

ON SHORT TIME EXISTENCE FOR THE PLANAR NETWORK FLOW

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Abstract

We prove the existence of the flow by curvature of regular planar networks starting from an initial network which is non-regular. The proof relies on a monotonicity formula for expanding solutions and a local regularity result for the network flow in the spirit of B. White's local regularity theorem for mean curvature flow. We also show a pseudolocality theorem for mean curvature flow in any codimension, assuming only that the initial submanifold can be locally written as a graph with sufficiently small Lipschitz constant.

1. Introduction

A natural generalization of the flow of smooth hypersurfaces by mean curvature is the flow of surface clusters, where three hypersurfaces can meet under equal angles, forming a liquid edge. These edges then again can meet on lower dimensional strata. The simplest such configuration, which already includes many aspects of the situation in higher dimensions, is the flow by curvature of a network of curves in the plane.

In brief, we consider a planar network to be a finite union of embedded curves of non-zero length, which only intersect at their endpoints. We require that at each such point, called a multiple point, a finite number, but at least two endpoints come together. We call a network *regular* if at each multiple point three ends of segments meet, forming angles of $2\pi/3$. Without this condition, but requiring that the segments have mutually distinct exterior unit tangents at each multiple point, we call such a network *non-regular*. A solution to the planar network flow is a smooth family of regular, planar networks, such that the normal component of the speed under the evolution at every point on each segment is given by the curvature vector of the segment at the point. For a more precise definition see Section 2.

Since the evolution by curvature of a regular network is the gradient flow of the length functional it is natural to assume that at regular times only triple points are present and the angles formed by the segments are

balanced. This is supported by the fact that only the balanced configuration with three segments meeting infinitesimally minimizes length around each multiple point, if one allows as competitors connected networks with additional segments.

After the pioneering work of Brakke [3], whose definition of moving varifolds includes the evolution of networks described here, and the fundamental results on curve shortening flow of embedded closed curves by Gage/Hamilton and Grayson [9, 10], the first thorough analytical treatment of the flow of networks was undertaken by Mantegazza, Novaga and Tortorelli [15]. Aside from establishing short time existence of the network flow starting from a regular initial network, their focus is mainly on the evolution of three arcs with three fixed endpoints, meeting at one interior triple point. In this special setting they obtain long time existence and convergence under certain hypotheses. In a recent preprint by Magni, Mantegazza and Novaga [13] it is shown that these hypotheses are actually fulfilled, provided none of the arcs contracts to zero length. Existence and convergence properties of the network flow in other special configurations have been studied in [2, 11, 16, 22, 21].

It is conjectured that at a singular time of the flow no tangent flow which is a static line of higher multiplicity can develop. An immediate consequence of this conjecture is that at any singular time, the length of one of the segments shrinks to zero, and at least two triple junctions collide. It, thus, can be expected that at the singular time a non-regular network forms.

In the present paper we show that starting from a non-regular initial network, a smooth evolution of regular networks exists. In this evolution it might happen that out of non-regular initial multiple points new segments are created.

Theorem 1.1. *Let γ_0 be a non-regular, connected planar network with bounded curvature as in Definition 2.1. Then there exists $T > 0$ and a smooth connected solution of the planar network flow of regular networks $(\gamma_t)_{0 < t < T}$ such that $\gamma_t \rightarrow \gamma_0$ in the varifold topology as $t \searrow 0$. Away from the non-regular multiple points of γ_0 the convergence is in C_{loc}^∞ . Furthermore, there exists a constant $C > 0$ such that*

$$\sup_{\gamma_t} |k| \leq \frac{C}{\sqrt{t}},$$

and the length of the shortest segment of γ_t is bounded from below by $C^{-1}\sqrt{t}$.

REMARKS 1.2. i) The proof uses a local monotonicity formula, which only works if the network is locally tree-like, i.e., contains locally no loops. In the proof we glue small tree-like self-similarly expanding networks into γ_0 around non-regular multiple points. These tree-like, connected, self-similarly expanding networks always exist. Nevertheless,

there are also non tree-like self-similarly expanding networks, which would correspond to the creation of new bounded regions in the complement of the network out of non-regular multiple points. Our proof would not work gluing in this type of expanders.

ii) The proof of this result presented here yields, only with minor modifications, also the corresponding statement for non-regular initial networks with bounded curvature and fixed endpoints.

iii) Using a relaxation scheme via the Allen–Cahn equation Sáez [20] has shown that regular, smooth solutions starting from a non-regular initial network, which are tree-like, and satisfy the estimates on the curvature and the length of the shortest segment as above, are unique in their topological class. This yields a corresponding uniqueness statement for our constructed solutions in this case.

The method of proof relies on a monotone integral quantity, which implies that self-similarly expanding solutions are attractive under the flow. This monotone integral quantity has already been applied by the second author in the setting of Lagrangian Mean Curvature Flow in several places, see [17, 18, 19]. The second main ingredient is a local regularity result in the spirit of White’s local regularity theorem for smooth mean curvature flow [26].

In the statement of the following local regularity result, Θ_{S^1} is the Gaussian density of the self-similarly shrinking circle. Note that $\Theta_{S^1} = \sqrt{(2\pi/e)} > 3/2$. The quantity $\Theta(x, t, r)$ is the Gaussian density at scale $r > 0$, centered at the point (x, t) . For details and the definition of proper flows, see Section 8.

Theorem 1.3. *Let $(\gamma_t)_{t \in [0, T]}$ be a smooth, proper and regular planar network flow in $B_\rho(x_0) \times (t_0 - \rho^2, t_0)$ which reaches the point x_0 at time $t_0 \in (0, T]$. Let $0 < \varepsilon, \eta < 1$. There exist $C = C(\varepsilon, \eta)$ such that if*

$$(1.1) \quad \Theta(x, t, r) \leq \Theta_{S^1} - \varepsilon$$

for all $(x, t) \in B_\rho(x_0) \times (t_0 - \rho^2, t_0)$ and $0 < r < \eta\rho$ for some $\eta > 0$, where $(1 + \eta)\rho^2 \leq t_0 < T$, then

$$|k|^2(x, t) \leq \frac{C}{\sigma^2 \rho^2}$$

for $(x, t) \in (\gamma_t \cap B_{(1-\sigma)\rho}(x_0)) \times (t_0 - (1 - \sigma)^2 \rho^2, t_0)$ and all $\sigma \in (0, 1)$.

REMARKS 1.4. i) One can, furthermore, show that there is a constant $\kappa = \kappa(\varepsilon, \eta) > 0$ such that the length of each segment which intersects $B_{(1-\sigma)\rho}(x_0) \times (t_0 - (1 - \sigma)^2 \rho^2, t_0)$ is bounded from below by $\kappa \cdot \sigma\rho$. This implies corresponding scaling invariant estimates on all higher derivatives of the curvature.

ii) We also prove a corresponding result if the evolving network is locally tree-like, i.e., does not contain any closed loops of length less than $\delta > 0$

and the Gaussian density ratios are bounded from above by $2 - \varepsilon$.

iii) Note that the result implies that any regular smooth flow, which is sufficiently close in measure to the static configuration consisting of three half-lines meeting under equal angles, is smoothly close. Recently Tonegawa and Wickramasekera [24] have shown that this is also true for integer Brakke flows.

To get sufficiently good local control away from the multiple junctions we also show the following pseudolocality theorem. Since it also holds for mean curvature flow, we formulate it in full generality. A similar estimate assuming initial control on the second fundamental form has been shown by Chen and Yin [5] and assuming control on up to fourth derivatives by Brendle and Huisken [4].

In the following, for any point $x \in \mathbb{R}^{n+k}$ we write $x = (\hat{x}, \tilde{x})$ where \hat{x} is the orthogonal projection of x on the \mathbb{R}^n -factor and \tilde{x} the orthogonal projection on the \mathbb{R}^k factor. We define the cylinder $C_R(x_0) \subset \mathbb{R}^{n+k}$ by

$$C_r(x) = \{x \in \mathbb{R}^{n+k} \mid |\hat{x} - \hat{x}_0| < r, |\tilde{x} - \tilde{x}_0| < r\}.$$

Furthermore, we write $B_r^n(x_0) = \{(\hat{x}, \tilde{x}_0) \in \mathbb{R}^{n+k} \mid |\hat{x} - \hat{x}_0| < r\}$.

Theorem 1.5. *Let $(M_t^n)_{0 \leq t < T}$ be a smooth mean curvature flow of embedded n -dimensional submanifolds in \mathbb{R}^{n+k} with area ratios bounded by D . Then for any $\eta > 0$, there exists $\varepsilon, \delta > 0$, depending only on n, k, η, D , such that if $x_0 \in M_0$ and $M_0 \cap C_1(x_0)$ can be written as $\text{graph}(u)$, where $u : B^n(x_0) \rightarrow \mathbb{R}^k$ with Lipschitz constant less than ε , then*

$$M_t \cap C_\delta(x_0) \quad t \in [0, \delta^2] \cap [0, T)$$

is a graph over $B_\delta^n(x_0)$ with Lipschitz constant less than η and height bounded by $\eta\delta$.

REMARKS 1.6. i) In codimension one the local estimates of Ecker and Huisken [8] yield that a local bound on the second fundamental form or higher derivatives thereof on $M_0 \cap C_\delta(x_0)$ imply a corresponding bound in $M_t \cap C_{\delta/2}(x_0)$ for $t \in [0, \delta^2/4] \cap [0, T)$.

ii) By localizing Huisken's monotonicity formula, see, for example, [7] or [25], the result is still true for local mean curvature flows without an assumption on the area ratios.

iii) The proof of this result uses the local regularity theorem of White [26]. By replacing this with Brakke's local regularity theorem for Brakke flows [3], see also [12, 23], the above statement is still true if one only assumes initially that $(M_t^n)_{0 \leq t < T}$ is an integer Brakke flow, provided that the flow has no sudden mass loss in $C_1(x_0)$.

Proof outline. The cone-like structure at the non-regular multiple points suggests that the regular evolution out of such a point should be close to a self-similarly expanding solution. Given such a non-regular

initial network γ_0 we glue in, around each non-regular multiple point, a tree-like, self-similarly expanding, regular solution at scale $s^{1/2}$ to obtain an approximating network γ_0^s . Since the curvature of γ_0^s is of the scale $s^{-1/2}$ and the shortest segment of length proportional to $s^{1/2}$ we obtain from standard short-time existence a solution γ_t^s only up to a time proportional to s .

To show that these solutions exist for a time $T_0 > 0$ independent of s we use the local monotonicity formula to show that the solutions γ_t^s are close in an integrated sense to a self-similarly expanding solution around each non-regular multiple point. The uniqueness of self-similarly solutions in their ‘topological class’, together with a compactness argument then yields that there are many times such that γ_t^s is close to the corresponding self-similarly expanding solution in $C^{1,\alpha}$ around each of the non-regular multiple points. This in turn gives that the Gaussian density ratios on the appropriate scale are less than $3/2 + \varepsilon$. Theorem 1.3 then gives estimates on the curvature which are independent of s , together with lower bounds on the length of the shortest segment which yields existence up to a time T_0 , independent of s . Passing to the limit $s \rightarrow 0$ we obtain the desired solution.

The proof of the local regularity result, Theorem 1.3, follows the proof of White [26] and the alternative proof Ecker [7]. To make this proof work in the case of networks, in a first part we show that the only self-similarly shrinking networks with Gaussian density less than Θ_{S^1} are a constant line through the origin, or three half-lines meeting at equal angles at the origin. The second part is that we show that any smooth network flow which is weakly close to three half-lines meeting at equal angles is also smoothly close. To do this we localize the interior integral estimates in [15].

Structure of the paper. In Section 2, we give the basic definitions. The monotonicity formula for expanding solutions is presented in Section 3. In Section 4, we give a proof for the uniqueness of self-expanders in their topological class, together with a lemma showing that networks which are in a weak integral sense close to the self-expander are actually $C^{1,\alpha}$ -close. Stating the necessary conditions for an approximating sequence γ_0^s we show the estimates on the Gaussian density ratios in Section 5. In Section 6, we give the omitted proofs of some technical lemmas from Section 5. Following this, we show that we can construct such an approximating sequence by gluing in a self-expander at the right scale into γ_0 in Section 7 and give the proof of Theorem 1.1.

In Section 8, we first localize the higher order integral estimates from [15] and then investigate tangent flows to the network flow and self-similarly shrinking solutions. We complete this section by proving the local regularity result, Theorem 1.3.

In Section 9, we prove the pseudolocality result Theorem 1.5.

We finish the paper with an appendix containing several helpful technical results.

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2. Definitions and set-up

Definition 2.1 (Regular and non-regular network). We define a *regular*, planar network γ as follows.

- i) There is a finite number of points $S = \{a_i\}_{i=1}^n$ on γ such that $\gamma \setminus S$ is a finite union of smooth, embedded curves of positive length (*branches*).
- ii) If σ is a non-compact branch of γ then it approaches a half-line P at infinity, i.e.,

$$\lim_{R \rightarrow \infty} \text{dist}(\sigma \setminus B_R(0), P \setminus B_R(0)) = 0.$$

We will, furthermore, assume that the curvature of such a non-compact branch is uniformly bounded.

- iii) Each point in S , called a *triple point*, is the endpoint of three curves $\{\sigma_j\}_{j=1}^3$ satisfying the following condition: If T_j denotes the exterior unit tangent vector induced by each σ_j , then

$$T_1 + T_2 + T_3 = 0.$$

We call a network *non-regular* if each point in S is an endpoint of at least two line segments $\{\sigma_j\}_{j=1}^k$ $k \geq 2$, and the induced exterior unit tangent vectors are mutually distinct

$$T_i \neq T_j \quad \text{for } i \neq j.$$

We will call such a point a non-regular multiple point.

Consider a smooth family of regular, planar networks $(\gamma_t)_{0 \leq t < T}$, i.e., γ_{t_2} is a smooth deformation of γ_{t_1} . This implies that the number of triple points in $S_t = \{a_i(t)\}_{i=1}^n$ stays fixed. So we can assume that there exists a smooth family of regular parameterizations $(N_t)_{0 \leq t < T}$ of the evolving network. We will call $(\gamma_t)_{0 \leq t < T}$ a solution to the *network flow* if the deformation vector

$$\frac{dN}{dt} = X \quad \text{satisfies} \quad X^\perp = \vec{k}$$

at each non-singular point.

REMARKS 2.2. i) As a consequence, using the above notation, at each triple point

$$\sum_{j=1}^3 \langle \vec{k}_j, JT_j \rangle = \langle X, JT_1 + JT_2 + JT_3 \rangle = 0,$$

where J is the complex structure.

ii) Note that for a network without triple points, i.e., a union of curves, this is curve shortening flow.

iii) A network flow still satisfies the avoidance principle when comparing to smooth solutions of curve shortening flow. Comparing with big shrinking circles it is easy to see that any half-line P at infinity will remain fixed under the flow.

iv) Note that ii) in Definition 2.1 implies that such a network has bounded length ratios, i.e., there exists $D > 0$ such that

$$\mathcal{H}^1(\gamma \cap B_r(0)) \leq Dr.$$

v) By the work of Mantegazza, Novaga and Tortorelli [15, 14] it is known that for a given smooth, regular, planar network a smooth solution to the network flow exists, at least for a short time, provided it is compact with possible fixed endpoints. It is shown there that the solution exists as long as the curvature of the evolving network stays bounded, and none of the lengths of the branches goes to zero. This statement can be easily extended to the case of regular networks with non-compact branches as in Definition 2.1, see the beginning of Section 7.

vi) It would also be possible to study networks with fixed or moving endpoints. To avoid the non-conceptual, but technical difficulties arising from the contribution of the endpoints, we do not consider this case.

Definition 2.3. Let χ be a regular network with finitely many triple points. We say that χ is of class $C^{k,\alpha}$ where $k \geq 1, 0 \leq \alpha \leq 1$ if there exists $\delta > 0$ and a collection of points $(p_i) \subset \chi$, either finitely or countably many, such that

- a) the collection of balls $(B_{3\delta/4}(p_i))$ covers χ ,
- b) each ball $B_\delta(p_i)$ contains at most one triple point. If it contains no triple point, then $B_{3\delta/4}(p_i) \cap \chi$ can be written as a graph over its affine tangent line at p_i , where the graph function has $C^{k,\alpha}$ -norm less than one,
- c) if $B_\delta(p_i)$ contains a triple point, then p_i is the triple point, and $\chi \cap B_\delta(p_i)$ consists of three curves meeting at p_i . Each of the curves in $B_{3\delta/4}(p_i)$ can be written as a graph over the corresponding affine tangent half-line at p_i where the graph function has $C^{k,\alpha}$ -norm less than one.

We say that another regular network σ is ε -close to χ in $C^{k,\alpha}$, if σ is contained in the $\delta/2$ -neighborhood of χ and the triple junctions of σ are in one to one correspondence with the triple junctions of χ , with the triple junctions of σ being in a $\delta/2$ -neighborhood of the triple junctions of χ . Furthermore, in case b) in the above local graph representation, $\sigma \cap B_{3\delta/4}(p_i)$ can be written as a graph as well, where the difference of the graph functions is less than ε in $C^{k,\alpha}$. In case c) we assume that there exist unit vectors N_i such that $\sigma \cap B_{3\delta/4}(p_i)$ can be written

as $\chi + u_i N_i$, where the u_i are defined on a connected sub-domain of $B_\delta(p_i) \cap \chi$ and continuous. Restricted to each of the three local branches of χ we assume that the $C^{k,\alpha}$ -norm of u_i is less than ε with respect to arc-length parametrization on each branch.

3. Monotonicity formulas

Let $\tilde{\theta}_t$ be the angle that the tangent vector of γ_t makes with the x -axis. This is a well defined function up to multiple of π away from the triple junction points. Because at each of these points the angle jumps by $2\pi/3$, there is a well defined function θ_t which is continuous on γ_t and coincides with $\tilde{\theta}_t$ up to a multiple of $\pi/3$. An important observation is that $\vec{k} = J\nabla\theta_t$, where J is the complex structure.

Set $\lambda = xdy - ydx$. We assume that the planar network γ_t has no loops, so we can define β_t to be such that

$$d\beta_t = \lambda_{\gamma_t}.$$

Note that β_t is Lipschitz because its gradient is bounded linearly and, thus, β_t grows at most quadratically.

Finally, for any $x_0 \in \mathbb{R}^2$ and t_0 , define for $t < t_0$ the backwards heat kernel centered at (x_0, t_0) :

$$(3.1) \quad \rho_{x_0, t_0}(x, t) = \frac{1}{\sqrt{4\pi(t_0 - t)}} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}.$$

Lemma 3.1. *The following evolution equations hold away from the triple junction points:*

$$\begin{aligned} (i) \quad & \frac{d\theta_t}{dt} = \Delta\theta_t + \langle \nabla\theta_t, X \rangle; \\ (ii) \quad & \frac{d\beta_t}{dt} = \Delta\beta_t + \langle \nabla\beta_t, X \rangle - 2\theta_t; \\ (iii) \quad & \frac{d\rho_{x_0, t_0}}{dt} = -\Delta\rho_{x_0, t_0} + \langle \nabla\rho_{x_0, t_0}, X \rangle \\ & \quad - \left| \vec{k} + \frac{(x-x_0)^\perp}{2(t_0-t)} \right|^2 \rho_{x_0, t_0} + |\vec{k}|^2 \rho_{x_0, t_0}. \end{aligned}$$

Proof. The derivation of these equations proceeds as in the smooth case except that now we have a tangential motion that needs to be taken into account. For this reason we will only show the second formula.

