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ON ETA-FUNCTIONS FOR NILMANIFOLDS

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To the memory of Friedrich Hirzebruch

Abstract

Motivated by index formulas for Dirac-type operators over negatively curved Riemannian manifolds of finite volume, we study η-functions of certain differential operators on nilmanifolds.

1. Introduction

The index theorem of Atiyah, Patodi, and Singer for elliptic differential operators of first order over closed manifolds M with boundary involves the η-invariant of an associated formally self-adjoint elliptic operator of first order over ∂M. By definition, the η-invariant of such an operator A is the value at 0 of the η-function of A, for $s \in \mathbb{C}$ with sufficiently large real part given by the absolutely convergent series

$$\eta(A, s) := \sum_{\lambda} \operatorname{sign}(\lambda)|\lambda|^{-s}, \tag{1.1}$$

where the summation is over the non-zero eigenvalues of A, each eigenvalue occuring as often as its multiplicity requires. The η-function of A is a meromorphic function in the (whole) complex plane; see [**3**, p. 74]. It is a priori not clear whether $\eta(A) = \eta(A, 0)$ is finite. However, in relevant cases it is, by the work of Atiyah, Patodi, and Singer; see for example [**3**, Theorem 4.5].

Our work on the η-function was motivated by index problems for generalized Dirac operators over non-compact Riemannian manifolds with pinched negative sectional curvature and finite volume. A neighborhood of infinity of such manifolds is of the form $(0, \infty) \times M_0$ with Riemannian metric of the form $dt^2 + g_t$, where g_t is a family of Riemannian metrics on M_0; see [**6**]. The connected components of the cross sections $M_t = \{t\} \times M_0$ are infra-nilmanifolds; for so-called neat lattices in symmetric spaces of negative sectional curvature they are of the form $\Gamma\backslash N$, where N is a nilpotent Lie group of a specific Heisenberg type. The η-invariant of importance here is the limit, as $t \to \infty$, of the η-invariants of the induced operators over M_t; see [**4**, Theorem 8.10]. Up to sign, its

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so-called high-energy part is given by the asymptotic η-invariant of associated operators over the connected components of the cross sections M_t, by [**4**, Theorem 9.29], and the η-function of such operators is the objective of our study.

To set the stage, let N be a simply connected nilpotent Lie group of dimension n, endowed with a left-invariant Riemannian metric and the spin structure induced by the Lie algebra $\mathfrak{n}$ of left-invariant vector fields on N. Denote by $\mathrm{Cliff}(\mathfrak{n})$ and $\Sigma_{\mathfrak{n}}$ the complex Clifford algebra and the complex vector space of spinors associated to $\mathfrak{n}$, respectively, and recall that $\Sigma_{\mathfrak{n}}$ is a $\mathrm{Cliff}(\mathfrak{n})$-module.

Let $\Gamma \subseteq N$ be a lattice, and let $\tau : \Gamma \to \mathrm{U}(V)$ be a unitary representation of Γ on a finite-dimensional Hermitian vector space V. We refer to τ as the *twist*. For ease of notation, we extend τ trivially to a unitary representation on $\Sigma_{\mathfrak{n}} \otimes V$,

$$\tau \colon \Gamma \to U(\Sigma_{\mathfrak{n}} \otimes V), \quad \tau(\gamma) := \mathrm{id} \otimes \tau(\gamma). \tag{1.2}$$

