is the kernel vector bundle of the tangent $\mathbf{B^v}$-morphism $(T\pi,\pi)$, then we obtain the short exact sequence:

$$\begin{tikzcd} 0 \arrow[r, hook] & VTE \arrow[r, hook] \arrow[d, "\tau_E"] & TE \arrow[r, "\pi!"] \arrow[d, "\tau_E"] & \pi^*TM \arrow[r] \arrow[d, "\pi^*\tau_M"] & 0 \\ & E \arrow[r, "\mathrm{Id}_E"] & E \arrow[r, "\mathrm{Id}_E"] & E & \end{tikzcd} \tag{1.1}$$

where $\pi!$ is the projection of TE onto π^*TM.

We know that a split to the right in the previous short exact sequence—i.e., a smooth map $\Gamma \in \mathbf{Man}(\pi^*TM,TE)$ so that $\pi!\circ\Gamma = \mathrm{Id}_{\pi^*TM}$—is a *connection in the Ehresmann sense*.

If (HTE,τ_E,E) is the image vector bundle of the $\mathbf{B^v}$-morphism (Γ,Id_E), then the tangent vector bundle (TE,τ_E,E) is a Whitney sum between the *horizontal vector bundle* (HTE,τ_E,E) and the *vertical vector bundle* (VTE,τ_E,E).

From the above notion of connection, one can easily derive more specific types of connections by imposing additional conditions. In the literature, one can find several generalizations of the concept of Ehresmann connection obtained by relaxing the conditions on the horizontal vector bundle.

- First of all, we are thinking here of the so-called *partial connections*, where (HTE,τ_E,E) does not determine a full complement of (VTE,τ_E,E). More precisely, $\Gamma(HTE,\tau_E,E)$ has zero intersection with $\Gamma(VTE,\tau_E,E)$, but (HTE,τ_E,E) projects onto a vector subbundle of (TM,τ_M,M) (see [3]).
- Second, there also exists a notion of *pseudo-connection*, introduced under the name of *quasi-connection* in a paper by Wong [21]. Linear pseudo-connections and generalization of it have been studied by many authors (see [5]).

Popescu built the *relative tangent space*, and using that, he obtained a new *generalized connection* on a vector bundle [12] (see also [13]).

In the paper [6] by Fernandes, a *contravariant connection* in the framework of Poisson geometry is presented. Given a Poisson manifold M with tensor Λ, which does not have to be of constant rank, a *covariant connection* on the principal bundle (P,π,M) is a G-invariant bundle map $\Gamma \in \mathbf{Man}(\pi^*(T^*M),TP)$ so that the diagram is commutative:

$$\begin{tikzcd} \pi^*\left(T^*M\right) \arrow[r, "\Gamma"] \arrow[d, "\pi^*(\overset{*}{\tau}_M)"'] & TP \arrow[d, "T\pi"] \\ T^*M \arrow[r, "\sharp_\Lambda"] & TM \end{tikzcd}$$

where $(\sharp_\Lambda,\mathrm{Id}_M)$ is the natural vector bundle morphism induced by the Poisson tensor. In the paper [6], Fernandez has extended this theory by replacing the cotangent bundle of a Poisson manifold by a Lie algebroid over an arbitrary manifold and the map $\sharp_\Lambda$ by the anchor map of the Lie algebroid. This resulted into a notion of *Lie algebroid connection*, which, in particular, turns out to be appropriate for studying the geometry of singular distributions.

Langerock and Cantrijn [2] proposed a *general notion of connection* on a fiber bundle (E,π,M) as being a smooth linear bundle map $\Gamma \in \mathbf{Man}(\pi^*(F),TE)$ so that the diagram is commutative:

$$\begin{tikzcd} \pi^*(F) \arrow[r, "\Gamma"] \arrow[d] & TE \arrow[d, "T\pi"] \\ F \arrow[r, "\rho"] & TM \end{tikzcd}$$

where (F,ν,M) is an arbitrary vector bundle and (ρ,Id_M) is a vector bundle morphism of (F,ν,M) source and (TM,τ_M,M) target.

Different equivalent definitions of a (linear) connection on a vector bundle are known, and there are in current usage. We know the following:

Theorem 1. *If we have a short exact sequence of vector bundles over a paracompact manifold M*

$$\begin{tikzcd} 0 \arrow[r, hook] & E' \arrow[r, "f"] \arrow[d, "\pi'"] & E \arrow[r, "g"] \arrow[d, "\pi"] & E'' \arrow[r] \arrow[d, "\pi''"] & 0 \\ & M \arrow[r, "\mathrm{Id}_M"] & M \arrow[r, "\mathrm{Id}_M"] & M & \end{tikzcd}$$

then there exists a right split if and only if there exists a left split.

So a split to the left in the short exact sequence (1.1)—i.e. a smooth map $\Gamma \in \mathbf{Man}(TE, VTE)$ so that $\Gamma \circ i = \mathrm{Id}_{TE}$—is an equivalent definition with the Ehresmann connection.

We know that a Lie algebroid is a vector bundle $(F, \nu, N) \in |\mathbf{B}^{\mathbf{v}}|$ such that there exists:

$$\left(\rho, \mathrm{Id}_N\right) \in \mathbf{B}^{\mathbf{v}}\left((F, \nu, N), \left(TN, \tau_N, N\right)\right)$$

and an operation:

$$\begin{array}{rcl} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[,]_F} & \Gamma(F, \nu, N) \\ (u, v) & \longmapsto & [u, v]_F \end{array}$$

with the following properties:

LA$_1$ the equality holds good

$$[u, f \cdot v]_F = f[u, v]_F + \Gamma(\rho, \mathrm{Id}_N)(u) f \cdot v,$$

for all $u, v \in \Gamma(F, \nu, N)$ and $f \in \mathcal{F}(N)$,

LA$_2$ the 4-tuple $(\Gamma(F, \nu, N), +, \cdot, [,]_F)$ is a Lie $\mathcal{F}(N)$-algebra,

LA$_3$ the **Mod**-morphism $\Gamma(\rho, \mathrm{Id}_N)$ is a **LieAlg**-morphism of

$$\left(\Gamma(F, \nu, N), +, \cdot, [,]_F\right)$$

source and

$$\left(\Gamma\left(TN, \tau_N, N\right), +, \cdot, [,]_{TN}\right)$$

target.

The triple

$$\left((F, \nu, N), [,]_F, \left(\rho, \mathrm{Id}_N\right)\right)$$

is called *Lie algebroid*, and the couple $([,]_F, (\rho, \mathrm{Id}_N))$ is called *Lie algebroid structure.*

We remark that the secret of the Ehresmann connection is given by the following diagrams:

$$\begin{tikzcd} E \arrow[d, "\pi"] & \left(TM, [,]_{TM}\right) \arrow[r, "\mathrm{Id}_{TM}"] \arrow[d, "\tau_M"] & \left(TM, [,]_{TM}\right) \arrow[d, "\tau_M"] \\ M \arrow[r, "\mathrm{Id}_M"] & M \arrow[r, "\mathrm{Id}_M"] & M \end{tikzcd}$$

where (E, π, M) is a fiber bundle and $((TM, \tau_M, M), [,]_{TM}, (\mathrm{Id}_{TM}, \mathrm{Id}_M))$ is the standard Lie algebroid.

It was the first time that there appeared an idea to change the standard Lie algebroid with an arbitrary Lie algebroid as in the following diagrams:

$$\begin{tikzcd} E \arrow[d, "\pi"] & \left(F, [,]_F\right) \arrow[r, "\rho"] \arrow[d, "\nu"] & \left(TM, [,]_{TM}\right) \arrow[d, "\tau_M"] \\ M \arrow[r, "\mathrm{Id}_M"] & M \arrow[r, "\mathrm{Id}_M"] & M \end{tikzcd}$$

For the second time there appeared an idea to change in the previous diagrams the identities morphisms with arbitrary **Man**-isomorphisms h and η as in the following diagrams:

$$\begin{array}{ccccccc} E & & \big(F,[\,,]_{F,h}\big) & \xrightarrow{\rho} & \big(TM,[\,,]_{TM}\big) & \xrightarrow{Th} & \big(TN,[\,,]_{TN}\big) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\tau_M} & & \downarrow{\scriptstyle\tau_N} \\ M & \xrightarrow{h} & N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \end{array}$$

where

$$(\rho,\eta) \in \mathbf{B^v}\big((F,\nu,M),\big(TM,\tau_M,M\big)\big)$$

and

$$\begin{array}{ccc} \Gamma(F,\nu,N)\times\Gamma(F,\nu,N) & \xrightarrow{[,]_{F,h}} & \Gamma(F,\nu,N) \\ (u,v) & \longmapsto & [u,v]_{F,h} \end{array}$$

is an operation with the following properties:

GLA$_1$ *the equality holds good*

$$[u,f\cdot v]_{F,h} = f[u,v]_{F,h} + \Gamma(Th\circ\rho,h\circ\eta)(u)f\cdot v,$$

for all $u,v\in\Gamma(F,\nu,N)$ *and* $f\in\mathcal{F}(N)$.

GLA$_2$ *the* 4*-tuple* $(\Gamma(F,\nu,N),+,\cdot,[\,,]_{F,h})$ *is a Lie* $\mathcal{F}(N)$*-algebra*,

GLA$_3$ *the* ***Mod****-morphism* $\Gamma(Th\circ\rho,h\circ\eta)$ *is a* ***LieAlg****-morphism of*

$$\big(\Gamma(F,\nu,N),+,\cdot,[\,,]_{F,h}\big)$$

source and

$$\big(\Gamma\big(TN,\tau_N,N\big),+,\cdot,[\,,]_{TN}\big)$$

target.

We will say that *the triple*

$$\big((F,\nu,N),[\,,]_{F,h},(\rho,\eta)\big)$$

is a generalized Lie algebroid. The couple $([\,,]_{F,h},(\rho,\eta))$ will be called *generalized Lie algebroid structure*.

So we extend the notion of Lie algebroid from one base manifold to a pair of diffeomorphic base manifolds, and we obtain the notion of *generalized Lie algebroid*.

Remark 2. In the particular case, $(\eta,h) = (\mathrm{Id}_M,\mathrm{Id}_M)$, we obtain the definition of Lie algebroid.

We can define the set of morphisms of

$$\big((F,\nu,N),[\,,]_{F,h},(\rho,\eta)\big)$$

source and

$$\big((F',\nu',N'),[\,,]_{F',h'},(\rho',\eta')\big)$$

target as being the set

$$\big\{\big(\varphi,\varphi_0\big)\in\mathbf{B^v}\big((F,\nu,N),(F',\nu',N')\big)\big\}$$

such that $\varphi_0\in\mathrm{Iso}_{\mathbf{Man}}(N,N')$ and the **Mod**-morphism $\Gamma(\varphi,\varphi_0)$ is a **LieAlg**-morphism of

$$\big(\Gamma(F,\nu,N),+,\cdot,[\,,]_{F,h}\big)$$

source and

$$\left(\Gamma(F',\nu',N'),+,\cdot,[\,,]_{F',h'}\right)$$

target.

So we can discuss about *the category* **GLA** *of generalized Lie algebroids*. Examples of objects of this category are presented in Section 2. We remark that **GLA** is a subcategory of the category $\mathbf{B^v}$.

Using this new notion, we build the *Lie algebroid generalized tangent bundle* in Theorems 7 and 10. Particularly, if $((F,\nu,N),[\,,]_F,(\rho,\mathrm{Id}_N))$ is a Lie algebroid, $(E,\pi,M)=(F,\nu,N)$ and $h=\mathrm{Id}_M$, then we obtain a similar Lie algebroid with the *prolongation Lie algebroid* (see [7]). New and important results are presented in [8,10,11,17,19]. See also [14,15,16,18]. Using this general framework, in Section 4, we propose and develop a (linear) connections theory of Ehresmann type for fiber bundles in general and for vector bundles in particular. It covers all types of connections mentioned. In this general framework, we can define the covariant derivatives of sections of a fiber bundle (E,π,M) with respect to sections of a generalized Lie algebroid

$$\left((F,\nu,N),[\,,]_{F,h},(\rho,\eta)\right).$$

In particular, if we use the generalized Lie algebroid structure:

$$\left([\,,]_{TM,\mathrm{Id}_M},\left(\mathrm{Id}_{TM},\mathrm{Id}_M\right)\right)$$

for the tangent bundle (TM,τ_M,M) in our theory, then the linear connections obtained are similar with the classical linear connections.

It is known that in Yang-Mills theory, the set

$$\mathrm{Cov}^0_{(E,\pi,M)}$$

of covariant derivatives for the vector bundle (E,π,M) such that

$$X\left(\langle u,v\rangle_E\right)=\left\langle D_X(u),v\right\rangle_E+\left\langle u,D_X(v)\right\rangle_E,$$

for any $X\in\mathcal{X}(M)$ and $u,v\in\Gamma(E,\pi,M)$, is very important, because the Yang-Mills theory is a variational theory that use (see [1]) the Yang-Mills functional:

$$\begin{array}{ccc}\mathrm{Cov}^0_{(E,\pi,M)} & \xrightarrow{\mathcal{YM}} & \mathbb{R}\\ D_X & \longmapsto & \dfrac{1}{2}\displaystyle\int_M \left\|\mathbb{R}^{D_X}\right\|^2 v_g\end{array}$$

where $\mathbb{R}^{D_X}$ is the curvature.

Using our linear connections theory, we succeed to extend the set $\mathrm{Cov}^0_{(E,\pi,M)}$ of Yang-Mills theory, because using all generalized Lie algebroid structures for the tangent bundle (TM,τ_M,M), we obtain all possible linear connections for the vector bundle (E,π,M).

More importantly, it may bring within the reach of connection theory certain geometric structures that have not yet been considered from such a point of view. Finally, using our theory of linear connections, the formulas of Ricci and Bianchi type and linear connections of Levi-Civita type are presented.

2 Preliminaries

Let $((F,\nu,N),[\,,]_{F,h},(\rho,\eta))\in|\mathbf{GLA}|$ be.

- Locally, for any $\alpha,\beta\in\overline{1,p}$, we set $[t_\alpha,t_\beta]_{F,h}=L^\gamma_{\alpha\beta}t_\gamma$. We easily obtain that $L^\gamma_{\alpha\beta}=-L^\gamma_{\beta\alpha}$, for any $\alpha,\beta,\gamma\in\overline{1,p}$.

The real local functions $L^\gamma_{\alpha\beta}$, $\alpha,\beta,\gamma\in\overline{1,p}$ will be called the *structure functions of the generalized Lie algebroid* $((F,\nu,N),[\,,]_{F,h},(\rho,\eta))$.

- We assume the following diagrams:

$$\begin{array}{ccccc}F & \xrightarrow{\rho} & TM & \xrightarrow{Th} & TN\\ \downarrow\nu & & \downarrow\tau_M & & \downarrow\tau_N\\ N & \xrightarrow{\eta} & M & \xrightarrow{h} & N\\ (\chi^{\tilde{\imath}},z^\alpha) & & (x^i,y^i) & & (\chi^{\tilde{\imath}},z^{\tilde{\imath}})\end{array}$$

where $i,\tilde{\imath}\in\overline{1,m}$ and $\alpha\in\overline{1,p}$.

If

$$(\chi^{\tilde{i}}, z^{\alpha}) \longrightarrow (\chi^{\tilde{i}'}(\chi^{\tilde{i}}), z^{\alpha'}(\chi^{\tilde{i}}, z^{\alpha})), \quad (x^i, y^i) \longrightarrow (x^{i'}(x^i), y^{i'}(x^i, y^i))$$

and

$$(\chi^{\tilde{i}}, z^{\tilde{i}}) \longrightarrow (\chi^{\tilde{i}\prime}(\chi^{\tilde{i}}), z^{\tilde{i}\prime}(\chi^{\tilde{i}}, z^{\tilde{i}})),$$

then

$$z^{\alpha'} = \Lambda_{\alpha}^{\alpha'} z^{\alpha}, \quad y^{i'} = \frac{\partial x^{i'}}{\partial x^i} y^i$$

and

$$z^{\tilde{i}'} = \frac{\partial \chi^{\tilde{i}'}}{\partial \chi^{\tilde{i}}} z^{\tilde{i}}.$$

We assume that $(\theta, \mu) \overset{put}{=} (Th \circ \rho, h \circ \eta)$. If $z^{\alpha} t_{\alpha} \in \Gamma(F, \nu, N)$ is arbitrary, then

$$\Gamma(Th \circ \rho, h \circ \eta)(z^{\alpha} t_{\alpha}) f(h \circ \eta(\varkappa)) = \left(\theta_{\alpha}^{\tilde{i}} z^{\alpha} \frac{\partial f}{\partial \varkappa^{\tilde{i}}}\right)(h \circ \eta(\varkappa)) = \left((\rho_{\alpha}^{i} \circ h)(z^{\alpha} \circ h) \frac{\partial f \circ h}{\partial x^i}\right)(\eta(\varkappa)),$$

for any $f \in \mathcal{F}(N)$ and $\varkappa \in N$.

The coefficients ρ_{α}^{i} and $\theta_{\alpha}^{\tilde{i}}$ change to $\rho_{\alpha'}^{i'}$ and $\theta_{\alpha'}^{\tilde{i}'}$, respectively, according to the rule:

$$\rho_{\alpha'}^{i'} = \Lambda_{\alpha'}^{\alpha} \rho_{\alpha}^{i} \frac{\partial x^{i'}}{\partial x^i},$$

and

$$\theta_{\alpha'}^{\tilde{i}'} = \Lambda_{\alpha'}^{\alpha} \theta_{\alpha}^{\tilde{i}} \frac{\partial \varkappa^{\tilde{i}'}}{\partial \varkappa^{\tilde{i}}},$$

where

$$\left\| \Lambda_{\alpha'}^{\alpha} \right\| = \left\| \Lambda_{\alpha}^{\alpha'} \right\|^{-1}.$$

Remark 3. The following equalities hold good:

$$\rho_{\alpha}^{i} \circ h \frac{\partial f \circ h}{\partial x^i} = \left(\theta_{\alpha}^{\tilde{i}} \frac{\partial f}{\partial \varkappa^{\tilde{i}}}\right) \circ h, \quad \forall f \in \mathcal{F}(N).$$

and

$$\left(L_{\alpha\beta}^{\gamma} \circ h\right)\left(\rho_{\gamma}^{k} \circ h\right) = \left(\rho_{\alpha}^{i} \circ h\right) \frac{\partial\left(\rho_{\beta}^{k} \circ h\right)}{\partial x^i} - \left(\rho_{\beta}^{j} \circ h\right) \frac{\partial\left(\rho_{\alpha}^{k} \circ h\right)}{\partial x^j}.$$

Theorem 4. *Let $M, N \in |\mathbf{Man}|$, $h \in \mathrm{Iso}_{\mathbf{Man}}(M, N)$ and $\eta \in \mathrm{Iso}_{\mathbf{Man}}(N, M)$ be. Using the tangent $\mathbf{B}^{\mathbf{v}}$-morphism $(T\eta, \eta)$ and the operation*

$$\begin{array}{ccc} \Gamma(TN, \tau_N, N) \times \Gamma(TN, \tau_N, N) & \xrightarrow{[,]_{TN,h}} & \Gamma(TN, \tau_N, N) \\ (u, v) & \longmapsto & [u, v]_{TN,h} \end{array}$$

where

$$[u, v]_{TN,h} = \Gamma\left(T(h \circ \eta)^{-1}, (h \circ \eta)^{-1}\right)\left(\left[\Gamma\left(T(h \circ \eta), h \circ \eta\right) u, \Gamma\left(T(h \circ \eta), h \circ \eta\right) v\right]_{TN}\right),$$

for any $u, v \in \Gamma(TN, \tau_N, N)$, we obtain that

$$\left((TN, \tau_N, N), (T\eta, \eta), [,]_{TN,h}\right)$$

is a generalized Lie algebroid.

