

## THE DYNAMICS THEOREM FOR *CMC* SURFACES IN $\mathbb{R}^3$

WILLIAM H. MEEKS, III & GIUSEPPE TINAGLIA

### Abstract

In this paper, we study the space of translational limits  $\mathcal{T}(M)$  of a surface  $M$  properly embedded in  $\mathbb{R}^3$  with nonzero constant mean curvature and bounded second fundamental form. There is a natural map  $\mathcal{T}$  which assigns to any surface  $\Sigma \in \mathcal{T}(M)$  the set  $\mathcal{T}(\Sigma) \subset \mathcal{T}(M)$ . Among various dynamics type results we prove that surfaces in minimal  $\mathcal{T}$ -invariant sets of  $\mathcal{T}(M)$  are chord-arc. We also show that if  $M$  has an infinite number of ends, then there exists a nonempty minimal  $\mathcal{T}$ -invariant set in  $\mathcal{T}(M)$  consisting entirely of surfaces with planes of Alexandrov symmetry. Finally, when  $M$  has a plane of Alexandrov symmetry, we prove the following characterization theorem:  $M$  has finite topology if and only if  $M$  has a finite number of ends greater than one.

### 1. Introduction

A general problem in classical surface theory is to describe the asymptotic geometric structure of a connected, noncompact, properly embedded, nonzero constant mean curvature (*CMC*) surface  $M$  in  $\mathbb{R}^3$ . In this paper, we will show that when  $M$  has bounded second fundamental form, for any divergent sequence of points  $p_n \in M$ , a subsequence of the translated surfaces  $M - p_n$  converges to a properly immersed surface of the same constant mean curvature which bounds a smooth open subdomain on its mean convex side. The collection  $\mathbb{T}(M)$  of all these limit surfaces sheds light on the geometry of  $M$  at infinity.

We will focus our attention on the subset  $\mathcal{T}(M) \subset \mathbb{T}(M)$  consisting of the connected components of surfaces in  $\mathbb{T}(M)$  which pass through the origin in  $\mathbb{R}^3$ . Given a surface  $\Sigma \in \mathcal{T}(M)$ , we will prove that  $\mathcal{T}(\Sigma)$  is always a subset of  $\mathcal{T}(M)$ . In particular, we can consider  $\mathcal{T}$  to represent a function:

$$\mathcal{T}: \mathcal{T}(M) \rightarrow \mathcal{P}(\mathcal{T}(M)),$$

---

This material is based upon work for the NSF under Award No. DMS-1004003. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.

Partially supported by The Leverhulme Trust.

Received 10/22/2008.

where  $\mathcal{P}(\mathcal{T}(M))$  denotes the power set of  $\mathcal{T}(M)$ . Using the fact that  $\mathcal{T}(M)$  has a natural compact metric space topology, we obtain classical dynamics type results on  $\mathcal{T}(M)$  with respect to the mapping  $\mathcal{T}$ . These dynamics results include the existence of nonempty minimal  $\mathcal{T}$ -invariant sets in  $\mathcal{T}(M)$  and are described in Theorem 2.3, which we refer to as the *CMC Dynamics Theorem* in  $\mathbb{R}^3$ , or more simply as just the *Dynamics Theorem*.

Assume  $M \subset \mathbb{R}^3$  is a connected, noncompact, properly embedded *CMC* surface with bounded second fundamental form. In section 3, we demonstrate various properties of the minimal  $\mathcal{T}$ -invariant sets in  $\mathcal{T}(M)$ . For example, we prove:

*Surfaces in minimal  $\mathcal{T}$ -invariant sets in  $\mathcal{T}(M)$  are chord-arc.*

*If  $M$  has an infinite number of ends, then  $\mathcal{T}(M)$  contains a minimal  $\mathcal{T}$ -invariant set in which every element has a plane of Alexandrov symmetry.*

*If  $M$  has finite genus, then any element in a minimal  $\mathcal{T}$ -invariant set is a Delaunay surface<sup>1</sup>.*

In the special case that  $M$  has finite topology<sup>2</sup>, this last result follows from the main theorem in [13], however the full generality of this result is needed in applications in [21, 23, 24, 26].