Associated to τ, we obtain a Hermitian vector bundle

$$E_\tau = N \times_\tau (\Sigma_{\mathfrak{n}} \otimes V) \to \Gamma \backslash N, \tag{1.3}$$

where the elements of E_τ are Γ-orbits $\{(\gamma x, \tau(\gamma)w)\}$ in $N \times (\Sigma_{\mathfrak{n}} \otimes V)$. Sections of E_τ correspond to maps

$$\sigma \colon N \to \Sigma_{\mathfrak{n}} \otimes V \quad \text{such that} \quad \sigma(\gamma x) = \tau(\gamma)\sigma(x), \tag{1.4}$$

for all $\gamma \in \Gamma$ and $x \in N$. Clifford multiplication by vector fields on the factor $\Sigma_{\mathfrak{n}}$ commutes with τ, since τ acts trivially on $\Sigma_{\mathfrak{n}}$. Hence Clifford multiplication on E_τ is well-defined. The Levi–Civita connection and the left-invariant flat connection on N induce Hermitian connections on E_τ, and, with respect to both, E_τ turns into a Dirac bundle in the sense of Gromov and Lawson; see [**8**].

Example 1.5 (Spinor bundles)**.** Spin structures of $\Gamma \backslash N$ are determined by representations $\tau : \Gamma \to \{\pm 1\} \subseteq \mathrm{U}(1)$. The corresponding spinor bundles are given as $E_\tau = N \times_\tau (\Sigma_{\mathfrak{n}} \otimes \mathbb{C})$. The left-invariant spin structure $N \times \mathrm{Spin}(\mathfrak{n})$ corresponds to the trivial representation $\tau \equiv 1$.

Fix an orthonormal frame $X_1, \dots, X_n$ of $\mathfrak{n}$. Then the (flat) Dirac operator A on sections of E_τ induced by the left-invariant flat connection on N can be written as

$$A\sigma = \sum X_j \cdot d\sigma(X_j), \tag{1.6}$$

where the dot indicates Clifford multiplication. In the case where N is the Heisenberg group and τ is the trivial representation, this operator occurs in the work [**5**] of Deninger and Singhof on e-invariants. Ideas from their article were important for the determination of the asymptotic high-energy η-invariant in [**4**, Section 9].

It is easy to see that A is a formally self-adjoint and elliptic differential operator of order 1 with symbol

$$\sigma_A(d\varphi)\sigma = \operatorname{grad}\varphi \cdot \sigma. \tag{1.7}$$

Denote by $L^2(E_\tau)$ the space of square integrable sections of E_τ. We consider A as an unbounded self-adjoint operator in $L^2(E_\tau)$ with $H^1(E_\tau)$ as domain of definition, where $H^1(E_\tau)$ denotes the space of all H^1-sections of E_τ, that is, of square integrable sections σ of E_τ with square integrable weak derivatives. If $F \subseteq N$ is a fundamental domain for the action of Γ, then

$$\|\sigma\|_{L^2}^2 = \int_F |\sigma|^2 \quad \text{and} \quad \|\sigma\|_{H^1}^2 = \int_F (|\sigma|^2 + |d\sigma|^2)$$

if we identify sections σ of E_τ with maps $N \to \Sigma_{\mathfrak{n}} \otimes V$ as in (1.4).

We are concerned with the η-function $\eta(A, s)$ of A as an unbounded self-adjoint operator in $L^2(E_\tau)$. Note that the η-function of A is the sum of the corresponding η-functions for the decomposition of V into irreducible representations of Γ. Thus we may assume throughout that τ is irreducible.

It is shown in [**4**, Theorem 9.31] that the η-function of A vanishes identically if the center C_N of N has dimension at least 2. Thus we can restrict our attention to the case where C_N is of dimension 1. Note that this is precisely the interesting case in the representation theory of nilpotent Lie groups.

We choose X_1 as a generator of C_N. The center $C_\Gamma = \Gamma \cap C_N$ of Γ is infinite cyclic and is generated by $\zeta := \exp(\ell X_1)$, for some $\ell > 0$. Then $\Gamma\backslash N$ is foliated by closed geodesics of equal length ℓ, the translates of $C_\Gamma\backslash C_N$. For convenience, we rescale the metric so that $\ell = 2\pi$.