For any **Man**-isomorphisms η and h, new and interesting generalized Lie algebroid structures for the tangent vector bundle (TN, τ_N, N) are obtained. For any base $\{t_\alpha,\ \alpha \in \overline{1,m}\}$ of the module of sections $(\Gamma(TN, \tau_N, N), +, \cdot)$, we obtain the structure functions

$$L_{\alpha\beta}^{\gamma} = \left(\theta_\alpha^i \frac{\partial \theta_\beta^j}{\partial x^i} - \theta_\beta^i \frac{\partial \theta_\alpha^j}{\partial x^i}\right) \tilde{\theta}_j^\gamma, \quad \alpha, \beta, \gamma \in \overline{1,m}$$

where

$$\theta_\alpha^i, \quad i, \alpha \in \overline{1,m}$$

are real local functions so that

$$\Gamma\big(T(h \circ \eta), h \circ \eta\big)(t_\alpha) = \theta_\alpha^i \frac{\partial}{\partial x^i}$$

and

$$\tilde{\theta}_j^\gamma, \quad i, \gamma \in \overline{1,m}$$

are real local functions so that

$$\Gamma\big(T(h \circ \eta)^{-1}, (h \circ \eta)^{-1}\big)\left(\frac{\partial}{\partial x^j}\right) = \tilde{\theta}_j^\gamma t_\gamma.$$

In particular, using arbitrary isometries (symmetries, translations, rotations, etc.) for the Euclidean three-dimensional space Σ, and arbitrary basis for the module of sections, we obtain a lot of generalized Lie algebroid structures for the tangent vector bundle $(T\Sigma, \tau_\Sigma, \Sigma)$.

Remark 5. If $(E, \pi, M) \in |\mathbf{B}|$, then we obtain the $\mathbf{B^v}$-morphism

$$\begin{tikzcd} \pi^*(h^*F) \arrow[r, hook] \arrow[d, "\pi^*(h^*\nu)"'] & F \arrow[d, "\nu"] \\ E \arrow[r, "h\circ\pi"] & N \end{tikzcd} \tag{2.1}$$

In particular, if $(E, \pi, M) \in |\mathbf{B^v}|$ and $(\overset{*}{E}, \overset{*}{\pi}, M)$ is its dual, then we obtain the $\mathbf{B^v}$-morphism:

$$\begin{tikzcd} \overset{*}{\pi}{}^*(h^*F) \arrow[r, hook] \arrow[d, "\overset{*}{\pi}{}^*(h^*\nu)"'] & F \arrow[d, "\nu"] \\ \overset{*}{E} \arrow[r, "h\circ\overset{*}{\pi}"] & N \end{tikzcd} \tag{2.2}$$

3 The Lie algebroid generalized tangent bundle

We consider the following diagram:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & (F, [\,,]_{F,h}, (\rho, \eta)) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

where $(E, \pi, M) \in |\mathbf{B}|$ and $((F, \nu, N), [\,,]_{F,h}, (\rho, \eta)) \in |\mathbf{GLA}|$.

We take (x^i, y^a) as canonical local coordinates on (E, π, M), where $i \in \overline{1,m}$ and $a \in \overline{1,r}$. Let

$$(x^i, y^a) \longrightarrow (x^{i'}(x^i), y^{a'}(x^i, y^a))$$

be a change of coordinates on (E, π, M). Then the coordinates y^a change to $y^{a'}$ according to the rule:

$$y^{a'} = \frac{\partial y^{a'}}{\partial y^a} y^a.$$

In particular, if (E,π,M) is vector bundle, then the coordinates y^a change to $y^{a'}$ according to the rule:

$$y^{a'} = M_a^{a'} y^a.$$

Easily, we obtain the following

Theorem 6. *Let* $(\overset{\pi^*(h^*F)}{\rho}, \mathrm{Id}_E)$ *be the* $\mathbf{B^v}$*-morphism of* $(\pi^*(h^*F),\pi^*(h^*\nu),E)$ *source and* (TE,τ_E,E) *target, where*

$$\begin{array}{rcl} \pi^*(h^*F) & \xrightarrow{\overset{\pi^*(h^*F)}{\rho}} & TE \\ Z^\alpha T_\alpha(u_x) & \longmapsto & (Z^\alpha \cdot \rho_\alpha^i \circ h \circ \pi) \dfrac{\partial}{\partial x^i}(u_x) \end{array}$$

Using the operation

$$\Gamma(\pi^* \ (h^*F),\pi^*(h^*\nu),E)^2 \xrightarrow{[,]_{\pi^*(h^*F)}} \Gamma(\pi^*(h^*F),\pi^*(h^*\nu),E)$$

defined by

$$\begin{gathered} \left[T_\alpha,T_\beta\right]_{\pi^*(h^*F)} = L_{\alpha\beta}^\gamma \circ h \circ \pi \cdot T_\gamma, \quad \left[T_\alpha, fT_\beta\right]_{\pi^*(h^*F)} = fL_{\alpha\beta}^\gamma \circ h \circ \pi T_\gamma + \rho_\alpha^i \circ h \circ \pi \frac{\partial f}{\partial x^i} T_\beta, \\ \left[fT_\alpha,T_\beta\right]_{\pi^*(h^*F)} = -\left[T_\beta, fT_\alpha\right]_{\pi^*(h^*F)}, \end{gathered}$$

for any $f \in \mathcal{F}(E)$, it results that

$$\left(\left(\pi^*(h^*F),\pi^*\left(h^*\nu\right),E\right),[,]_{\pi^*(h^*F)},\left(\overset{\pi^*(h^*F)}{\rho},\mathrm{Id}_E\right)\right)$$

is a Lie algebroid that is called the pull-back Lie algebroid of the generalized Lie algebroid

$$\big((F,\nu,N),[,]_{F,h},(\rho,\eta)\big).$$

If $z = z^\alpha t_\alpha \in \Gamma(F,\nu,N)$, then we obtain the section

$$Z = (z^\alpha \circ h \circ \pi) T_\alpha \in \Gamma(\pi^*(h^*F),\pi^*(h^*\nu),E)$$

so that $Z(u_x) = z(h(x))$, for any $u_x \in \pi^{-1}(U \cap h^{-1}V)$.

Let

$$(\partial_i, \dot{\partial}_a) \overset{put}{=} \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}\right)$$

be the base sections for the Lie $\mathcal{F}(E)$-algebra

$$(\Gamma(TE,\tau_E,E),+,\cdot,[,]_{TE}).$$

For any sections

$$Z^\alpha T_\alpha \in \Gamma\big(\pi^*(h^*F),\pi^*(h^*F),E\big)$$

and

$$Y^a \dot{\partial}_a \in \Gamma\big(VTE,\tau_E,E\big)$$

we obtain the section

$$\begin{aligned} Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a &=: Z^\alpha (T_\alpha \oplus (\rho_\alpha^i \circ h \circ \pi)\partial_i) + Y^a (0_{\pi^*(h^*F)} \oplus \dot{\partial}_a) \\ &= Z^\alpha T_\alpha \oplus (Z^\alpha(\rho_\alpha^i \circ h \circ \pi)\partial_i + Y^a \dot{\partial}_a) \in \Gamma(\pi^*(h^*F) \oplus TE, \overset{\oplus}{\pi}, E). \end{aligned}$$

Since we have

$$Z^{\alpha}\tilde{\partial}_{\alpha}+Y^{a}\dot{\tilde{\partial}}_{a}=0$$
$$\Updownarrow$$
$$Z^{\alpha}T_{\alpha}=0\ \wedge Z^{\alpha}(\rho_{\alpha}^{i}\circ h\circ\pi)\partial_{i}+Y^{a}\dot{\partial}_{a}=0,$$

it implies $Z^{\alpha}=0$, $\alpha\in\overline{1,p}$ and $Y^{a}=0$, $a\in\overline{1,r}$.

Therefore, the sections $\tilde{\partial}_1,\ldots,\tilde{\partial}_p,\dot{\tilde{\partial}}_1,\ldots,\dot{\tilde{\partial}}_r$ are linearly independent.

We consider the vector subbundle $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ of the vector bundle $(\pi^*(h^*F)\oplus TE,\overset{\oplus}{\pi},E)$, for which the $\mathcal{F}(E)$-module of sections is the $\mathcal{F}(E)$-submodule of $(\Gamma(\pi^*(h^*F)\oplus TE,\overset{\oplus}{\pi},E),+,\cdot)$, generated by the set of sections $(\tilde{\partial}_{\alpha},\dot{\tilde{\partial}}_a)$.

The base sections $(\tilde{\partial}_{\alpha},\dot{\tilde{\partial}}_a)$ will be called the *natural* (ρ,η)-base.

The matrix of coordinate transformation on $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ at a change of fibred charts is

$$\left\Vert\begin{array}{cc}\Lambda_{\alpha}^{\alpha'}\circ h\circ\pi & 0\\ (\rho_{\alpha}^{i}\circ h\circ\pi)\dfrac{\partial y^{a'}}{\partial x^{i}} & \dfrac{\partial y^{a'}}{\partial y^{a}}\end{array}\right\Vert.$$

In particular, if (E,π,M) is a vector bundle, then the matrix of coordinate transformation on $((\rho,\eta)TE,$ $(\rho,\eta)\tau_E,E)$ at a change of fibred charts is

$$\left\Vert\begin{array}{cc}\Lambda_{\alpha}^{\alpha'}\circ h\circ\pi & 0\\ (\rho_{a}^{i}\circ h\circ\pi)\frac{\partial M_{b}^{a'}\circ\pi}{\partial x_{i}}y^{b} & M_{a}^{a'}\circ\pi\end{array}\right\Vert.$$

Easily, we obtain

Theorem 7. *Let $(\tilde{\rho},\mathrm{Id}_E)$ be the* $\mathbf{B}^{\mathbf{v}}$*-morphism of $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ source and (TE,τ_E,E) target, where*

$$\begin{array}{ccc}(\rho,\eta)TE & \xrightarrow{\tilde{\rho}} & TE\\ (Z^{\alpha}\tilde{\partial}_{\alpha}+Y^{a}\dot{\tilde{\partial}}_{a})(u_x) & \longmapsto & (Z^{\alpha}(\rho_{\alpha}^{i}\circ h\circ\pi)\partial_{i}+Y^{a}\dot{\partial}_{a})(u_x).\end{array}$$

Using the operation

$$\Gamma\big((\rho,\eta)TE,(\rho,\eta)\tau_E,E\big)^2\xrightarrow{[,]_{(\rho,\eta)TE}}\Gamma\big((\rho,\eta)TE,(\rho,\eta)\tau_E,E\big)$$

defined by

$$\begin{aligned}&[Z_1^{\alpha}\tilde{\partial}_{\alpha}+Y_1^{a}\dot{\tilde{\partial}}_{a},Z_2^{\beta}\tilde{\partial}_{\beta}+Y_2^{b}\dot{\tilde{\partial}}_{b}]_{(\rho,\eta)TE}\\ &\quad=[Z_1^{\alpha}T_{\alpha},Z_2^{\beta}T_{\beta}]_{\pi^*(h^*F)}\oplus[Z_1^{\alpha}(\rho_{\alpha}^{i}\circ h\circ\pi)\partial_{i}+Y_1^{a}\dot{\partial}_{a},Z_2^{\beta}(\rho_{\beta}^{j}\circ h\circ\pi)\partial_{j}+Y_2^{b}\dot{\partial}_{b}]_{TE},\end{aligned}$$

for any $Z_1^{\alpha}\tilde{\partial}_{\alpha}+Y_1^{a}\dot{\tilde{\partial}}_{a}$ and $Z_2^{\beta}\tilde{\partial}_{\beta}+Y_2^{b}\dot{\tilde{\partial}}_{b}$, we obtain that the couple $([,]_{(\rho,\eta)TE},(\tilde{\rho},\mathrm{Id}_E))$ is a Lie algebroid structure for the vector bundle $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$.

Remark 8. In particular, if $h=\mathrm{Id}_M$, then the Lie algebroid

$$(((\mathrm{Id}_{TM},\mathrm{Id}_M)TE,(\mathrm{Id}_{TM},\mathrm{Id}_M)\tau_E,E),[,]_{(\mathrm{Id}_{TM},\mathrm{Id}_M)TE},(\widetilde{\mathrm{Id}_{TM}},\mathrm{Id}_E))$$

is isomorphic with the usual Lie algebroid

$$((TE,\tau_E,E),[,]_{TE},(\mathrm{Id}_{TE},\mathrm{Id}_E)).$$

This is a reason for which the Lie algebroid

$$(((\rho,\eta)TE,(\rho,\eta)\tau_E,E),[,]_{(\rho,\eta)TE},(\tilde{\rho},\mathrm{Id}_E)),$$

will be called the *Lie algebroid generalized tangent bundle*.

The vector bundle $((\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ will be called the *generalized tangent bundle*.

3.1 The Lie algebroid generalized tangent bundle of dual vector bundle

Let $(E,\pi,M)\in|\mathbf{B^v}|$ be. We build the generalized tangent bundle of dual vector bundle $(\overset{*}{E},\overset{*}{\pi},M)$ using the diagram:

$$\begin{array}{ccc}\overset{*}{E} & & (F,[,]_{F,h},(\rho,\eta)) \\ \downarrow{\scriptstyle\overset{*}{\pi}} & & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow{\quad h\quad} & N,\end{array}$$

where $((F,\nu,N),[,]_{F,h},(\rho,\eta))\in|\mathbf{GLA}|$.

We take (x^i,p_a) as canonical local coordinates on $(\overset{*}{E},\overset{*}{\pi},M)$, where $i\in\overline{1,m}$ and $a\in\overline{1,r}$.

Consider

$$(x^i,p_a)\longrightarrow(x^{i'}(x^i),p_{a'}(x^i,p_a))$$

a change of coordinates on $(\overset{*}{E},\overset{*}{\pi},M)$. Then the coordinates p_a change to $p_{a'}$ according to the rule:

$$p_{a'}=M^a_{a'}p_a.$$

Easily, we obtain the following

Theorem 9. *Let $(\overset{\overset{*}{\pi}^*(h^*F)}{\rho},\mathrm{Id}_{\overset{*}{E}})$ be the $\mathbf{B^v}$-morphism of $(\overset{*}{\pi}^*(h^*F),\overset{*}{\pi}^*(h^*\nu),\overset{*}{E})$ source and $(T\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E})$ target, where*

$$\begin{array}{rcl}\overset{*}{\pi}^*(h^*F) & \xrightarrow{\overset{\overset{*}{\pi}^*(h^*F)}{\rho}} & T\overset{*}{E}\\ Z^\alpha T_\alpha(\overset{*}{u}_x) & \longmapsto & (Z^\alpha\cdot\rho^i_\alpha\circ h\circ\overset{*}{\pi})\dfrac{\partial}{\partial x^i}(\overset{*}{u}_x)\end{array}$$

Using the operation

$$\Gamma(\overset{*}{\pi}^*(h^*F),\overset{*}{\pi}^*(h^*\nu),\overset{*}{E})^2\xrightarrow{[,]_{\overset{*}{\pi}^*(h^*F)}}\Gamma(\overset{*}{\pi}^*(h^*F),\overset{*}{\pi}^*(h^*\nu),\overset{*}{E})$$

defined by

$$[T_\alpha,T_\beta]_{\overset{*}{\pi}^*(h^*F)}=L^\gamma_{\alpha\beta}\circ h\circ\overset{*}{\pi}\cdot T_\gamma,\quad [T_\alpha,fT_\beta]_{\overset{*}{\pi}^*(h^*F)}=f\cdot L^\gamma_{\alpha\beta}\circ h\circ\overset{*}{\pi}\cdot T_\gamma+\rho^i_\alpha\circ h\circ\overset{*}{\pi}\cdot\frac{\partial f}{\partial x^i}\cdot T_\beta,$$
$$[fT_\alpha,T_\beta]_{\overset{*}{\pi}^*(h^*F)}=-[T_\beta,fT_\alpha]_{\overset{*}{\pi}^*(h^*F)},$$

for any $f\in\mathcal{F}(\overset{*}{E})$, it results that

$$\left((\overset{*}{\pi}^*(h^*F),\overset{*}{\pi}^*(h^*\nu),\overset{*}{E}),[,]_{\overset{*}{\pi}^*(h^*F)},\left(\overset{\overset{*}{\pi}^*(h^*F)}{\rho},\mathrm{Id}_{\overset{*}{E}}\right)\right)$$

is a Lie algebroid that is called the pull-back Lie algebroid of the generalized Lie algebroid

$$\big((F,\nu,N),[,]_{F,h},(\rho,\eta)\big).$$

If $z=z^\alpha t_\alpha\in\Gamma(F,\nu,N)$, then we obtain the section

$$Z=(z^\alpha\circ h\circ\overset{*}{\pi})T_\alpha\in\Gamma(\overset{*}{\pi}^*(h^*F),\overset{*}{\pi}^*(h^*\nu),\overset{*}{E})$$

so that $Z(\overset{*}{u}_x)=z(h(x))$, for any $\overset{*}{u}_x\in\overset{*}{\pi}^{-1}(U\cap h^{-1}V)$.

Let

$$(\overset{*}{\partial}_i,\dot{\partial}^a)\overset{put}{=}\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial p_a}\right)$$

be the base sections for the Lie $\mathcal{F}(\overset{*}{E})$-algebra

$$\left(\Gamma(T\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}),+,\cdot,[,]_{T\overset{*}{E}}\right).$$

For any sections

$$Z^{\alpha}T_{\alpha}\in\Gamma(\overset{*}{\pi}^{*}(h^{*}F),\overset{*}{\pi}^{*}(h^{*}\nu),\overset{*}{E})$$

and

$$Y_{a}\dot{\partial}^{a}\in\Gamma\left(VT\overset{*}{E},\tau_{\overset{*}{E}},\overset{*}{E}\right),$$

we obtain the section

$$\begin{aligned}Z^{\alpha}\overset{*}{\tilde{\partial}}_{\alpha}+Y_{a}\overset{\cdot\,a}{\tilde{\partial}} &=: Z^{\alpha}(T_{\alpha}\oplus(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi})\overset{*}{\partial}_{i})+Y_{a}(0_{\overset{*}{\pi}^{*}(h^{*}F)}\oplus\dot{\partial}^{a})\\ &= Z^{\alpha}T_{\alpha}\oplus(Z^{\alpha}(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi})\overset{*}{\partial}_{i}+Y_{a}\dot{\partial}^{a})\in\Gamma(\overset{*}{\pi}^{*}(h^{*}F)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}).\end{aligned}$$

Since we have

$$\begin{gathered}Z^{\alpha}\overset{*}{\tilde{\partial}}_{\alpha}+Y_{a}\overset{\cdot\,a}{\tilde{\partial}}=0_{\overset{*}{\pi}^{*}(h^{*}F)\oplus T\overset{*}{E}}\\ \Updownarrow\\ Z^{\alpha}T_{\alpha}=0_{\overset{*}{\pi}^{*}(h^{*}F)}\wedge Z^{\alpha}(\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi})\overset{*}{\partial}_{i}+Y_{a}\dot{\partial}^{a}=0_{T\overset{*}{E}},\end{gathered}$$

it implies $Z^{\alpha}=0$, $\alpha\in\overline{1,p}$ and $Y_{a}=0$, $a\in\overline{1,r}$.