The family of functions β_t can be chosen so that its time derivative is continuous. Then, denoting by \mathcal{L}_X the Lie derivative in the X direction, we obtain from Cartan's formula

$$d(d\beta_t/dt) = \mathcal{L}_X \lambda = d(X \lrcorner \lambda) + X \lrcorner d\lambda = d(X \lrcorner \lambda) - 2d\theta_t,$$

where in the last equality we use the fact that

$$(X \lrcorner d\lambda)_{\gamma_t} = (X^\perp \lrcorner d\lambda)_{\gamma_t} = (J\nabla\theta_t \lrcorner d\lambda)_{\gamma_t} = -2d\theta_t.$$

Therefore,

$$d(d\beta_t/dt + 2\theta_t - X \lrcorner \lambda) = 0.$$

Note that the function which has differential zero is continuous on γ_t and so we can add a time dependent constant to each β_t to obtain that

$$\frac{d\beta_t}{dt} = \langle X, Jx \rangle - 2\theta_t.$$

The desired formula follows from $\langle X, Jx \rangle = \Delta\beta_t + \langle X, \nabla\beta_t \rangle$. q.e.d.

Lemma 3.2. *Let f be in $C^2(\mathbb{R})$. The following identities hold:*

$$(i) \quad \begin{aligned} \frac{d}{dt} \int_{\gamma_t} f(\theta_t) \rho_{x_0, t_0} d\mu &= - \int_{\gamma_t} f''(\theta_t) |\vec{k}|^2 \rho_{x_0, t_0} d\mu \\ &\quad - \int_{\gamma_t} f(\theta_t) \left| \vec{k} + \frac{(x - x_0)^\perp}{2(t_0 - t)} \right|^2 \rho_{x_0, t_0} d\mu. \end{aligned}$$

$$(ii) \quad \begin{aligned} \frac{d}{dt} \int_{\gamma_t} f(\beta_t + 2t\theta_t) \rho_{x_0, t_0} d\mu &= \\ &\quad - \int_{\gamma_t} f''(\beta_t + 2t\theta_t) |x^\perp - 2t\vec{k}|^2 \rho_{x_0, t_0} d\mu \\ &\quad - \int_{\gamma_t} f(\beta_t + 2t\theta_t) \left| \vec{k} + \frac{(x - x_0)^\perp}{2(t_0 - t)} \right|^2 \rho_{x_0, t_0} d\mu. \end{aligned}$$

Proof. We prove the second identity. Set $\alpha_t = \beta_t + 2t\theta_t$. Then

$$\frac{df(\alpha_t)}{dt} = \Delta f(\alpha_t) - f''(\alpha_t) |\nabla\alpha_t|^2 + \langle \nabla f(\alpha_t), X \rangle,$$

and so

$$\begin{aligned} \frac{d}{dt} \int_{\gamma_t} f(\alpha_t) \rho_{x_0, t_0} d\mu &= \int_{\gamma_t} \rho_{x_0, t_0} \Delta f(\alpha_t) - f(\alpha_t) \Delta \rho_{x_0, t_0} d\mu \\ &\quad + \int_{\gamma_t} \operatorname{div}(X^\top f(\alpha_t) \rho_{x_0, t_0}) d\mu - \int_{\gamma_t} f''(\alpha_t) |x^\perp - 2t\vec{k}|^2 \rho_{x_0, t_0} d\mu \\ &\quad - \int_{\gamma_t} f(\alpha_t) \left| \vec{k} + \frac{(x - x_0)^\perp}{2(t_0 - t)} \right|^2 \rho_{x_0, t_0} d\mu. \end{aligned}$$

We need to show that the first two integral terms vanish. Decompose γ_t into k smooth curves $\{\sigma_j\}_{j=1}^k$. Then, we obtain from Green's formulas

$$\begin{aligned} \int_{\gamma_t} \rho_{x_0, t_0} \Delta f(\alpha_t) - f(\alpha_t) \Delta \rho_{x_0, t_0} d\mu \\ = \sum_{j=1}^k \int_{\partial\sigma_j} \rho_{x_0, t_0} \langle \nabla f(\alpha_t), \nu \rangle - f(\alpha_t) \langle \nabla \rho_{x_0, t_0}, \nu \rangle d\mu, \end{aligned}$$

where ν is the exterior unit normal to each σ_j . It is straightforward to see that if σ_j is non-compact then the boundary term at ‘‘infinity’’

vanishes. Let x_1 be a triple junction point meeting three line segments, which we relabel as σ_1, σ_2 , and σ_3 . Then, at the point x_1

$$\begin{aligned} \sum_{i=1}^3 \langle \nabla f(\alpha_t), \nu_i \rangle &= \sum_{i=1}^3 f' \langle Jx_1 - 2tJ\vec{k}_i, \nu_i \rangle \\ &= f' \langle Jx_1 - 2tJX, \nu_1 + \nu_2 + \nu_3 \rangle = 0. \end{aligned}$$

The same argument shows that

$$\sum_{i=1}^3 \langle \nabla \rho_{x_0, t_0}, \nu_i \rangle = 0,$$

and so

$$\int_{\gamma_t} \rho_{x_0, t_0} \Delta f(\alpha_t) - f(\alpha_t) \Delta \rho_{x_0, t_0} d\mu = 0.$$

For the second integral term we use again the decomposition

$$\int_{\gamma_t} \operatorname{div}(X^\top f(\alpha_t) \rho_{x_0, t_0}) d\mu = \sum_{j=1}^k \int_{\sigma_j} \operatorname{div}(X^\top f(\alpha_t) \rho_{x_0, t_0}) d\mu,$$

and one can argue as before to conclude that

$$\int_{\gamma_t} \operatorname{div}(X^\top f(\alpha_t) \rho_{x_0, t_0}) d\mu = 0. \quad \text{q.e.d.}$$

In the applications later, the evolving network will only be *locally* tree-like, i.e., only locally without loops. To apply the above monotonicity formula we have to localize it. We assume that $(\gamma_t)_{0 \leq t < T}$ is a smooth solution to the network flow such that $\gamma_t \cap B_4$ does not contain any closed loop for all $0 \leq t < T$. As before, we define β locally on $\gamma_t \cap B_4$.

Let φ be a smooth cut-off function such that $\varphi = 1$ on B_2 , $\varphi = 0$ on $\mathbb{R}^2 \setminus B_3$ and $0 \leq \varphi \leq 1$.

Lemma 3.3. *The following estimate holds:*

$$\begin{aligned} \frac{d}{dt} \int_{\gamma_t} \varphi |\beta_t + 2t\theta_t|^2 \rho_{x_0, t_0} d\mu &\leq - \int_{\gamma_t} \varphi |x^\perp - 2t\vec{k}|^2 \rho_{x_0, t_0} d\mu \\ &\quad + C \int_{\gamma_t \cap (B_3 \setminus B_2)} |\beta_t + 2t\theta_t|^2 \rho_{x_0, t_0} d\mu. \end{aligned}$$

Proof. We have

$$\begin{aligned} \left(\frac{d}{dt} - \Delta_{\gamma_t} \right) \varphi &= -\Delta_{\mathbb{R}^2} \varphi + D^2 \varphi(\nu, \nu) + \langle \nabla \varphi, X^T \rangle \\ &\leq C \chi_{B_3 \setminus B_2} + \langle \nabla \varphi, X^T \rangle. \end{aligned}$$

As in the proof of Lemma 3.2 we set $\alpha_t = \beta_t + 2t\theta_t$. Then

$$\begin{aligned} \left(\frac{d}{dt} - \Delta\right)\varphi\alpha_t^2 &\leq -2\langle\nabla\varphi, \nabla\alpha_t^2\rangle - 2\varphi|\nabla\alpha_t|^2 + \langle\nabla(\varphi\alpha_t^2), X^T\rangle \\ &\quad + C\chi_{B_3\setminus B_2}\alpha_t^2 \\ &\leq -\varphi|\nabla\alpha_t|^2 + \langle\nabla(\varphi\alpha_t^2), X^T\rangle + C\chi_{B_3\setminus B_2}\alpha_t^2, \end{aligned}$$

where we estimated

$$\begin{aligned} |\langle\nabla\varphi, \nabla\alpha_t^2\rangle| &\leq 2|D\varphi||\alpha_t||\nabla\alpha_t| \leq 4\frac{|D\varphi|^2}{\varphi}\alpha_t^2 + \frac{1}{2}\varphi|\nabla\alpha_t|^2 \\ &\leq C\chi_{B_3\setminus B_2}\alpha_t^2 + \frac{1}{2}\varphi|\nabla\alpha_t|^2. \end{aligned}$$

The rest follows as in the proof of the previous lemma. q.e.d.

4. Uniqueness of self-expanders

Consider the negatively curved metric

$$g = \exp(|x|^2)(dx_1^2 + dx_2^2).$$

A network ψ is said to be a *geodesic* for g if, when parametrized proportionally to arc-length, is a critical point for the length functional when restricted to variations with compact support. The network is said to be a *self-expander* if $\vec{k} = \psi^\perp$ on each branch and we say that the self-expander is *regular* if it has only triple junctions and the angles at each triple junction are $2\pi/3$.

Given a function u or a curve ψ , we denote by $u', u'', \psi',$ and ψ'' , the correspondent derivatives with respect to the space parameter.

In this section, we show that regular self-expanders are unique in their topological class.

Lemma 4.1 (Ilmanen and White). *A network ψ is a regular self-expander if and only if it is a geodesic for g .*

Proof. Let $(\psi_s)_{0 \leq s \leq \varepsilon}$ be a compactly supported continuous deformation of ψ which is a C^1 deformation when restricted to each branch. Each network ψ_s has only triple junctions. If we set

$$X = \frac{d\psi_s}{ds} \quad \text{and} \quad T = \frac{\psi'_s}{|\psi'_s|},$$

then, assuming parametrization proportional to arc-length, we have for each branch

$$\begin{aligned} (4.1) \quad \frac{d}{ds} \int_a^b (g(\psi'_s, \psi'_s))^{1/2} dt &= \frac{d}{ds} \int_a^b \exp(|x|^2/2) |\psi'_s| dt \\ &= \langle T, X \rangle \exp(|x|^2/2) \Big|_a^b + \int_a^b \langle x^\perp - \vec{k}, X \rangle |\psi'_s| \exp(|x|^2/2) dt. \end{aligned}$$

If ψ is a geodesic then by choosing variations X compactly supported on each branch we obtain that, indeed, $x^\perp = \vec{k}$ on each branch. Choosing variations supported on a neighborhood of each triple junction we obtain that

$$T_1 + T_2 + T_3 = 0,$$

where T_1, T_2, T_3 denote the outward unit tangent vectors at each triple junction and so ψ is a regular self-expander. Likewise, if ψ is a regular self-expander, it is simple to see that it is a critical point for the length functional. q.e.d.

Definition 4.2. We say that a self-expander ψ has an end asymptotic to a half-line $L = \{xe^{i\alpha} \mid x \geq 0\}$ if, for R large enough, a connected component $\bar{\psi}$ of $\psi \setminus B_R$ can be parametrized as

$$\bar{\psi} = \{xe^{i\alpha} + u(x)e^{i(\alpha+\pi/2)} \mid \text{for all } x \geq R\}, \quad \text{where } \lim_{x \rightarrow \infty} u(x) = 0.$$

Lemma 4.3. *Let P be a union of half-lines meeting at the origin and ψ a self-expander for which*

$$\lim_{r \rightarrow \infty} \text{dist}(\psi \setminus B_r, P) = 0.$$

Then ψ is asymptotic to P in the sense of Definition 4.2. Moreover, the decay of u for $x \geq R$ is given by

$$|u| \leq C_1 e^{-x^2/2}, \quad |u'| \leq C_1 x^{-1} e^{-x^2/2}, \quad |u''| \leq C_2 e^{-x^2/2},$$

and

$$|u^{(3)}| \leq C_3 x e^{-x^2/2}, \quad |u^{(4)}| \leq C_4 x^2 e^{-x^2/2},$$

where each C_i depends only on R , $u(R)$ and $u'(R)$.

Proof. In [22] it is shown that each asymptotic end of a self-expander is asymptotic to a half-line. Even more the graph function u decays exponentially. Since ψ is a self-expander the function u satisfies

$$u'' = (1 + (u')^2)(u - xu').$$

By possibly changing orientation, a simple application of the maximum principle (see [22]) implies that we can assume without loss of generality that $u > 0$ and $u' < 0$, if the self expander is not identical with the half-line. The function $v = u - xu'$ is strictly positive and satisfies

$$v' = -x(1 + (u')^2)v < -xv.$$

Integrating this inequality yields the first two estimates. Inserting that into the equation for u'' we get the third estimate and the fourth and fifth estimates come from computing $u^{(3)}$, $u^{(4)}$, and using the previous derived estimates. q.e.d.

We say that two self-expanders ψ_0 and ψ_1 are *asymptotic to each other* if their ends are asymptotic to the same half-lines. In this setting, we say they have the *same topological class* if there is a smooth family of maps

$$F_t : \psi_0 \longrightarrow \mathbb{R}^2, \quad 0 \leq t \leq 1,$$

such that F_0 is the identity, $F_1(\psi_0) = \psi_1$, the distance between any two triple junctions of $F_t(\psi_0)$ is uniformly bounded below, and

$$\lim_{r \rightarrow \infty} \sup \left\{ |dF_t|_{F_t(x)} \mid x \in \gamma_0 \setminus B_r(0) \right\} = 0 \text{ for every } 0 \leq t \leq 1.$$

Theorem 4.4. *If ψ_0 and ψ_1 are two regular self-expanders asymptotic to each other and in the same topological class, then they coincide.*

Proof. Let $(x_i^0)_{i \in A}$ and $(x_i^1)_{i \in A}$ denote the triple junctions (finite set) of ψ_0 and ψ_1 respectively. Because the networks are in the same topological class, we can rearrange the elements of $(x_i^0)_{i \in A}$ so that each x_i^0 is connected to x_i^1 by the existing deformation of ψ_0 into ψ_1 . Denote by $(x_i^s)_{0 \leq s \leq 1}$ the unique geodesic connecting these points.

For each s we consider the network ψ_s such that if x_i^0 is connected to x_j^0 by a geodesic, then x_i^s is connected to x_j^s through a geodesic as well. To handle the non-compact branches we proceed as follows. Let P denote a common asymptotic half-line to ψ_0 and ψ_1 , which means that there are geodesics $\gamma_0 \subset \psi_0$, $\gamma_1 \subset \psi_1$ asymptotic to P at infinity and starting at some points x_i^0 and x_i^1 respectively. Define γ_s to be the unique geodesic starting at x_i^s and asymptotic to P . Because these are geodesics with respect to a negatively curved metric it is easy to see that if γ_s intersects $\gamma_{s'}$ then they must coincide.

Hence, we have constructed a smooth family of triple-junction networks $(\psi_s)_{0 \leq s \leq 1}$ connecting ψ_0 and ψ_1 and such that:

- i) The triple-junctions $(x_i^s)_{i \in A}$ of ψ_s connect the triple-junctions of ψ_0 to the ones of ψ_1 and, for each index i fixed, the path $(x_i^s)_{0 \leq s \leq 1}$ is a geodesic with respect to the metric g .
- ii) Each branch of ψ_s is a geodesic for g .
- iii) There is R large enough so that $\psi_s \setminus B_R(0)$ has N connected components, each asymptotic to a half-line L_j , $j = 1, \dots, N$. We can find angles α_j such that each end of ψ_s becomes parametrized as

$$\psi_s(x) = x e^{i\alpha_j} + u_{j,s}(x) e^{i(\alpha_j + \pi/2)} \quad \text{for all } x \geq R.$$

This follows from Lemma 4.3.

- iv) The vector

$$X = \frac{d}{ds} \psi_s$$

is continuous, C^1 when restricted to each branch, and

$$X = O(e^{-r^2/2}), \quad \nabla X = O(r^{-1} e^{-r^2/2}).$$

Moreover,

$$\alpha_{j,s} = \frac{du_{j,s}}{ds}$$

satisfies

$$|\alpha_{j,s}| = O(e^{-x^2/2}) \quad |\alpha'_{j,s}| = O(x^{-1}e^{-x^2/2}).$$

It is enough to provide justification for the second set of estimates. For ease of notation we omit the indexes s and j on $\alpha_{j,s}$ and $u_{j,s}$. We have

$$\alpha'' = (1 + (u')^2)(\alpha - x\alpha') + 2u'\alpha'(u - xu').$$

We can assume without loss of generality that $\alpha(R) \geq 0$. Moreover, it follows from our construction that

$$\lim_{x \rightarrow \infty} |\alpha(x)| + |\alpha'(x)| = 0.$$

A simple application of the maximum principle shows that α cannot have negative local minimum or a positive local maximum. Hence, $\alpha \geq 0$ and $\alpha' \leq 0$. The function $\beta = \alpha - x\alpha'$ satisfies

$$\beta' = -x(1 + (u')^2)\beta - 2xu'\alpha'(u - xu') \leq -x\beta,$$

because $u'(u - xu') \leq 0$ (see proof of Lemma 4.3), and integration of this inequality implies property iv).

Denote by L the length function with respect to the metric g and consider the family of functions

$$\begin{aligned} F_t(s) &= L(\psi_s \cap B_{2R}(0)) \\ &+ \sum_{j=1}^N \int_{2R}^t \exp((x^2 + u_{j,s}^2)/2) \sqrt{1 + (u'_{j,s})^2} dx \\ &- N \int_{2R}^t \exp(x^2/2) dx. \end{aligned}$$

The decays given in Lemma 4.3 imply the existence of a constant C such that for every $t \leq t'$

$$(4.2) \quad \|F_t - F_{t'}\|_{C^3} \leq C \exp(-t),$$

and so when t tends to infinity F_t converges uniformly in C^2 to a function F . Furthermore, if $s = 0$ or $s = 1$, we have from combining (4.1) with property iv) that

$$\lim_{t \rightarrow \infty} \frac{dF_t}{ds}(s) = 0,$$

and, thus, F has a critical point when $s = 0$ or $s = 1$.