Theorem 1.8. *Up to the normalization $\ell = 2\pi$, the η-function of A does not depend on the left-invariant Riemannian metric on N.*

Remarks 1.9. (1) Via Malcev polynomials, Γ determines N. Thus we may consider the η-function of A as an invariant of the pair (Γ, τ).
(2) In [**2**], Atiyah, Patodi, and Singer discuss the stability of η-invariants of twisted versions of the standard Dirac operators; see [**2**, Theorems 2.4 and 3.3]. They normalize by considering differences of such η-invariants and get strong stability properties. For the operators considered here, we do not need to take differences, but the stability property is much more restricted.

The only simply connected two-step nilpotent Lie groups with one-dimensional center are the standard Heisenberg groups H_m. We think of H_m as $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ with group law given by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \langle x, y' \rangle). \tag{1.10}$$

Let D_m be the set of m-tupels $d = (d_1, \dots, d_m)$ of natural numbers such that d_i divides d_{i+1}, $1 \le i < m$. Then, for any $d \in D_m$,

$$\Gamma_d := \{(x, y, z) \mid x, y \in \mathbb{Z}^m, z \in \mathbb{Z}, d_i \text{ divides } x_i\} \tag{1.11}$$

is a lattice in H_m. Gordon and Wilson showed that the isomorphism type of Γ_d is determined by d and that, up to automorphism of H_m, any lattice in H_m is equal to some Γ_d; see [**7**, Section 2]. Note that $(0, 0, 1)$ is in the center of N and that, for any irreducible unitary representation of Γ_d, $\tau(0, 0, 1)$ acts by multiplication with $e^{2\pi i c}$, for some constant $c \in (0, 1]$.

Theorem 1.12. [**4**, Theorem 10.47] *For any irreducible unitary representation τ of Γ_d and any left-invariant Riemannian metric on H_m with $\ell = 2\pi$ as above, we have*

$$\eta(A, s) = d_1 \cdots d_m \dim V \sum_{w \equiv c, w \neq 0} \varepsilon(w)|w|^{m-s},$$

for all $s \in \mathbb{C}$ with sufficiently large real part, where $\tau(0, 0, 1) = e^{2\pi i c}\,\mathrm{id}$ and where $\varepsilon(w) = \mathrm{sign}(w)$ if m is even and $\varepsilon(w) = -1$ if m is odd.

The results of the present article can be used to simplify the proof of the above theorem in [**4**]. We explain this in Section 4, below.

For $c > 0$ and $\Re s > 1$, the *Hurwitz zeta function* ζ_c is given by the infinite sum

$$\zeta_c(s) = \sum\nolimits_{k \ge 0} (k + c)^{-s}. \tag{1.13}$$

For each $c > 0$, ζ_c can be extended to a meromorphic function on the complex plane, defined for all $s \neq 1$ and with a simple pole at $s = 1$, where the residue is equal to 1. We have $\zeta_1 = \zeta$, the *Riemann zeta function*. Setting $\zeta_0 := \zeta$, the formula in Theorem 1.12 turns into

$$\eta(A, s) = d_1 \cdots d_m \dim V \{(-1)^m \zeta_c(s - m) - \zeta_{1-c}(s - m)\}.$$

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2. First steps

Recall that we consider the case where the center C_N of N has dimension 1. Throughout, we choose the first vector X_1 in the orthonormal frame $X_1, \dots, X_n$ of $\mathfrak{n}$ as a generator of the Lie algebra of C_N. The center $C_\Gamma = \Gamma \cap C_N$ of Γ is infinite cyclic and is generated by $\zeta := \exp(\ell X_1)$, for some $\ell > 0$. Then $\Gamma \backslash N$ is fibered by closed geodesics of equal length ℓ, the translates of $C_\Gamma \backslash C_N$. We multiply the Riemannian metric of N by $(2\pi/\ell)^2$, and then $\ell = 2\pi$. This changes the spectrum and the η-function of A by a factor of $(\ell/2\pi)^2$ and $(2\pi/\ell)^{2s}$, respectively.