Therefore, the sections

$$\overset{*}{\tilde{\partial}}_{1},\ldots,\overset{*}{\tilde{\partial}}_{p},\overset{\cdot\,1}{\tilde{\partial}},\ldots,\overset{\cdot\,r}{\tilde{\partial}}$$

are linearly independent.

We consider the vector subbundle

$$((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E})$$

of vector bundle

$$(\overset{*}{\pi}^{*}(h^{*}F)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}),$$

for which the $\mathcal{F}(\overset{*}{E})$ module of sections is the $\mathcal{F}(\overset{*}{E})$-submodule of

$$\left(\Gamma\left(\overset{*}{\pi}^{*}(h^{*}F)\oplus T\overset{*}{E},\overset{\oplus}{\overset{*}{\pi}},\overset{*}{E}\right),+,\cdot\right),$$

generated by the family of sections $(\overset{*}{\tilde{\partial}}_{\alpha},\overset{\cdot\,a}{\tilde{\partial}})$, which is called the *natural* (ρ,η)*-base*.

The matrix of coordinate transformation on $((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E})$ at a change of fibred charts is

$$\left\|\begin{array}{cc}\Lambda_{\alpha}^{\alpha'}\circ h\circ\overset{*}{\pi} & 0\\ (\rho_{\alpha}^{i}\circ h\circ\overset{*}{\pi})\dfrac{\partial M_{a'}^{b}\circ\overset{*}{\pi}}{\partial x_{i}}p_{b} & M_{a'}^{a}\circ\overset{*}{\pi}\end{array}\right\|.$$

Easily, we obtain

Theorem 10. *Let* $(\overset{*}{\tilde{\rho}}, \mathrm{Id}_{\overset{*}{E}})$ *be the* $\mathbf{B^v}$*-morphism of* $((\rho,\eta)T\overset{*}{E}, (\rho,\eta)\tau_{\overset{*}{E}}, \overset{*}{E})$ *source and* $(T\overset{*}{E}, \tau_{\overset{*}{E}}, \overset{*}{E})$ *target, where*

$$\begin{array}{ccc} (\rho,\eta)T\overset{*}{E} & \xrightarrow{\overset{*}{\tilde{\rho}}} & T\overset{*}{E} \\ (Z^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a \overset{\cdot}{\tilde{\partial}}{}^a)(\overset{*}{u}_x) & \longmapsto & (Z^\alpha(\rho^i_\alpha \circ h \circ \overset{*}{\pi})\overset{*}{\partial}_i + Y_a \dot{\partial}^a)(\overset{*}{u}_x). \end{array}$$

Using the operation

$$\Gamma((\rho,\eta)T\overset{*}{E}, (\rho,\eta)\tau_{\overset{*}{E}}, \overset{*}{E})^2 \xrightarrow{[,]_{(\rho,\eta)T\overset{*}{E}}} \Gamma((\rho,\eta)T\overset{*}{E}, (\rho,\eta)\tau_{\overset{*}{E}}, \overset{*}{E})$$

defined by

$$\begin{aligned} &[Z_1^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a^1 \overset{\cdot}{\tilde{\partial}}{}^a, Z_2^\beta \overset{*}{\tilde{\partial}}_\beta + Y_b^2 \overset{\cdot}{\tilde{\partial}}{}^b]_{(\rho,\eta)T\overset{*}{E}} \\ &\quad = [Z_1^\alpha T_\alpha, Z_2^\beta T_\beta]_{\overset{*}{\pi}{}^*(h^*F)} \oplus [Z_1^\alpha(\rho^i_\alpha \circ h \circ \overset{*}{\pi})\overset{*}{\partial}_i + Y_a^1 \dot{\partial}^a, Z_2^\beta(\rho^j_\beta \circ h \circ \overset{*}{\pi})\overset{*}{\partial}_j + Y_b^2 \dot{\partial}^b]_{T\overset{*}{E}}, \end{aligned}$$

for any $Z_1^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a^1 \overset{\cdot}{\tilde{\partial}}{}^a$ *and* $Z_2^\beta \overset{*}{\tilde{\partial}}_\beta + Y_b^2 \overset{\cdot}{\tilde{\partial}}{}^b$*, we obtain that the couple* $([,]_{(\rho,\eta)T\overset{*}{E}}, (\overset{*}{\tilde{\rho}}, \mathrm{Id}_{\overset{*}{E}}))$ *is a Lie algebroid structure for the vector bundle* $((\rho,\eta)T\overset{*}{E}, (\rho,\eta)\tau_{\overset{*}{E}}, \overset{*}{E})$.

The Lie algebroid generalized tangent bundle of the dual vector bundle $(\overset{*}{E}, \overset{*}{\pi}, M)$ will be denoted:

$$\left(\left((\rho,\eta)T\overset{*}{E}, (\rho,\eta)\tau_{\overset{*}{E}}, \overset{*}{E}\right), [,]_{(\rho,\eta)T\overset{*}{E}}, \left(\overset{*}{\tilde{\rho}}, \mathrm{Id}_{\overset{*}{E}}\right)\right).$$

4 (Linear) (ρ,η)-connections

We consider the diagram:

$$\begin{array}{ccc} E & & (F,[,]_{F,h},(\rho,\eta)) \\ \pi\downarrow & & \downarrow\nu \\ M & \xrightarrow{h} & N \end{array}$$

where $(E,\pi,M) \in |\mathbf{B}|$ and $((F,\nu,N),[,]_{F,h},(\rho,\eta)) \in |\mathbf{GLA}|$.

Let

$$(((\rho,\eta)TE, (\rho,\eta)\tau_E, E), [,]_{(\rho,\eta)TE}, (\tilde{\rho}, \mathrm{Id}_E))$$

be the Lie algebroid generalized tangent bundle of the fiber bundle (E,π,M).

We consider the $\mathbf{B^v}$-morphism $((\rho,\eta)\pi!, \mathrm{Id}_E)$ given by the commutative diagram:

$$\begin{array}{ccc} (\rho,\eta)TE & \xrightarrow{(\rho,\eta)\pi!} & \pi^*(h^*F) \\ {\scriptstyle (\rho,\eta)\tau_E}\downarrow & & \downarrow{\scriptstyle \pi^*(h^*\nu)} \\ E & \xrightarrow{\mathrm{Id}_E} & E \end{array}$$

This is defined as:

$$(\rho,\eta)\pi!((Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a)(u_x)) = (Z^\alpha T_\alpha)(u_x),$$

for any $Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \in \Gamma((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$.

Using the $\mathbf{B^v}$-morphism $((\rho,\eta)\pi!, \mathrm{Id}_E)$, and the $\mathbf{B^v}$-morphism (2.1), we obtain the *tangent* (ρ,η)*-application* $((\rho,\eta)T\pi, h\circ\pi)$ of $((\rho,\eta)TE, (\rho,\eta)\tau_E, E)$ source and (F,ν,N) target.

Definition 11. The kernel of the tangent (ρ,η)-application is written:

$$(V(\rho,\eta)TE,(\rho,\eta)\tau_E,E)$$

and it is called *the vertical subbundle*.

We remark that the set $\{\overset{\cdot}{\tilde{\partial}}_a,\ a\in\overline{1,r}\}$ is a base of the $\mathcal{F}(E)$-module:

$$(\Gamma(V(\rho,\eta)TE,(\rho,\eta)\tau_E,E),+,\cdot).$$

Proposition 12. *The short sequence of vector bundles*

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d] & V(\rho,\eta)TE \arrow[r, hook] \arrow[d] & (\rho,\eta)TE \arrow[r, "(\rho,\eta)\pi!"] \arrow[d] & \pi^*(h^*F) \arrow[r] \arrow[d] & 0 \arrow[d] \\
E \arrow[r, "\mathrm{Id}_E"] & E \arrow[r, "\mathrm{Id}_E"] & E \arrow[r, "\mathrm{Id}_E"] & E \arrow[r, "\mathrm{Id}_E"] & E
\end{tikzcd}$$

is exact.

Definition 13. A **Man**-morphism $(\rho,\eta)\Gamma$ of $(\rho,\eta)TE$ source and $V(\rho,\eta)TE$ target defined by:

$$(\rho,\eta)\Gamma\Big(Z^\gamma\overset{*}{\tilde{\partial}}_\gamma+Y^a\overset{\cdot}{\tilde{\partial}}_a\Big)(u_x)=\big(Y^a+(\rho,\eta)\Gamma^a_\gamma Z^\gamma\big)\overset{\cdot}{\tilde{\partial}}_a(u_x),$$

so that the $\mathbf{B^v}$ morphism $((\rho,\eta)\Gamma,\mathrm{Id}_E)$ is a split to the left in the previous exact sequence, will be called (ρ,η)-*connection for the fiber bundle* (E,π,M).

The (ρ,Id_M)-connection will be called *ρ-connection* and will be denoted $\rho\Gamma$ and the $(\mathrm{Id}_{TM},\mathrm{Id}_M)$-connection will be called *connection* and will be denoted Γ.

Definition 14. If $(\rho,\eta)\Gamma$ is a (ρ,η)-connection for the fiber bundle (E,π,M), then the kernel of the $\mathbf{B^v}$-morphism $((\rho,\eta)\Gamma,\mathrm{Id}_E)$ is written $(H(\rho,\eta)TE,(\rho,\eta)\tau_E,E)$ and will be called the *horizontal vector subbundle*.

Definition 15. If $(E,\pi,M)\in|\mathbf{B}|$, then the **B**-morphism (Π,π) defined by the commutative diagram

$$\begin{tikzcd}
V(\rho,\eta)TE \arrow[r, "\Pi"] \arrow[d, "(\rho,\eta)\tau_E"'] & E \arrow[d, "\pi"] \\
E \arrow[r, "\pi"] & M
\end{tikzcd}$$

so that the components of the image of the vector $Y^a\overset{\cdot}{\tilde{\partial}}_a(u_x)$ are the real numbers $Y^1(u_x),\ldots,Y^r(u_x)$ will be called the *canonical projection* ***B**-morphism*.

In particular, if $(E,\pi,M)\in|\mathbf{B^v}|$ and $\{s_a,a\in\overline{1,r}\}$ is a base of the $\mathcal{F}(M)$-module of sections $(\Gamma(E,\pi,M),+,\cdot)$, then Π is defined by:

$$\Pi(Y^a\overset{\cdot}{\tilde{\partial}}_a(u_x))=Y^a(u_x)s_a(x).$$

Theorem 16. *If $(\rho,\eta)\Gamma$ is a (ρ,η)-connection for the fiber bundle (E,π,M), then its components satisfy the law of transformation:*

$$(\rho,\eta)\Gamma^{a'}_{\gamma'}=\frac{\partial y^{a'}}{\partial y^a}\left[\rho^k_\gamma\circ h\circ\pi\frac{\partial y^a}{\partial x^k}+(\rho,\eta)\Gamma^a_\gamma\right]\Lambda^\gamma_{\gamma'}\circ h\circ\pi. \tag{4.1}$$

If $(\rho,\eta)\Gamma$ is a (ρ,η)-connection for the vector bundle (E,π,M), then its components satisfy the law of transformation:

$$(\rho,\eta)\Gamma^{a'}_{\gamma'}=M^{a'}_a\circ\pi\left[\rho^k_\gamma\circ h\circ\pi\frac{\partial M^a_{b'}\circ\pi}{\partial x^k}y^{b'}+(\rho,\eta)\Gamma^a_\gamma\right]\Lambda^\gamma_{\gamma'}\circ h\circ\pi. \tag{4.1$'$}$$

In the particular case of Lie algebroids, $(\eta,h)=(\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.1′) *become:*

$$\rho\Gamma^{a'}_{\gamma'}=M^{a'}_a\circ\pi\left[\rho^k_\gamma\circ\pi\frac{\partial M^a_{b'}\circ\pi}{\partial x^k}y^{b'}+\rho\Gamma^a_\gamma\right]\Lambda^\gamma_{\gamma'}\circ\pi. \tag{4.1$''$}$$

In the classical case, $(\rho,\eta,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.1″) *become:*

$$\Gamma^{i'}_{k'}=\frac{\partial x^{i'}}{\partial x^i}\circ\tau_M\left[\frac{\partial}{\partial x^k}\left(\frac{\partial x^i}{\partial x^{j'}}\circ\tau_M\right)y^{j'}+\Gamma^i_k\right]\frac{\partial x^k}{\partial x^{k'}}\circ\tau_M. \tag{4.1$'''$}$$

Proof. Let (Π, π) be the canonical projection **B**-morphism.

Obviously, the components of

$$\Pi \circ (\rho,\eta)\Gamma(Z^{\gamma'}\overset{*}{\tilde{\partial}}_{\gamma'} + Y^{a'}\dot{\tilde{\partial}}_{a'})(u_x)$$

are the real numbers

$$(Y^{a'} + (\rho,\eta)\Gamma^{a'}_{\gamma'}Z^{\gamma'})(u_x).$$

Since

$$\big(Z^{\gamma'}\overset{*}{\tilde{\partial}}_{\gamma'} + Y^{a'}\dot{\tilde{\partial}}_{a'}\big)\big(u_x\big) = Z^{\gamma'}\Lambda^{\gamma}_{\gamma'} \circ h \circ \pi \overset{*}{\tilde{\partial}}_{\gamma}\big(u_x\big) + \left(Z^{\gamma'}\rho^{i'}_{\gamma'} \circ h \circ \pi \frac{\partial y^a}{\partial x^{i'}} + \frac{\partial y^a}{\partial y^{a'}}Y^{a'}\right)\dot{\tilde{\partial}}_a\big(u_x\big),$$

it results that the components of

$$\Pi \circ (\rho,\eta)\Gamma(Z^{\gamma'}\overset{*}{\tilde{\partial}}_{\gamma'} + Y^{a'}\dot{\tilde{\partial}}_{a'})(u_x)$$

are the real numbers

$$(Z^{\gamma'}\rho^{i'}_{\gamma'} \circ h \circ \pi \frac{\partial y^a}{\partial x^{i'}} + \frac{\partial y^a}{\partial y^{a'}}Y^{a'} + (\rho,\eta)\Gamma^a_{\gamma}Z^{\gamma'}\Lambda^{\gamma}_{\gamma'} \circ h \circ \pi)(u_x)\frac{\partial y^{a'}}{\partial y^a},$$

where

$$\left\| \frac{\partial y^a}{\partial y^{a'}} \right\| = \left\| \frac{\partial y^{a'}}{\partial y^a} \right\|^{-1}.$$

Therefore, we have:

$$\left(Z^{\gamma'}\rho^{i'}_{\gamma'} \circ h \circ \pi \frac{\partial y^a}{\partial x^{i'}} + \frac{\partial y^a}{\partial y^{a'}}Y^{a'} + (\rho,\eta)\Gamma^a_{\gamma}Z^{\gamma'}\Lambda^{\gamma}_{v'} \circ h \circ \pi\right)\frac{\partial y^{a'}}{\partial y^a} = Y^{a'} + (\rho,\eta)\Gamma^{a'}_{\gamma'}Z^{\gamma'}.$$

After some calculations, we obtain:

$$(\rho,\eta)\Gamma^{a'}_{\gamma'} = \frac{\partial y^{a'}}{\partial y^a}\left(\rho^i_{\gamma} \circ (h \circ \pi)\frac{\partial y^a}{\partial x^i} + (\rho,\eta)\Gamma^a_{\gamma}\right)\Lambda^{\gamma}_{\gamma'} \circ h \circ \pi. \qquad \square$$

Remark 17. If we have a set of real local functions $(\rho,\eta)\Gamma^a_{\gamma}$ that satisfies the relations of passing (4.1), then we have a (ρ,η)-connection $(\rho,\eta)\Gamma$ for the fiber bundle (E,π,M)

Example 18. If Γ is an Ehresmann connection for the vector bundle (E,π,M) on components Γ^a_k, then the differentiable real local functions $(\rho,\eta)\Gamma^a_{\gamma} = (\rho^k_{\gamma} \circ h \circ \pi)\Gamma^a_k$ are the components of a (ρ,η)-connection $(\rho,\eta)\Gamma$ for the vector bundle (E,π,M). This (ρ,η)-connection will be called the (ρ,η)-connection associated to the connection Γ.

Definition 19. If $(\rho,\eta)\Gamma$ is a (ρ,η)-connection for the vector bundle (E,π,M) and $z = z^{\gamma}t_{\gamma} \in \Gamma(F,\nu,M)$, then the application

$$\begin{array}{rcl} \Gamma(E,\pi,M) & \xrightarrow{(\rho,\eta)D_z} & \Gamma(E,\pi,M) \\ u = u^a s_a & \longmapsto & (\rho,\eta)D_z u \end{array}$$

where

$$(\rho,\eta)D_z u = z^{\gamma} \circ h\left(\rho^k_{\gamma} \circ h\frac{\partial u^a}{\partial x^k} + (\rho,\eta)\Gamma^a_{\gamma} \circ u\right)s_a \tag{4.2}$$

will be called the *covariant* (ρ,η)*-derivative associated to* (ρ,η)*-connection* $(\rho,\eta)\Gamma$ *with respect to the section* z.

Remark 20. In the particular case of Lie algebroids, $(\eta,h)=(\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.2) become:

$$\rho D_z u = z^\gamma\left(\rho_\gamma^k\frac{\partial u^a}{\partial x^k}+\rho\Gamma_\gamma^a\circ u\right)s_a. \tag{4.2$'$}$$

In the classical case, $(\rho,\eta,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.2′) become:

$$D_X Y = X^k\left(\frac{\partial Y^i}{\partial x^k}+\Gamma_k^i\circ Y\right)\partial_i. \tag{4.2$''$}$$

Definition 21. Let $(\rho,\eta)\Gamma$ be a (ρ,η)-connection for the fiber bundle (E,π,M). If for each local vector $(m+r)$-chart (U,s_U) and for each local vector $(n+p)$-chart (V,t_V) so that $U\cap h^{-1}(V)\neq\phi$, it exists the differentiable real functions $(\rho,\eta)\Gamma_{b\gamma}^a$ defined on $U\cap h^{-1}(V)$ such that

$$(\rho,\eta)\Gamma_\gamma^a\circ u=(\rho,\eta)\Gamma_{b\gamma}^a\cdot u^b,\quad \forall u=u^b s_b\in\Gamma(E,\pi,M),$$

then we say that $(\rho,\eta)\Gamma$ is linear.