In section 4, we deal with *CMC* surfaces with a plane of Alexandrov symmetry. In particular we obtain the following characterization result:

*If  $M$  is a complete, connected, noncompact embedded *CMC* surface with a plane of Alexandrov symmetry and bounded second fundamental form, then  $M$  has finite topology if and only if it has a finite number of ends greater than one.*

The collection of properly embedded *CMC* surfaces with bounded second fundamental form is quite large and varied (see [4, 10, 11, 15, 16, 17]). Many of these examples appear as doubly and singly periodic surfaces. The techniques of Kapouleas [10] and Mazzeo-Pacard [16] can be applied to obtain many nonperiodic examples of finite and infinite topology. Some theoretical aspects of the study of these special surfaces have been developed previously in works of Meeks [18], Korevaar-Kusner-Solomon [13] and Korevaar-Kusner [12]; results from all of these three key papers are applied here. More generally, the broader theory of properly embedded *CMC* surfaces in homogeneous three-manifolds is an active field of research with many interesting recent results [2, 5, 9]. In [22], we will generalize the ideas contained in this paper to obtain

---

<sup>1</sup>In this manuscript, “Delaunay surfaces” refers to the embedded *CMC* surfaces of revolution discovered by Delaunay [3] in 1841.

<sup>2</sup>A surface has *finite topology* if it is homeomorphic to a closed surface minus a finite number of points.

related theoretical results for properly embedded separating *CMC* hypersurfaces of bounded second fundamental form in homogeneous  $n$ -manifolds.

In subsequent papers, [21, 23, 24, 26], we apply the results contained in this manuscript. In [26], we prove that the existence of a Delaunay surface in  $\mathcal{T}(M)$  implies  $M$  does not admit any other noncongruent isometric immersion into  $\mathbb{R}^3$  with the same constant mean curvature (see also [14, 29]). In [23], we show that *any complete, embedded, noncompact, simply-connected CMC surface  $M$  in a fixed homogeneous three-manifold  $N$  has the appearance of a suitably scaled helicoid nearby any point of  $M$  where the second fundamental form is sufficiently large* (see [30, 31] for related results).

**Acknowledgements.** We thank Rob Kusner and Joaquin Perez for their helpful comments on the results and proofs contained in this paper. We also thank Joaquin Perez for making the figures that appear here. The second author would also like to thank the Mathematics Department at University of Notre Dame, where this work was initiated.

## 2. The Dynamics theorem for *CMC* surfaces of bounded curvature

In this section, motivated by previous work of Meeks, Perez, and Ros in [19], we prove a dynamics type result for the space  $\mathcal{T}(M)$  of certain translational limit surfaces of a properly embedded, *CMC* surface  $M \subset \mathbb{R}^3$  with bounded second fundamental form. All of these limit surfaces satisfy the almost-embedded property described in the next definition.

**Definition 2.1.** Suppose  $W$  is a complete flat three-manifold with boundary  $\partial W = \Sigma$  together with an isometric immersion  $f: W \rightarrow \mathbb{R}^3$  such that  $f$  restricted to the interior of  $W$  is injective. This being the case, if  $f(\Sigma)$  is a *CMC* surface and  $f(W)$  lies on the mean convex side of  $f(\Sigma)$ , we call the image surface  $f(\Sigma)$  a *strongly Alexandrov embedded CMC surface*.

We note that, by elementary separation properties, any properly embedded *CMC* surface in  $\mathbb{R}^3$  is always strongly Alexandrov embedded. Furthermore, by item 1 of Theorem 2.3 below, any strongly Alexandrov embedded *CMC* surface in  $\mathbb{R}^3$  with bounded second fundamental form is properly immersed in  $\mathbb{R}^3$ .