We may assume that the representation τ of Γ is irreducible. Then there is a constant $c \in \mathbb{R}$ such that, for $\zeta = \exp(2\pi X_1)$ as above,

$$\tau(\zeta)v = e^{2\pi i c}v, \tag{2.1}$$

for all $v \in V$. Arguing as in [**4**, Section 10.1], we get that c is a rational number. We obtain a corresponding Fourier decomposition,

$$L^2(E_\tau) \cong \oplus_{w \equiv c} L^2(E_\tau, w), \tag{2.2}$$

where $\equiv$ stands for congruence modulo integers and where $L^2(E_\tau, w)$ denotes the space of maps σ in $L^2(E_\tau)$ such that

$$\sigma(xe^{tX_1}) = e^{iwt}\sigma(x), \tag{2.3}$$

for all $x \in N$. Now $L^2(E_\tau, w)$ is invariant under Clifford multiplication with left-invariant vector fields. In particular, A is well-defined on $L^2(E_\tau, w)$ with domain $H^1(E_\tau, w) = L^2(E_\tau, w) \cap H^1(E_\tau)$. For a section σ in $H^1(E_\tau, w)$, we have

$$\begin{aligned} A\sigma &= X_1 \cdot d\sigma(X_1) + \sum_{j>1} X_j \cdot d\sigma(X_j) \\ &= iwX_1 \cdot \sigma + \sum_{j>1} X_j \cdot d\sigma(X_j) \\ &= w\omega_0\sigma + \sum_{j>1} X_j \cdot d\sigma(X_j), \end{aligned} \tag{2.4}$$

by (2.3), where ω_0 denotes the unitary involution given by Clifford multiplication with iX_1. We obtain

$$\begin{aligned} A(\omega_0\sigma) &= -w\sigma - iX_1 \cdot \sum_{j>1} X_j \cdot d\sigma(X_j) \\ &= 2w\sigma - \omega_0 A\sigma. \end{aligned} \tag{2.5}$$

Therefore, the anti-commutator of A and ω_0 on $H^1(E_\tau, w)$ is $2w\,\mathrm{id}$, or, in other words, $A - w\omega_0$ and ω_0 anti-commute on $H^1(E_\tau, w)$. The crucial point in (2.4) and (2.5) is that X_j is parallel with respect to the flat connection, and we actually need this only in the X_1-direction.

Denote by $L(w, \alpha)$ the eigenspace of A in $L^2(E_\tau, w)$ with respect to α, and set

$$L_\pm(w, \alpha) = \{\sigma \in L(w, \alpha) \mid \omega_0\sigma = \pm\sigma\}. \tag{2.6}$$

For $\sigma \in L(w, \alpha)$, we have

$$A\sigma = \alpha\sigma \quad \text{and} \quad A(\omega_0\sigma) = 2w\sigma - \alpha\omega_0\sigma. \tag{2.7}$$

There are three cases with respect to possible contributions of $\pm\alpha$ to the η-function of A.

Proposition 2.8. *We have*

(1) $L_+(w, \alpha) = 0$ *if* $\alpha \neq w$ *and* $L(w, w) = L_+(w, w)$ *if* $w \neq 0$;
(2) $L_-(w, \alpha) = 0$ *if* $\alpha \neq -w$ *and* $L(w, -w) = L_-(w, -w)$ *if* $w \neq 0$;
(3) $\dim L(w, \alpha) = \dim L(w, -\alpha)$ *if* $\alpha \neq \pm w$.

Proof. Let $\sigma \in L_+(w, \alpha)$ be non-zero. Then $\alpha\sigma = (2w - \alpha)\sigma$, by (2.7), and hence $\alpha = w$. Hence $L_+(w, \alpha) = 0$ if $\alpha \neq w$. Conversely, assume that $w \neq 0$, and let $\sigma \in L(w, w)$ be non-zero. Then

$$A(\sigma - \omega_0\sigma) = -w(\sigma - \omega_0\sigma),$$

by (2.7), and hence $\sigma - \omega_0\sigma \in L(w, -w)$. Since $w \neq -w$, $L(w, w)$ and $L(w, -w)$ are orthogonal, and hence $\sigma - \omega_0\sigma$ is orthogonal to σ. Since ω_0 is unitary, we have $\|\omega_0\sigma\| = \|\sigma\|$ and conclude that $\sigma = \omega_0\sigma$. This proves (2.8), and the proof of (2.8) is analogous.