The differentiable real local functions $(\rho,\eta)\Gamma_{b\gamma}^a$ will be called the *Christoffel coefficients of linear* (ρ,η)-*connection* $(\rho,\eta)\Gamma$.

Theorem 22. *If $(\rho,\eta)\Gamma$ is a linear (ρ,η)-connection for the fiber bundle (E,π,M), then its components satisfy the law of transformation*

$$(\rho,\eta)\Gamma_{b'\gamma'}^{a'}=\frac{\partial y^{a'}}{\partial y^a}\left[\rho_\gamma^k\circ h\frac{\partial}{\partial x^k}\left(\frac{\partial y^a}{\partial y^{b'}}\right)+(\rho,\eta)\Gamma_{b\gamma}^a\frac{\partial y^b}{\partial y^{b'}}\right]\Lambda_{\gamma'}^\gamma\circ h. \tag{4.3}$$

If $(\rho,\eta)\Gamma$ is a linear (ρ,η)-connection for the vector bundle (E,π,M), then its components satisfy the law of transformation

$$(\rho,\eta)\Gamma_{b'\gamma'}^{a'}=M_a^{a'}\left[\rho_\gamma^k\circ h\frac{\partial M_{b'}^a}{\partial x^k}+(\rho,\eta)\Gamma_{b\gamma}^a M_{b'}^b\right]\Lambda_{\gamma'}^\gamma\circ h. \tag{4.3$'$}$$

In the particular case of Lie algebroids, $(\eta,h)=(\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.3′) *become:*

$$\rho\Gamma_{b'\gamma'}^{a'}=M_a^{a'}\left[\rho_\gamma^k\frac{\partial M_{b'}^a}{\partial x^k}+\rho\Gamma_{b\gamma}^a M_{b'}^b\right]\Lambda_{\gamma'}^\gamma. \tag{4.3$''$}$$

In the classical case, $(\rho,\eta,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.3″) *become:*

$$\Gamma_{j'k'}^{i'}=\frac{\partial x^{i'}}{\partial x^i}\left[\frac{\partial}{\partial x^k}\left(\frac{\partial x^i}{\partial x^{j'}}\right)+\Gamma_{jk}^i\frac{\partial x^j}{\partial x^{j'}}\right]\frac{\partial x^k}{\partial x^{k'}}. \tag{4.3$'''$}$$

Theorem 23. *If $(\rho,\eta)\Gamma$ is a linear (ρ,η)-connection for the vector bundle (E,π,M), then, for any $z=z^\gamma t_\gamma\in\Gamma(F,\nu,M)$, we obtain the covariant (ρ,η)-derivative associated to the linear (ρ,η)-connection $(\rho,\eta)\Gamma$ with respect to the section z*

$$\begin{array}{ccc}\Gamma(E,\pi,M) & \xrightarrow{(\rho,\eta)D_z} & \Gamma(E,\pi,M)\\ u=u^a s_a & \longmapsto & (\rho,\eta)D_z u\end{array}$$

defined by

$$(\rho,\eta)D_z u = z^\gamma\circ h\left(\rho_\gamma^k\circ h\frac{\partial u^a}{\partial x^k}+(\rho,\eta)\Gamma_{b\gamma}^a\cdot u^b\right)s_a. \tag{4.4}$$

In the particular case of Lie algebroids, $(\eta,h)=(\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.4) *become:*

$$\rho D_z u = z^\gamma\left(\rho_\gamma^k\frac{\partial u^a}{\partial x^k}+\rho\Gamma_{b\gamma}^a\cdot u^b\right)s_a. \tag{4.4$'$}$$

In the classical case, $(\rho,\eta,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.4′) *become:*

$$D_X Y = X^k\left(\frac{\partial Y^i}{\partial x^k}+\Gamma_{jk}^i\cdot Y^j\right)\partial_i. \tag{4.4$''$}$$

4.1 (Linear) (ρ,η)-connections for dual vector bundle

Let $(E,\pi,M)\in|\mathbf{B^v}|$ be. We consider the following diagram:

$$\begin{tikzcd} \overset{*}{E} \arrow[d, "\overset{*}{\pi}"'] & (F,[,]_{F,h},(\rho,\eta)) \arrow[d, "\nu"] \\ M \arrow[r, "h"] & N, \end{tikzcd}$$

where $((F,\nu,N),[,]_{F,h},(\rho,\eta))\in|\mathbf{GLA}|$.

Let

$$\left(\left((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right),[,]_{(\rho,\eta)T\overset{*}{E}},\left(\overset{*}{\tilde{\rho}},\mathrm{Id}_{\overset{*}{E}}\right)\right)$$

be the Lie algebroid generalized tangent bundle of the vector bundle $(\overset{*}{E},\overset{*}{\pi},M)$.

We consider the $\mathbf{B^v}$-morphism $((\rho,\eta)\overset{*}{\pi}!,\mathrm{Id}_{\overset{*}{E}})$ given by the commutative diagram:

$$\begin{tikzcd} (\rho,\eta)T\overset{*}{E} \arrow[r, "(\rho,\eta)\overset{*}{\pi}!"] \arrow[d, "(\rho,\eta)\tau_{\overset{*}{E}}"'] & \overset{*}{\pi}^*(h^*F) \arrow[d, "\overset{*}{\pi}^*(h^*\nu)"] \\ \overset{*}{E} \arrow[r, "\mathrm{Id}_{\overset{*}{E}}"] & \overset{*}{E} \end{tikzcd}$$

Using the components, this is defined as:

$$(\rho,\eta)\overset{*}{\pi}!\left(Z^\alpha\overset{*}{\tilde{\partial}}_\alpha+Y_a\overset{\cdot\, a}{\tilde{\partial}}\right)(\overset{*}{u}_x)=\left(Z^\alpha T_\alpha\right)(\overset{*}{u}_x),$$

for any $Z^\alpha\overset{*}{\tilde{\partial}}_\alpha+Y_a\overset{\cdot\, a}{\tilde{\partial}}\in((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E})$.

Using the $\mathbf{B^v}$-morphism $((\rho,\eta)\overset{*}{\pi}!,\mathrm{Id}_{\overset{*}{E}})$ and the $\mathbf{B^v}$-morphism (2.2), we obtain the *tangent* (ρ,η)*-application* $((\rho,\eta)T\overset{*}{\pi},h\circ\overset{*}{\pi})$ of $((\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E})$ source and (F,ν,N) target.

Definition 24. The kernel of the tangent (ρ,η)-application

$$\left((\rho,\eta)T\overset{*}{\pi},h\circ\overset{*}{\pi}\right)$$

is written as

$$\left(V(\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}\right)$$

and will be called the *vertical subbundle*.

The set $\{\overset{\cdot\, a}{\tilde{\partial}},\ a\in\overline{1,r}\}$ is a base for the $\mathcal{F}(\overset{*}{E})$-module

$$\left(\Gamma(V(\rho,\eta)T\overset{*}{E},(\rho,\eta)\tau_{\overset{*}{E}},\overset{*}{E}),+,\cdot\right).$$

Proposition 25. *The short sequence of vector bundles*

$$\begin{tikzcd} 0 \arrow[r, hook] \arrow[d] & V(\rho,\eta)T\overset{*}{E} \arrow[r, hook] \arrow[d] & (\rho,\eta)T\overset{*}{E} \arrow[r, "(\rho,\eta)\overset{*}{\pi}!"] \arrow[d] & \overset{*}{\pi}^*(h^*F) \arrow[r] \arrow[d] & 0 \arrow[d] \\ \overset{*}{E} \arrow[r, "\mathrm{Id}_{\overset{*}{E}}"] & \overset{*}{E} \arrow[r, "\mathrm{Id}_{\overset{*}{E}}"] & \overset{*}{E} \arrow[r, "\mathrm{Id}_{\overset{*}{E}}"] & \overset{*}{E} \arrow[r, "\mathrm{Id}_{\overset{*}{E}}"] & \overset{*}{E} \end{tikzcd}$$

is exact.

Definition 26. A **Man**-morphism $(\rho,\eta)\Gamma$ of $(\rho,\eta)T\overset{*}{E}$ source and $V(\rho,\eta)T\overset{*}{E}$ target defined by:

$$(\rho,\eta)\Gamma\Big(Z^{\gamma}\overset{*}{\tilde{\partial}}_{\gamma}+Y_a\overset{\cdot\,a}{\tilde{\partial}}\Big)\big(\overset{*}{u}_x\big)=\big(Y_b-(\rho,\eta)\Gamma_{b\gamma}Z^{\gamma}\big)\overset{\cdot\,b}{\tilde{\partial}}\big(\overset{*}{u}_x\big),$$

such that the $\mathbf{B}^{\mathbf{v}}$-morphism $((\rho,\eta)\Gamma,\mathrm{Id}_{\overset{*}{E}})$ is a split to the left in the previous exact sequence, will be called (ρ,η)-*connection for the dual vector bundle* $(\overset{*}{E},\overset{*}{\pi},M)$.

The differentiable real local functions $(\rho,\eta)\Gamma_{b\gamma}$ will be called the *components of* (ρ,η)*-connection* $(\rho,\eta)\Gamma$.

The (ρ,Id_M)-connection for the dual vector bundle $(\overset{*}{E},\overset{*}{\pi},M)$ will be called *ρ-connection for the dual vector bundle* $(\overset{*}{E},\overset{*}{\pi},M)$ and will be denoted $\rho\Gamma$.

The $(\mathrm{Id}_{TM},\mathrm{Id}_M)$-connection for the dual vector bundle $(\overset{*}{E},\overset{*}{\pi},M)$ will be called *connection for the dual vector bundle* $(\overset{*}{E},\overset{*}{\pi},M)$ and will be denoted Γ.

Let $\{s^a,\ a\in\overline{1,r}\}$ be the dual base of the base $\{s_a,\ a\in\overline{1,r}\}$.

The $\mathbf{B}^{\mathbf{v}}$-morphism $(\overset{*}{\Pi},\overset{*}{\pi})$ defined by the commutative diagram

$$\begin{array}{ccc} V(\rho,\eta)T\overset{*}{E} & \xrightarrow{\overset{*}{\Pi}} & \overset{*}{E} \\ {\scriptstyle (\rho,\eta)\tau_{\overset{*}{E}}}\downarrow & & \downarrow{\scriptstyle \overset{*}{\pi}} \\ \overset{*}{E} & \xrightarrow{\overset{*}{\pi}} & M, \end{array}$$

where, $\overset{*}{\Pi}$ is defined by

$$\overset{*}{\Pi}\Big(Y_a\overset{\cdot\,a}{\tilde{\partial}}\big(\overset{*}{u}_x\big)\Big)=Y_a\big(\overset{*}{u}_x\big)s^a(x),$$

is canonical projection $\mathbf{B}^{\mathbf{v}}$-morphism.

Theorem 27. *If $(\rho,\eta)\Gamma$ is a (ρ,η)-connection for the vector bundle $(\overset{*}{E},\overset{*}{\pi},M)$, then its components satisfy the law of transformation*

$$(\rho,\eta)\Gamma_{b'\gamma'}=M_{b'}^{b}\circ\overset{*}{\pi}\Big[-\rho_{\gamma}^{k}\circ h\circ\overset{*}{\pi}\cdot\frac{\partial M_b^{a'}\circ\overset{*}{\pi}}{\partial x^k}p_{a'}+(\rho,\eta)\Gamma_{b\gamma}\Big]\Lambda_{\gamma'}^{\gamma}\circ(h\circ\overset{*}{\pi}).\tag{4.5}$$

In the particular case of Lie algebroids, $(\eta,h)=(\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.5) *become:*

$$\rho\Gamma_{b'\gamma'}=M_{b'}^{b}\circ\overset{*}{\pi}\Big[-\rho_{\gamma}^{k}\circ\overset{*}{\pi}\cdot\frac{\partial M_b^{a'}\circ\overset{*}{\pi}}{\partial x^k}p_{a'}+\rho\Gamma_{b\gamma}\Big]\Lambda_{\gamma'}^{\gamma}\circ\overset{*}{\pi}.\tag{4.5$'$}$$

In the classical case, $(\rho,\eta,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.5′) *become:*

$$\Gamma_{j'k'}=\frac{\partial x^j}{\partial x^{j'}}\circ\overset{*}{\tau}_M\Big[-\frac{\partial}{\partial x^k}\Big(\frac{\partial x^{i'}}{\partial x^j}\circ\overset{*}{\tau}_M\Big)p_{i'}+\Gamma_{jk}\Big]\frac{\partial x^k}{\partial x^{k'}}\circ\overset{*}{\tau}_M.\tag{4.5$''$}$$

Proof. Let $(\overset{*}{\Pi},\overset{*}{\pi})$ be the canonical projection $\mathbf{B}^{\mathbf{v}}$-morphism.

Obviously, the components of

$$\overset{*}{\Pi}\circ(\rho,\eta)\Gamma\Big(Z^{\gamma}\overset{*}{\tilde{\partial}}_{\gamma}+Y_a\overset{\cdot\,a}{\tilde{\partial}}\Big)\big(\overset{*}{u}_x\big)$$

are the real numbers

$$\big(Y_{b'}-(\rho,\eta)\Gamma_{b'\gamma'}Z^{\gamma'}\big)\big(\overset{*}{u}_x\big).$$

Since

$$\Big(Z^{\gamma'}\overset{*}{\tilde{\partial}}_{\gamma'}+Y_{b'}\overset{\cdot\,b'}{\tilde{\partial}}\Big)\big(\overset{*}{u}_x\big)=Z^{\gamma'}\Lambda_{\gamma'}^{\gamma}\circ h\circ\overset{*}{\pi}\cdot\overset{*}{\tilde{\partial}}_{\alpha}\big(\overset{*}{u}_x\big)+\Big(Z^{\gamma'}\rho_{\gamma'}^{i'}\circ h\circ\overset{*}{\pi}\frac{\partial M_b^{a'}\circ\pi}{\partial x^{i'}}p_{a'}+M_b^{b'}Y_{b'}\Big)\overset{\cdot\,b}{\tilde{\partial}}\big(\overset{*}{u}_x\big),$$

it results that the components of:

$$\overset{*}{\Pi}\circ(\rho,\eta)\Gamma\Big(Z^{\gamma'}\overset{*}{\tilde{\partial}}_{\gamma'}+Y_{b'}\dot{\tilde{\partial}}^{b'}\Big)(\overset{*}{u}_x)$$

are the real numbers:

$$\left(Z^{\gamma'}\rho_{\gamma'}^{k'}\circ h\circ\overset{*}{\pi}\frac{\partial M_b^{a'}\circ\overset{*}{\pi}}{\partial x^{k'}}p_{a'}+M_b^{b'}\circ\overset{*}{\pi}Y_{b'}-(\rho,\eta)\Gamma_{b\gamma}Z^{\gamma'}\Lambda_{\gamma'}^{\gamma}\circ h\circ\overset{*}{\pi}\right)M_{b'}^b\circ\overset{*}{\pi}(\overset{*}{u}_x),$$

where $\|M_{b'}^b\|=\|M_b^{b'}\|^{-1}$.

Therefore, we have:

$$\left(Z^{\gamma'}\rho_{\gamma'}^{k'}\circ h\circ\overset{*}{\pi}\frac{\partial M_b^{a'}\circ\overset{*}{\pi}}{\partial x^{k'}}p_{a'}+M_b^{b'}\circ\overset{*}{\pi}Y_{b'}-(\rho,\eta)\Gamma_{b\gamma}Z^{\gamma'}\Lambda_{\gamma'}^{\gamma}\circ h\circ\overset{*}{\pi}\right)M_{b'}^b\circ\overset{*}{\pi}=Y_{b'}-(\rho,\eta)\Gamma_{b'\gamma'}Z^{\gamma'}.$$

After some calculations we obtain:

$$(\rho,\eta)\Gamma_{b'\gamma'}=M_{b'}^b\circ\overset{*}{\pi}\left(-\rho_\gamma^k\circ h\circ\overset{*}{\pi}\cdot\frac{\partial M_b^{a'}\circ\overset{*}{\pi}}{\partial x^k}p_{a'}+(\rho,\eta)\Gamma_{b\gamma}\right)\Lambda_{\gamma'}^{\gamma}\circ h\circ\overset{*}{\pi}. \qquad \square$$

Remark 28. If we have a set of real local functions $(\rho,\eta)\Gamma_{b\gamma}$ that satisfies the relations of passing (4.5), then we have a (ρ,η)-connection $(\rho,\eta)\Gamma$ for the dual vector bundle $(\overset{*}{E},\overset{*}{\pi},M)$.

Example 29. If Γ is an Ehresmann connection for the vector bundle $(\overset{*}{E},\overset{*}{\pi},M)$ on components Γ_{bk}, then the differentiable real local functions

$$(\rho,\eta)\Gamma_{b\gamma}=(\rho_\gamma^k\circ h\circ\overset{*}{\pi})\Gamma_{bk}$$

are the components of a (ρ,η)-connection $(\rho,\eta)\Gamma$ for the vector bundle $(\overset{*}{E},\overset{*}{\pi},M)$, which will be called the (ρ,η)-*connection associated to the connection* Γ.

Definition 30. If $(\rho,\eta)\Gamma$ is a (ρ,η)-connection for the vector bundle $(\overset{*}{E},\overset{*}{\pi},M)$, then for any

$$z=z^\gamma t_\gamma\in\Gamma(F,\nu,N)$$

the application

$$\begin{array}{ccc}\Gamma(\overset{*}{E},\overset{*}{\pi},M) & \xrightarrow{(\rho,\eta)D_z} & \Gamma(\overset{*}{E},\overset{*}{\pi},M)\\ \overset{*}{u}=u_as^a & \longmapsto & (\rho,\eta)D_z\overset{*}{u}\end{array}$$

defined by

$$(\rho,\eta)D_z\overset{*}{u}=z^\gamma\circ h(\rho_\gamma^k\circ h\frac{\partial u_b}{\partial x^k}-(\rho,\eta)\Gamma_{b\gamma}\circ\overset{*}{u})s^b, \tag{4.6}$$

will be called the *covariant* (ρ,η)*-derivative associated to* (ρ,η)*-connection* $(\rho,\eta)\Gamma$ *with respect to section* z.

Remark 31. In the particular case of Lie algebroids, $(\eta,h)=(\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.6) become:

$$\rho D_z\overset{*}{u}=z^\gamma\left(\rho_\gamma^k\frac{\partial u_b}{\partial x^k}-\rho\Gamma_{b\gamma}\circ\overset{*}{u}\right)s^b. \tag{4.6$'$}$$

In the classical case, $(\rho,\eta,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.6$'$) become:

$$D_X\omega=X^k\left(\frac{\partial\omega_j}{\partial x^k}-\Gamma_{jk}\circ\omega\right)dx^j. \tag{4.6$''$}$$

Definition 32. Let $(\rho,\eta)\Gamma$ be a linear (ρ,η)-connection for the vector bundle $(\overset{*}{E},\overset{*}{\pi},M)$. If for each local vector $(m+r)$-chart $(U,\overset{*}{s}_U)$ and for each local vector $(n+p)$-chart (V,t_V) such that $U\cap h^{-1}(V)\neq\phi$, there exists the differentiable real functions $\rho\Gamma^a_{b\gamma}$ defined on $U\cap h^{-1}(V)$ such that

$$(\rho,\eta)\Gamma_{b\gamma}\circ\overset{*}{u}=(\rho,\eta)\Gamma^a_{b\gamma}\cdot u_a,\quad \forall\overset{*}{u}=u_as^a\in\Gamma(\overset{*}{E},\overset{*}{\pi},M)$$

then we say that $(\rho,\eta)\Gamma$ *is linear*.