A standard computation shows that on each compact branch we have (assuming parametrization proportional to arc-length)

$$\begin{aligned}
 \frac{d^2}{ds^2} \int_a^b (g(\psi'_s, \psi'_s))^{1/2} dt &= \int_a^b |\psi'_s|^{-1} (|\nabla_{\psi'_s} X^\perp|^2 \\
 &\quad - \text{Rm}(X, \psi'_s, \psi'_s, X)) dt + (|\psi'_s|^{-1} g(\nabla_X X, \psi'_s)) \Big|_a^b \\
 &= \int_a^b |\psi'_s|^{-1} (|\nabla_{\psi'_s} X^\perp|^2 - \text{Rm}(X, \psi'_s, \psi'_s, X)) dt,
 \end{aligned}$$

where we used property i) on the second equality and all the geometric quantities are computed with respect to the metric g . Combining this identity with property iv) we have

$$\frac{d^2 F_t}{ds^2} = \int_{\psi_s \cap B_t(0)} |\psi'_s|^{-2} (|\nabla_{\psi'_s} X^\perp|^2 - \text{Rm}(X, \psi'_s, \psi'_s, X)) dl + O(e^{-t}).$$

The Gaussian curvature of g is equal to $-e^{-|x|^2}$ and so the integrals above are bounded independently of t . Therefore, we obtain from (4.2) that

$$\frac{d^2 F}{ds^2}(s) = \int_{\psi_s} |\psi'_s|^{-2} (|\nabla_{\psi'_s} X^\perp|^2 - \text{Rm}(X, \psi'_s, \psi'_s, X)) dl \geq 0,$$

where the last inequality comes from the fact that g has strictly negative Gaussian curvature. As a result, F is a convex function with two critical points and, hence, identically constant. The formula above implies that X must be a constant multiple of ψ'_s and, thus, it must vanish at all triple-junction points. The fact that X is continuous implies that X is identically zero and this proves the desired result. q.e.d.

Before using this Theorem to prove a compactness result we need one more definition.

Definition 4.5. Two regular networks σ_0 and σ_1 are in the same (ν, η, r, R, C) topological class if there is a smooth family $(\hat{\sigma}_t)_{0 \leq t \leq 1}$ of networks, with possible boundary points, such that for every $0 \leq t \leq 1$

- a) the distance between any two triple junctions of $\hat{\sigma}_t$ is bigger or equal to η ;
- b) all the triple junctions of $\hat{\sigma}_t$ are contained in $B_r(0)$ and the boundary points of $\hat{\sigma}_t$ are contained outside $B_R(0)$, with $r \leq R$;
- c) For every $R \geq s \geq r$

$$\text{dist}(\hat{\sigma}_t \setminus B_s, P) \leq \nu + C \exp(-s^2/C);$$

- d) $\sigma_0 \cap B_R \subseteq \hat{\sigma}_0$ and $\sigma_1 \cap B_R \subseteq \hat{\sigma}_1$.

We can now state the following corollary.

Corollary 4.6. *Let ψ be a regular self-expander with ends asymptotic to a union of half-lines P . Fix $r_1, \eta, C_1, D_1, \alpha < 1/2$, and R .*

For every ε , there are $R_1 \geq R, \beta$, and ν , all depending on $\varepsilon, r_1, \eta, C_1, D_1, \alpha, P, R$, so that if σ is a regular network that satisfies:

i)

$$\mathcal{H}^1(\sigma \cap B_r(x)) \leq D_1 r \text{ for all } x \text{ and } r;$$

ii)

$$\int_{\sigma \cap B_{R_1}(0)} |\vec{k} - x^\perp|^2 d\mathcal{H}^1 \leq \beta;$$

iii) σ and ψ are in the same $(\nu, \eta_1, r_1, R_1, C_1)$ topological class then σ must be ε -close in $C^{1,\alpha}(B_{R_1}(0))$ to ψ .

Proof. We start by finding $R_1 \geq R$ and ν so that if σ is a regular self-expander in the same $(2\nu, \eta/2, r_1 + 1, R_1 - 1, C_1)$ topological class as ψ then σ must be $\varepsilon/2$ -close in $C^{1,\alpha}(B_{R_1}(0))$ to ψ .

Suppose not. Then we can find a sequence of self-expanders σ_i with R_i tending to infinity, ν_i tending to zero, and such that σ_i is not $\varepsilon/2$ -close in $C^{1,\alpha}(B_{R_i}(0))$ to ψ . Let b_i denote a smooth branch of σ_i that connects $\sigma_i \cap \{|x| = R_i\}$ to one of the triple junctions inside $B_{2r_1}(0)$. Because of c) and Lemma 4.3, there is some r_2 such that, for every i large enough, $b_i \setminus B_{r_2}(0)$ can be written as a graph of a function with $C^{1,\alpha}$ norm less than $\varepsilon/4$ and defined over part of $P \setminus B_{r_2}$. As a result, if σ_i is not $\varepsilon/2$ -close in $C^{1,\alpha}(B_{R_i}(0))$ to ψ , we can find r_3 such that σ_i is not $\varepsilon/2$ -close in $C^{1,\alpha}(B_{r_3}(0))$ to ψ for every i large enough. Because each branch of σ_i is a geodesic of g , it is simple to see that we have uniform length bounds for σ_i . Standard compactness arguments show that a subsequence of σ_i converges in $C^{1,\alpha}$ to regular self-expander σ which, in virtue of property c), is asymptotic to P at infinity. If we can show that ψ and σ are in the same topological class, then Theorem 4.4 implies that they have to coincide and this is a contradiction.

Arguing as in Theorem 4.4, we can change the family $(\hat{\sigma}_t^i)_{0 \leq t \leq 1}$ of networks given by hypothesis iii) and construct a family $(\hat{\psi}_t^i)_{0 \leq t \leq 1}$ of networks connecting $\psi \cap B_{R_i-1}(0)$ to $\sigma_i \cap B_{R_i-1}(0)$ such that all the branches are geodesics for g , and those which intersect $\{|x| = R_i - 1\}$ have a uniform decay towards the half-lines of P ($\hat{\psi}_t^i$ should satisfy, with obvious modifications, properties i)–iv) described in the proof of Theorem 4.4). Making i tending to infinity, it is easy to recognize that $(\hat{\psi}_t^i)_{0 \leq t \leq 1}$ converges to a family of networks $(\hat{\psi}_t)_{0 \leq t \leq 1}$ connecting ψ to σ and satisfying properties i)–iv) mentioned in Theorem 4.4. Hence, the self-expanders must be in the same topological class.

Set $\varepsilon_1 = \min\{\varepsilon/2, \nu, \eta/2, r_1, 1/2\}$, and let σ be a regular network satisfying the hypothesis of the lemma. Condition i) and ii) imply that

$$\int_{B_{R_1}(0) \cap \sigma} |\vec{k}|^2 d\mathcal{H}^1 \leq \beta + \int_{B_{R_1}(0) \cap \sigma} |x^\perp|^2 d\mathcal{H}^1 \leq \beta + D_1 R_1^3.$$

Thus, we have uniform $C^{1,1/2}$ estimates for σ in $B_{R_1}(0)$. A standard compactness argument shows that by taking β small enough, we can assume that σ is ε_1 -close in $C^{1,\alpha}$ to a regular self-expander ψ' in $B_{R_1}(0)$.

By the reasoning before, we, thus, get that ψ' is $\varepsilon/2$ -close in $C^{1,\alpha}$ to ψ in $B_{R_1}(0)$ and this implies the desired result. q.e.d.

5. Main theorem

To show the short-time existence result for non-regular initial networks, we will use a special family of approximating regular networks. We will state the needed properties of such an approximating family below and show in the sequel the needed estimates for the proof of the short-time existence result. We will show in Section 7 that for any non-regular initial network such an approximating family exists.

Fix a regular self expander ψ which is asymptotic to a union of half-lines denoted by P . Note that, by Lemma 4.3, P coincides with the blow-down of ψ . For any $x_0 \in \mathbb{R}^2$ and $t > 0$ denote

$$\Phi(x_0, t)(x) = \rho_{x_0, 0}(x, -t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x - x_0|^2}{4t}\right).$$

We also use the notation

$$A(r, R) = \{x \in \mathbb{R}^2 : r \leq |x| \leq R\}.$$

Let $(\gamma^s)_{0 < s \leq c}$ be a family of regular networks on \mathbb{R}^2 such that for every $0 < s \leq c$:

H1) There is a constant D_1 such that

$$\mathcal{H}^1(\gamma^s \cap B_r(x)) \leq D_1 r \text{ for all } x \text{ and } r.$$

H2) There is a constant D_2 such that for every s and x in γ^s

$$|\theta^s(x)| + |\beta^s(x)| \leq D_2(|x|^2 + 1).$$

H3) $\tilde{\gamma}^s = \frac{\gamma^s}{\sqrt{2s}}$ converges in $C_{\text{loc}}^{1,\alpha}$ to ψ . Without loss of generality we assume that

$$\lim_{s \rightarrow 0} (\theta^s + \tilde{\beta}^s) = 0,$$

where $\tilde{\beta}^s$ is primitive for the Liouville form of $\tilde{\gamma}^s$.

H4) The connected components of $P \cap A(r_0\sqrt{s}, 4)$ are in one-to-one correspondence with the connected components of

$$\gamma^s \cap A(r_0\sqrt{s}, 4),$$

and if θ is the angle that a half-line in P makes with the x -axis, there is a function u_s such that a connected component σ of $\gamma^s \cap A(r_0\sqrt{s}, 4)$ can be parametrized as

$$\sigma = \{x e^{i\theta} + u_s(x) e^{i(\theta + \pi/2)} \mid \text{for all } r_0\sqrt{s} \leq x \leq 4\}.$$

Moreover, the function u_s satisfies

$$|u_s(x)| + |x| \left| \frac{du_s}{dx} \right| + |x|^2 \left| \frac{d^2u_s}{dx^2} \right| \leq D_3 \left(|x|^2 + (2s)^{1/2} \exp(-|x|^2/4s) \right)$$

for some constant D_3 .

Assume that $(\gamma_t^s)_{t \geq 0}$ is a smooth solution to network flow with initial condition γ^s and denote by $\Theta_t^s(x, r)$ the Gaussian density of γ_t^s

$$(5.1) \quad \Theta_t^s(x_0, r) = \int_{\gamma_t^s} \Phi(x_0, r^2) d\mathcal{H}^1.$$

Note that in our previous notation we have $\Theta_t^s(x_0, r) = \Theta(x_0, t + r^2, r)$ with respect to the flow (γ_t^s) . We will show

Theorem 5.1. *There are s_1, δ_1 , and τ_1 depending on $\alpha < 1/2$, D_1, D_2, D_3, ψ, r_0 , and ε_0 , so that if*

$$t \leq \delta_1, r^2 \leq \tau_1 t, \text{ and } s \leq s_1,$$

then

$$\Theta_t^s(x_0, r) \leq 3/2 + \varepsilon_0$$

for every x_0 in $B_1(0)$.

Proof. Throughout the proof it will be understood that, unless stated, all constants will depend only on $\alpha < 1/2$, D_1, D_2, D_3, ψ, r_0 , and ε_0 . All the lemmas will be proven in Section 6.

Set

$$\tilde{\gamma}_t^s = \frac{1}{(2(s+t))^{1/2}} \gamma_t^s.$$

We start by proving estimates that hold either for short-time or far from the origin. They will be simple consequences of Huisken's monotonicity formula.

Lemma 5.2. *[Far from origin estimate] There are δ_1 and K_0 so that if $r^2 \leq t \leq \delta_1$, then*

$$\Theta_t^s(x_0, r) \leq 3/2 + \varepsilon_0$$

for every x_0 with $1 \geq |x_0| \geq K_0 \sqrt{2t}$.

[Short-time estimate] There are s_1 and q_1 such that if $s \leq s_1, r^2, t \leq q_1 s$, then

$$(5.2) \quad \Theta_t^s(x, r) \leq 3/2 + \varepsilon_0$$

for every x in $B_1(0)$.

REMARK 5.3. 1) It follows from the second estimate that we need only to prove Theorem 5.1 when $t \geq q_1 s$.

2) Setting

$$\tilde{\Theta}_t^s(x, r) = \int_{\tilde{\gamma}_t^s} \Phi(x, r^2) d\mathcal{H}^1,$$

and in virtue of

$$\Theta_t^s(x_0, r) = \tilde{\Theta}_t^s \left(\frac{x_0}{(2(s+t))^{1/2}}, \frac{r}{(2(s+t))^{1/2}} \right),$$

in order to prove Theorem 5.1 it suffices to find s_1, δ_1 , and τ_1

such that for every $s \leq s_1$, $q_1 s \leq t \leq \delta_1$, $r^2 \leq \tau_1$, and y_0 with $|y_0| \leq (2(s+t))^{-1/2}$, we have

$$\tilde{\Theta}_t^s(y_0, r) \leq 3/2 + \varepsilon_0.$$

3) Set

$$\tau_1 = q_1/(2(q_1 + 1)).$$

The second estimate in the lemma implies that for $s \leq s_1$, $t \leq q_1 s$, and $r^2 \leq \tau_1$ we have

$$\tilde{\Theta}_t^s(y_0, r) \leq 3/2 + \varepsilon_0$$

for every $|y_0| \leq (2(s+t))^{-1/2}$. The first estimate in Lemma 5.2 implies that for $r^2 \leq \tau_1$, $s \leq s_1$ and $q_1 s \leq t \leq \delta_1$,

$$\tilde{\Theta}_t^s(y_0, r) \leq 3/2 + \varepsilon_0$$

for every y_0 with $K_0 \leq |y_0| \leq (2(s+t))^{-1/2}$.

From now on, consider K_0, q_1, s_1 , and δ_1 , given by Lemma 5.2 and set $\tau_1 = q_1/(2(q_1 + 1))$.

In the next two lemmas we control the asymptotic behavior of $\tilde{\gamma}_t^s$. The proof will be a bit involving because it is important that r_1 does not depend on ν .

Lemma 5.4 (Proximity to P). *There are C_1 and r_1 so that for every ν we can find s_2 , and δ_2 for which the following holds. If $s \leq s_2$, $t \leq \delta_2$, and $r \leq 2$, then*

$$\text{dist}(y_0, P) \leq \nu + C_1 \exp(-|y_0|^2/C_1) \quad \text{if} \quad y_0 \in \tilde{\gamma}_t^s \cap A\left(r_1, (s+t)^{-1/8}\right),$$

and

$$\tilde{\Theta}_t^s(y_0, r) \leq 1 + \varepsilon_0/2 + \nu \quad \text{if} \quad y_0 \in A\left(r_1, (s+t)^{-1/8}\right).$$

Denote by F_t^s the normal deformation

$$F_t^s : \gamma^s \longrightarrow \mathbb{R}^2,$$

such that $\gamma_t^s = F_t^s(\gamma^s)$ and set $\tilde{F}_t^s = (2(s+t))^{-1/2} F_t^s$ so that $\tilde{\gamma}_t^s = \tilde{F}_t^s(\gamma^s)$. Using the previous lemma with $\nu = \varepsilon_0/2$ we obtain, as we shall see in Section 6,

Lemma 5.5. *There are r_2, δ_3, s_3 , and L , such that if $t \leq \delta_3$ and $s \leq s_3$ then*

$$|\tilde{F}_0^s(x) - \tilde{F}_t^s(x)| \leq L \quad \text{whenever} \quad \tilde{F}_0^s(x) \in A(r_2, (s+t)^{-1/8}/2).$$

Consider C_1 and r_1 given by Lemma 5.4, r_2, δ_3, s_3 , and L given by Lemma 5.5, and choose $\eta_1 = \eta_1(\tau_1)$ given by Lemma 10.2. We then set $r_3 = \max\{r_0, r_1, r_2, 1\}$. Apply Corollary 4.6 where we consider $R = \sqrt{1 + 2q_1 K_0} + r_3$, $\varepsilon = \varepsilon(\psi, \alpha)$ to be the one given by Lemma 10.1, r_1 to be r_3 , and η_1, C_1, D_1, α , and P to be the constants already defined. Then, we get the existence of R_1, β , and ν for which Corollary 4.6 holds.

Consider now $s_2 = s_2(\nu)$, $\delta_2 = \delta_2(\nu)$ given by Lemma 5.4 and set $s_4 = \min\{s_1, s_2, s_3\}$, $\delta_4 = \min\{\delta_1, \delta_2, \delta_3\}$. Finally, decrease s_4, δ_4 if necessary so that

$$(s_4 + \delta_4)^{-1/8} \geq 2R_1.$$

The next lemma is essential to prove Theorem 5.1 and its content is that the proximity of $\tilde{\gamma}_t^s$ to a self-expander can be controlled in an integral sense. It is the only place where we use the evolution equations derived in Section 3.

Choose $a > 1$ such $(1 + 2q_1)/a > 1$ and set $q = q_1/a$.

Lemma 5.6. *There are δ_0 , and s_0 so that for every*

$$qs \leq T \leq \delta_0 \quad \text{and} \quad s \leq s_0,$$

we have

$$\frac{1}{(a-1)T} \int_T^{aT} \int_{\tilde{\gamma}_t^s \cap B_{R_1}} |\vec{k} - x^\perp|^2 d\mathcal{H}^1 dt \leq \beta.$$

Consider δ_0, s_0 for which the lemma holds and set $s_5 = \min\{s_0, s_4\}$, $\delta_5 = \min\{\delta_0, \delta_4\}$. Decrease s_5 if necessary so that $q_1 s_5 \leq \delta_5$.

Having all the constants properly defined, we can now finish the proof. Set

$$T_0 = \sup\{T \mid \tilde{\Theta}_t^s(x, r) \leq 3/2 + \varepsilon_0 \quad \text{for all } x \in B_{K_0}(0), r^2 \leq \tau_1, t \leq T\}.$$

It suffices to show that $T_0 \geq \delta_5$ for every $s \leq s_5$. Remark 5.3 1) implies that $T_0 \geq q_1 s$.

Suppose that $T_0 < \delta_5$ and set $T = T_0/a$. Lemma 5.6 implies the existence of $T \leq t_1 \leq T_0$ so that

$$\int_{\tilde{\gamma}_{t_1}^s \cap B_{R_1}} |\vec{k} - x^\perp|^2 d\mathcal{H}^1 \leq \beta.$$

We now check that Corollary 4.6 can be applied with σ being $\tilde{\gamma}_{t_1}^s$. Conditions i) and ii) are trivially satisfied. For every $0 \leq t \leq t_1$ set

$$\hat{\sigma}_t = \tilde{F}_t^s(\gamma^s \cap B_{R_1+L}(0)).$$

During the proof of Lemma 5.5 we chose r_2 so that

$$\tilde{\Theta}_t^s(x, r) \leq 1 + \varepsilon_0$$

for every $r \leq 2$ and x in $A(r_2, (s+t)^{-1/8})$. This implies that all the triple junctions of $\hat{\sigma}_t$ are inside $B_{r_3}(0)$. Lemma 5.5 implies that the boundary points of $\hat{\sigma}_t$ lie outside $B_{R_1}(0)$, and so condition iii) b) is met. Condition iii) a) holds because Remark 5.3 3) implies that, for every x in $B_{R_1}(0)$, $r^2 \leq \tau_1$, and $t \leq t_1$

$$\tilde{\Theta}_t^s(x, r) \leq 3/2 + \varepsilon_0,$$

and so Lemma 10.2 can be applied with $R = R_1$. Condition iii) c) is satisfied because of Lemma 5.4. Condition iii) d) is not immediately satisfied because $\hat{\sigma}_0$ coincides with part of $(2s)^{-1/2}\gamma^s$ instead of ψ .