For the proof of (2.8), we may assume $\alpha \neq 0$. We get, for $\sigma \in L(w, \alpha)$,

$$\begin{aligned} A((\omega_0 - w/\alpha)\sigma) &= 2w\sigma - \alpha\omega_0\sigma - w\sigma \\ &= -\alpha(\omega_0 - w/\alpha)\sigma, \end{aligned}$$

by (2.7), and hence $(\omega_0 - w/\alpha)\sigma \in L(w, -\alpha)$. Applying this to $\pm\alpha$, we obtain linear maps

$$\begin{aligned} (\omega_0 - w/\alpha)&\colon L(w, \alpha) \to L(w, -\alpha), \\ (\omega_0 + w/\alpha)&\colon L(w, -\alpha) \to L(w, \alpha), \end{aligned} \tag{2.9}$$

which satisfy

$$\begin{aligned} (\omega_0 - w/\alpha)(\omega_0 + w/\alpha) &= (\omega_0 + w/\alpha)(\omega_0 - w/\alpha) \\ &= 1 - w^2/\alpha^2. \end{aligned}$$

If $\alpha \neq \pm w$, then the right-hand side is non-zero and, therefore, the above linear maps are isomorphisms. q.e.d.

Corollary 2.10. *For all* $s \in \mathbb{C}$ *with sufficiently large real part,*

$$\eta(A, s) = \sum_{w \equiv c, w \neq 0} \{\dim L(w, |w|) - \dim L(w, -|w|)\}|w|^{-s}.$$

Remark 2.11. Recall the normalization of the Riemannian metric from the beginning of the section. Without that normalization, there are factors of appropriate powers of $\ell/2\pi$ in our formulas.

3. The inert η-function

For $\sigma\colon N \to \Sigma_{\mathfrak{n}} \otimes V$ and $X \in \mathfrak{n}$, we write $X(\sigma) := d\sigma(X)$. With this notation, we have

$$A\sigma = X_1 \cdot X_1(\sigma) + B\sigma, \tag{3.1}$$

where B is a formally self-adjoint differential operator.

Proposition 3.2. *For all smooth sections σ of E_τ, we have*

$$A^2\sigma = -X_1(X_1(\sigma)) + B^2\sigma.$$

Proof. Straightforward, using that X_1 is in the center of $\mathfrak{n}$. q.e.d.

Let $\bar{N} := C_N\backslash N = N/C_N$, a nilpotent Lie group of dimension $n-1$. Since the Riemannian metric on N is right-invariant under the center C_N of N, $\bar{N}$ carries a left-invariant Riemannian metric such that the projection

$$N \to \bar{N} \tag{3.3}$$

is a Riemannian submersion. The projection factors through the action of Γ and results in a Riemannian submersion and principal S^1-bundle

$$\Gamma\backslash N \to \bar{\Gamma}\backslash\bar{N} \tag{3.4}$$

with closed geodesics of length 2π as fibers, where $\bar{\Gamma} = C_\Gamma\backslash\Gamma$.