Recall that the only compact Alexandrov-embedded<sup>3</sup> *CMC* surfaces in  $\mathbb{R}^3$  are spheres by the classical result of Alexandrov [1]. Hence, from this point on, we will only consider surfaces  $M$  which are noncompact and connected.

<sup>3</sup>A compact surface  $\Sigma$  immersed in  $\mathbb{R}^3$  is *Alexandrov embedded* if  $\Sigma$  is the boundary of a compact three-manifold immersed in  $\mathbb{R}^3$ .

**Definition 2.2.** Suppose  $M \subset \mathbb{R}^3$  is a connected, noncompact, strongly Alexandrov-embedded *CMC* surface with bounded second fundamental form.

- 1)  $\mathcal{T}(M)$  is the set of all connected, strongly Alexandrov-embedded *CMC* surfaces  $\Sigma \subset \mathbb{R}^3$ , which are obtained in the following way.  
There exists a sequence of points  $p_n \in M$ ,  $\lim_{n \rightarrow \infty} |p_n| = \infty$ , such that the translated surfaces  $M - p_n$  converge  $C^2$  on compact sets of  $\mathbb{R}^3$  to a strongly Alexandrov-embedded *CMC* surface  $\Sigma'$ , and  $\Sigma$  is a connected component of  $\Sigma'$  passing through the origin. Actually, we consider the immersed surfaces in  $\mathcal{T}(M)$  to be *pointed* in the sense that if such a surface is not embedded at the origin, then we consider the surface to represent two different elements in  $\mathcal{T}(M)$  depending on a choice of one of the two preimages of the origin.
- 2)  $\Delta \subset \mathcal{T}(M)$  is called  *$\mathcal{T}$ -invariant* if  $\Sigma \in \Delta$  implies  $\mathcal{T}(\Sigma) \subset \Delta$ .
- 3) A nonempty subset  $\Delta \subset \mathcal{T}(M)$  is called a *minimal  $\mathcal{T}$ -invariant set* if it is  $\mathcal{T}$ -invariant and contains no smaller nonempty  $\mathcal{T}$ -invariant sets.
- 4) If  $\Sigma \in \mathcal{T}(M)$  and  $\Sigma$  lies in a minimal  $\mathcal{T}$ -invariant set of  $\mathcal{T}(M)$ , then  $\Sigma$  is called a *minimal element* of  $\mathcal{T}(M)$ .

Throughout the remainder of this paper,  $\mathbb{B}(p, R)$  denotes the open ball in  $\mathbb{R}^3$  of radius  $R$  centered at the point  $p$ , and  $\mathbb{B}(R)$  denotes the open ball of radius  $R$  centered at the origin in  $\mathbb{R}^3$ . Furthermore, we will always orient surfaces so that their mean curvature  $H$  is positive.

With these definitions in hand, we now state our Dynamics Theorem.

**Theorem 2.3** (*CMC Dynamics Theorem*). *Let  $M \subset \mathbb{R}^3$  be a connected, noncompact, strongly Alexandrov-embedded *CMC* surface with bounded second fundamental form. Let  $W$  be the associated complete flat three-manifold on the mean convex side of  $M$ . Then the following statements hold:*

- 1)  $M$  is properly immersed in  $\mathbb{R}^3$ .
- 2) There exist positive constants  $c_1, c_2$  depending only on the mean curvature of  $M$  and on an upper bound for the norm of its second fundamental form, such that for any  $p \in M$  and  $R \geq 1$ ,

$$(1) \quad c_1 \leq \frac{\text{Area}(M \cap \mathbb{B}(p, R))}{\text{Volume}(W \cap \mathbb{B}(p, R))} \leq c_2.$$

*In particular, for  $R \geq 1$ ,  $\text{Area}(M \cap \mathbb{B}(R)) \leq \frac{4\pi c_2}{3} R^3$ . Furthermore,  $M$  has a regular neighborhood of radius  $\varepsilon$  in  $W$ , where  $\varepsilon > 0$  only depends on the mean curvature of  $M$  and on an upper bound for the norm of its second fundamental form.*