Nonetheless, using hypothesis H3) and picking s_5 smaller if necessary, one can extend the family $(\hat{\sigma}_t)_{0 \leq t \leq t_1}$ so that condition iii) d), indeed, holds.

Therefore, we get from Corollary 4.6 that $\tilde{\gamma}_{t_1}^s$ is ε -close in $C^{1,\alpha}(B_{R_1}(0))$ to ψ . Denote by $(\hat{\gamma}_l^s)_{l \geq 0}$ the solution to network flow with initial condition $\tilde{\gamma}_{t_1}^s$. A simple computation shows that

$$\hat{\gamma}_l^s = \sqrt{1 + 2l} \tilde{\gamma}_{t_1 + l\lambda^2}^s,$$

where $\lambda^2 = 2(s + t_1)$. Applying Lemma 10.1 we conclude that for every $l \leq q_1$

$$\tilde{\Theta}_{t_1 + l\lambda^2}^s(x, r) = \hat{\Theta}_l^s(\sqrt{1 + 2l}x, \sqrt{1 + 2l}r) \leq 3/2 + \varepsilon_0$$

provided

$$\sqrt{1 + 2l}|x| \leq R_1 - 1 \quad \text{and} \quad (1 + 2l)r^2 \leq q_1.$$

Hence, for all $t_1 \leq t \leq t_1(1 + 2q_1)$,

$$\tilde{\Theta}_t^s(x, r) \leq 3/2 + \varepsilon_0$$

for every x in $B_{K_0}(0)$ and $r^2 \leq \tau_1$, which implies that $T_0 \geq t_1(1 + 2q_1)$. This is a contradiction because

$$t_1(1 + 2q_1) \geq T(1 + 2q_1) = T_0(1 + 2q_1)/a > T_0. \quad \text{q.e.d.}$$

6. Omitted proofs from Section 5

We prove the various lemmas used in the previous section.

Proof of Lemma 5.2. We start by showing the existence of K_0 so that for every y_0 in \mathbb{R}^2 with $|y_0| \geq K_0$ and $\lambda > 0$

$$\int_{\lambda(\gamma^s \cap B_3(0))} \Phi(y_0, 1) d\mathcal{H}^1 \leq 3/2 + \varepsilon_0/2.$$

We argue by contradiction and assume the existence of y_i tending to infinity, λ_i , and s_i , for which

$$(6.1) \quad \int_{\lambda_i(\gamma^{s_i} \cap B_3(0))} \Phi(y_i, 1) d\mathcal{H}^1 \geq 3/2 + \varepsilon_0/2.$$

The first remark is that $(\lambda_i)_{i \in \mathbb{N}}$ has to be an unbounded sequence, because for some universal constant C and for all i sufficiently large

$$\int_{\lambda_i(\gamma^{s_i} \cap B_3(0))} \Phi(y_i, 1) d\mathcal{H}^1 \leq C\lambda_i \exp(-|y_i|^2/8 + C\lambda_i^2) \mathcal{H}^1(\gamma^{s_i} \cap B_3(0)).$$

The second remark is that from hypothesis H3) and H4) it follows the existence of D_4 depending only on ψ, r_0 and D_3 , so that on $\gamma^s \cap B_3(0)$

$$|\vec{k}| \leq D_4 \left(1 + s^{-1/2} e^{-\frac{|x|^2}{4s}} \right),$$

and, thus, setting $\sigma_i = \lambda_i \gamma^{s_i}$ and $l_i = \lambda_i^2 s_i$, we have on $\sigma_i \cap B_{3\lambda_i}(0)$

$$|\vec{k}| \leq D_4 \left(\lambda_i^{-1} + l_i^{-1/2} e^{-\frac{|x|^2}{4l_i}} \right).$$

Because y_i is tending to infinity, it is easy to recognize that the curvature goes to zero uniformly on compact sets centered around y_i . As a result, $\sigma_i - y_i$ converges to either a line, or a union of half-lines. We first note that one only needs to consider the case $\lim_{i \rightarrow \infty} l_i = \infty$, otherwise H3) and $|y_i| \rightarrow \infty$ yield a contradiction to (6.1). Furthermore, all the triple junctions of σ_i are inside a ball of radius proportional to $l_i^{1/2}$ and the shortest distance between them is also proportional to $l_i^{1/2}$. Hence, because y_i is getting arbitrarily large, we see that $\sigma_i - y_i$ converges to either plane or a union of three half-lines. This contradicts inequality (6.1).

Hypothesis H1) ensures us that we can choose δ_1 so that for every x_0 in $B_1(0)$ and $l \leq 2\delta_1$

$$\int_{\gamma^s \setminus B_3(0)} \Phi(x_0, l) d\mathcal{H}^1 \leq \varepsilon_0/2.$$

The monotonicity formula implies that for $r^2, t \leq \delta_1$.

$$\begin{aligned} \Theta_t^s(x_0, r) &\leq \int_{\gamma^s} \Phi(x_0, r^2 + t) d\mathcal{H}^1 \\ &= \int_{\gamma^s \setminus B_3(0)} \Phi(x_0, r^2 + t) d\mathcal{H}^1 + \int_{\gamma^s \cap B_3(0)} \Phi(x_0, r^2 + t) d\mathcal{H}^1 \\ &\leq \varepsilon_0/2 + \int_{(r^2+t)^{-1/2}(\gamma^s \cap B_3(0))} \Phi(x_0/\sqrt{r^2+t}, 1) d\mathcal{H}^1 \\ &\leq 3/2 + \varepsilon_0, \end{aligned}$$

provided $|x_0| \geq K_0 \sqrt{r^2+t}$. This proves the first statement.

Pick

$$\varepsilon = \varepsilon(\psi, \alpha), \quad q_1 = q_1(\psi, \alpha)$$

given by Lemma 10.1 and apply this lemma with

$$\sigma_t = (2s)^{-1/2} \gamma_{2st}^s \quad \text{and} \quad R = K_0 \sqrt{q_1} + 1.$$

Note that by hypothesis H3) we can choose s_1 so that, for every $s \leq s_1$, σ_s is ε -close to ψ in $C^{1,\alpha}(B_R(0))$ and $s_1 q_1 \leq \delta_1$. Scale invariance implies that for every $s \leq s_1$, $r^2 \leq t \leq q_1 s$, and x in $B_{\sqrt{2sq_1}K_0}(0)$,

$$\Theta_t^s(x, r) \leq 3/2 + \varepsilon_0.$$

This proves the second statement because the ball $B_{\sqrt{2sq_1}K_0}(0)$ contains $B_{\sqrt{2t}K_0}(0)$ if $t \leq q_1 s$. q.e.d.

Proof of Lemma 5.4. Set

$$l = t(2(s+t))^{-1} \quad \text{and} \quad \sigma^s = (2(s+t))^{-1/2} \gamma^s.$$

Note that $l \leq 1$. Moreover, for $s_2 = s_2(r_0)$ and $\delta_2 = \delta_2(r_0)$ small we have that

$$\sigma^s \cap A \left(r_0, 3(s+t)^{-1/8} \right)$$

is graphical over $P \cap A \left(r_0, 3(s+t)^{-1/8} \right)$ and if v_s is a function arising from the graphical decomposition then

$$\begin{aligned} |v_s(x)| + |x| |dv_s/dx| + |x|^2 |d^2v_s/dx^2| \\ \leq D_3 \left(2(t+s)^{1/2} |x|^2 + \exp(-|x|^2/2) \right), \end{aligned}$$

which means that, by choosing $s_2 = s_2(D_3, r_0)$, $\delta_2 = \delta_2(D_3, r_0)$ small enough and choosing $r_1 = r_1(r_0, D_3) \geq \max\{r_0, 1\}$ large enough, we can ensure that

$$(6.2) \quad |v_s(x)| + |x| |dv_s/dx| \leq D_3 \left(2(t+s)^{1/2} |x|^2 + \exp(-|x|^2/2) \right) \leq 1$$

on $A \left(r_1, 3(s+t)^{-1/8} \right)$.

From now on pick

$$y_0 \in \tilde{\gamma}_t^s \cap A \left(3r_1 + 1, (s+t)^{-1/8} \right).$$

From the monotonicity formula we have that

$$1 \leq \Theta_0^s(y_0(2(s+t))^{1/2}, \sqrt{t}) = \int_{\sigma^s} \Phi(y_0, l) d\mathcal{H}^1 = A + B + C,$$

where

$$\begin{aligned} A &= \int_{\sigma^s \setminus B_{3(s+t)^{-1/8}}} \Phi(y_0, l) d\mathcal{H}^1, \\ B &= \int_{\sigma^s \cap B_{r_1}} \Phi(y_0, l) d\mathcal{H}^1, \\ C &= \int_{\sigma^s \cap A \left(r_1, 3(s+t)^{-1/8} \right)} \Phi(y_0, l) d\mathcal{H}^1. \end{aligned}$$

For every x with $|x| \geq 3(s+t)^{-1/8}$, the bounds for y_0 imply that

$$|x - y_0|^2 \geq |x|^2/3 + |y_0|^2,$$

and so

$$\Phi(y_0, l) \leq \sqrt{3} \exp(-|y_0|^2/(4l)) \Phi(0, 3l).$$

Thus, we can find $C_1 = C_1(D_1)$ for which

$$\begin{aligned} A &= \int_{\sigma^s \setminus B_{3(s+t)^{-1/8}}} \Phi(y_0, l) d\mathcal{H}^1 \\ &\leq \sqrt{3} \exp(-|y_0|^2/(4l)) \int_{\sigma^s \setminus B_{3(s+t)^{-1/8}}} \Phi(0, 3l) d\mathcal{H}^1 \\ &\leq C_1 \exp(-|y_0|^2/C_1). \end{aligned}$$

To estimate the second term we proceed in the same way. For every $|x| \leq r_1$, the bounds for y_0 imply that

$$|x - y_0|^2 \geq |x|^2 + |y_0|^2/3,$$

and so

$$\Phi(y_0, l) \leq \exp(-|y_0|^2/(12l))\Phi(0, l) \quad \text{for every } |x| \leq r_1.$$

Thus, we can find $C_1 = C_1(D_1)$ for which

$$B \leq \exp(-|y_0|^2/(12l)) \int_{\sigma^s \cap B_{r_1}} \Phi(0, l) d\mathcal{H}^1 \leq C_1 \exp(-|y_0|^2/C_1).$$

Finally, we estimate the third term. Denote by P_i the half-lines such that $P = \{P_i\}_{i=1}^N$, by a_i the orthogonal projection of y_0 on the line determined by P_i , and by b_i the projection of y_0 onto the normal space of P_i so that

$$\text{dist}(y_0, P) = \min\{|b_i|\} = |b_1|.$$

Furthermore, denote by σ_i^s the component of

$$\sigma^s \cap A\left(r_1, 3(s+t)^{-1/8}\right),$$

which is graphical over $P_i \cap A\left(r_1, 3(s+t)^{-1/8}\right)$ and by v_s^i the correspondent graph function. It is easy to recognize that for $i = 2, \dots, N$, we have $|b_i| \geq c|y_0|$, where $c = c(P)$ is some positive constant. Relabel $r_1 = r_1(r_0, D_3, P)$ such that (6.2) holds with $c/\sqrt{2}$ instead of 1 on the right hand side. Hence,

$$(v_s^i - b_i)^2 \geq (c^2/2)|y_0|^2,$$

and so there is $C_1 = C_1(D_1, P)$ such that

$$\begin{aligned} \int_{\sigma_i^s} \Phi(y_0, l) d\mathcal{H}^1 &\leq 2(4\pi l)^{-1/2} \int \exp\left(-\frac{(c^2/2)|y_0|^2 + (x - a_i)^2}{4l}\right) dx \\ &\leq C_1 \exp(-|y_0|^2/C_1). \end{aligned}$$

As a result, we combine all these estimates and obtain that for some $C_1 = C_1(D_1, P)$

$$1 \leq \int_{\sigma_1^s} \Phi(y_0, l) d\mathcal{H}^1 + C_1 \exp(-|y_0|^2/C_1).$$

We relabel r_1 one last time and find $r_1 = r_1(D_1, D_3, r_0, \psi, \alpha)$ so that

$$C_1 \exp(-r_1^2/C_1) \leq 1/2.$$

This combined with H1) implies that there exists a constant $c = c(D_1)$ such that

$$1/4 \leq \int_{\sigma_1^s \cap B_{c\sqrt{l}}(y_0)} \Phi(y_0, l) d\mathcal{H}^1 \leq 2 \sup_{|x-a_1| \leq c\sqrt{l}} \exp\left(-\frac{|v_s^1 - b_1|^2}{4l}\right).$$

By (6.2), the variation of $v_s^1(x)$ over the interval $|x - a_1| \leq c\sqrt{l}$ is $O(\sqrt{l})$ and, thus, there exists a constant $\tilde{c} = \tilde{c}(D_1, P)$ such that

$$\sup_{|x-a_1| \leq c\sqrt{l}} \frac{|v_s^1 - b_1|^2}{4l} \leq \tilde{c}.$$

Therefore, using that $\lambda \leq (1 - \exp(-\lambda)) \exp(\tilde{c})$ for $\lambda \in [0, \tilde{c}]$, we see that we can find $C_1 = C_1(D_1, P)$ for which

$$\begin{aligned} & \int_{|x-a_1| \leq c\sqrt{l}} \frac{(v_s^1 - b_1)^2 \exp(-(x - a_1)^2(4l)^{-1})}{4l (4\pi l)^{1/2}} dx \\ & \leq C \int_{|x-a_1| \leq c\sqrt{l}} \left(1 - \exp\left(-\frac{(v_s^1 - b_1)^2}{4l}\right) \right) \frac{\exp(-(x - a_1)^2(4l)^{-1})}{(4\pi l)^{1/2}} dx \\ & \leq C \left(\int_{\{x \geq r_1\}} \sqrt{1 + (dv_s^1/dx)^2} \frac{\exp(-(x - a_1)^2(4l)^{-1})}{(4\pi l)^{1/2}} dx \right. \\ & \quad \left. - \int_{\sigma_1^s \setminus B_{r_1}(0)} \Phi(y_0, l) d\mathcal{H}^1 \right) \\ & \leq C \left(\int_{\{x \geq r_1\}} \sqrt{1 + (dv_s^1/dx)^2} \frac{\exp(-(x - a_1)^2(4l)^{-1})}{(4\pi l)^{1/2}} dx - 1 \right) \\ & \quad + C_1 \exp(-|y_0|^2/C_1) \\ & \leq \int_{\{x \geq r_1\}} C (dv_s^1/dx)^2 \frac{\exp(-(x - a_1)^2(4l)^{-1})}{(4\pi l)^{1/2}} dx + C_1 \exp(-|y_0|^2/C_1), \end{aligned}$$

and, thus,

$$\begin{aligned} b_1^2 & \leq C_1 \int_{\{x \geq r_1\}} \left((v_s^1)^2 + (dv_s^1/dx)^2 \right) \frac{\exp(-(x - a_1)^2(4l)^{-1})}{(4\pi l)^{1/2}} dx \\ & \quad + C_1 \exp(-|y_0|^2/C_1). \end{aligned}$$

We observe that $|a_1| \geq c|y_0|$ for some constant $c = c(P)$ and that for every $0 \leq l \leq 1$ we have

$$\frac{(x + a)^2}{2} + \frac{x^2}{4l} \geq \frac{a^2}{8} + \frac{x^2}{8l}.$$

Thus, we obtain from (6.2) that, for some constant $C_1 = C_1(D_1, D_3, P)$,

$$\begin{aligned} & \int (dv_s^1/dx)^2 \frac{\exp(-(x - a_1)^2(4l)^{-1})}{(4\pi l)^{1/2}} dx \\ & \leq D_3 \int (\sqrt{s + t}|x| + \exp(-x^2/2)) \frac{\exp(-(x - a_1)^2(4l)^{-1})}{(4\pi l)^{1/2}} dx \\ & \leq C_1 \sqrt{s + t} + D_3 \int \exp(-x^2/2) \frac{\exp(-(x - a_1)^2(4l)^{-1})}{(4\pi l)^{1/2}} dx \end{aligned}$$

$$\begin{aligned}
&\leq C_1\sqrt{s+t} + D_3 \int \exp(-(x+a_1)^2/2) \frac{\exp(-x^2(4l)^{-1})}{(4\pi l)^{1/2}} dx \\
&\leq C_1\sqrt{s+t} + D_3 \exp(-a_1^2/8) \int \frac{\exp(-x^2(8l)^{-1})}{(4\pi l)^{1/2}} dx \\
&\leq C_1\sqrt{s+t} + C_1 \exp(-|y_0|^2/C_1).
\end{aligned}$$

The same type of estimate holds for the term

$$\int (v_s^1)^2 \frac{\exp(-(x-a_1)^2(4l)^{-1})}{(4\pi l)^{1/2}} dx,$$

and so we can choose s_2 and δ_2 both depending on $D_1, D_3, \psi, r_0, \alpha$, and ν , such that for every $s \leq s_2$ and $t \leq \delta_2$ we have

$$b_1 = \text{dist}(y_0, P) \leq \nu + C_1 \exp(-|y_0|^2/C_1).$$

We now show that, by relabeling r_1, s_2 , and δ_2 if necessary, we also have

$$\tilde{\Theta}_t^s(y_0, r) \leq 1 + \varepsilon_0/2 + \nu$$

for every $r \leq 1$. The argument is almost identical to what we have just done and so we will just point out the differences. We keep the same notation and assumptions. Arguing in the very same way as we did before, we obtain the existence of $C_1 = C_1(D_1, D_3, P)$ and $r_1 = r_1(D_1, D_3, r_0, \psi, \alpha)$ for which

$$\begin{aligned}
\tilde{\Theta}_t^s(y_0, r) &\leq \int_{\sigma^s} \Phi(y_0, l+r^2) d\mathcal{H}^1 \\
&\leq \int_{\sigma_1^s} \Phi(y_0, l+r^2) d\mathcal{H}^1 + C_1 \exp(-|y_0|^2/C_1) \\
&\leq \int \sqrt{1+(dv_s^1/dx)^2} \frac{\exp(-(x-a_1)^2(4(l+r^2))^{-1})}{(4\pi(l+r^2))^{1/2}} dx \\
&\quad + C_1 \exp(-|y_0|^2/C_1) \\
&\leq 1 + C_1 \int |dv_s^1/dx| \frac{\exp(-(x-a_1)^2(4(l+r^2))^{-1})}{(4\pi(l+r^2))^{1/2}} dx \\
&\quad + C_1 \exp(-|y_0|^2/C_1) \\
&\leq 1 + C_1\sqrt{s+t} + C_1 \int \exp(-x^2/2) \frac{\exp(-(x-a_1)^2(4(l+r^2))^{-1})}{(4\pi(l+r^2))^{1/2}} dx \\
&\quad + C_1 \exp(-|y_0|^2/C_1).
\end{aligned}$$

Using the fact that $0 \leq l \leq 1$ we obtain

$$\begin{aligned}
\tilde{\Theta}_t^s(y_0, r) &\leq 1 \\
&\quad + C_1\sqrt{s+t} + C_1 \int \exp(-(x+a_1)^2/2) \frac{\exp(-x^2(4(l+r^2))^{-1})}{(4\pi(l+r^2))^{1/2}} dx
\end{aligned}$$

$$\begin{aligned}
 & + C_1 \exp(-|y_0|^2/C_1) \\
 \leq & 1 + C_1 \sqrt{s+t} + C_1 \exp(-a_1^2/40) \int \frac{\exp(-x^2(8(l+r^2))^{-1})}{(4\pi(l+r^2))^{1/2}} dx \\
 & + C_1 \exp(-|y_0|^2/C_1) \\
 \leq & 1 + C_1 \sqrt{s+t} + C_1 \exp(-|y_0|^2/C_1) \\
 \leq & 1 + \varepsilon_0/2 + C_1 \sqrt{s+t}.
 \end{aligned}$$

Thus, like before, we can choose s_2, δ_2 for which the result holds. q.e.d.