The differentiable real local functions $(\rho,\eta)\Gamma^a_{b\gamma}$ will be called the *Christoffel coefficients of linear* (ρ,η)-*connection* $(\rho,\eta)\Gamma$.

Theorem 33. *If $(\rho,\eta)\Gamma$ is a linear (ρ,η)-connection for the vector bundle $(\overset{*}{E},\overset{*}{\pi},M)$, then its components satisfy the law of transformation*

$$(\rho,\eta)\Gamma^{a'}_{b'\gamma'}=M^b_{b'}\left[-\rho^k_\gamma\circ h\frac{\partial M^{a'}_b}{\partial x^k}+(\rho,\eta)\Gamma^a_{b\gamma}M^{a'}_a\right]\Lambda^\gamma_{\gamma'}\circ h. \tag{4.7}$$

In the particular case of Lie algebroids, $(\eta,h)=(\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.7) *become:*

$$\rho\Gamma^{a'}_{b'\gamma'}=M^b_{b'}\left[-\rho^k_\gamma\frac{\partial M^{a'}_b}{\partial x^k}+\rho\Gamma^a_{b\gamma}M^{a'}_a\right]\Lambda^\gamma_{\gamma'}. \tag{4.7$'$}$$

In the classical case, $(\rho,\eta,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.7$'$) *become:*

$$\Gamma^{i'}_{j'k'}=\frac{\partial x^j}{\partial x^{j'}}\left[-\frac{\partial}{\partial x^k}\left(\frac{\partial x^{i'}}{\partial x^j}\right)+\Gamma^i_{jk}\frac{\partial x^{i'}}{\partial x^i}\right]\frac{\partial x^k}{\partial x^{k'}}. \tag{4.7$''$}$$

Remark 34. Since

$$\frac{\partial M^{a'}_b}{\partial x^i}M^b_{b'}+\frac{\partial M^b_{b'}}{\partial x^i}M^{a'}_b=0,$$

it results that the relations (4.7) are equivalent with the relations (4.3$'$).

Theorem 35. *If $(\rho,\eta)\Gamma$ is a linear (ρ,η)-connection for the dual vector bundle $(\overset{*}{E},\overset{*}{\pi},M)$, then, for any $z=z^\gamma t_\gamma\in\Gamma(F,\nu,M)$, we obtain the covariant (ρ,η)-derivative associated to the linear (ρ,η)-connection $(\rho,\eta)\Gamma$ with respect to the section z*

$$\begin{array}{ccc}\Gamma(\overset{*}{E},\overset{*}{\pi},M) & \xrightarrow{(\rho,\eta)D_z} & \Gamma(\overset{*}{E},\overset{*}{\pi},M)\\ \overset{*}{u}=u_as^a & \longmapsto & (\rho,\eta)D_z\overset{*}{u}\end{array}$$

defined by

$$(\rho,\eta)D_z\overset{*}{u}=z^\gamma\circ h\left(\rho^k_\gamma\circ h\frac{\partial u_b}{\partial x^k}-(\rho,\eta)\Gamma^a_{b\gamma}\cdot u_a\right)s^b. \tag{4.8}$$

In the particular case of Lie algebroids, $(\eta,h)=(\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.8) *become:*

$$\rho D_z\overset{*}{u}=z^\gamma\left(\rho^k_\gamma\frac{\partial u_b}{\partial x^k}-\rho\Gamma^a_{b\gamma}\cdot u_a\right)s^b. \tag{4.8$'$}$$

In the classical case, $(\rho,\eta,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.8$'$) *become:*

$$D_X\omega=X^k\left(\frac{\partial\omega_j}{\partial x^k}-\Gamma^i_{jk}\cdot\omega_i\right)dx^j. \tag{4.8$''$}$$

In the next section, we use the same notation $(\rho,\eta)\Gamma$ for the linear (ρ,η)-connection for the vector bundle (E,π,M) or for its dual $(\overset{*}{E},\overset{*}{\pi},M)$.

Remark 36. If $(\rho,\eta)\Gamma$ is a linear (ρ,η)-connection for the vector bundle (E,π,M) or for its dual $(\overset{*}{E},\overset{*}{\pi},M)$ then, the tensor fields algebra:

$$(\mathcal{T}(E,\pi,M),+,\cdot,\otimes)$$

is endowed with the (ρ,η)-derivative:

$$\begin{array}{rcl}\Gamma(F,\nu,N)\times\mathcal{T}(E,\pi,M) & \xrightarrow{(\rho,\eta)D} & \mathcal{T}(E,\pi,M)\\ (z,T) & \longmapsto & (\rho,\eta)D_zT\end{array}$$

defined for a tensor field $T\in\mathcal{T}^p_q(E,\pi,M)$ by the relation:

$$\begin{aligned}&(\rho,\eta)D_zT(\overset{*}{u}_1,\ldots,\overset{*}{u}_p,u_1,\ldots,u_q)\\ &\quad=\Gamma(\rho,\eta)(z)(T(\overset{*}{u}_1,\ldots,\overset{*}{u}_p,u_1,\ldots,u_q))-T((\rho,\eta)D_z\overset{*}{u}_1,\ldots,\overset{*}{u}_p,u_1,\ldots,u_q)-\cdots\\ &\quad\quad-T(\overset{*}{u}_1,\ldots,(\rho,\eta)D_z\overset{*}{u}_p,u_1,\ldots,u_q)-T(\overset{*}{u}_1,\ldots,\overset{*}{u}_p,(\rho,\eta)D_zu_1,\ldots,u_q)-\cdots\\ &\quad\quad-T(\overset{*}{u}_1,\ldots,\overset{*}{u}_p,u_1,\ldots,(\rho,\eta)D_zu_q).\end{aligned}\tag{4.9}$$

Moreover, it satisfies the condition

$$(\rho,\eta)D_{f_1z_1+f_2z_2}T=f_1(\rho,\eta)D_{z_1}T+f_2(\rho,\eta)D_{z_2}T.\tag{4.10}$$

Consequently, if the tensor algebra $(\mathcal{T}(E,\pi,M),+,\cdot,\otimes)$ is endowed with a (ρ,η)-derivative defined for a tensor field $T\in\mathcal{T}^p_q(E,\pi,M)$ by (4.9), which satisfies the condition (4.10), then we can endowed (E,π,M) with a linear (ρ,η)-connection $(\rho,\eta)\Gamma$ such that its components are defined by the equality:

$$(\rho,\eta)D_{t_\gamma}s_b=(\rho,\eta)\Gamma^a_{b\gamma}s_a$$

or

$$(\rho,\eta)D_{t_\gamma}s^a=-(\rho,\eta)\Gamma^a_{b\gamma}s^b.$$

The (ρ,η)-derivative defined by (4.9) will be called the *covariant* (ρ,η)*-derivative*.

After some calculations, we obtain:

$$\begin{aligned}&(\rho,\eta)D_z\left(T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}s_{a_1}\otimes\cdots\otimes s_{a_p}\otimes s^{b_1}\otimes\cdots\otimes s^{b_q}\right)\\ &\quad=z^\gamma\circ h\Bigg(\rho^k_\gamma\circ h\frac{\partial T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}}{\partial x^k}+(\rho,\eta)\Gamma^{a_1}_{a\gamma}T^{a,a_2,\ldots,a_p}_{b_1,\ldots,b_q}+(\rho,\eta)\Gamma^{a_2}_{a\gamma}T^{a_1,a,\ldots,a_p}_{b_1,\ldots,b_q}+\cdots\\ &\quad\quad+(\rho,\eta)\Gamma^{a_p}_{a\gamma}T^{a_1,a_2,\ldots,a}_{b_1,\ldots,b_q}-\cdots-(\rho,\eta)\Gamma^b_{b_1\gamma}T^{a_1,a_2,\ldots,a_p}_{b,b_2,\ldots,b_q}-(\rho,\eta)\Gamma^b_{b_2\gamma}T^{a_1,a_2,\ldots,a_p}_{b_1,b,\ldots,b_q}-\cdots\\ &\quad\quad-(\rho,\eta)\Gamma^b_{b_q\gamma}T^{a_1,a_2,\ldots,a_p}_{b_1,b_2,\ldots,b}\Bigg)s_{a_1}\otimes\cdots\otimes s_{a_p}\otimes s^{b_1}\otimes\cdots\otimes s^{b_q}\\ &\quad\overset{\text{put}}{=}z^\gamma\circ hT^{a_1,\ldots,a_p}_{b_1,\ldots,b_q|\gamma}s_{a_1}\otimes\cdots\otimes s_{a_p}\otimes s^{b_1}\otimes\cdots\otimes s^{b_q}.\end{aligned}\tag{4.11}$$

We remark that if $(\rho,\eta)\Gamma$ is the linear (ρ,η)-connection associated to the Ehresmann linear connection Γ, namely $(\rho,\eta)\Gamma^a_{b\alpha}=(\rho^k_\alpha\circ h)\Gamma^a_{bk}$, then

$$T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q|\gamma}=(\rho^k_\gamma\circ h)T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q|k}.$$

In the particular case of Lie algebroids, $(\eta,h)=(\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.11) become:

$$\begin{aligned}&\rho D_z\left(T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}s_{a_1}\otimes\cdots\otimes s_{a_p}\otimes s^{b_1}\otimes\cdots\otimes s^{b_q}\right)\\ &=z^\gamma\Bigg(\rho^k_\gamma\frac{\partial T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}}{\partial x^k}+\rho\Gamma^{a_1}_{a\gamma}T^{a,a_2,\ldots,a_p}_{b_1,\ldots,b_q}+\rho\Gamma^{a_2}_{a\gamma}T^{a_1,a,\ldots,a_p}_{b_1,\ldots,b_q}+\cdots+\rho\Gamma^{a_p}_{a\gamma}T^{a_1,a_2,\ldots,a}_{b_1,\ldots,b_q}-\cdots\\ &\quad-\rho\Gamma^b_{b_1\gamma}T^{a_1,a_2,\ldots,a_p}_{b,b_2,\ldots,b_q}-\rho\Gamma^b_{b_2\gamma}T^{a_1,a_2,\ldots,a_p}_{b_1,b,\ldots,b_q}-\cdots-\rho\Gamma^b_{b_q\gamma}T^{a_1,a_2,\ldots,a_p}_{b_1,b_2,\ldots,b}\Bigg)s_{a_1}\otimes\cdots\otimes s_{a_p}\otimes s^{b_1}\otimes\cdots\otimes s^{b_q}\\ &\overset{\text{put}}{=}z^\gamma T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q|\gamma}s_{a_1}\otimes\cdots\otimes s_{a_p}\otimes s^{b_1}\otimes\cdots\otimes s^{b_q}.\end{aligned}\tag{4.11$'$}$$

In the classical case, $(\rho,\eta,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M,\mathrm{Id}_M)$, the relations (4.11′) become:

$$
\begin{aligned}
&D_X\left(T^{i_1\dots i_p}_{j_1\dots j_q}\partial_{i_1}\otimes\dots\otimes\partial_{i_p}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_q}\right)\\
&=X^k\left(\frac{\partial T^{i_1\dots i_p}_{j_1\dots j_q}}{\partial x^k}+\Gamma^{i_1}_{ik}T^{ii_2\dots i_p}_{j_1\dots j_q}+\dots+\Gamma^{i_p}_{ik}T^{i_1\dots i_{p-1}i}_{j_1\dots j_q}\right.\\
&\left.-\Gamma^{j}_{j_1k}T^{i_1\dots i_p}_{jj_2\dots j_q}-\dots-\Gamma^{j}_{j_qk}T^{i_1\dots i_p}_{j_1\dots j_{q-1}j}\right)\partial_{i_1}\otimes\dots\otimes\partial_{i_p}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_q}\\
&\stackrel{\text{put}}{=}X^kT^{i_1\dots i_p}_{j_1\dots j_q|k}\partial_{i_1}\otimes\dots\otimes\partial_{i_p}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_q}.
\end{aligned}
\tag{4.11″}
$$

5 Torsion and curvature. Formulas of Ricci and Bianchi type

We apply our theory for the diagram:

$$
\begin{tikzcd}
E \arrow[d, "\pi"'] & \big(F,[,]_{F,h},(\rho,\mathrm{Id}_N)\big) \arrow[d, "\nu"'] \\
M \arrow[r, "h"] & N,
\end{tikzcd}
$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $((F,\nu,N),[,]_{F,h},(\rho,\mathrm{Id}_N))\in|\mathbf{GLA}|$.

Let $\rho\Gamma$ be a linear ρ-connection for the vector bundle (E,π,M) by components $\rho\Gamma^a_{b\alpha}$.

Using the components of linear ρ-connection $\rho\Gamma$, then we obtain a linear ρ-connection $\rho\dot{\Gamma}$ for the vector bundle (E,π,M) given by the diagram:

$$
\begin{tikzcd}
E \arrow[d, "\pi"'] & \big(h^*F,[,]_{h^*F},(\overset{h^*F}{\rho},\mathrm{Id}_M)\big) \arrow[d, "h^*\nu"'] \\
M \arrow[r, "\mathrm{Id}_M"] & M
\end{tikzcd}
$$

If $(E,\pi,M)=(F,\nu,N)$, then, using the components of linear ρ-connection $\rho\Gamma$, we can consider a linear ρ-connection $\rho\ddot{\Gamma}$ for the vector bundle $(h^*E,h^*\pi,M)$ given by the diagram:

$$
\begin{tikzcd}
h^*E \arrow[d, "h^*\pi"'] & (h^*E,[,]_{h^*E},(\overset{h^*E}{\rho},\mathrm{Id}_M)) \arrow[d, "h^*\pi"] \\
M \arrow[r, "\mathrm{Id}_M"] & M
\end{tikzcd}
$$

Definition 37. If $(E,\pi,M)=(F,\nu,N)$, then the application

$$
\begin{array}{rcl}
\Gamma(h^*E,h^*\pi,M)^2 & \xrightarrow{(\rho,h)\mathbb{T}} & \Gamma(h^*E,h^*\pi,M)\\
(U,V) & \longrightarrow & \rho\mathbb{T}(U,V)
\end{array}
$$

defined by:

$$(\rho,h)\mathbb{T}(U,V)=\rho\ddot{D}_UV-\rho\ddot{D}_VU-[U,V]_{h^*E},$$

for any $U,V\in\Gamma(h^*E,h^*\pi,M)$, will be called *$(\rho,h)$-torsion associated to the linear ρ-connection* $\rho\ddot{\Gamma}$.

In the particular case of Lie algebroids, $h=\mathrm{Id}_M$, we obtain the application:

$$
\begin{array}{rcl}
\Gamma(E,\pi,M)^2 & \xrightarrow{\rho\mathbb{T}} & \Gamma(E,\pi,M)\\
(u,v) & \longrightarrow & \rho\mathbb{T}(u,v)
\end{array}
$$

defined by:

$$\rho\mathbb{T}(u,v)=\rho D_uv-\rho D_vu-[u,v]_E,$$

for any $u,v\in\Gamma(E,\pi,M)$, which will be called the *ρ-torsion associated to the linear ρ-connection* $\rho\Gamma$.

In the classical case, $(\rho,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M)$, we obtain the torsion $\mathbb{T}$ associated to the linear connection Γ.

Proposition 38. *The (ρ,h)-torsion $(\rho,h)\mathbb{T}$ associated to the linear ρ-connection $\rho\ddot{\Gamma}$ is $\mathbb{R}$-bilinear and antisymmetric.*

If

$$(\rho,h)\mathbb{T}(S_a,S_b)\overset{put}{=}(\rho,h)\mathbb{T}^c_{ab}S_c$$

then

$$(\rho,h)\mathbb{T}^c_{ab}=\rho\Gamma^c_{ab}-\rho\Gamma^c_{ba}-L^c_{ab}\circ h. \tag{5.1}$$

In the particular case of Lie algebroids, $h=\mathrm{Id}_M$, we have $\rho\mathbb{T}(s_a,s_b)\overset{put}{=}\rho\mathbb{T}^c_{ab}s_c$ and

$$\rho\mathbb{T}^c_{ab}=\rho\Gamma^c_{ab}-\rho\Gamma^c_{ba}-L^c_{ab}. \tag{5.1'}$$

In the classical case, $(\rho,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M)$, the equality $(5.1')$ becomes:

$$\mathbb{T}^i_{jk}=\Gamma^i_{jk}-\Gamma^i_{kj}. \tag{5.1''}$$

Definition 39. The application

$$\begin{array}{rcl}\Gamma(h^*F,h^*\nu,M)^2\times\Gamma(E,\pi,M) & \xrightarrow{(\rho,h)\mathbb{R}} & \Gamma(E,\pi,M)\\ ((Z,V),u) & \longrightarrow & (\rho,h)\mathbb{R}(Z,V)u\end{array}$$

defined by

$$(\rho,h)\mathbb{R}(Z,V)u=\rho\dot{D}_Z(\rho\dot{D}_Vu)-\rho\dot{D}_V(\rho\dot{D}_Zu)-\rho\dot{D}_{[Z,V]_{h^*F}}u,$$

for any $Z,V\in\Gamma(h^*F,h^*\nu,M)$ and $u\in\Gamma(E,\pi,M)$, will be called (ρ,h)-*curvature associated to the linear ρ-connection $\rho\dot{\Gamma}$*.

In the particular case of Lie algebroids, $h=\mathrm{Id}_M$, we obtain the application:

$$\begin{array}{rcl}\Gamma(F,\nu,M)^2\times\Gamma(E,\pi,M) & \xrightarrow{\rho\mathbb{R}} & \Gamma(E,\pi,M)\\ ((z,v),u) & \longrightarrow & \rho\mathbb{R}(z,v)u\end{array}$$

defined by

$$\rho\mathbb{R}(z,v)u=\rho\dot{D}_z(\rho\dot{D}_vu)-\rho\dot{D}_v(\rho\dot{D}_zu)-\rho\dot{D}_{[z,v]_F}u,$$

for any $z,v\in\Gamma(F,\nu,M)$ and $u\in\Gamma(E,\pi,M)$, which will be called *ρ-curvature associated to the linear ρ-connection* $\rho\Gamma$.

In the classical case, $(\rho,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M)$, we obtain the curvature $\mathbb{R}$ associated to the linear connection Γ.