For any $w \equiv c$, we can extend the representation τ of Γ to a unitary representation of the subgroup G of N generated by Γ and C_N by

$$\tau_w(\exp(tX_1)) := e^{iwt}\,\mathrm{id}\,. \tag{3.5}$$

Since C_N commutes with all $\gamma \in \Gamma$ and $w \equiv c$, τ_w is well-defined. The set $E_{\tau,w}$ of G-orbits in $N \times (\Sigma_{\mathfrak{n}} \otimes V)$ is a vector bundle over $\bar{\Gamma}\backslash\bar{N}$. Sections of $E_{\tau,w}$ correspond to maps

$$\sigma\colon N \to \Sigma_{\mathfrak{n}} \otimes V \tag{3.6}$$

satisfying both (1.4) and (2.3). Considered in this way, the space of square integrable sections of $E_{\tau,w}$ is equal to $L^2(E_\tau, w)$. Furthermore, B descends to an elliptic differential operator B_w on $E_{\tau,w}$, up to homothety unitarily equivalent to B on $L^2(E_\tau, w)$. The following result is immediate from Proposition 3.2 or also from (2.5).

Proposition 3.7. *Under the identification of $L^2(E_{\tau,w})$ with $L^2(E_\tau, w)$, we have*

$$A^2\sigma = w^2\sigma + B_w^2\sigma.$$

In particular, $\ker B_w = L(w,w) \oplus L(w,-w)$.

Now we observe that ω_0 is a super-symmetry of $E_{\tau,w}$ that anticommutes with B_w and, hence, gives rise to an operator B_w^+ from (sections of) $E^+_{\tau,w}$ to $E^-_{\tau,w}$, where $E^+_{\tau,w}$ and $E^-_{\tau,w}$ denote the eigenbundles of ω_0 for the eigenvalues 1 and -1, respectively. For the Fredholm index of B_w, we have $\operatorname{ind} B_w^+ = \dim L(w,w) - \dim L(w,-w)$, by Proposition 2.8. Hence we arrive at a formula that expresses the stability of the η-function:

Theorem 3.8. *For all $s \in \mathbb{C}$ with sufficiently large real part,*

$$\eta(A,s) = \sum_{w\equiv c, w\neq 0} \operatorname{sign}(w) \operatorname{ind} B_w^+ \,|w|^{-s}.$$

Proof of Theorem 1.8. Under a change of the left-invariant Riemannian metric on N, the associated Fredholm operators B_w^+ vary continuously so that their index remains unchanged. q.e.d.

4. The case of the Heisenberg lattices

We discuss now the proof of Theorem 1.12. Simplifying the corresponding discussion in [**4**, Section 10.2], we can assume from the outset that the standard basis of the Lie algebra $\mathfrak{h}_m$ of H_m is orthonormal, by Theorem 1.8. We label the standard basis such that the non-vanishing Lie brackets between the basis vectors are given by

$$[X_{2j}, X_{2j+1}] = X_1, \quad \text{for } 1 \leq j \leq m, \tag{4.1}$$

so that X_1 generates the center of $\mathfrak{h}_m$ as above.

By the choice of orthonormal basis $(X_1, \dots, X_{2m+1})$ of $\mathfrak{n}$, we obtain an identification $\Sigma_{\mathfrak{n}} = \Sigma_{2m+1}$. Adding a perpendicular line to $\mathfrak{n}$, spanned by a unit vector X_0, we get a further identification $\Sigma_{2m+1} = \Sigma_{2m+2}^+$. Clifford multiplication ω_j with $iX_{2j}X_{2j+1}$, $1 \leq j \leq m$, is a unitary Hermitian involution of $\Sigma_{\mathfrak{n}}$. The involutions ω_j commute pairwise, and hence we have an orthogonal decomposition into simultaneous eigenspaces,

$$\Sigma_{\mathfrak{n}} = \oplus \Sigma_\varepsilon \tag{4.2}$$

with $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \{\pm 1\}^m$, where $\dim \Sigma_\varepsilon = 1$ and where ω_j acts on Σ_ε by multiplication with ε_j. We obtain a corresponding orthogonal decomposition of E_τ,

$$E_\tau = \oplus_\varepsilon E_{\tau,\varepsilon}, \tag{4.3}$$

where ω_j acts by multiplication with ε_j on $E_{\tau,\varepsilon}$. We also obtain Fourier decompositions

$$L^2(E_{\tau,\varepsilon}) \cong \oplus_{w \equiv c} L^2(E_{\tau,\varepsilon}, w), \tag{4.4}$$

where $L^2(E_{\tau,\varepsilon}, w) = L^2(E_{\tau,\varepsilon}) \cap L^2(E_\tau, w)$.