Proof of Lemma 5.5. From scale invariance and applying Lemma 5.4 with $\nu = \varepsilon/2$, we can find $r_2 \geq 1, \delta_3$, and s_3 such that if $t \leq \delta_3$, and $s \leq s_3$, then

$$\Theta_t^s(x, r) \leq 1 + \varepsilon_0,$$

whenever $r \leq 2(2(s+t))^{1/2}$ and

$$x \in A\left(r_2(2(s+t))^{1/2}, (2(s+t))^{1/2}(s+t)^{-1/8}\right).$$

Hence, from White's regularity Theorem [26], we obtain the existence of a universal constant C for which

$$\left| \frac{dF_t^s}{dt}(p) \right| = |\vec{k}| \leq Ct^{-1/2},$$

whenever

$$F_t^s(p) \in A\left(3r_2(2(s+t))^{1/2}/2, 3(2(s+t))^{1/2}(s+t)^{-1/8}/4\right).$$

Choosing a larger r_2 (depending on C and the previous r_2) and δ_3, s_3 smaller if necessary, we obtain after integrating the previous inequality that

$$|F_t^s(p) - F_0^s(p)| \leq 2C\sqrt{t},$$

whenever

$$F_0^s(p) \in A\left(r_2(2(s+t))^{1/2}, (2(s+t))^{1/2}(s+t)^{-1/8}/2\right).$$

This finishes the proof. q.e.d.

Lemma 6.1. *There exists $\delta_5 > 0$, s.t. for $0 < s, t < \delta_5$ it holds that*

$$(6.3) \quad |\vec{k}(x)| + |\theta_t^s(x)| + |\beta_t^s(x)| \leq D_4 \quad \forall x \in \gamma_t^s \cap A(1/3, 3).$$

Proof. By assumption H4) the estimate is true for $t = 0$ and s sufficiently small. H4), furthermore, implies that for s sufficiently small, each component of $\gamma^s \cap A(1/8, 8)$ is a graph, uniformly small C^2 -norm over a half-line P . By Theorem 1.5 this implies that there exists $\delta_5 > 0$ such that $\gamma_t^s \cap A(1/6, 6)$ remains a graph with small gradient over P for $0 \leq t \leq \delta_5$. This already implies the first two estimates of the statement, since θ_t^s is continuous in t .

The estimates of Ecker and Huisken, [8], for graphical mean curvature flow then imply that

$$\gamma_t^s \cap A(1/5, 5)$$

remains is a graph over P with small C^2 -norm for $0 \leq t \leq \delta_5$. Let $(N_t^s)_{0 \leq t \leq T}$ be a smooth parametrization of the evolving network. Since $\gamma_t^s \cap A(1/5, 5)$ is free of triple junctions we can locally reparametrize $(N_t^s)_{0 \leq t \leq \min(T, \delta_5)}$ such that

$$\left(\frac{\partial}{\partial t} N \right)^T = X^T = 0$$

on $A(1/4, 4)$. Since $X = \vec{k}$, we have by the evolution equation for β_t that,

$$\left| \frac{d}{dt} \beta_t \right| \leq |\langle X, Jx \rangle| + 2|\theta_t| \leq C.$$

Decreasing δ_5 further if necessary, this implies the second part of statement. q.e.d.

Proof of Lemma 5.6. Set $T_0 = R^2(aT + s) + aT$. During this proof C denotes a constant which is allowed to depend also on a, R , and q (but not T and s). We have from the localized monotonicity formula applied to $2(s+t)\theta^s + \beta^s$ (see Lemma 3.3) that

$$\begin{aligned} & \frac{1}{(a-1)T} \int_T^{aT} \int_{\tilde{\gamma}_t^s \cap B_R(0)} |\vec{k} - x^\perp|^2 d\mathcal{H}^1 dt \\ &= \frac{1}{(a-1)T} \int_T^{aT} (2(s+t))^{-3/2} \int_{\gamma_t^s \cap B_{R\sqrt{2(s+t)}}(0)} |2(s+t)\vec{k} - x^\perp|^2 d\mathcal{H}^1 dt \\ &= \frac{1}{(a-1)T} \int_T^{aT} (2(s+t))^{-3/2} \int_{\gamma_t^s \cap B_{R\sqrt{2(s+t)}}(0)} |\nabla(2(s+t)\theta^s + \beta^s)|^2 d\mathcal{H}^1 dt \\ &\leq \frac{C}{T} \int_T^{aT} (s+t)^{-3/2} (T_0 - t)^{1/2} \int_{\gamma_t^s} \varphi |\nabla(2(s+t)\theta^s + \beta^s)|^2 \rho(0, T_0 - t) d\mathcal{H}^1 dt \\ &\leq \frac{C}{T} (s+T)^{-3/2} (T_0 - T)^{1/2} \int_{\gamma_T^s} \varphi (2(s+T)\theta^s + \beta^s)^2 \rho(0, T_0 - T) d\mathcal{H}^1 \\ &+ \frac{C}{T} (s+T)^{-3/2} (T_0 - T)^{1/2} \int_T^{aT} \int_{\gamma_t^s \cap A(2,3)} (2(s+t)\theta_t^s + \beta_t^s)^2 \rho(0, T_0 - t) d\mathcal{H}^1 dt \\ &\leq \frac{C}{T} (s+T)^{-3/2} (T_0 - T)^{1/2} \int_{\gamma_0^s} \varphi (2s\theta^s + \beta^s)^2 \rho(0, T_0) d\mathcal{H}^1 \\ &+ \frac{C}{T} (s+T)^{-3/2} (T_0 - T)^{1/2} \int_0^{aT} \int_{\gamma_t^s \cap A(2,3)} (2(s+t)\theta_t^s + \beta_t^s)^2 \rho(0, T_0 - t) d\mathcal{H}^1 dt \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{C}{T(s+T)} \int_{\gamma_0^s} \varphi(2s\theta^s + \beta^s)^2 \rho(0, T_0) d\mathcal{H}^1 \\
 &+ \frac{C}{T(s+T)} \int_0^{aT} \int_{\gamma_t^s \cap A(2,3)} (2(s+t)\theta_t^s + \beta_t^s)^2 \rho(0, T_0 - t) d\mathcal{H}^1 dt \\
 &= A + B.
 \end{aligned}$$

For the second term, using Lemma 6.1, we have

$$\begin{aligned}
 B &\leq \frac{C((s+aT)+1)}{T(s+T)} \int_0^{aT} \int_{\gamma_t^s \cap A(2,3)} |x|^4 \rho(0, T_0 - t) d\mathcal{H}^1 dt \\
 &\leq \frac{C((s+aT)+1)}{T(s+T)} \int_0^{aT} (T_0 - t)^2 \int_{(T_0-t)^{-1/2}(\gamma_t^s \cap A(2,3))} |x|^4 \rho(0, 1) d\mathcal{H}^1 dt \\
 &\leq \frac{C((s+aT)+1)T_0^3}{T(s+T)} \sup_{0 < t < aT} \int_{(T_0-t)^{-1/2}(\gamma_t^s \cap A(2,3))} |x|^4 e^{-|x|^2/4} d\mathcal{H}^1 \\
 &\leq CT_0 \sup_{0 < t < aT} \int_{(T_0-t)^{-1/2}(\gamma_t^s \cap A(2,3))} |x|^4 e^{-|x|^2/4} d\mathcal{H}^1.
 \end{aligned}$$

Note that

$$T_0 \leq R^2 \delta_0 (a + 1/q) + a\delta_0,$$

and so we can choose δ_0 small enough so that $B \leq \beta/2$.

We now estimate the first term. Recall that if β is primitive for the Liouville form on the network γ , then $\beta_t = l^{-2}\beta$ is primitive for the Liouville form on $l^{-1}\gamma$. Set

$$\lambda = \frac{s}{T+s} \quad \text{and} \quad l = \sqrt{2(T+s)}.$$

Then

$$\begin{aligned}
 A &\leq \frac{C}{T(s+T)} \int_{\gamma^s \cap B_3} (2s\theta^s + \beta^s)^2 \rho(0, T_0) d\mathcal{H}^1 \\
 &= \frac{C(s+T)}{T} \int_{l^{-1}(\gamma^s \cap B_3)} (\lambda\theta^s + \beta_t^s)^2 \rho(0, l^{-2}T_0) d\mathcal{H}^1 \\
 &\leq C \int_{l^{-1}(\gamma^s \cap B_3)} (\lambda\theta^s + \beta_t^s)^2 \rho(0, l^{-2}T_0) d\mathcal{H}^1,
 \end{aligned}$$

where the last equality follows because $T \geq qs$. Consider

$$F(T, s) = \int_{l^{-1}(\gamma^s \cap B_3)} (\lambda\theta^s + \beta_t^s)^2 \rho(0, l^{-2}T_0) d\mathcal{H}^1,$$

where we remark the existence of a constant C (independent of T and s) such that

$$C^{-1} \leq l^{-2}T_0 \leq C.$$

Given any β_1 small it is enough to show the existence of s_0 and δ_0 so that if $qs \leq T \leq \delta_0$ and $s \leq s_0$ then

$$F(T, s) \leq \beta_1.$$

We now argue by contradiction and assume the existence of s_i and T_i (with $q_{s_i} \leq T_i$) converging to zero for which $F(T_i, s_i) \geq \beta_1$. We also assume that $l_i^{-2}T_0$ converges to T_1 .

Suppose first that λ_i (as defined above) has a subsequence converging to a positive number λ . In that case

$$l_i^{-1}\gamma^{s_i} = \lambda_i^{1/2}\tilde{\gamma}^{s_i}$$

converges in $C_{\text{loc}}^{1,\alpha}$ to $\lambda^{1/2}\psi$. Hypothesis H3) implies that

$$\lim_i F(T_i, s_i) \leq \lim_i \lambda_i^2 \int_{\tilde{\gamma}^{s_i}} (\theta^s + \tilde{\beta}^{s_i})^2 \rho(0, l_i^{-2}\lambda_i^{-1}T_0) d\mathcal{H}^1 = 0.$$

Suppose now that λ_i has a subsequence converging to zero. It follows at once that

$$\lim_i \int_{l_i^{-1}(\gamma^{s_i} \cap B_{r_0\sqrt{s_i}})} (\lambda_i\theta^{s_i} + \beta_{l_i}^{s_i})^2 \rho(0, l_i^{-2}T_0) d\mathcal{H}^1 = 0.$$

Note that by hypothesis H4)

$$l_i^{-1}\gamma^{s_i} \cap A(r_0(\lambda_i/2)^{1/2}, 3l_i^{-1})$$

is graphical over P and if v_i is the function arising from the graphical decomposition of $l_i^{-1}\gamma^{s_i}$ then

$$|v_i(x)| + |x||dv_i/dx| + |x|^2 |d^2v_i/dx^2| \leq D_3 \left(l_i|x|^2 + (\lambda_i)^{1/2} e^{-\frac{|x|^2}{2\lambda_i}} \right).$$

Therefore, we have that

$$(6.4) \quad |\nabla\beta_{l_i}^{s_i}| = |x^\perp| = \frac{|xv_i' - v_i|}{\sqrt{1 + (v_i')^2}} \leq D_3 \left(l_i|x|^2 + (\lambda_i)^{1/2} \right).$$

We now argue that for any connected component of

$$l_i^{-1}\gamma^{s_i} \cap A(r_0(\lambda_i/2)^{1/2}, 3l_i^{-1}),$$

there is x_i converging to zero for which $b_i = \beta_{l_i}^{s_i}(x_i)$ also converges to zero. From hypothesis H3) we see that any connected components of $\tilde{\psi}^{s_i} \cap A(2r_0, 3r_0)$ contains y_i such that

$$\lim_i (\theta^{s_i}(y_i) + \tilde{\beta}^{s_i}(y_i)) = 0.$$

Setting $x_i = \lambda_i y_i$, it is simple to see that $b_i = \lambda_i \tilde{\beta}^{s_i}(y_i)$ tends to zero.

Therefore, we can use gradient estimate (6.4) and the graphical decomposition to conclude the existence of a constant C independent of i such that

$$|\beta_{l_i}^{s_i}(x)| \leq C \left(l_i|x|^3 + (\lambda_i)^{1/2}|x| \right) + b_i \quad \text{on} \quad A(r_0(\lambda_i/2)^{1/2}, 3l_i^{-1}).$$

Hence,

$$\lim_i F(T_i, s_i) = \lim_i \int_{l_i^{-1}\gamma^{s_i} \cap A(r_0(\lambda_i/2)^{1/2}, 3l_i^{-1})} (\lambda_i\theta^{s_i} + \beta_{l_i}^{s_i})^2 \rho(0, l_i^{-2}T_0) d\mathcal{H}^1$$

$$\begin{aligned}
 &= \lim_i \int_{l_i^{-1}\gamma^{s_i} \cap A(r_0(\lambda_i/2)^{1/2}, 3l_i^{-1})} (\beta_{l_i}^{s_i})^2 \rho(0, l_i^{-2}T_0) d\mathcal{H}^1 \\
 &\leq \lim_i C(l_i^2 + \lambda_i + b_i^2) \int_{l_i^{-1}\gamma^{s_i}} (|x|^6 + |x|^2 + 1) \rho(0, l_i^{-2}T_0) d\mathcal{H}^1 = 0.
 \end{aligned}$$

This is a contradiction.

q.e.d.

7. Short-time existence

We will show, in this section, that one can glue a scaled self-expander at scale s into the initial network around a non-regular multiple point to obtain a sequence of regular networks γ^s which satisfy the hypotheses H1)–H4) in Section 5. We will show that combining Theorem 5.1 and 1.3 then proves short time existence of the network flow for non-regular initial networks, Theorem 1.1.

We will first discuss the question of short-time existence for regular networks with unbounded branches. By the definition of a regular network there exists an $R_0 > 0$ such that outside of B_{R_0} the initial network γ_0 consists of a finite number of non-compact branches γ_0^i , $i = 1, \dots, n$ which can be written as graphs over corresponding half-lines P_i . Since the curvature of γ_0 is bounded and the γ_0^i approach the half-lines P_i at infinity, we assume that each γ_0^i can be written as a normal graph over P_i with small C^1 -norm. We define the points $q_k^i := \gamma_0^i \cap \partial B_k$ for $k \geq k_0 > R_0$; $k, k_0 \in \mathbb{N}$. By the results in [15, 14] there exists a maximal solution $(\gamma_{k,t})_{0 \leq t < T_k}$ of the network flow, starting at $\gamma_0 \cap B_k$ with fixed endpoints q_k^i . Using Proposition 8.1 in B_{k_0} and Theorem 1.5 to control the boundary points as well as estimates of Ecker and Huisken for graphical mean curvature flow to control the parts outside of B_{k_0} to see that there is $T > 0$ such that $T_k \geq T$ for all $k \geq k_0$, together with uniform estimates on the curvature. We, thus, take a limit $k \rightarrow \infty$ to obtain a solution to the network flow, starting at γ_0 .

Now let γ be a non-regular initial network with bounded curvature. For simplicity let us assume that γ has only one non-regular multiple point at the origin.

If the multiple point consists only of two branches meeting at an angle different than π , then smoothing the cone point and using estimates for graphical mean curvature flow, see, for example, the proof of Lemma 6.1, one easily constructs a solution starting at γ as claimed in Theorem 1.1.

So we can assume that at the origin at least three branches meet, and let T_j , $j = 1, \dots, n$, be the exterior unit normals. We denote with

$$P_j = \{-tT_j \mid t \geq 0\}$$

be the corresponding half-lines. Since γ has bounded curvature, we can assume, by scaling γ if necessary, that $\gamma \cap B_5$ consists of n branches γ_j corresponding to the half-lines P_j ; and if θ_j is the angle that P_j makes

with the x -axis, there is a function w^j such that γ_j can be parametrized as

$$\gamma_j = \{xe^{i\theta_j} + w^j(x)e^{i(\theta_j+\pi/2)} \mid 0 \leq x \leq 5\}.$$

Note that the assumption that γ has bounded curvature implies

$$(7.1) \quad |w^j(x)| \leq Cx^2 \quad \text{and} \quad \left| \frac{d}{dx}w^j(x) \right| \leq Cx.$$

In [22] it was shown that for $n = 3$ there exists a unique tree-like self-expander ψ asymptotic to $P := \cup_{j=1}^n P_j$. In the case $n > 3$ the existence of tree-like, connected self-expanders was shown by Mazzeo–Saez [16]. Note that Lemma 4.3 gives the asymptotics of ψ outside a large ball B_{r_0} .

We now aim to glue $\psi_s := \sqrt{2s} \cdot \psi$ into γ to get a family satisfying the conditions H1)–H4). Let v_s^j be the graph function corresponding to the branch ψ_s^j on $A(r_0\sqrt{2s}, 4)$. By Lemma 4.3 we have the estimate

$$(7.2) \quad \begin{aligned} |v_s^j| &\leq (2s)^{1/2}Ce^{-x^2/4s}, \quad |(v_s^j)'| \leq x^{-1}(2s)^{1/2}Ce^{-x^2/4s}, \\ |(v_s^j)''| &\leq (2s)^{-1/2}Ce^{-x^2/4s}. \end{aligned}$$

Let $\varphi : \mathbb{R}^+ \rightarrow [0, 1]$ be a cut-off function s.t. $\varphi = 1$ on $[0, 1]$ and $\varphi = 0$ on $[2, \infty)$. We define γ^s via the graph function u_s^j in the gluing region $A(r_0\sqrt{2s}, 4)$ by

$$u_s^j := \varphi(s^{-1/4}x)v_s^j(x) + (1 - \varphi(s^{-1/4}x))w^j(x).$$

It can easily be checked that γ^s satisfies the assumptions H1)–H3). From (7.1) and (7.2) we see that

$$|u_s^j| \leq C\left(x^2 + (2s)^{1/2}e^{-x^2/4s}\right).$$

Furthermore,

$$\begin{aligned} (u_s^j)' &= s^{-1/4}\varphi'(s^{-1/4}x)v_s^j(x) + \varphi(s^{-1/4}x)(v_s^j)'(x) \\ &\quad - s^{-1/4}\varphi'(s^{-1/4}x)w^j(x) + (1 - \varphi(s^{-1/4}x))(w^j)'(x). \end{aligned}$$

We have $x^{-1} \leq s^{-1/4} \leq 2x^{-1}$ on $\{\varphi'(s^{-1/4}x) \neq 0\}$ and so we can estimate

$$\begin{aligned} x|(u_s^j)'| &\leq C(|v_s^j(x)| + x|(v_s^j)'(x)| + |w^j(x)| + x|(w^j)'(x)|) \\ &\leq C\left(x^2 + (2s)^{1/2}e^{-x^2/4s}\right). \end{aligned}$$

The estimate for $(u_s^j)''$ follows similarly, which shows that also H4) is satisfied.