Proposition 40. *The (ρ,h)-curvature $(\rho,h)\mathbb{R}$ associated to the linear ρ-connection $\rho\dot{\Gamma}$, is $\mathbb{R}$-linear in each argument and antisymmetric in the first two arguments.*

If

$$(\rho,h)\mathbb{R}(T_\beta,T_\alpha)s_b\overset{put}{=}(\rho,h)\mathbb{R}^a_{b\alpha\beta}s_a,$$

then

$$(\rho,h)\mathbb{R}^a_{b\alpha\beta}=\rho^j_\beta\circ h\frac{\partial\rho\Gamma^a_{b\alpha}}{\partial x^j}+\rho\Gamma^a_{e\beta}\rho\Gamma^e_{b\alpha}-\rho^i_\alpha\circ h\frac{\partial\rho\Gamma^a_{b\beta}}{\partial x^i}-\rho\Gamma^a_{e\alpha}\rho\Gamma^e_{b\beta}+\rho\Gamma^a_{b\gamma}L^\gamma_{\alpha\beta}\circ h. \tag{5.2}$$

In the particular case of Lie algebroids, $h=\mathrm{Id}_M$, we obtain $\rho\mathbb{R}(t_\beta,t_\alpha)s_b\overset{put}{=}\rho\mathbb{R}^a_{b\alpha\beta}s_a$, and

$$\rho\mathbb{R}^a_{b\ \alpha\beta}=\rho^j_\beta\frac{\partial\rho\Gamma^a_{b\alpha}}{\partial x^j}+\rho\Gamma^a_{e\beta}\rho\Gamma^e_{b\alpha}-\rho^i_\alpha\frac{\partial\rho\Gamma^a_{b\beta}}{\partial x^i}-\rho\Gamma^a_{e\alpha}\rho\Gamma^e_{b\beta}+\rho\Gamma^a_{b\gamma}L^\gamma_{\alpha\beta}. \tag{5.2'}$$

In the classical case, $(\rho,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M)$, we obtain $\mathbb{R}(\partial_k,\partial_h)s_b\overset{put}{=}\mathbb{R}^a_{bhk}s_a$, and the equality $(5.2')$ becomes:

$$\mathbb{R}^a_{b\ hk}=\frac{\partial\Gamma^a_{bh}}{\partial x^k}+\Gamma^a_{ek}\Gamma^e_{bh}-\frac{\partial\Gamma^a_{bk}}{\partial x^h}-\Gamma^a_{eh}\Gamma^e_{bk}. \tag{5.2''}$$

Theorem 41. *For any $u^a s_a \in \Gamma(E,\pi,M)$, we will use the notation*

$$u^a_{|\alpha\beta} = \rho^j_\beta \circ h \frac{\partial}{\partial x^j}(u^{a_1}_{|\alpha}) + \rho\Gamma^{a_1}_{b_\beta} u^b_{|\alpha},$$

and we verify the equality:

$$u^{a_1}_{|\alpha\beta} - u^{a_1}_{|\beta\alpha} = u^a (\rho,h)\mathbb{R}^{a_1}_{a\alpha\beta} - u^{a_1}_{|\gamma} L^\gamma_{\alpha\beta} \circ h.$$

After some calculations, we obtain:

$$(\rho,h)\mathbb{R}^{a_1}_{a\alpha\beta} = u_a\Big(u^{a_1}_{|\alpha\beta} - u^{a_1}_{|\beta\alpha} + u^{a_1}_{|\gamma} L^\gamma_{\alpha\beta} \circ h\Big), \tag{5.3}$$

where $u_a s^a \in \Gamma(\overset{*}{E},\overset{*}{\pi},M)$ such that $u_a u^b = \delta^b_a$.

In the particular case of Lie algebroids, $h = \mathrm{Id}_M$, the relations (5.3) *become:*

$$\rho\mathbb{R}^{a_1}_{a\alpha\beta} = u_a\Big(u^{a_1}_{|\alpha\beta} - u^{a_1}_{|\beta\alpha} + u^{a_1}_{|\gamma} L^\gamma_{\alpha\beta}\Big). \tag{5.3$'$}$$

In the classical case, $(\rho,h) = (\mathrm{Id}_{TM},\mathrm{Id}_M)$, the relations (5.3$'$) *become:*

$$\mathbb{R}^{a_1}_{aij} = u_a\Big(u^{a_1}_{|ij} - u^{a_1}_{|ji}\Big). \tag{5.3$''$}$$

Proof. Since

$$\begin{aligned} u^{a_1}_{|\alpha\beta} &= \rho^j_\beta \circ h\bigg(\frac{\partial}{\partial x^j}\bigg(\rho^i_\alpha \circ h \frac{\partial u^{a_1}}{\partial x^i} + \rho\Gamma^{a_1}_{a\alpha} u^a\bigg)\bigg) + \rho\Gamma^{a_1}_{b\beta}\bigg(\rho^i_\alpha \circ h \frac{\partial u^b}{\partial x^i} + \rho\Gamma^b_{a\alpha} u^a\bigg) \\ &= \rho^j_\beta \circ h \frac{\partial \rho^i_\alpha \circ h}{\partial x^j}\frac{\partial u^{a_1}}{\partial x^i} + \rho^j_\beta \circ h\rho^i_\alpha \circ h \frac{\partial}{\partial x^j}\bigg(\frac{\partial u^{a_1}}{\partial x^i}\bigg) + \rho^j_\beta \circ h \frac{\partial \rho\Gamma^{a_1}_{a\alpha}}{\partial x^j} u^a + \rho^j_\beta \circ h \rho\Gamma^{a_1}_{a\alpha}\frac{\partial u^a}{\partial x^j} \\ &\quad + \rho^i_\alpha \circ h\rho\Gamma^{a_1}_{b\beta}\frac{\partial u^b}{\partial x^i} + \rho\Gamma^{a_1}_{b\beta}\rho\Gamma^b_{a\alpha} u^a \end{aligned}$$

and

$$\begin{aligned} u^{a_1}_{|\beta\alpha} &= \rho^i_\alpha \circ h\bigg(\frac{\partial}{\partial x^i}\bigg(\rho^j_\beta \circ h \frac{\partial u^{a_1}}{\partial x^j} + \rho\Gamma^{a_1}_{a\beta} u^a\bigg)\bigg) + \rho\Gamma^{a_1}_{b\alpha}\bigg(\rho^j_\beta \circ h \frac{\partial u^b}{\partial x^j} + \rho\Gamma^b_{a\beta} u^a\bigg) \\ &= \rho^i_\alpha \circ h \frac{\partial \rho^j_\beta \circ h}{\partial x^i}\frac{\partial u^{a_1}}{\partial x^j} + \rho^j_\beta \circ h\rho^i_\alpha \circ h \frac{\partial}{\partial x^i}\bigg(\frac{\partial u^{a_1}}{\partial x^j}\bigg) + \rho^i_\alpha \circ h \frac{\partial \rho\Gamma^{a_1}_{a\beta}}{\partial x^i} u^a + \rho^i_\alpha \circ h \rho\Gamma^{a_1}_{a\beta}\frac{\partial u^a}{\partial x^i} \\ &\quad + \rho^j_\beta \circ h\rho\Gamma^{a_1}_{b\alpha}\frac{\partial u^b}{\partial x^j} + \rho\Gamma^{a_1}_{b\alpha}\rho\Gamma^b_{a\beta} u^a, \end{aligned}$$

it results that

$$\begin{aligned} u^{a_1}_{|\alpha\beta} - u^{a_1}_{|\beta\alpha} &= \rho^j_\beta \circ h \frac{\partial \rho^i_\alpha \circ h}{\partial x^j}\frac{\partial u^{a_1}}{\partial x^i} - \rho^i_\alpha \circ h \frac{\partial \rho^j_\beta \circ h}{\partial x^i}\frac{\partial u^{a_1}}{\partial x^j} \\ &\quad + \bigg(\rho^j_\beta \circ h\rho^i_\alpha \circ h \frac{\partial^2 u^{a_1}}{\partial x^i \partial x^j} - \rho^j_\beta \circ h\rho^i_\alpha \circ h \frac{\partial^2 u^{a_1}}{\partial x^j \partial x^i}\bigg) \\ &\quad + \bigg(\rho^j_\beta \circ h \frac{\partial \rho\Gamma^{a_1}_{a\alpha}}{\partial x^j} u^a - \rho^i_\alpha \circ h \frac{\partial \rho\Gamma^{a_1}_{a\beta}}{\partial x^i} u^a\bigg) \\ &\quad + \bigg(\rho^j_\beta \circ h\rho\Gamma^{a_1}_{a\alpha}\frac{\partial u^a}{\partial x^j} - \rho^j_\beta \circ h\rho\Gamma^{a_1}_{b\alpha}\frac{\partial u^b}{\partial x^j}\bigg) \\ &\quad + \bigg(\rho^i_\alpha \circ h\rho\Gamma^{a_1}_{b\beta}\frac{\partial u^b}{\partial x^i} - \rho^i_\alpha \circ h\rho\Gamma^{a_1}_{a\beta}\frac{\partial u^a}{\partial x^i}\bigg) \\ &\quad + \rho\Gamma^{a_1}_{b\beta}\rho\Gamma^b_{a\alpha} u^a - \rho\Gamma^{a_1}_{b\alpha}\rho\Gamma^b_{a\beta} u^a. \end{aligned}$$

After some calculations, we obtain:

$$u^{a_1}_{|\alpha\beta} - u^{a_1}_{|\beta\alpha} = L^\gamma_{\beta\alpha} \circ h\rho^k_\gamma \circ h \frac{\partial u^{a_1}}{\partial x^k} + \bigg(\rho^j_\beta \circ h \frac{\partial \rho\Gamma^{a_1}_{a\alpha}}{\partial x^j} u^a - \rho^i_\alpha \circ h \frac{\partial \rho\Gamma^{a_1}_{a\beta}}{\partial x^i} u^a\bigg) + \rho\Gamma^{a_1}_{b\beta}\rho\Gamma^b_{a\alpha} u^a - \rho\Gamma^{a_1}_{b\alpha}\rho\Gamma^b_{a\beta} u^a.$$

Since

$$u^a(\rho,h)\mathbb{R}^{a_1}_{a\alpha\beta} = u^a\left(\rho^j_\beta\circ h\frac{\partial\rho\Gamma^{a_1}_{a\alpha}}{\partial x^j}+\rho\Gamma^{a_1}_{e\beta}\rho\Gamma^e_{a\alpha}-\rho^i_\alpha\circ h\frac{\partial\rho\Gamma^{a_1}_{a\beta}}{\partial x^i}-\rho\Gamma^{a_1}_{e\alpha}\rho\Gamma^e_{a\beta}-\rho\Gamma^{a_1}_{a\gamma}L^\gamma_{\beta\alpha}\circ h\right).$$

and

$$u^{a_1}_{|\gamma}L^\gamma_{\alpha\beta}\circ h=\left(\rho^k_\gamma\circ h\frac{\partial u^{a_1}}{\partial x^k}+\rho\Gamma^{a_1}_{a\gamma}u^a\right)L^\gamma_{\alpha\beta}\circ h$$

it results that

$$\begin{aligned}&u^a(\rho,h)\mathbb{R}^{a_1}_{a\alpha\beta}-u^{a_1}_{|\gamma}L^\gamma_{\alpha\beta}\circ h\\&=-L^\gamma_{\alpha\beta}\circ h\rho^k_\gamma\circ h\frac{\partial u^{a_1}}{\partial x^k}+\left(\rho^j_\beta\circ h\frac{\partial\rho\Gamma^{a_1}_{a\alpha}}{\partial x^j}u^a-\rho^i_\alpha\circ h\frac{\partial\rho\Gamma^{a_1}_{a\beta}}{\partial x^i}u^a\right)+\rho\Gamma^{a_1}_{b\beta}\rho\Gamma^b_{a\alpha}u^a-\rho\Gamma^{a_1}_{b\alpha}\rho\Gamma^b_{a\beta}u^a.\end{aligned}$$ □

Lemma 42. *If $(E,\pi,M)=(F,\nu,N)$, then, for any $u^as_a\in\Gamma(E,\pi,M)$, we have that $u^a_{|c}$, $a,c\in\overline{1,n}$ are the components of a tensor field of $(1,1)$ type.*

Proof. Let U and U' be two vector local $(m+n)$ charts such that $U\cap U'\neq\phi$.

Since $u^{a'}(x)=M^{a'}_a(x)u^a(x)$, for any $x\in U\cap U'$, it results that

$$\rho^{k'}_{c'}\circ h(x)\frac{\partial u^{a'}(x)}{\partial x^{k'}}=\rho^{k'}_{c'}\circ h(x)\frac{\partial}{\partial x^{k'}}(M^{a'}_a(x))u^a(x)+M^{a'}_a(x)\rho^{k'}_{c'}\circ h(x)\frac{\partial u^a(x)}{\partial x^{k'}}. \tag{5.4}$$

Since, for any $x\in U\cap U'$, we have

$$\rho\Gamma^{a'}_{b'c'}(x)=M^{a'}_a(x)(\rho^k_c\circ h(x)\frac{\partial}{\partial x^k}(M^a_{b'}(x))+\rho\Gamma^a_{bc}(x)M^b_{b'}(x))M^c_{c'}(x),$$

and

$$0=\frac{\partial}{\partial x^{k'}}(M^{a'}_a(x)M^a_{b'}(x))=\frac{\partial}{\partial x^{k'}}(M^{a'}_a(x))M^a_{b'}(x)+M^{a'}_a(x)\frac{\partial}{\partial x^{k'}}(M^a_{b'}(x))$$

it results that

$$\rho\Gamma^{a'}_{b'c'}(x)u^{b'}(x)=-\rho^{k'}_{c'}\circ h(x)\frac{\partial}{\partial x^{k'}}(M^{a'}_a(x))u^a(x)+M^{a'}_a(x)\rho\Gamma^a_{bc}(x)u^b(x)M^c_{c'}(x). \tag{5.5}$$

Summing the equalities (5.4) and (5.5), it results the conclusion of lemma. □

Theorem 43. *If $(E,\pi,M)=(F,\nu,N)$, then, for any*

$$u^as_a\in\Gamma(E,\pi,M),$$

we will use the notation

$$u^{a_1}_{|a|b}=u^{a_1}_{|ab}-\rho\Gamma^d_{ab}u^{a_1}_{|d}$$

and we verify the formulas of Ricci type

$$u^{a_1}_{|a|b}-u^{a_1}_{|b|a}+(\rho,h)\mathbb{T}^d_{ab}u^{a_1}_{|d}=u^d(\rho,h)\mathbb{R}^{a_1}_{dab}-u^{a_1}{}_{|c}L^c_{ab}\circ h. \tag{5.6}$$

In the particular case of Lie algebroids, $h=\mathrm{Id}_M$, the relations (5.6) *become:*

$$u^{a_1}_{|a|b}-u^{a_1}_{|b|a}+\rho\mathbb{T}^d_{ab}u^{a_1}_{|d}=u^d\rho\mathbb{R}^{a_1}_{dab}-u^{a_1}_{|c}L^c_{ab} \tag{5.6$'$}$$

In the classical case, $(\rho,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M)$, the relations (5.6′) *become:*

$$u^{i_1}_{|i|j}-u^{i_1}_{|i|j}+\mathbb{T}^k_{ij}u^{i_1}_{|k}=u^k\mathbb{R}^{i_1}_{kij} \tag{5.6$''$}$$

Theorem 44. *For any $u_a s^a \in \Gamma(\overset{*}{E}, \overset{*}{\pi}, M)$ we will use the notation:*

$$u_{b_1|\alpha\beta} = \rho_\beta^j \circ h \frac{\partial}{\partial x^j}(u_{b_1|\alpha}) - \rho\Gamma^b_{b_1\beta} u_{b|\alpha}$$

and we verify the equality:

$$u_{b_1|\alpha\beta} - u_{b_1|\beta\alpha} = -u_b(\rho,h)\mathbb{R}^b_{b_1\alpha\beta} - u_{b_1|\gamma} L^\gamma_{\alpha\beta} \circ h.$$

After some calculations, we obtain:

$$(\rho,h)\mathbb{R}^b_{b_1\alpha\beta} = u^b(-u_{b_1|\alpha\beta} + u_{b_1|\beta\alpha} - u_{b_1|\gamma} L^\gamma_{\alpha\beta} \circ h), \tag{5.7}$$

where $u^a s_a \in \Gamma(E,\pi,M)$ such that $u_a u^b = \delta^b_a$.

In the particular case of Lie algebroids, $h = \mathrm{Id}_M$, the relations (5.7) *become:*

$$\rho\mathbb{R}^b_{b_1\alpha\beta} = u^b(-u_{b_1|\alpha\beta} + u_{b_1|\beta\alpha} - u_{b_1|\gamma} L^\gamma_{\alpha\beta}). \tag{5.7$'$}$$

In the classical case, $(\rho,h) = (\mathrm{Id}_{TM}, \mathrm{Id}_M)$, the relations (5.7′) *become:*

$$\mathbb{R}^b_{b_1\,ij} = u^b(-u_{b_1|ij} + u_{b_1|ji}). \tag{5.7$''$}$$

Proof. Since

$$\begin{aligned}
u_{b_1|\alpha\beta} &= \rho_\beta^j \circ h \left(\frac{\partial}{\partial x^j} \left(\rho^i_\alpha \circ h \frac{\partial u_{b_1}}{\partial x^i} - \rho\Gamma^b_{b_1\alpha} u_b \right) \right) - \rho\Gamma^b_{b_1\beta} \left(\rho^i_\alpha \circ h \frac{\partial u_b}{\partial x^i} - \rho\Gamma^a_{b\alpha} u_a \right) \\
&= \rho^j_\beta \circ h \frac{\partial \rho^i_\alpha \circ h}{\partial x^j} \frac{\partial u_{b_1}}{\partial x^i} + \rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial}{\partial x^j} \left(\frac{\partial u_{b_1}}{\partial x^i} \right) - \rho^j_\beta \circ h \frac{\partial \rho\Gamma^b_{b_1\alpha}}{\partial x^j} u_b - \rho^j_\beta \circ h \rho\Gamma^b_{b_1\alpha} \frac{\partial u_b}{\partial x^j} \\
&\quad - \rho^i_\alpha \circ h \rho\Gamma^b_{b_1\beta} \frac{\partial u_b}{\partial x^i} + \rho\Gamma^b_{b_1\beta} \rho\Gamma^a_{b\alpha} u_a
\end{aligned}$$

and

$$\begin{aligned}
u_{b_1|\beta\alpha} &= \rho_\alpha^i \circ h \left(\frac{\partial}{\partial x^i} \left(\rho^j_\beta \circ h \frac{\partial u_{b_1}}{\partial x^j} - \rho\Gamma^b_{b_1\beta} u_b \right) \right) - \rho\Gamma^b_{b_1\alpha} \left(\rho^j_\beta \circ h \frac{\partial u_b}{\partial x^j} - \rho\Gamma^a_{b\beta} u_a \right) \\
&= \rho^i_\alpha \circ h \frac{\partial \rho^j_\beta \circ h}{\partial x^i} \frac{\partial u_{b_1}}{\partial x^i} + \rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial}{\partial x^i} \left(\frac{\partial u_{b_1}}{\partial x^j} \right) - \rho^i_\alpha \circ h \frac{\partial \rho\Gamma^b_{b_1\beta}}{\partial x^i} u_b - \rho^i_\alpha \circ h \rho\Gamma^b_{b_1\beta} \frac{\partial u_b}{\partial x^i} \\
&\quad - \rho^j_\beta \circ h \rho\Gamma^b_{b_1\alpha} \frac{\partial u_b}{\partial x^i} + \rho\Gamma^b_{b_1\alpha} \rho\Gamma^a_{b\beta} u_a
\end{aligned}$$

it results that

$$\begin{aligned}
u_{b_1|\alpha\beta} - u_{b_1|\beta\alpha} &= \rho^j_\beta \circ h \frac{\partial \rho^i_\alpha \circ h}{\partial x^j} \frac{\partial u_{b_1}}{\partial x^i} - \rho^i_\alpha \circ h \frac{\partial \rho^j_\beta \circ h}{\partial x^i} \frac{\partial u_{b_1}}{\partial x^j} + \rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial}{\partial x^j} \left(\frac{\partial u_{b_1}}{\partial x^i} \right) \\
&\quad - \rho^j_\beta \circ h \rho^i_\alpha \circ h \frac{\partial}{\partial x^i} \left(\frac{\partial u_{b_1}}{\partial x^j} \right) + \rho^i_\alpha \circ h \frac{\partial \rho\Gamma^b_{b_1\beta}}{\partial x^i} u_b - \rho^j_\beta \circ h \frac{\partial \rho\Gamma^b_{b_1\alpha}}{\partial x^j} u_b + \rho^j_\beta \circ h \rho\Gamma^b_{b_1\alpha} \frac{\partial u_b}{\partial x^j} \\
&\quad - \rho^j_\beta \circ h \rho\Gamma^b_{b_1\alpha} \frac{\partial u_b}{\partial x^j} + \rho^i_\alpha \circ h \rho\Gamma^b_{b_1\alpha} \frac{\partial u_b}{\partial x^i} - \rho^i_\alpha \circ h \rho\Gamma^b_{b_1\alpha} \frac{\partial u_b}{\partial x^i} + \rho\Gamma^b_{b_1\beta} \rho\Gamma^a_{b\alpha} u_a - \rho\Gamma^b_{b_1\alpha} \rho\Gamma^a_{b\beta} u_a.
\end{aligned}$$