By straightforward calculation and (4.1), the square of A is given by

$$\begin{aligned} A^2(\sigma) &= \Delta(\sigma) + \sum_{j<k} X_j \cdot X_k \cdot d\sigma([X_j, X_k]) \\ &= \Delta(\sigma) + \sum_{j \geq 1} X_{2j} \cdot X_{2j+1} \cdot d\sigma(X_1) \\ &= \Delta(\sigma) - i(\omega_1 + \dots + \omega_m) \cdot d\sigma(X_1), \\ &= \Delta(\sigma) + w(\omega_1 + \dots + \omega_m) \cdot d\sigma(X_1), \end{aligned} \tag{4.5}$$

where $\Delta = -\operatorname{tr} \operatorname{Hess}$ denotes the standard Laplace operator of N, here acting on maps from N to $\Sigma_{\mathfrak{n}} \otimes V$.

Now it is shown (along standard lines) in [**4**, Formula 10.30] that $L^2(E_{\tau,\varepsilon})$ is $d_1 \cdots d_m \dim V |w|^m$ times the standard representation of H_m associated to the linear maps $\mathfrak{h}_m \to \mathbb{R}$, which sends X_1 to w. Hence Δ

has eigenvalues $w^2+|w|(2p_1+\cdots+2p_m+m)$ on these, labeled by integers $p_1,\dots,p_m \geq 0$, and all with multiplicity $d_1\cdots d_m \dim V|w|^m$.

By our discussion further up, we only need to consider the possible eigenvalue w^2 of A^2, thus simplifying the corresponding discussion on page 1952 in [**4**]. By what we just found and (4.5), w^2 is an eigenvalue of A^2 precisely for the choices

$$p_1 = \cdots = p_m = 0 \quad \text{and} \quad \varepsilon_1 = \cdots = \varepsilon_m = -\operatorname{sign} w.$$

The rest of the proof is along the lines in [**4**]: Since ω_0 commutes with the ω_j, it leaves the subspaces Σ_ε invariant. Moreover, since

$$\omega_0 \cdots \omega_m = i^{m+1} X_1 \cdots X_{2m+1} \tag{4.6}$$

acts as the identity on Σ_{2m+1}, ω_0 acts by multiplication with $\varepsilon_1\cdots\varepsilon_m$ on Σ_ε. Now $X_1(\sigma) = iw\sigma$, for any σ in $L^2(\tau,w)$. Hence the eigenspace for A^2 in $L^2(\tau,w)$ with eigenvalue w^2 is an eigenspace of A with eigenvalue w if m is odd and $|w|$ if m is even. Since the multiplicity is $d_1\cdots d_m \dim V|w|^m$, we obtain, for all $s\in\mathbb{C}$ with sufficiently large real part and even m,

$$\eta(A,s) = d_1\cdots d_m \dim V \sum_{w\equiv c,\, w\neq 0} \operatorname{sign}(w)|w|^{m-s}. \tag{4.7}$$

For odd m, we get

$$\eta(A,s) = -d_1\cdots d_m \dim V \sum_{w\equiv c,\, w\neq 0} |w|^{m-s}. \tag{4.8}$$

This finishes the proof of Theorem 1.12.

Remark 4.9. In terms of Theorem 3.8, we get

$$\operatorname{ind} B_w^+ = \varepsilon(w) d_1\cdots d_m \dim V|w|^m,$$

where $\varepsilon(w) = \operatorname{sign}(w)$ for even m and $\varepsilon(w) = -1$ for odd m. Are there formulas of a similar nature in the general case? What this comes down to is the discussion of the kernels of the operators B_w, by Theorem 3.8. I suspect that the kernels are trivial in many cases.

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