Proof of Theorem 1.1. As discussed at the beginning of this section there exists a smooth solution to the network flow $(\gamma_t^s)_{0 \leq t \leq T_s}$ for some $T_s > 0$. We now aim to show that there exists a $T_0 > 0$ such that

$T_s \geq T_0$ for all $s > 0$ and that there are time interior estimates on k and all its higher derivatives for all positive times, independent of s .

Using Theorem 1.5 and interior estimates for higher derivatives of the curvature we see that we can pick a smooth family of points $P_j(t, s) \in \gamma_s^j \cap A(1/3, 1/2)$ such that assumptions (8.7) and (8.8) are satisfied, with constants independent of s for $0 \leq t < \min\{T_s, \delta\}$, where $\delta > 0$ does not depend on s . Then Proposition 8.1 gives estimates on the curvature and its derivatives, independent of s on $\mathbb{R}^2 \setminus B_{1/2} \times (0, \min\{T_s, \delta\})$.

To get the desired estimates on $B_{1/2}$ we aim to apply Theorem 5.1 and Theorem 1.3. Fix $\varepsilon_0 > 0$ such that $3/2 + \varepsilon_0 < \Theta_{S^1}$, and let s_1, δ_1, τ_1 be determined by Theorem 5.1.

Pick $0 < t_0 < \min\{T_s, \delta_1, \delta\}$ and $x_0 \in B_{1/2}$. Let $\rho := (t_0/2)^{1/2}$. Note that $B_\rho(x_0) \subset B_1$. Theorem 5.1 then implies that the Gaussian density ratios

$$\Theta(x, t, r) \leq 3/2 + \varepsilon_0$$

for all $(x, t) \in B_\rho(x_0) \times (t_0 - \rho^2, t_0)$ and $r \leq \sqrt{\tau_1} \rho$. Thus, by Theorem 1.3 with $\sigma = 1$, there exists C , depending only on ε_0, τ_1 such that

$$|k|(x_0, t_0) \leq \frac{C}{t_0^{1/2}},$$

together with the corresponding estimates on all higher derivatives. By Remark 1.4 there is a $\kappa > 0$, depending only on ε_0, τ_1 such that the length of the shortest segment is bounded from below by $\kappa \cdot t_0^{1/2}$.

Together with the estimate on $(\mathbb{R}^2 \setminus B_{1/2}) \times (0, \min\{T_s, \delta\})$ this implies that $T_s \geq T_0 := \min\{\delta, \delta_1\}$. By the estimates on the curvature, which are independent of s we can take a subsequential limit of the flows $(\gamma_t^s)_{0 < t < \bar{T}}$ as $s \rightarrow 0$ to obtain a limiting flow $(\gamma_t)_{0 < t < \bar{T}}$ starting at the non-regular network γ .

Note that by Theorem 1.5 and the interior estimates of Ecker/Huisken, away from any triple and multiple point, the flow $(\gamma_t)_{0 < t < \bar{T}}$ attains the initial network γ in C^∞ . Furthermore, by the above estimate in B_1 and Proposition 8.1 we have

$$|k| \leq \frac{C}{t^{1/2}}.$$

The estimate on the length of the shortest segment passes to the limit as well. q.e.d.

8. Local regularity

In this section, we will prove some local regularity results for the network flow.

Integral estimates. We will need to localize the integral estimates in the work of Mantegazza, Novaga and Tortorelli [15]. In the following

we will outline what modifications of the original proofs are needed to obtain the estimates in the local case. The setup is as follows.

Let $(\gamma_t)_{t \in [0, T]}$ be a regular, smooth solution of the network flow on \mathbb{R}^2 . Let $N : \gamma_0 \times [0, T] \rightarrow \mathbb{R}^2$ be a smooth and regular parametrization of the flow. We denote the tangential component of the deformation vector by

$$(8.1) \quad \vec{\lambda} = X^T.$$

As defined before we denote with T_j the exterior unit tangent vector induced by each σ_j at each triple point. We then define

$$(8.2) \quad k_i = \langle \vec{k}_i, JT_i \rangle \quad \text{and} \quad \lambda_i = \langle \vec{\lambda}, T_i \rangle.$$

The balancing condition at each triple point then implies

$$(8.3) \quad k_1 + k_2 + k_3 = 0 \quad \text{and} \quad \lambda_1 + \lambda_2 + \lambda_3 = 0.$$

We would like to point out that our setup differs to the one in [15] in that we do not want to prescribe the tangential component $\vec{\lambda}$ of the deformation vector. If one aims to prove a short-time existence result, one has to specify the tangential velocity. Nevertheless, and this is important in the following discussion, the integral estimates on the curvature and higher derivatives of the curvature of the evolving network do not depend on the choice of tangential velocity. Another point is that the calculations in [15] are done only for a network consisting of three curves, meeting at one common triple point, and with three fixed endpoints. As already mentioned there, see Remark 3.24 in [15], these calculations generalize without any changes to networks with more than one triple point, but with fixed endpoints. In the following we will explain how to generalize these estimates to networks with arbitrary tangential speed, more than one triple point and any number of moving endpoints.

We assume that along each segment σ_i we have fixed an orientation and, thus, the unit tangent vector field τ along is well defined. Note that at each endpoint p of σ_i we have

$$\tau(p) = \pm T_i,$$

depending on the chosen orientation. We fix the unit normal vector field ν along N by requiring that

$$J\tau = \nu.$$

This convention implies that the curvature k of N is given by

$$k = \langle \vec{k}, \nu \rangle = \langle \partial_s \tau, \nu \rangle = -\langle \partial_s \nu, \tau \rangle,$$

where s is the arc-length parameter along $\sigma_i(t)$. Similarly we define

$$\lambda = \langle X, \tau \rangle = \langle \vec{\lambda}, \tau \rangle.$$

Note that this implies again that at an endpoint p of σ_i it holds that

$$k(p) = \pm k_i \quad \text{and} \quad \lambda(p) = \pm \lambda_i.$$

It then can be easily checked that the evolution equations for τ, ν and k do not depend on the choice of the tangential speed $\vec{\lambda}$ and are given by

$$(8.4) \quad \partial_t \tau = (\langle \nabla k, \tau \rangle + k\lambda)\nu,$$

$$(8.5) \quad \partial_t \nu = -(\langle \nabla k, \tau \rangle + k\lambda)\tau,$$

$$(8.6) \quad \partial_t k = \Delta k + \langle \nabla k, \vec{\lambda} \rangle + k^3,$$

see (2.4), (2.5) and (2.6) in [15]. Furthermore, the estimates and relations between the curvatures k_i and the tangential speeds λ_i at a triple point, following (2.6) until (2.10) in [15], remain valid. As well the evolution equation for higher derivatives of the curvature and the relations between time and spacial derivatives given in Lemma 3.7 and the calculus rules in Remark 3.9 in [15] are not affected.

This ensures that all the calculations for integrals of the curvature and its derivatives are identical up to contributions from the boundary points. To control the influence of the boundary points we make the following assumption.

Assumption. We assume that the evolving network $(\gamma_t)_{t \in [0, T]}$ has boundary points $Q_l(t)$, where $l = 1, \dots, N$. We assume that these boundary points are all disjoint and at each of this points it holds that

$$(8.7) \quad X^T|_{(P(t), t)} = \vec{\lambda}(Q_l(t), t) = 0$$

for all $t \in [0, T]$. Furthermore, we assume that there are positive constants C_j such that

$$(8.8) \quad \sup_{l \in \{1, \dots, N\}} |\nabla^j k|_{(Q_l(t), t)} \leq C_j$$

for all $j = 0, 1, \dots, j_0$, where $j_0 \in \mathbb{N}$, $t \in [0, T]$.

With this assumption the additional terms in the evolution of the integral of the square of $\partial_s^j k$ can be controlled. To demonstrate this, and for the reader's convenience, we will do this calculation explicitly, compare with (3.4) in [15].

$$\begin{aligned} \frac{d}{dt} \int_{\gamma_t} |\nabla^j k|^2 ds &= 2 \int_{\gamma_t} \nabla^j k \partial_t \nabla^j k ds \\ &\quad + \int_{\gamma_t} |\nabla^j k|^2 (\operatorname{div}(\vec{\lambda}) - k^2) ds \\ &= 2 \int_{\gamma_t} \nabla^j k \Delta \nabla^j k + \nabla_{\vec{\lambda}} \nabla^j k \nabla^j k ds \\ &\quad + \int_{\gamma_t} \mathfrak{p}_{j+3}(\nabla^j k) \nabla^j k ds + \int_{\gamma_t} |\nabla^j k|^2 (\operatorname{div}(\vec{\lambda}) - k^2) ds \end{aligned}$$

$$\begin{aligned}
(8.9) \quad &= -2 \int_{\gamma_t} |\nabla^{j+1} k|^2 ds + \int_{\gamma_t} \operatorname{div}(\vec{\lambda} |\nabla^j k|^2) ds \\
&+ \int_{\gamma_t} \mathfrak{p}_{2j+4}(\nabla^j k) ds + \sum_{\text{3-points}} \sum_{i=1}^3 \langle T_i, \nabla |\nabla^j k|^2 \rangle \Big|_{\text{3-point}} \\
&+ \sum_{l=1}^N \langle T_l, \nabla |\nabla^j k|^2 \rangle \Big|_{Q_l} \\
&= -2 \int_{\gamma_t} |\nabla^{j+1} k|^2 ds + \int_{\gamma_t} \mathfrak{p}_{2j+4}(\nabla^j k) ds \\
&\quad \sum_{\text{3-points}} \sum_{i=1}^3 \langle T_i, \nabla |\nabla^j k|^2 \rangle + \lambda_i |\nabla^j k|^2 \Big|_{\text{3-point}} \\
&\quad + \sum_{l=1}^N \langle T_l, \nabla |\nabla^j k|^2 \rangle \Big|_{Q_l}.
\end{aligned}$$

In the special case $j = 0$ one gets

$$\begin{aligned}
\frac{d}{dt} \int_{\gamma_t} k^2 ds &= -2 \int_{\gamma_t} |\nabla k|^2 ds + \int_{\gamma_t} k^4 ds \\
&+ \sum_{\text{3-points}} \sum_{i=1}^3 \langle T_i, \nabla(k^2) \rangle + \lambda_i k^2 \Big|_{\text{3-point}} + \sum_{l=1}^N \langle T_l, \nabla(k^2) \rangle \Big|_{Q_l}.
\end{aligned}$$

The relations at the triple points, see (2.10) in [15], imply that at each triple point

$$\sum_{i=1}^3 2 \langle T_i, \nabla(k^2) \rangle + \lambda_i k^2 \Big|_{\text{3-point}} = 0.$$

Thus, the order of differentiation at the triple point can be lowered by one order, and one gets

$$\begin{aligned}
(8.10) \quad &\frac{d}{dt} \int_{\gamma_t} k^2 ds = -2 \int_{\gamma_t} |\nabla k|^2 ds + \int_{\gamma_t} k^4 ds \\
&- \sum_{\text{3-points}} \sum_{i=1}^3 \lambda_i k^2 \Big|_{\text{3-point}} + \sum_{l=1}^N \langle T_l, \nabla(k^2) \rangle \Big|_{Q_l}.
\end{aligned}$$

Following verbatim the computations in [15] one can use interpolation inequalities for L^p -norms of k and higher derivatives of k to absorb the term

$$\int_{\gamma_t} k^4 ds,$$

and the boundary terms at the triple-points. Note that the contributions at the boundary points Q_l are bounded by NC_0C_1 . This leads to the

estimate, compare (3.10) in [15],

$$(8.11) \quad \frac{d}{dt} \int_{\gamma_t} k^2 ds \leq C \left(1 + \int_{\gamma_t} k^2 ds \right)^3,$$

where C depends only on a bound for the inverses of the lengths of the segments the evolving network and NC_0C_1 . This inequality implies that the L^2 -norm of k cannot grow too quickly. It can be, furthermore, shown that an estimate for the L^2 -norm of every even derivative $\nabla^j k$ is true, which depends only on the L^2 -norm of k , a bound for the inverses of the lengths of the segments of the evolving network and NC_jC_{j+1} . Compare here the proof of Proposition 3.13 in [15].

A bound for the inverses of the lengths of the segments $l(\sigma_i)$ of the evolving network, depending on the initial network and $\int k^2$ is also true. Note that since at the endpoints Q_l we have $\lambda_l = 0$ there is no extra contribution there. As in the proof of Proposition 3.15 in [15] one obtains

$$(8.12) \quad \frac{d}{dt} \left(1 + \int_{\gamma_t} k^2 + \sum_i \frac{1}{l(\sigma_i)} \right) \leq C \left(1 + \int_{\gamma_t} k^2 + \sum_i \frac{1}{l(\sigma_i)} \right)^3,$$

where C depends only on NC_0C_1 . Thus, also the length of the shortest segment remains bounded from below for a short time. Thus, there exists a $T_0 > 0$, depending only on the L^2 -norm of the curvature of γ_0 , the inverses of the lengths of the segments of γ_0 , and N, C_1, C_2 such that on $[0, T_0]$ the L^2 -norm of k and the inverse of the length of the shortest segment remains uniformly bounded.

To obtain estimates for higher derivatives of k which are interior in time, Mantegazza, Novaga and Tortorelli look, for j even, at the evolution of integrals of the form,

$$\int_{\gamma_t} k^2 + \frac{t}{2!} |\nabla k|^2 + \cdots + \frac{t^j}{j!} |\nabla^j k|^2 ds.$$

By (8.9) we get for the time derivative of such a quantity in our case only the additional boundary term

$$\sum_{l=1}^N \sum_{i=0}^j \frac{t^i}{i!} \langle T_l, \nabla |\nabla^i k|^2 \rangle \Big|_{Q_l},$$

which by our assumption (8.8) is bounded on finite time intervals. So arguing as in [15] we obtain, compare p.273 there, that on $[0, T_0]$

$$\int_{\gamma_t} k^2 + \frac{t}{2!} |\nabla k|^2 + \cdots + \frac{t^j}{j!} |\nabla^j k|^2 ds \leq \tilde{C}_j.$$

Here the constants \tilde{C}_j depend only on the L^2 -norm of the curvature of γ_0 , the inverses of the lengths of the segments of γ_0 and the constants

C_1, \dots, C_{j+1} . Using interpolation inequalities, see Remark 3.12 in [15], we can, thus, state the following Proposition.

Proposition 8.1. *Let $(\gamma_t)_{t \in [0, T]}$ be a smooth solution to the network flow, with N endpoints, satisfying the assumptions (8.7) and (8.8). Then there exists $T_0 > 0$, depending only on the L^2 -norm of the curvature of γ_0 , the inverses of the lengths of the segments of γ_0 , and N, C_1, C_2 such that for all $0 < t < \min\{T, T_0\}$ it holds for all $j > 0$ that*

$$|\nabla^j k| \leq \hat{C}_j \cdot t^{-\frac{j}{2} - \frac{1}{4}},$$

where \hat{C}_j depends only on the L^2 -norm of the curvature of γ_0 , the inverses of the lengths of the segments of γ_0 , N and the constants C_1, \dots, C_{j+1} .

Generalized self-similarly shrinking networks. In the following we define a *degenerate regular network*. It can be seen as a C^1 -limit of regular networks, where it is allowed that the lengths of some segments go to zero.

Definition 8.2 (Degenerate regular network). We consider a connected graph G consisting of a finite number of edges e_i , $1 \leq i \leq N_1$ and vertices v_j , $1 \leq j \leq N_2$. We assume that the edges are either homeomorphic to the interval $[0, 1]$, with two boundary points, or homeomorphic to $[0, \infty)$, with one boundary point. We assume that at the vertices always three such boundary points meet. We, furthermore, assume that there exists a continuous map $\Psi : G \rightarrow T\mathbb{R}^2$, $x \mapsto (\Psi(x), \Psi'(x))$ such that if e_i is homeomorphic to a finite interval, then either

- i) Ψ restricted to e_i is the smooth, regular parametrization of a curve in \mathbb{R}^2 up to the endpoints, with self-intersections possibly only at the endpoints, or
- ii) Ψ is degenerate, i.e., it maps to a fixed point $(p, v) \in T_p\mathbb{R}^2$ with $|v| = 1$, for some $p \in \mathbb{R}^2$.

If the edge is homeomorphic to a half-line we assume the first case. At each vertex we assume that the three tangent directions of the curves meeting there form 120-degree angles. We call (G, Ψ) as above a *degenerate regular network* if there exists a sequence of homeomorphisms $\Psi_k : G \rightarrow \mathbb{R}^2$ as above, such that $\Psi_k \rightarrow \Psi$ in C^1 , where we assume that the Ψ_k are actually embeddings, i.e., $\Psi_k(G)$ are regular networks. If one or several edges are mapped under Ψ to a single point p in \mathbb{R}^2 , we call this sub-network the core at p . Note that for a degenerate regular network, the core at a point p is always a connected sub-network.

We call (G, Ψ) a generalized self-similarly shrinking network, if Ψ is a degenerate embedding in the sense above, and $\Psi|_{e_i}$ satisfies the self-shrinker equation

$$\vec{k} = -\frac{x^\perp}{2}$$

for all $1 \leq i \leq N_1$. The evolving self-similar solution for $t \in (-\infty, 0)$ is then given by

$$N_t = \sqrt{-t}\Psi.$$

The most basic example of a generalized self-similarly shrinking solution with triple points is the union of three half-lines, meeting at the origin under a 120-degree condition. We will call this solution the *standard triod*.

The following Lemma is from [11], for the convenience of the reader we give the proof in here.

Lemma 8.3 (Hättenschweiler). *Let (G, Ψ) be a generalized self-similarly shrinking network, such that G is a tree. Then $\Psi(G)$ consists of half-lines emanating from the origin, with possibly a core at the origin.*

Proof. First note that any non-degenerate self-similar shrinking curve is a member of the one-parameter family of curves classified by Abresch and Langer in [1]. Their classification result implies the following. If the curve contains the origin, then it is a straight line through the origin. Otherwise it is contained in a compact subset of \mathbb{R}^2 , but it is still diffeomorphic to a line. In the latter case, any such curve has a constant winding direction with respect to the origin. Aside from the circle, any other solution has a countable, non-vanishing number of self-intersections.