After some calculations, we obtain:

$$u_{b_1|\alpha\beta} - u_{b_1|\beta\alpha} = L^\gamma_{\beta\alpha} \circ h \rho^k_\gamma \circ h \frac{\partial u_{b_1}}{\partial x^k} + \left(\rho^i_\alpha \circ h \frac{\partial \rho\Gamma^b_{b_1\beta}}{\partial x^i} u_b - \rho^j_\beta \circ h \frac{\partial \rho\Gamma^b_{b_1\alpha}}{\partial x^j} u_b \right) + \rho\Gamma^b_{b_1\beta} \rho\Gamma^a_{b\alpha} u_a - \rho\Gamma^b_{b_1\alpha} \rho\Gamma^a_{b\beta} u_a.$$

Since

$$u_b(\rho,h)\mathbb{R}^b_{b_1\alpha\beta} = u_b \left(\rho^j_\beta \circ h \frac{\partial \rho\Gamma^b_{b_1\alpha}}{\partial x^j} + \rho\Gamma^b_{e\beta} \rho\Gamma^e_{b_1\alpha} - \rho^i_\alpha \circ h \frac{\partial \rho\Gamma^b_{b_1\beta}}{\partial x^i} - \rho\Gamma^b_{e\alpha} \rho\Gamma^e_{b_1\beta} - \rho\Gamma^b_{b_1\gamma} L^\gamma_{\beta\alpha} \circ h \right)$$

and

$$u_{b_1|\gamma}L^{\gamma}_{\alpha\beta}\circ h=\left(\rho^k_\gamma\circ h\frac{\partial u_{b_1}}{\partial x^k}-\rho\Gamma^b_{b_1\gamma}u_b\right)L^{\gamma}_{\alpha\beta}\circ h$$

it results that

$$\begin{aligned}-u_b(\rho,h)\mathbb{R}^b_{b_1,\alpha\beta}-u_{b_1|\gamma}L^{\gamma}_{\alpha\beta}\circ h=&-L^{\gamma}_{\alpha\beta}\circ h\rho^k_\gamma\circ h\frac{\partial u_{b_1}}{\partial x^k}+\left(\rho^i_\alpha\circ h\frac{\partial\rho\Gamma^b_{b_1\beta}}{\partial x^i}u_b-\rho^j_\beta\circ h\frac{\partial\rho\Gamma^b_{b_1\alpha}}{\partial x^j}u_b\right)\\&+\rho\Gamma^b_{b_1\beta}\rho\Gamma^a_{b\alpha}u_a-\rho\Gamma^b_{b_1\alpha}\rho\Gamma^a_{b\beta}u_a.\end{aligned}$$

□

Lemma 45. *If $(E,\pi,M)=(F,\nu,N)$, then, for any*

$$u_b s^b\in\Gamma(\overset{*}{E},\overset{*}{\pi},M),$$

we have that $u_{b|c}$, $b,c\in\overline{1,n}$ are the components of a tensor field of $(0,2)$ type.

Proof. Let U and U' be two vector local $(m+n)$ charts such that $U\cap U'\neq\phi$.
Since $u_{b'}(x)=M^b_{b'}(x)u_b(x)$, for any $x\in U\cap U'$, it results that

$$\rho^{k'}_{c'}\circ h(x)\frac{\partial u_{b'}(x)}{\partial x^{k'}}=\rho^{k'}_{c'}\circ h(x)\frac{\partial}{\partial x^{k'}}(M^b_{b'}(x))u_b(x)+M^b_{b'}(x)\rho^{k'}_{c'}\circ h(x)\frac{\partial u_b(x)}{\partial x^{k'}}. \tag{5.8}$$

Since, for any $x\in U\cap U'$, we have

$$\rho\Gamma^{a'}_{b'c'}(x)=M^{a'}_a(x)\left(\rho^k_c\circ h(x)\frac{\partial}{\partial x^k}(M^a_{b'}(x))+\rho\Gamma^a_{bc}(x)M^b_{b'}(x)\right)M^c_{c'}(x),$$

and

$$0=\frac{\partial}{\partial x^{k'}}(M^{a'}_a(x)M^a_{b'}(x))=\frac{\partial}{\partial x^{k'}}(M^{a'}_a(x))M^a_{b'}(x)+M^{a'}_a(x)\frac{\partial}{\partial x^{k'}}(M^a_{b'}(x))$$

it results that

$$\rho\Gamma^{a'}_{b'c'}(x)u_{a'}(x)=-\rho^{k'}_{c'}\circ h(x)\frac{\partial}{\partial x^{k'}}(M^b_{b'}(x))u_b(x)+M^b_{b'}(x)\rho\Gamma^a_{bc}(x)u_a(x)M^c_{c'}(x). \tag{5.9}$$

Summing the equalities (5.8) and (5.9), it results the conclusion of lemma. □

Theorem 46. *If $(E,\pi,M)=(F,\nu,N)$, then, for any*

$$u_b s^b\in\Gamma(\overset{*}{E},\overset{*}{\pi},M),$$

we will use the notation

$$u_{b_1|a|b}=u_{b_1|ab}-\rho\Gamma^d_{ab}u_{b_1|d}$$

and we verify the formulas of Ricci type

$$u_{b_1|a|b}-u_{b_1|b|a}+(\rho,h)\mathbb{T}^d_{ab}u_{b_1|d}=-u_d(\rho,h)\mathbb{R}^d_{b_1\,ab}-u_{b_1|d}L^d_{ab}\circ h. \tag{5.10}$$

In the particular case of Lie algebroids, $h=\mathrm{Id}_M$, the relations (5.10) *become:*

$$u_{b_1\,|a|b}-u_{b_1\,|b|a}+\rho\mathbb{T}^d_{ab}u_{b_1|d}=-u_d\rho\mathbb{R}^d_{b_1\,ab}-u_{b_1|d}L^d_{ab}. \tag{5.10$'$}$$

In the classical case, $(\rho,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M)$, the relations (5.10$'$) *become:*

$$u_{j_1|i|j}-u_{j_1|j|i}+\mathbb{T}^h_{ij}u_{j_1|h}=u_h\mathbb{R}^h_{j_1\,ij}. \tag{5.10$''$}$$

Theorem 47. *For any tensor field:*

$$T^{a_1\dots a_p}_{b_1\dots b_q} s_{a_1} \otimes \cdots \otimes s_{a_p} \otimes s^{b_1} \otimes \cdots \otimes s^{b_q},$$

we verify the equality:

$$\begin{aligned} T^{a_1\dots a_p}_{b_1\dots b_q|\alpha\beta} - T^{a_1\dots a_p}_{b_1\dots b_q|\beta\alpha} = T^{aa_2\dots a_p}_{b_1\dots b_q}(\rho,h)\mathbb{R}^{a_1}_{a\alpha\beta} + \cdots + T^{a_1\dots a_{p-1}a}_{b_1\dots b_q}(\rho,h)\mathbb{R}^{a_p}_{a\alpha\beta} - T^{a_1\dots a_p}_{bb_2\dots b_q}(\rho,h)\mathbb{R}^{b}_{b_1\alpha\beta} - \cdots \\ - T^{a_1\dots a_p}_{b_1\dots b_{q-1}b}(\rho,h)\mathbb{R}^{b}_{b_q\alpha\beta} - T^{a_1\dots a_p}_{b_1\dots b_q|\gamma} L^{\gamma}_{\alpha\beta}\circ h. \end{aligned} \tag{5.11}$$

In the particular case of Lie algebroids, $h = \mathrm{Id}_M$, the relations (5.11) *become:*

$$\begin{aligned} T^{a_1\dots a_p}_{b_1\dots b_q|\alpha\beta} - T^{a_1\dots a_p}_{b_1\dots b_q|\beta\alpha} = T^{aa_2\dots a_p}_{b_1\dots b_q}\rho\mathbb{R}^{a_1}_{a\alpha\beta} + \cdots + T^{a_1\dots a_{p-1}a}_{b_1\dots b_q}\rho\mathbb{R}^{a_p}_{a\alpha\beta} - T^{a_1\dots a_p}_{bb_2\dots b_q}\rho\mathbb{R}^{b}_{b_1\,\alpha\beta} - \cdots \\ - T^{a_1\dots a_p}_{b_1\dots b_{q-1}b}\rho\mathbb{R}^{b}_{b_q\,\alpha\beta} - T^{a_1\dots a_p}_{b_1\dots b_q|\gamma} L^{\gamma}_{\alpha\beta}. \end{aligned} \tag{5.11$'$}$$

In the classical case, $(\rho,h) = (\mathrm{Id}_{TM}, \mathrm{Id}_M)$, the relations (5.11$'$) *become:*

$$T^{i_1\dots i_p}_{j_1\dots j_q|hk} - T^{i_1\dots i_p}_{j_1\dots j_q|kh} = T^{ii_2\dots i_p}_{j_1\dots j_q}\mathbb{R}^{i_1}_{i\,hk} + \cdots + T^{i_1\dots i_{p-1}i}_{j_1\dots j_q}\mathbb{R}^{i_p}_{i\,hk} - T^{i_1\dots i_p}_{jj_2\dots j_q}\mathbb{R}^{j}_{j_1\,hk} - \cdots - T^{i_1\dots i_p}_{j_1\dots j_{q-1}j}\mathbb{R}^{j}_{j_q\,hk}. \tag{5.11$''$}$$

Theorem 48. *If $(E,\pi,M) = (F,\nu,N)$, then we obtain the following formulas of Ricci type:*

$$\begin{aligned} & T^{a_1\dots a_p}_{b_1\dots b_q|b|c} - T^{a_1\dots a_p}_{b_1\dots b_q|c|b} + (\rho,h)\mathbb{T}^{d}_{bc} T^{a_1\dots a_p}_{b_1\dots b_q|d} \\ &\quad = T^{aa_2\dots a_p}_{b_1\dots b_q}(\rho,h)\mathbb{R}^{a_1}_{a\,bc} + \cdots + T^{a_1\dots a_{p-1}a}_{b_1\dots b_q}(\rho,h)\mathbb{R}^{a_p}_{a\,bc} - T^{a_1\dots a_p}_{bb_2\dots b_q}(\rho,h)\mathbb{R}^{b}_{b_1\,bc} - \cdots \\ &\quad - T^{a_1\dots a_p}_{b_1\dots b_{q-1}b}(\rho,h)\mathbb{R}^{b}_{b_q\,bc} - T^{a_1\dots a_p}_{b_1\dots b_q|d} L^{d}_{bc}\circ h. \end{aligned} \tag{5.12}$$

In the particular case of Lie algebroids, $h = \mathrm{Id}_M$, the relations (5.12) *become:*

$$\begin{aligned} & T^{a_1\dots a_p}_{b_1\dots b_q|b|c} - T^{a_1\dots a_p}_{b_1\dots b_q|c|b} + \rho\mathbb{T}^{d}_{bc} T^{a_1\dots a_p}_{b_1\dots b_q|d} \\ &\quad = T^{aa_2\dots a_p}_{b_1\dots b_q}\rho\mathbb{R}^{a_1}_{a\,bc} + \cdots + T^{a_1\dots a_{p-1}a}_{b_1\dots b_q}\rho\mathbb{R}^{a_p}_{a\,bc} - T^{a_1\dots a_p}_{bb_2\dots b_q}\rho\mathbb{R}^{b}_{b_1\,bc} - \cdots \\ &\quad - T^{a_1\dots a_p}_{b_1\dots b_{q-1}b}\rho\mathbb{R}^{b}_{b_q\,bc} - T^{a_1\dots a_p}_{b_1\dots b_q|d} L^{d}_{bc}. \end{aligned} \tag{5.12$'$}$$

In the classical case, $(\rho,h) = (\mathrm{Id}_{TM}, \mathrm{Id}_M)$, the relations (5.12$'$) *become:*

$$\begin{aligned} & T^{i_1\dots i_p}_{j_1\dots j_q|h|k} - T^{i_1\dots i_p}_{j_1\dots j_q|k|h} + \mathbb{T}^{m}_{hk} T^{i_1\dots i_p}_{j_1\dots j_q|m} \\ &\quad = T^{ii_2\dots i_p}_{j_1\dots j_q}\mathbb{R}^{i_1}_{i\,hk} + \cdots + T^{i_1\dots i_{p-1}i}_{j_1\dots j_q}\mathbb{R}^{i_p}_{i\,hk} - T^{i_1\dots i_p}_{jj_2\dots j_q}\mathbb{R}^{j}_{j_1\,hk} - \cdots - T^{i_1\dots i_p}_{j_1\dots j_{q-1}j}\mathbb{R}^{j}_{j_q\,hk}. \end{aligned} \tag{5.12$''$}$$

We observe that if the structure functions of generalized Lie algebroid:

$$((F,\nu,M),[,]_{F,h},(\rho,\mathrm{Id}_M)),$$

the (ρ,h)-torsion associated to linear ρ-connection $\rho\ddot{\Gamma}$, and the (ρ,h)-curvature associated to linear ρ-connection $\rho\dot{\Gamma}$ are null, then we have the equality:

$$T^{a_1\dots a_p}_{b_1\dots b_q|b|c} = T^{a_1\dots a_p}_{b_1\dots b_q|c|b},$$

which generalizes the Schwartz equality.

Theorem 49. *If $(E,\pi,M) = (F,\nu,N)$, then the following relations hold good:*

$$\sum_{\mathrm{cyclic}(U_1,U_2,U_3)} \left\{(\rho\ddot{D}_{U_1}(\rho,h)\mathbb{T})(U_2,U_3) - (\rho,h)\mathbb{R}(U_1,U_2)U_3 + (\rho,h)\mathbb{T}((\rho,h)\mathbb{T}(U_1,U_2),U_3)\right\} = 0, \tag{$\tilde{B}_1$}$$

and

$$\sum_{\mathrm{cyclic}(U_1,U_2,U_3,U)} \left\{(\rho\ddot{D}_{U_1}(\rho,h)\mathbb{R})(U_2,U_3)U + (\rho,h)\mathbb{R}((\rho,h)\mathbb{T}(U_1,U_2),U_3)U\right\} = 0. \tag{$\tilde{B}_2$}$$

respectively. This identities will be called the first and the second identity of Bianchi type, respectively.

In the particular case of Lie algebroids, $h = \mathrm{Id}_M$*, the identities* $(\tilde{B}_1)$ *and* $(\tilde{B}_2)$ *become:*

$$\sum_{\mathrm{cyclic}(u_1,u_2,u_3)} \left\{ (\rho D_{u_1}\rho\mathbb{T})(u_2,u_3) - \rho\mathbb{R}(u_1,u_2)u_3 + \rho\mathbb{T}(\rho\mathbb{T}(u_1,u_2),u_3) \right\} = 0, \qquad (\tilde{B}_1{}')$$

$$\sum_{\mathrm{cyclic}(u_1,u_2,u_3,u)} \left\{ (\rho D_{u_1}\rho\mathbb{R})(u_2,u_3)u + \rho\mathbb{R}(\rho\mathbb{T}(u_1,u_2),u_3)u \right\} = 0. \qquad (\tilde{B}_2{}')$$

In the classical case, $(\rho,h) = (\mathrm{Id}_{TM},\mathrm{Id}_M)$*, the identities* $(\tilde{B}_1{}')$ *and* $(\tilde{B}_2{}')$ *become:*

$$\sum_{\mathrm{cyclic}(X_1,X_2,X_3)} \left\{ (D_{X_1}\mathbb{T})(X_2,X_3) - \mathbb{R}(X_1,X_2)X_3 + \mathbb{T}(\,\mathbb{T}(X_1,X_2),X_3) \right\} = 0, \qquad (\tilde{B}_1{}'')$$

$$\sum_{\mathrm{cyclic}(X_1,X_2,X_3,X)} \left\{ (D_{X_1}\mathbb{R})(X_2,X_3)X + \mathbb{R}(\mathbb{T}(X_1,X_2),X_3)X \right\} = 0. \qquad (\tilde{B}_2{}'')$$

Proof. Using the equality:

$$(\rho\ddot{D}_{U_1}(\rho,h)\mathbb{T})(U_2,U_3) = \rho\ddot{D}_{U_1}((\rho,h)\mathbb{T}(U_2,U_3)) - (\rho,h)\mathbb{T}(\rho\ddot{D}_{U_1}U_2,U_3) - (\rho,h)\mathbb{T}(U_2,\rho\ddot{D}_{U_1}U_3)$$

and the Jacobi identity, we obtain the first identity of Bianchi type.

Using the equality:

$$\begin{aligned}(\rho\ddot{D}_{U_1}(\rho,h)\mathbb{R})(U_2,U_3)U &= \rho\ddot{D}_{U_1}((\rho,h)\mathbb{R}(U_2,U_3)U) - (\rho,h)\mathbb{R}(\rho\ddot{D}_{U_1}U_2,U_3)U \\ &\quad - (\rho,h)\mathbb{R}(U_2,\rho\ddot{D}_{U_1}U_3)U - (\rho,h)\mathbb{R}(U_2,U_3)\rho\ddot{D}_{U_1}U\end{aligned}$$

and the Jacobi identity, we obtain the second identity of Bianchi type. □

Remark 50. On components, the identities of Bianchi type become:

$$\sum_{\mathrm{cyclic}(a_1,a_2,a_3)} \left\{ (\rho,h)\mathbb{T}^b_{a_2a_3|a_1} + (\rho,h)\mathbb{T}^b_{ga_3}\cdot(\rho,h)\mathbb{T}^g_{a_1a_2} \right\} = \sum_{\mathrm{cyclic}(a_1,a_2,a_3)} (\rho,h)\mathbb{R}^a_{a_3a_1a_2}$$

and

$$\sum_{\mathrm{cyclic}(a,a_1,a_2,a_3)} \left\{ (\rho,h)\mathbb{R}^b_{a\,a_2a_3|a_1} + (\rho,h)\mathbb{R}^b_{a\,ga_3}\cdot(\rho,h)\mathbb{T}^g_{a_2a_1} \right\} = 0.$$

If the (ρ,h) torsion is null, then the identities of Bianchi type become:

$$\sum_{\mathrm{cyclic}(a_1a_2,a_3)} (\rho,h)\mathbb{R}^b_{a_3a_1a_2} = 0$$

and

$$\sum_{\mathrm{cyclic}(a,a_1,a_2,a_3)} (\rho,h)\mathbb{R}^b_{a\,a_2a_3|a_1} = 0.$$

6 (Pseudo)metrizable vector bundles. Formulas of Levi-Civita type

We will apply our theory for the diagram:

$$\begin{array}{ccc} E & & (F,[\,,]_{F,h},(\rho,\mathrm{Id}_N)) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow{h} & N, \end{array}$$

where $(E,\pi,M)\in|\mathbf{B^v}|$ and $((F,\nu,N),[\,,]_{F,h},(\rho,\mathrm{Id}_N))\in|\mathbf{GLA}|$.

Definition 51. We will say that the *vector bundle* (E,π,M) *is endowed with a pseudometrical structure* if it exists $g = g_{ab}s^a \otimes s^b \in \mathcal{T}_2^0(E,\pi,M)$ such that for each $x \in M$, the matrix $\|g_{ab}(x)\|$ is nondegenerate and symmetric.

Moreover, if for each $x \in M$ the matrix $\|g_{ab}(x)\|$ has constant signature, then we will say that *the vector bundle* (E,π,M) *is endowed with a metrical structure*.

If $g = g_{ab}s^a \otimes s^b \in \mathcal{T}_2^0(E,\pi,M)$ is a (pseudo)metrical structure, then, for any $a,b \in \overline{1,r}$ and for any vector local $(m+r)$-chart (U,s_U) of (E,π,M), we consider the real functions

$$U \xrightarrow{\tilde{g}^{ba}} \mathbb{R}$$

such that $\|\tilde{g}^{ba}(x)\| = \|g_{ab}(x)\|^{-1}$, for any $\forall x \in U$.

Definition 52. We admit that (E,π,M) is a vector bundle endowed with a (pseudo)metrical structure g and with a linear ρ-connection $\rho\Gamma$.

We will say that the *linear ρ-connection* $\rho\Gamma$ is compatible with the (pseudo)metrical structure g if:

$$\rho D_z g = 0, \quad \forall z \in \Gamma(F,\nu,N).$$

Definition 53. We will say that the vector bundle (E,π,M) is *ρ-(pseudo)metrizable*, if it exists a (pseudo)metrical structure $g \in \mathcal{T}_2^0(E,\pi,M)$ and a linear ρ-connection $\rho\Gamma$ for (E,π,M) compatible with g. The Id_{TM}-(pseudo)metrizable vector bundles will be called *(pseudo)metrizable vector bundles*.

In particular, if (TM,τ_M,M) is a (pseudo)metrizable vector bundle, then we will say that (TM,τ_M,M) is a *(pseudo)Riemannian space*, and the manifold M will be called *(pseudo)Riemannian manifold.*

The linear connection of a (pseudo)Riemannian space will be called *(pseudo)Riemannian linear connection.*

Theorem 54. *If $(E,\pi,M) = (F,\nu,N)$ and $g \in \mathcal{T}_2^0(h^*E,h^*\pi,M)$ is a (pseudo)metrical structure, then the local real functions:*

$$\rho\Gamma^a_{bc} = \frac{1}{2}\tilde{g}^{ad}\Big(\rho^k_c \circ h\frac{\partial g_{bd}}{\partial x^k} + \rho^j_b \circ h\frac{\partial g_{dc}}{\partial x^j} - \rho^l_d \circ h\frac{\partial g_{bc}}{\partial x^l} - (L^e_{bc}\circ h)g_{ed} - (L^e_{bd}\circ h)g_{ec} + (L^e_{dc}\circ h)g_{eb}\Big) \tag{6.1}$$

are the components of a linear ρ-connection $\rho\ddot{\Gamma}$ for the vector bundle $(h^*E,h^*\pi,M)$ such that $(\rho,h)\mathbb{T} = 0$ and the vector bundle $(h^*E,h^*\pi,M)$ becomes ρ-(pseudo)metrizable. This linear ρ-connection $\rho\Gamma$ will be called the linear ρ-connection of Levi-Civita type.

In the particular case of Lie algebroids, $h = \mathrm{Id}_M$, the relations (6.1) *become:*

$$\rho\Gamma^a_{bc} = \frac{1}{2}\tilde{g}^{ad}\left(\rho^k_c\frac{\partial g_{bd}}{\partial x^k} + \rho^j_b\frac{\partial g_{dc}}{\partial x^j} - \rho^l_d\frac{\partial g_{bc}}{\partial x^l} - L^e_{bc}g_{ed} - L^e_{bd}g_{ec} + L^e_{dc}g_{eb}\right). \tag{6.1$'$}$$

In the classical case, $(\rho,h) = (\mathrm{Id}_{TM},\mathrm{Id}_M)$, the relations (6.1$'$) *become:*

$$\Gamma^i_{jk} = \frac{1}{2}\tilde{g}^{ih}\left(\frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{hk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h}\right). \tag{6.1$''$}$$

Proof. Since

$$(\rho\ddot{D}_U g)V\otimes Z = \Gamma\left(\overset{h^*E}{\rho},\mathrm{Id}_M\right)(U)(g(V\otimes Z)) - g((\rho\ddot{D}_U V)\otimes Z) - g(V\otimes(\rho\ddot{D}_U Z)), \quad \forall U,V,Z \in \Gamma(h^*E,h^*\pi,M).$$

it results that, for any $U,V,Z \in \Gamma(h^*E,h^*\pi,M)$, we obtain the equalities:

$$\Gamma(\overset{h^*E}{\rho},\mathrm{Id}_M)(U)(g(V\otimes Z)) = g((\rho\ddot{D}_U V)\otimes Z) + g(V\otimes(\rho\ddot{D}_U Z)), \tag{6.2}$$

$$\Gamma(\overset{h^*E}{\rho},\mathrm{Id}_M)(Z)(g(U\otimes V)) = g((\rho\ddot{D}_Z U)\otimes V) + g(U\otimes(\rho\ddot{D}_Z V)), \tag{6.3}$$

$$\Gamma(\overset{h^*E}{\rho},\mathrm{Id}_M)(V)(g(Z\otimes U)) = g((\rho\ddot{D}_V Z)\otimes U) + g(Z\otimes(\rho\ddot{D}_V U)). \tag{6.4}$$

We observe that $(6.2) + (6.4) - (6.3)$ is equivalent with the equality:

$$\begin{aligned} &g((\rho\ddot{D}_U V + \rho\ddot{D}_V U)\otimes Z) + g((\rho\ddot{D}_V Z - \rho\ddot{D}_Z V)\otimes U) + g((\rho\ddot{D}_U Z - \rho\ddot{D}_Z U)\otimes V) \\ &\quad = \Gamma\left(\overset{h^*E}{\rho},\mathrm{Id}_M\right)(U)(g(V\otimes Z)) + \Gamma\left(\overset{h^*E}{\rho},\mathrm{Id}_M\right)(V)(g(Z\otimes U)) - \Gamma\left(\overset{h^*E}{\rho},\mathrm{Id}_M\right)(Z)(g(U\otimes V)). \end{aligned}$$

Using the condition $(\rho,h)\mathbb{T}=0$, which is equivalent with the equality:

$$\rho\ddot{D}_U V-\rho\ddot{D}_V U-[U,V]_{h^*E}=0,$$

we obtain the equality:

$$\begin{aligned}&2g((\rho\ddot{D}_U V)\otimes Z)+g([V,U]_{h^*E}\otimes Z)+g([V,Z]_{h^*E}\otimes U)+g([U,Z]_{h^*E}\otimes V)\\&=\Gamma\left(\overset{h^*E}{\rho},\mathrm{Id}_M\right)(U)(g(V\otimes Z))+\Gamma\left(\overset{h^*E}{\rho},\mathrm{Id}_M\right)(V)(g(Z\otimes U))\\&-\Gamma\left(\overset{h^*E}{\rho},\mathrm{Id}_M\right)(Z)(g(U\otimes V)),\quad \forall U,V,Z\in\Gamma(h^*E,h^*\pi,M).\end{aligned}$$

This equality is equivalent with the following equality:

$$\begin{aligned}2g((\rho\ddot{D}_U V)\otimes Z)&=\Gamma\left(\overset{h^*E}{\rho},\mathrm{Id}_M\right)(U)\cdot(g(V\otimes Z))+\Gamma\left(\overset{h^*E}{\rho},\mathrm{Id}_M\right)(V)(g(Z\otimes U))\\&-\Gamma\left(\overset{h^*E}{\rho},\mathrm{Id}_M\right)(Z)(g(U\otimes V))+g([U,V]_{h^*E}\otimes Z)-g([V,Z]_{h^*E}\otimes U)+g([Z,U]_{h^*E}\otimes V)\end{aligned}$$

for any $U,V,Z\in\Gamma(h^*E,h^*\pi,M)$.

If $U=S_c, V=S_b$ and $Z=S_d$, then we obtain the equality:

$$\begin{aligned}2g((\rho\Gamma^a_{bc}S_a)\otimes S_d)&=\rho^k_c\circ h\frac{\partial g(S_b\otimes S_d)}{\partial xk}+\rho^j_b\circ h\frac{\partial g(S_d\otimes S_c)}{\partial x^j}-\rho^l_d\circ h\frac{\partial g(S_b\otimes S_c)}{\partial x^l}+g((L^e_{cb}\circ h)S_e\otimes S_d)\\&-g((L^e_{bd}\circ h)S_e\otimes S_c)+g((L^e_{dc}\circ h)S_e\otimes S_b),\end{aligned}$$

which is equivalent with:

$$2g_{da}\rho\Gamma^a_{bc}=\rho^k_c\circ h\frac{\partial g_{bd}}{\partial xk}+\rho^j_b\circ h\frac{\partial g_{dc}}{\partial x^j}-\rho^l_d\circ h\frac{\partial g_{bc}}{\partial x^l}-(L^e_{bc}\circ h)g_{ed}-(L^e_{bd}\circ h)g_{ec}+(L^e_{dc}\circ h)g_{eb}$$

Finally, we obtain:

$$\rho\Gamma^a_{bc}=\frac{1}{2}\tilde{g}^{ad}\left(\rho^k_c\circ h\frac{\partial g_{bd}}{\partial xk}+\ \rho^j_b\circ h\frac{\partial g_{dc}}{\partial x^j}-\rho^l_d\circ h\frac{\partial g_{bc}}{\partial x^l}-(L^e_{bc}\circ h)g_{ed}-(L^e_{bd}\circ h)g_{ec}+(L^e_{dc}\circ h)g_{eb}\right),$$

where $\|\tilde{g}^{ad}(x)\|=\|g_{da}(x)\|^{-1}$, for any $x\in M$. □

Theorem 55. *If $(E,\pi,M)=(F,\nu,N)$, $g\in\mathcal{T}^0_2(h^*E,h^*\pi,M)$ is a (pseudo)metrical structure and $\mathbb{T}\in\mathcal{T}^1_2(h^*E,h^*\pi,M)$ such that its components are skew symmetric in the lover indices, then the local real functions*

$$\rho\mathring{\Gamma}^a_{bc}=\rho\Gamma^a_{bc}+\frac{1}{2}\tilde{g}^{ad}(g_{de}\mathbb{T}^e_{bc}-g_{be}\mathbb{T}^e_{dc}+g_{ec}\mathbb{T}^e_{bd})\tag{6.5}$$

are the components of a linear ρ-connection compatible with the (pseudo)metrical structure g, where $\rho\Gamma^a_{bc}$ are the components of linear ρ-connection of Levi-Civita type (6.1)*. Therefore, the vector bundle $(h^*E,h^*\pi,M)$ becomes ρ-(pseudo)metrizable and the tensor field $\mathbb{T}$ is the (ρ,h)-torsion tensor field.*

In the particular case of Lie algebroids, $h=\mathrm{Id}_M$, $g\in\mathcal{T}^0_2(E,\pi,M)$ is a (pseudo)metrical structure and $\mathbb{T}\in\mathcal{T}^1_2(E,\pi,M)$ such that its components are skew symmetric in the lover indices, then the local real functions

$$\rho\mathring{\Gamma}^a_{bc}=\rho\Gamma^a_{bc}+\frac{1}{2}\tilde{g}^{ad}(g_{de}\mathbb{T}^e_{bc}-g_{be}\mathbb{T}^e_{dc}+g_{ec}\mathbb{T}^e_{bd})\tag{6.5'}$$

are the components of a linear ρ-connection compatible with the (pseudo)metrical structure g, where $\rho\Gamma^a_{bc}$ are the components of linear ρ-connection of Levi-Civita type $(6.1')$.

In the classical case, $(\rho,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M)$, $g\in\mathcal{T}^0_2(TM,\tau_M,M)$ is a (pseudo)metrical structure and $\mathbb{T}\in\mathcal{T}^1_2(TM,\tau_M,M)$ such that its components are skew symmetric in the lover indices, then the local real functions

$$\mathring{\Gamma}^i_{jk}=\Gamma^i_{jk}+\frac{1}{2}\tilde{g}^{ih}(g_{he}\mathbb{T}^e_{jk}-g_{je}\mathbb{T}^e_{hk}+g_{ek}\mathbb{T}^e_{jh})\tag{6.5''}$$

are the components of a linear connection compatible with the (pseudo)metrical structure g, where Γ^i_{jk} are the components of linear connection of Levi-Civita type $(6.1'')$.

Theorem 56. *If $(E,\pi,M)=(F,\nu,M)$, $g\in\mathcal{T}_2^0(h^*E,h^*\pi,M)$ is a (pseudo)metrical structure and $\rho\mathring{\Gamma}$ is the linear ρ-connection* (6.5) *for the vector bundle $(h^*E,h^*\pi,M)$, then the local real functions*

$$\rho\tilde{\Gamma}^a_{b\alpha}=\rho\mathring{\Gamma}^a_{b\alpha}+\frac{1}{2}\tilde{g}^{ac}g_{cb\overset{\circ}{|}\alpha} \tag{6.6}$$

are the components of a linear ρ-connection such that the vector bundle $(h^*E,h^*\pi,M)$ becomes ρ-(pseudo)metrizable.

In the particular case of Lie algebroids, $h=\mathrm{Id}_M$, $g\in\mathcal{T}_2^0(E,\pi,M)$ is a (pseudo)metrical structure and $\rho\mathring{\Gamma}$ is the linear ρ-connection (6.5') *for the vector bundle (E,π,M), then the local real functions:*

$$\rho\tilde{\Gamma}^a_{b\alpha}=\rho\mathring{\Gamma}^a_{b\alpha}+\frac{1}{2}\tilde{g}^{ac}g_{cb\overset{\circ}{|}\alpha} \tag{6.6'}$$

are the components of a linear ρ-connection such that the vector bundle (E,π,M) becomes ρ-(pseudo)metrizable.

In the classical case, $(\rho,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M)$, $g\in\mathcal{T}_2^0(TM,\tau_M,M)$ is a (pseudo)metrical structure and $\rho\mathring{\Gamma}$ is the linear ρ-connection (6.5") *for the vector bundle (TM,τ_M,M), then the local real functions:*

$$\tilde{\Gamma}^i_{jk}=\mathring{\Gamma}^i_{jk}+\frac{1}{2}\tilde{g}^{ih}g_{hj\overset{\circ}{|}k} \tag{6.6''}$$

are the components of a linear connection such that the vector bundle (TM,τ_M,M) becomes (pseudo)metrizable.

Theorem 57. *If $g\in\mathcal{T}_2^0(h^*E,h^*\pi,M)$ is a (pseudo)metrical structure, $\rho\tilde{\Gamma}$ is the linear ρ-connection* (6.6) *for the vector bundle $(h^*E,h^*\pi,M)$, $T=T^d_{c\alpha}S_d\otimes S^c\otimes t^\alpha$, and $O^{ca}_{bd}=\frac{1}{2}\delta^c_b\delta^a_d-g_{bd}\tilde{g}^{ca}$ is the Obata operator, then the local real functions*

$$\rho\hat{\Gamma}^a_{b\alpha}=\rho\tilde{\Gamma}^a_{b\alpha}+\frac{1}{2}O^{ca}_{bd}T^d_{c\alpha} \tag{6.7}$$

are the components of a linear ρ-connection such that the vector bundle $(h^*E,h^*\pi,M)$ becomes ρ-(pseudo)metrizable.

In the particular case of Lie algebroids, $h=\mathrm{Id}_M$, $g\in\mathcal{T}_2^0(E,\pi,M)$ is a (pseudo)metrical structure, $\rho\tilde{\Gamma}$ is the linear ρ-connection (6.6') *for the vector bundle (E,π,M), $T=T^d_{c\alpha}s_d\otimes s^c\otimes t^\alpha$ and and $O^{ca}_{bd}=\frac{1}{2}\delta^c_b\delta^a_d-g_{bd}\tilde{g}^{ca}$ is the Obata operator, then the local real functions*

$$\rho\hat{\Gamma}^a_{b\alpha}=\rho\tilde{\Gamma}^a_{b\alpha}+\frac{1}{2}O^{ca}_{bd}T^d_{c\alpha} \tag{6.7'}$$

are the components of a linear ρ-connection such that the vector bundle (E,π,M) becomes ρ-(pseudo)metrizable.

In the classical case, $(\rho,h)=(\mathrm{Id}_{TM},\mathrm{Id}_M)$, $g\in\mathcal{T}_2^0(TM,\tau_M,M)$ is a (pseudo)metrical structure, $\tilde{\Gamma}$ is the linear connection (6.6") *for the vector bundle (TM,τ_M,M), $T=T^l_{hk}\partial_l\otimes dx^h\otimes dx^k$ and and $O^{hi}_{jl}=\frac{1}{2}\delta^h_j\delta^i_l-g_{jl}\tilde{g}^{hi}$ is the Obata operator, then the local real functions*

$$\hat{\Gamma}^i_{jk}=\tilde{\Gamma}^i_{jk}+\frac{1}{2}O^{hi}_{jl}T^l_{hk} \tag{6.7''}$$

are the components of a linear connection such that the vector bundle (TM,τ_M,M) becomes (pseudo)metrizable.

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