Let us consider $\Gamma' \subset \Gamma := \psi(G)$, which consists of Γ with all half-lines going to infinity removed. For $\theta \in \mathbb{S}^1$ let $S(\theta)$ be the half-line, emanating from the origin in direction of θ . Consider

$$R(\theta) := \sup\{|x| \mid x \in \Gamma' \cap S(\theta)\}.$$

If Γ' is not only the core at the origin, there exists a φ_0 such that $R(\varphi_0) = |S(\varphi_0) \cap \gamma_i| > 0$, where γ_i is a non-degenerate curve of Ψ . Since the γ_i 's don't change their winding direction we have

$$R(\varphi) = |\gamma_i(\varphi)|$$

for all $\varphi \in \{\varphi \mid \gamma_i \cap S(\varphi) \neq \emptyset\}$. At an endpoint of γ_i we have $R(\cdot) > 0$ otherwise γ_i would have been a half-line, starting at the origin. At this endpoint, also if it has a core, there is always another $\gamma_{i'}$ which continues smoothly with the same winding direction, $R(\varphi)$ stays positive and

$$R(\varphi) = |\gamma_{i'}(\varphi)|$$

for all $\varphi \in \{\varphi \mid \gamma_{i'} \cap S(\varphi) \neq \emptyset\}$. This also implies that $R(\varphi)$ is continuous. Continuing until φ reaches again φ_0 we find a closed non-contractible loop in Γ' , which yields a contradiction. q.e.d.

Let us assume that $(\gamma_t)_{0 \leq t < T}$ is a network flow. Huisken's monotonicity formula implies that the function

$$\Theta_{x_0, t_0}(t) := \int_{\gamma_t} \rho_{x_0, t_0} ds$$

is decreasing in time for $t < t_0$, and the limit $\Theta(x_0, t_0) := \lim_{t \nearrow t_0} \Theta_{x_0, t_0}(t)$ is the Gaussian density at (x_0, t_0) . The function $\Theta_{x_0, t_0}(t)$ is constant in time if and only if the evolving network is a self-similarly shrinking network, centered at the space-time point (x_0, t_0) .

The Gaussian density of the shrinking sphere can easily be computed to be

$$\Theta_{S^1} = \sqrt{\frac{2\pi}{e}}.$$

Note that $\Theta_{S^1} > 3/2$. For a generalized self-similar shrinking network γ we denote $\Theta_\gamma := \int_\gamma \rho_{0,0}(\cdot, -1) ds$.

Lemma 8.4. *Let γ be a generalized self-similarly shrinking network and assume that $\Theta_\gamma < \Theta_{S^1}$. Then γ is tree-like, and, thus, either a multiplicity one line, or the standard triod.*

Proof. By the work of Colding–Minicozzi, [6], it holds that

$$(8.13) \quad \Theta_\gamma = \int_\gamma \rho_{0,0}(\cdot, -1) ds = \sup_{x_0 \in \mathbb{R}^2, t_0 > -1} \int_\gamma \rho_{x_0, t_0}(\cdot, -1) ds.$$

Assume that γ is not tree-like. Let us first assume that the complement of γ in \mathbb{R}^2 contains no bounded component. It is easy to see from the proof of the previous lemma that this implies that γ consists of at least six half-lines emanating from the origin, together with a core. Thus, would imply that $\Theta_\gamma \geq 3$, a contradiction.

Let B be a bounded component of the complement of γ and $\tilde{\gamma}$ the sub-network of γ which bounds B , counted with unit multiplicity. Since $\tilde{\gamma}$ is smooth with corners, and no triple junctions, we can evolve it by classical curve shortening flow until it shrinks at (x_0, t_0) to a ‘round’ point. By the monotonicity formula this implies that

$$\int_{\tilde{\gamma}} \rho_{x_0, t_0} ds \geq \Theta_{S^1}.$$

By (8.13) this implies

$$\Theta_\gamma \geq \int_\gamma \rho_{x_0, t_0} ds \geq \int_{\tilde{\gamma}} \rho_{x_0, t_0} ds \geq \Theta_{S^1}. \quad \text{q.e.d.}$$

Given a sequence $\lambda_i \nearrow \infty$ and a space-time point (x_0, t_0) , where $0 < t_0 \leq T$ the standard parabolic rescaling around (x_0, t_0) of the flow is given by

$$\gamma_\tau^i = \lambda_i (\gamma_{\lambda_i^{-2}\tau+t_0} - x_0),$$

where $\tau \in [-\lambda_i^2 t_0, \lambda_i^2 (T - t_0)]$. Recall that the monotonicity formula implies

$$\Theta_{x_0, t_0}(t) - \Theta(x_0, t_0) = \int_t^{t_0} \int_{\gamma_\sigma} \left| \vec{k} + \frac{(x - x_0)^\perp}{2(t_0 - \sigma)} \right|^2 \rho_{x_0, t_0}(\cdot, \sigma) ds d\sigma.$$

Changing variables according to the parabolic rescaling, we obtain

$$\Theta_{x_0, t_0}(t) - \Theta(x_0, t_0) = \int_{\lambda_i^2(t-t_0)}^0 \int_{\gamma_i^i} \left| \vec{k} - \frac{x^\perp}{2\tau} \right|^2 \rho_{0,0}(\cdot, \tau) ds d\tau,$$

or for fixed time $\tau_0 \in (-\lambda_i^2 t_0, 0)$,

$$(8.14) \quad \Theta_{x_0, t_0}(t_0 + \lambda_i^{-2} \tau_0) - \Theta(x_0, t_0) = \int_{\tau_0}^0 \int_{\gamma_\tau^i} \left| \vec{k} - \frac{x^\perp}{2\tau} \right|^2 \rho_{0,0}(\cdot, \tau) ds d\tau.$$

We now give a slightly modified version of the Blowup-Lemma in [11].

Lemma 8.5. *There exists a subsequence (λ_i) (relabelled again the same) such that for almost all $\tau \in (-\infty, 0)$ and for any $\alpha \in (0, 1/2)$*

$$\gamma_\tau^i \rightarrow \bar{\gamma}_\tau$$

in $C_{loc}^{1,\alpha} \cap W_{loc}^{2,2}$, where $\bar{\Gamma}_\tau$ is a generalized self-similarly shrinking network at time τ . This convergence also holds in the sense of radon measures for all τ . Note that the subsequence does not depend on τ and also not the limit (except for scaling).

Proof. We first choose a subsequence such that the rescalings converge as Brakke flows to a self-similarly shrinking tangent flow. Let

$$f_i(\tau) := \int_{\gamma_\tau^i} \left| \vec{k} - \frac{x^\perp}{2\tau} \right|^2 \rho_{0,0}(\cdot, \tau) ds.$$

Note that (8.14) implies that $f_i \rightarrow 0$ in $L_{loc}^1((-\infty, 0])$. Thus, there exists a subsequence such that f_i converges point-wise a.e. to zero. This implies that for any $R > 0$

$$\int_{\gamma_\tau^i \cap B_R(0)} |k|^2 ds \leq C,$$

independent of i . By choosing a further subsequence we can assume that Γ_τ^i converges in $C_{loc}^{1,\alpha}$ to a degenerate network. Note that each limiting segment, which is non-degenerate is in $W_{loc}^{2,2}$ and is a weak solution of

$$\vec{k} = \frac{x^\perp}{2\tau}.$$

By elliptic regularity, each such segment is actually smooth, and, thus, the limiting network is a generalized self-similarly shrinking network at time τ . But since this limit has to coincide in measure with the limiting Brakke-flow it is unique, and the whole sequence converges. The convergence in $W_{loc}^{2,2}$ is implied by the weak convergence in $W^{2,2}$ and the fact that $f_i(\tau) \rightarrow 0$. q.e.d.

This can be strengthened, if the limit has unit density.

Lemma 8.6. *Assume that a sequence of rescalings as above converges in the sense of Brakke flows to a regular self-similarly shrinking network,*

$$(\gamma_t^i) \rightarrow (\bar{\gamma}_t).$$

Then this convergence is smooth on all compact subsets of $\mathbb{R}^2 \times (-\infty, 0)$.

Proof. By the Lemma before we can choose a further subsequence such that we have $\Gamma_\tau^i \rightarrow \bar{\Gamma}_\tau$ in $C_{\text{loc}}^{1,\alpha} \cap W_{\text{loc}}^{2,2}$ for almost every τ . Now take any set of the form $\Omega = \bar{B}_R(0) \times [a, b]$, $a < b < 0$, where we choose R big enough, such that $\partial B_R(0)$ intersects $\bar{\Gamma}_\tau$ for $\tau \in [a - 2, b]$ only in the straight lines going out to infinity (if they exist). Since for almost every τ we have convergence in $C^{1,\alpha}$ we know that the Gaussian density ratios in this set are less than $1 + \varepsilon$ for all $\tau \in [a - 3/2, b]$. Thus, there can be no triple points present, and by the estimates of White [26], we can choose i_0 big enough such that $|\nabla^j k|$ is small on Γ_i for all $j \geq 0$ on $B_{R+1} \setminus B_R$ for all $\tau \in [a - 1, b]$ and all $i > i_0$. Now for any given $\epsilon, \delta > 0$ we choose i_0 even bigger such that there exists times τ_j , $j = 0, \dots, N := 2[(b - a + 1)/\delta] + 1$ such that

$$|\tau_0 - a + 1|, |\tau_N - b| \leq \delta, |\tau_{j+1} - \tau_j| \leq \delta$$

for all $0 \leq j \leq N - 1$ and

$$\|\Gamma_{\tau_j}^i - \bar{\Gamma}_{\tau_j}\|_{W^{2,2}(B_{R+1})} \leq \epsilon$$

for all $i > i_0$. We now fix $\epsilon > 0$ and adjust $\delta > 0$ accordingly such that we can ensure that by (8.12) that

$$\|\Gamma_\tau^i\|_{W^{2,2}(B_{R+1})} \leq C$$

for all $\tau \in [a, b]$ and $i > i_0$. The higher order interior estimates then prove smooth subsequential convergence. For this argument we had chosen a subsequence, but since we can always choose such a subsequence, the whole sequence converges. q.e.d.

Regularity results. In the following we will give some regularity results for ‘proper’ flows, where we say that a flow is a proper flow, given by its space-time track \mathcal{M} in an open subset U of space-time, if

$$\mathcal{M} = \overline{\mathcal{M}} \cap U,$$

compare with section 2.3 in [26].

Theorem 8.7. *Let $(\gamma_t^i)_{0 < t < T}$ be a sequence of smooth network flows with uniformly bounded length ratios which converges locally as Brakke flows to the standard triod. Then this convergence is smooth.*

Proof. We can assume that the triple point of the standard triod is at the origin. Fix $T' > T$. Then for any $0 < t_1 < t_2 < T$ we have

$$\int_{\gamma_{t_j}^i} \Phi_{0,T'} ds \rightarrow \frac{3}{2} \quad \text{where } j = 1, 2.$$

Then as in the proof of Lemma 8.5 we obtain that there is subsequence where we have $C^{1,\alpha}$ -convergence to the standard triod for a.e. $t \in (t_1, t_2)$. Note that by White's regularity theorem the convergence is smooth away from the triple point, and that out of combinatorial reasons no core can develop at the triple point of the standard triod. As in the proof of Lemma 8.6 we can then show that the convergence is actually smooth. q.e.d.

In the following we will prove a local regularity result in the spirit of Brian White's result for mean curvature flow [26]. We follow here the alternative proof of Ecker [7, Theorem 5.6]. The Gaussian density ratios are defined as

$$(8.15) \quad \Theta(x, t, r) := \Theta_{t-r^2}(x, t).$$

In the case of proper flows, which are only defined in an open subset of space-time one has to localize Huisken's monotonicity formula. Compare with section 10 in [25] and Remark 4.16 together with Proposition 4.17 in [7]. To keep the presentation simpler, we will only give proofs of the next two theorems for flows defined on all of \mathbb{R}^2 and leave the modifications in the case of proper flows to the reader.

Theorem 8.8. *Let $(\gamma_t)_{t \in [0, T]}$ be a smooth, proper and regular planar network flow in $B_\rho(x_0) \times (t_0 - \rho^2, t_0)$ which reaches the point x_0 at time $t_0 \in (0, T]$. Assume that for some $\varepsilon > 0$ it holds that*

$$(8.16) \quad \Theta(x, t, r) \leq 2 - \varepsilon$$

for all $(x, t) \in B_\rho(x_0) \times (t_0 - \rho^2, t_0)$ and $0 < r < \eta\rho$ for some $\eta > 0$, where $(1 + \eta)\rho^2 \leq t_0 < T$. Furthermore, assume that $\gamma_t \cap B_\rho(x_0)$ has no closed loops of length less than $\delta\rho > 0$ for all $t \in (t_0 - (1 + \eta)\rho^2, t_0)$ for some $\delta > 0$. Then there exists $C = C(\varepsilon, \eta, \delta)$ such that

$$|k|^2(x, t) \leq \frac{C}{\sigma^2 \rho^2}$$

for $(x, t) \in (\gamma_t \cap B_{(1-\sigma)\rho}(x_0)) \times (t_0 - (1 - \sigma)^2 \rho^2, t_0)$ and all $\sigma \in (0, 1)$.

REMARK 8.9. Note that the bound on the curvature, together with the balancing condition and (8.16), gives that there is a constant $\kappa = \kappa(\varepsilon, \eta, \delta) > 0$ such that the length of each segment which intersects $B_{(1-\sigma)\rho}(x_0) \times (t_0 - (1 - \sigma)^2 \rho^2, t_0)$ is bounded from below by $\kappa \cdot \sigma\rho$. This implies, using Theorem 8.1, corresponding scaling invariant estimates on all higher derivatives of the curvature.

Proof. We can first assume that $t_0 < T$, and pass to limits later. By translation and scaling we can, furthermore, assume that $x_0 = 0$ and $\rho = 1$. We can now follow more or less verbatim the proof of Theorem 5.6 in [7]. Supposing that the statement is not correct we can find a sequence of smooth, regular network flows (γ_t^j) , defined for $t \in [-1 - \eta, 0]$, reaching the point $(0, 0)$ and satisfying the above conditions, but

$$(8.17) \quad \zeta_j^2 := \sup_{\sigma \in (0,1)} \left(\sigma^2 \sup_{(-1-\sigma)^2, 0) \cap \gamma_t^j \cap B_{1-\sigma}} |k|^2 \right) \rightarrow \infty$$

as $j \rightarrow \infty$. We can find $\sigma_j \in (0, 1)$ such that

$$\zeta_j^2 = \sigma_j^2 \sup_{(-1-\sigma_j)^2, 0) \cap \gamma_t^j \cap B_{1-\sigma_j}} |k|^2,$$

and $y_j \in \gamma_{\tau_j}^j \cap \bar{B}_{1-\sigma_j}$ at a time $\tau_j \in [-(1 - \sigma_j)^2, 0]$ so that

$$(8.18) \quad \zeta_j^2 = \sigma_j^2 |k(y_j, \tau_j)|^2.$$

We now take

$$\lambda_j = |k(y_j, \tau_j)|^{-1},$$

and define

$$\tilde{\gamma}_s^j = \frac{1}{\lambda_j} \left(\gamma_{\lambda_j^2 s + \tau_j}^j - y_j \right)$$

for $s \in [-\lambda_j^2 \sigma_j^2 / 4, 0]$. As in the proof of Theorem 5.6 in [7] we see that

$$(8.19) \quad 0 \in \tilde{\gamma}_0^j, \quad |k(0, 0)| = 1,$$

and

$$\sup_{(-\lambda_j^{-2} \sigma_j^2 / 4, 0) \cap \tilde{\gamma}_s^j \cap B_{\lambda_j^{-1} \sigma_j / 2}} |k|^2 \leq 4$$

for every $j \geq 1$. Since $\lambda_j^{-2} \sigma_j^2 = \zeta_j^2 \rightarrow \infty$ we see that up to a subsequence, labeled again the same,

$$(8.20) \quad \tilde{\gamma}_s^j \rightarrow \tilde{\gamma}_s^\infty$$

converges locally uniformly $\mathbb{R}^2 \times \mathbb{R}$ in $C^{1,\alpha}$ to a limiting $C^{1,1}$ -solution $\tilde{\gamma}_s^\infty$ of the network flow. Note that this limiting solution is defined for $s \in (-\infty, 0]$ and possibly degenerate, i.e., cores and higher density lines can develop. But note that (8.16) implies that on $\tilde{\gamma}_s^\infty$

$$(8.21) \quad \Theta(x, t, r) \leq 2 - \epsilon$$

for all $r > 0$ and (x, t) in $\mathbb{R}^2 \times (-\infty, 0]$. Together with the fact that $\tilde{\gamma}_s^\infty$ is uniformly bounded in $C^{1,1}$, this implies that $\tilde{\gamma}_s^\infty$ is non-degenerate, i.e., there are no higher densities and no cores. Furthermore, the assumption on the lower bound for the length of closed loops implies that $\tilde{\gamma}_s^\infty$ is tree-like. The estimate (8.21) yields that the tangent flow at $-\infty$ is either a static unit density line, or the standard triod and $\lim_{r \rightarrow -\infty} \Theta(x, t, r)$

is either 1 or $3/2$. In the first case this implies that $\tilde{\gamma}^\infty$ is a static unit density line. But then White's local regularity theorem implies that the convergence in (8.20) is smooth. This gives a contradiction to (8.19). In the second case $\tilde{\gamma}^\infty$ has to have a triple point, and, thus, is the standard triod. Then Proposition 8.7 gives a contradiction as before. q.e.d.

Without the assumption on the length of the shortest loops, we prove a similar statement if the Gaussian densities are less than Θ_{S^1} :

Proof of Theorem 1.3. The proof is nearly identical to the proof of Theorem 8.8. Rescaling and translating such that $x_0 = 0$ and $\rho = 1$ we assume that as a contradiction we have a sequence of smooth, regular network flows (γ_t^j) defined for $t \in [-1 - \eta, 0]$, reaching $(0, 0)$, satisfying (8.17) and

$$(8.22) \quad \Theta(x, t, r) \leq \Theta_{S^1} - \varepsilon$$

for all $(x, t) \in B_1 \times (-1, 0)$ and $0 < r < \eta$. Rescaling as before we obtain a limiting $C^{1,1}$ solution $(\tilde{\gamma}_s^\infty)$ which satisfies

$$\Theta(x, t, r) \leq \Theta_{S^1} - \varepsilon$$

for all $r > 0$ and $(x, t) \in \mathbb{R}^2(-\infty, 0]$. By Lemma 8.4 this implies that the tangent flow at $-\infty$ is either a static unit density line, or the standard triod. We reach a contradiction as in the proof of the previous theorem. q.e.d.

9. A pseudolocality result for mean curvature flow

We recall the following setup from the introduction. For any point $x \in \mathbb{R}^{n+k}$ we write $x = (\hat{x}, \tilde{x})$ where \hat{x} is the orthogonal projection of x on the \mathbb{R}^n -factor and \tilde{x} the orthogonal projection on the \mathbb{R}^k factor. We define the cylinder $C_R(x_0) \subset \mathbb{R}^{n+k}$ by

$$C_r(x) = \{x \in \mathbb{R}^{n+k} \mid |\hat{x} - \hat{x}_0| < r, |\tilde{x} - \tilde{x}_0| < r\}.$$

Furthermore, we write $B_r^n(x_0) = \{(\hat{x}, \tilde{x}_0) \in \mathbb{R}^{n+k} \mid |\hat{x} - \hat{x}_0| < r\}$.

Proof of Theorem 1.5. We first assume that $T \geq 1$. Translating x_0 to 0 and rescaling with a factor $R > 0$ we can assume that $M_0 \cap C_R(0) = \text{graph}(u)$ where $u : B_R^n(0) \rightarrow \mathbb{R}^k$ with Lipschitz constant less than ε . We want to show that there exists $R \gg 1$ such that $M_t \cap C_1(0)$ is a graph with Lipschitz constant less than η and height bounded by $\eta/2$ for all $t \in [0, 1]$.

Recall that by [26], if all Gaussian density ratios up to scale 1 centered at $(x, t) \in B_2(0) \times [0, 1]$ are bounded above by $1 + \varepsilon_0$ then

$$|A|_{M_1}(x) \leq C(\varepsilon_0)$$

for all $x \in M_1 \cap C_1(0)$. Furthermore, a compactness argument implies that $C(\varepsilon_0) \rightarrow 0$ as $\varepsilon \rightarrow 0$. This implies that we can choose $0 < \varepsilon_1 < \varepsilon_0$

such that if the Gaussian density ratios up to scale 1 centered at $(x, t) \in B_2(0) \times [0, 1]$ are bounded by $1 + \varepsilon_1$ and

$$M_1 \cap C_1(0) \subset C_1 \cap \{|\tilde{y}| \leq \eta/2\},$$

then $M_1 \cap C_1(0)$ is a graph over $B_1^n(0)$ with Lipschitz constant bounded above by η .

Now assume that $y \in C_2(0) \cap \{|\hat{y}| \leq \eta/2\}$. We then have for $R \geq 2, r \leq 1$ that

$$\begin{aligned}
 \Theta_0(y, r) &= \int_{M_0 \cap C_R} \frac{1}{(4\pi r^2)^{n/2}} e^{-\frac{|x-y|^2}{4r^2}} d\mathcal{H}^n(x) \\
 &\quad + \int_{M_0 \setminus C_R} \frac{1}{(4\pi r^2)^{n/2}} e^{-\frac{|x-y|^2}{4r^2}} d\mathcal{H}^n(x) \\
 (9.1) \quad &\leq \int_{B_R^n(0)} \frac{1}{(4\pi r^2)^{n/2}} e^{-\frac{|\hat{x}-\hat{y}|^2}{4r^2}} \left(\det(\mathbf{1} + Du^\top \circ Du) \right)^{1/2} d\hat{x}^n \\
 &\quad + e^{-\frac{(R-3)^2}{8r^2}} \int_{M_0 \setminus C_R} \frac{1}{(4\pi r^2)^{n/2}} e^{-\frac{|x-y|^2}{8r^2}} d\mathcal{H}^n(x) \\
 &\leq (1 + \varepsilon^2)^{n/2} + Ce^{-\frac{(R-3)^2}{8r^2}} \leq 1 + \varepsilon_1,
 \end{aligned}$$

provided $\varepsilon \leq \varepsilon_2$ and $R \geq R_0 \geq 3$.

By assumption we have that $\sup_{B_R^n(0)} |u| \leq \varepsilon R$. Let us assume that ε is small enough, depending on R , such that

$$(9.2) \quad \varepsilon R \leq \eta/4.$$

Now let $y \in C_2(0) \setminus \{|\hat{y}| \leq \eta/2\}$. We can then estimate

$$\begin{aligned}
 \Theta_0(y, r) &= \int_{M_0 \cap C_R} \frac{1}{(4\pi r^2)^{n/2}} e^{-\frac{|x-y|^2}{4r^2}} d\mathcal{H}^n(x) \\
 &\quad + \int_{M_0 \setminus C_R} \frac{1}{(4\pi r^2)^{n/2}} e^{-\frac{|x-y|^2}{4r^2}} d\mathcal{H}^n(x) \\
 &\leq \int_{B_R^n(0)} \frac{1}{(4\pi r^2)^{n/2}} e^{-\frac{|u(\hat{x})-\hat{y}|^2 + |\hat{x}-\hat{y}|^2}{4r^2}} \left(\det(\mathbf{1} + Du^\top \circ Du) \right)^{1/2} d\hat{x}^n \\
 &\quad + e^{-\frac{(R-3)^2}{8r^2}} \int_{M_0 \setminus C_R} \frac{1}{(4\pi r^2)^{n/2}} e^{-\frac{|x-y|^2}{8r^2}} d\mathcal{H}^n(x) \\
 &\leq e^{-\frac{\eta^2}{64r^2}} (1 + \varepsilon^2)^{n/2} + Ce^{-\frac{(R-3)^2}{8r^2}}.
 \end{aligned}$$

Now we choose $R = R_1 \geq R_0$ such that

$$Ce^{-\frac{(R_1-2)^2}{8}} < 1 - e^{-\frac{\eta^2}{128}}.$$

Then choose $\varepsilon \leq \min\{\varepsilon_2, 4^{-1}\eta R_1^{-1}\}$ such that

$$e^{-\frac{\eta^2}{64}}(1 + \varepsilon^2)^{n/2} < e^{-\frac{\eta^2}{128}}.$$

Note that the first assumption on ε implies that (9.2) is satisfied. We see that that $\Theta_0(y, r) < 1$ for all $y \in C_2 \setminus \{|\hat{y}| \leq \eta/2\}$. Using the monotonicity of the Gaussian density ratios this yields that

$$M_t \cap C_2(0) \subset C_2(0) \cap \{|\hat{y}| \leq \eta/2\}$$

for all $t \in [0, 1]$ and by the choice of ε and estimate (9.1) that

$$\Theta(x, t, r) \leq 1 + \varepsilon_1$$

for all $(x, t) \in C_2(0) \times [0, 1]$ and scales r up to one. By the choice of ε_1 this gives that $M_1 \cap C_1(0)$ is a smooth graph over $B_1^n(0)$ with Lipschitz constant bounded above by η .

We want to show that this is also true for $M_t \cap C_1(0)$ for any $0 < t < 1$. Pick $t_0 \in (0, 1)$ and let $\lambda = t_0^{-1/2}$. Let $(M_t^\lambda)_{0 \leq t \leq \lambda^2 T}$ be the flow, parabolically rescaled by λ . Note that for any $x_0 \in M_0^\lambda \cap C_{(\lambda-1)R_1}(0)$ we can shift x_0 to 0 and see that our previous assumptions are satisfied for this flow. That yields that $M_1^\lambda \cap C_1(x_0)$ is a smooth graph over $B_1^n(\hat{x}_0)$ with Lipschitz constant bounded above by η . Note that this property is scaling invariant. Scaling back this implies that $M_{t_0} \cap C_{(1-t_0^{1/2})R_1+t_0^{1/2}}(0)$ is a graph over $B_{(1-t_0^{1/2})R_1+t_0^{1/2}}^n(0)$ with Lipschitz constant less than η .

Since $(1 - t_0^{1/2})R_1 + t_0^{1/2} \geq 2 - t_0^{1/2} \geq 1$ this implies the statement.

If $T < 1$ we can first rescale the flow by a factor $\lambda = T^{-1/2}$ as above and then scale back to get the result for $0 < t < T$. q.e.d.

10. Appendix

We derive some useful technical results. In what follows σ_0 is a regular network with $(\sigma_t)_{0 \leq t < T}$ being a regular solution for network flow. Moreover, χ is a fixed regular network in \mathbb{R}^2 and ε_0 is a universal constant less than $1/2$.

Lemma 10.1. *Fix ε_0 and α . There exist $\varepsilon = \varepsilon(\chi, \varepsilon_0, \alpha)$ and $q_1 = q_1(\chi, \varepsilon_0, \alpha)$ so that, for every $R \geq 2$, if σ_0 is ε -close to χ in $C^{1,\alpha}(B_R(0))$, then for every $r^2, t \leq q_1$ and $y \in B_{R-1}(0)$*

$$\Theta_t(y, r) \leq 3/2 + \varepsilon_0.$$

Proof. We argue by contradiction. Suppose there are sequences $(\varepsilon_i)_{i \in \mathbb{N}}$, $(r_i)_{i \in \mathbb{N}}$, $(t_i)_{i \in \mathbb{N}}$ all converging to zero, $(R_i)_{i \in \mathbb{N}}$ with $R_i \geq 2$ for all i , $(y_i)_{i \in \mathbb{N}}$ with $y_i \in B_{R_i-1}(0)$, and $(\sigma_t^i)_{0 \leq t \leq t_i}$ a sequence of regular

solutions to network flow for which σ_0^i is ε_i -close to χ in $C^{1,\alpha}(B_{R_i}(0))$ and

$$\Theta_{t_i}(y_i, r_i) > 3/2 + \varepsilon_0.$$

The fact that σ_i is ε_i -close to χ in $C^{1,\alpha}(B_{R_i}(0))$ means that

- a) there are functions u_i defined on $C^0(\chi \cap B_{R_i}(0))$ which are $C^{1,\alpha}$ when restricted to each branch and the $C^{1,\alpha}$ norm on each branch converges to zero;
- b) there are unit vectors N_i defined on $\chi \cap B_{R_i}(0)$ such that $\langle N_i, \nu \rangle$ is a smooth function on each branch that converges uniformly to one, where ν denotes a unit normal vector on the respective branch;
- c)

$$\sigma_0^i \cap B_{R_i}(0) = \chi \cap B_{R_i}(0) + u_i N_i.$$

Set $\lambda_i = \sqrt{r_i^2 + t_i}$. It is simple to recognize that we can find functions v_i defined on $\lambda_i^{-1}\chi$ such that

$$\lambda_i^{-1}\sigma_0^i \cap B_{R_i\lambda_i^{-1}}(0) = \lambda_i^{-1}\chi \cap B_{R_i\lambda_i^{-1}}(0) + v_i N_i.$$

Because the $C^{0,\alpha}$ norm of the first derivatives of v_i converges uniformly to zero on each branch of $\lambda_i^{-1}\chi \cap B_{R_i\lambda_i^{-1}}(0)$ we obtain that if the limit of $\lambda_i^{-1}(\sigma_0^i - y_i)$ in the varifold sense is not empty, then it must be either a line or three half-lines meeting at a common point. In any case we have

$$\lim_i \int_{\lambda_i^{-1}(\sigma_0^i - y_i)} (4\pi)^{-1/2} \exp(-|x|^2/4) d\mathcal{H}^1 \leq 3/2.$$

This contradicts the fact that, using the monotonicity formula,

$$\begin{aligned} 3/2 + \varepsilon_0 &< \Theta_{t_i}(y_i, r_i) \leq \Theta_0(y_i, \lambda_i) \\ &= \int_{\lambda_i^{-1}(\sigma_0^i - y_i)} (4\pi)^{-1/2} \exp(-|x|^2/4) d\mathcal{H}^1. \quad \text{q.e.d.} \end{aligned}$$

Assume that σ is non-compact, asymptotic to half-lines at infinity, and contains no closed loops.

Lemma 10.2. *Fix ε_0 , $R > 2$, and τ . There is $\eta = \eta(\tau)$ such that if*

$$\Theta(x, r) \leq 3/2 + \varepsilon_0 < 2$$

for every x in $B_R(0)$ and $r^2 \leq \tau$, then the distance between any two triple junctions in $B_R(0)$ is greater than η .

Proof. Choose $T > 0$ so that

$$\int_0^T (4\pi)^{-1/2} \exp(-t^2/4) dt = 1/2 + \varepsilon_0/16 - 1/32,$$

and $r_1 > 0$ so that

$$\int_0^{r_1} (4\pi)^{-1/2} \exp(-t^2/4) dt = 1/32 - \varepsilon_0/16.$$

Suppose that σ has two triple junctions in $B_R(0)$ at a distance η smaller than τr_1 . Denote the midpoint between the triple junctions by y and consider the network $\gamma = \tau^{-1}(\sigma - y)$ which has two triple junctions x_1 and $-x_1$ inside $B_{r_1}(0)$. Because γ has no closed loops we can find paths a_1, b_1, a_2 , and b_2 contained in γ such that a_1, a_2 and b_1, b_2 connect x_1 and $-x_1$ respectively to a point at a distance T from the origin and $a_1 \cap a_2 = \{x_1\}$, $b_1 \cap b_2 = \{-x_1\}$, $a_i \cap b_j = \emptyset$.

Consider the metric $g = (4\pi)^{-1} \exp(-|x|^2/2)(dx_1^2 + dx_2^2)$ on \mathbb{R}^2 and denote its distance function by d_g . Straight lines containing the origin are geodesics for g and so any point with $|p| = T$ has

$$d_g(x, p) \geq 1/2 + \varepsilon_0/8 - 1/16 \quad \text{for any } x \in B_{r_1}(0).$$

Thus,

$$\begin{aligned} 3/2 + \varepsilon_0 &\geq \Theta(y, \tau) = \int_{\gamma} (4\pi)^{-1/2} \exp(-|x|^2/4) d\mathcal{H}^1 \\ &\geq \sum_{i=1}^2 \left(\int_{a_i} (4\pi)^{-1/2} \exp(-|x|^2/4) d\mathcal{H}^1 + \int_{b_i} (4\pi)^{-1/2} \exp(-|x|^2/4) d\mathcal{H}^1 \right) \\ &= \sum_{i=1}^2 \left(\int_{a_i} dl_g + \int_{b_i} dl_g \right) \geq 4(1/2 + \varepsilon_0/8 - 1/16) = 7/4 + \varepsilon_0/2. \end{aligned}$$

This is impossible because $\varepsilon_0 < 1/2$.

q.e.d.

References

- [1] U. Abresch and J. Langer, *The normalized curve shortening flow and homothetic solutions*, J. Differential Geom. **23** (1986), no. 2, 175–196, MR0845704, Zbl 0592.53002.
- [2] G. Bellettini and M. Novaga, *Curvature evolution of nonconvex lens-shaped domains*, J. Reine Angew. Math. **656** (2011), 17–46, MR2818854, Zbl 1237.53063.
- [3] K. Brakke, *The motion of a surface by its mean curvature*, Princeton Univ. Press, 1978, MR0485012, Zbl 0386.53047.
- [4] S. Brendle and G. Huisken, *Mean curvature flow with surgery of mean convex surfaces in \mathbb{R}^3* , Invent. Math. **203** (2016), no. 2, 615–654, MR3455158, Zbl 06550066.
- [5] B.-L. Chen and L. Yin, *Uniqueness and pseudolocality theorems of the mean curvature flow*, Comm. Anal. Geom. **15** (2007), no. 3, 435–490, MR2379801, Zbl 1138.53055.
- [6] T. H. Colding and W. P. Minicozzi, II, *Generic mean curvature flow I: generic singularities*, Ann. of Math. (2) **175** (2012), no. 2, 755–833, MR2993752, Zbl 1239.53084.
- [7] K. Ecker, *Regularity theory for mean curvature flow*, Progress in Nonlinear Differential Equations and their Applications, 57, Birkhäuser Boston Inc., Boston, MA, 2004, MR2024995, Zbl 1058.53054.

- [8] K. Ecker and G. Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), no. 3, 547–569, MR1117150, Zbl 0707.53008.
- [9] M. E. Gage and R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Diff. Geom. **23** (1986), 69–96, MR0840401, Zbl 0621.53001.
- [10] M. A. Grayson, *The heat equation shrinks embedded plane curves to points*, J. Diff. Geom. **26** (1987), 285–314, MR0906392, Zbl 0667.53001.
- [11] J. Hättenschweiler, *Mean curvature flow of networks with triple junctions in the plane*, Diplomarbeit, Zürich: ETH Zürich, Department of Mathematics, 46 p., 2007.
- [12] K. Kasai and Y. Tonegawa, *A general regularity theory for weak mean curvature flow*, Calc. Var. Partial Differential Equations **50** (2014), nos 1–2, 1–68, MR3194675, Zbl 1298.53063.
- [13] A. Magni, C. Mantegazza, and M. Novaga, *Motion by curvature of planar networks, II*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **15** (2016), 117–144, MR3495423, Zbl 1339.53063.
- [14] C. Mantegazza, *Evolution by curvature of networks of curves in the plane*, Variational problems in Riemannian geometry, Progr. Nonlinear Differential Equations Appl., vol. 59, Birkhäuser, Basel, 2004, Joint project with Matteo Novaga and Vincenzo Maria Tortorelli, pp. 95–109, MR2076269, Zbl 1073.53085.
- [15] C. Mantegazza, M. Novaga, and V. Maria Tortorelli, *Motion by curvature of planar networks*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **3** (2004), no. 2, 235–324, MR2076269, Zbl 1170.53313.
- [16] R. Mazzeo and M. Sáez, *Self-similar expanding solutions for the planar network flow*, Analytic aspects of problems in Riemannian geometry: elliptic PDEs, solitons and computer imaging, Sémin. Congr., vol. 22, Soc. Math. France, Paris, 2011, pp. 159–173, MR3060453, Zbl 1280.53062.
- [17] A. Neves, *Singularities of Lagrangian mean curvature flow: zero-Maslov class case*, Invent. Math. **168** (2007), no. 3, 449–484, MR2299559, Zbl 1119.53052.
- [18] A. Neves, *Finite time singularities for Lagrangian mean curvature flow*, Ann. of Math. (2) **177** (2013), no. 3, 1029–1076, MR3034293, Zbl 1315.53074.
- [19] A. Neves and G. Tian, *Translating solutions to Lagrangian mean curvature flow*, Trans. Amer. Math. Soc. **365** (2013), no. 11, 5655–5680, MR3091260, Zbl 1282.53057.
- [20] M. Sáez Trumper, *Uniqueness of self-similar solutions to the network flow in a given topological class*, Comm. Partial Differential Equations **36** (2011), no. 2, 185–204, MR2763337, Zbl 1257.35003.
- [21] O. C. Schnürer, A. Azouani, M. Georgi, J. Hell, N. Jangle, A. Koeller, T. Marxen, S. Rittthaler, M. Sáez, F. Schulze, and B. Smith, *Evolution of convex lens-shaped networks under the curve shortening flow*, Trans. Amer. Math. Soc. **363** (2011), no. 5, 2265–2294, MR2763716, Zbl 1120.53083.
- [22] O. C. Schnürer and F. Schulze, *Self-similarly expanding networks to curve shortening flow*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **6** (2007), no. 4, 511–528, MR2394409, Zbl 1139.53031.
- [23] Y. Tonegawa, *A second derivative Hölder estimate for weak mean curvature flow*, Adv. Calc. Var. **7** (2014), no. 1, 91–138, MR3176585, Zbl 1283.53064.

- [24] Y. Tonegawa and N. Wickramasekera, *The blow up method for Brakke flows: networks near triple junctions*, Arch. Ration. Mech. Anal. **221** (2016), no. 3, 1161–1222, MR3508999, Zbl 1347.35048.
- [25] B. White, *Stratification of minimal surfaces, mean curvature flows, and harmonic maps*, J. reine angew. Math. **488** (1997), 1–35, MR1465365, Zbl 0874.58007.
- [26] B. White, *A local regularity theorem for mean curvature flow*, Ann. of Math. (2) **161** (2005), no. 3, 1487–1519, MR2180405, Zbl 1091.53045.

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