- 3)  $W$  is a handlebody<sup>4</sup> and every point in  $W$  is a distance of less than  $\frac{1}{H}$  from  $\partial W$ , where  $H$  is the mean curvature of  $M$ .
- 4)  $\mathcal{T}(M)$  is nonempty and  $\mathcal{T}$ -invariant.
- 5)  $\mathcal{T}(M)$  has a natural compact topological space structure given by a metric  $d_{\mathcal{T}(M)}$ . The metric  $d_{\mathcal{T}(M)}$  is induced by the Hausdorff distance between compact subsets of  $\mathbb{R}^3$ .
- 6) If  $M$  is an element of  $\mathcal{T}(M)$ , then  $\mathcal{T}(M)$  is a connected space. In particular, if  $M$  is invariant under a translation, then  $\mathcal{T}(M)$  is connected.
- 7) A nonempty set  $\Delta \subset \mathcal{T}(M)$  is a minimal  $\mathcal{T}$ -invariant set if and only if whenever  $\Sigma \in \Delta$ , then  $\mathcal{T}(\Sigma) = \Delta$ .
- 8) Every nonempty  $\mathcal{T}$ -invariant set of  $\mathcal{T}(M)$  contains a nonempty minimal  $\mathcal{T}$ -invariant set. In particular, since  $\mathcal{T}(M)$  is itself a nonempty  $\mathcal{T}$ -invariant set,  $\mathcal{T}(M)$  always contains nonempty minimal invariant sets.
- 9) Any minimal  $\mathcal{T}$ -invariant set in  $\mathcal{T}(M)$  is a compact connected subspace of  $\mathcal{T}(M)$ .

*Proof.* For the proofs of items 1 and 2, see Corollary 5.2 in [25] or see [20]. The key idea in the proof of Corollary 5.2 is to show that the immersed surface  $M$  has a fixed-size regular neighborhood on its mean convex side.

We now prove item 3. The proof that  $W$  is a handlebody is based on topological techniques used previously to study the topology of a complete, orientable flat three-manifold  $X$  with minimal surfaces as boundary. These techniques were first developed by Frohman and Meeks [8] and later generalized by Freedman [7]. An important consequence of the results and theory developed in these papers is that if  $\partial X$  is mean convex,  $X$  is not a handlebody, and  $X$  is not a Riemannian product of a flat surface with an interval, then  $X$  contains an orientable, noncompact, embedded, stable minimal surface  $\Sigma$  with compact boundary. Suppose now that  $M \subset \mathbb{R}^3$  is a strongly Alexandrov-embedded CMC surface with associated domain  $W$  on its mean convex side. Since  $M$  is not totally geodesic,  $W$  cannot be a Riemannian product of a flat surface with an interval. Therefore, if  $W$  is not a handlebody, there exists an orientable, noncompact, embedded stable minimal surface  $\Sigma \subset W$  with compact boundary. Since  $\Sigma$  is orientable and stable, a result of Fisher-Colbrie [6] implies  $\Sigma$  has finite total curvature. It is well known that such a  $\Sigma$  has an end  $E$  which is asymptotic to an end of a catenoid or a plane [28]. We will obtain a contradiction when  $E$  is a catenoidal-type end; the case where  $E$  is a planar-type end can be treated in the same manner. After a rotation of  $M$ , assume that the catenoid to which

---

<sup>4</sup>A *handlebody* is a three-manifold with boundary which is homeomorphic to a closed regular neighborhood of some connected, properly embedded simplicial one-complex in  $\mathbb{R}^3$ .

$E$  is asymptotic is vertical and  $E$  is a graph over the complement of a disk in the  $(x_1, x_2)$ -plane; assume the disk is  $\mathbb{B}(R) \cap \{x_3 = 0\}$  for some large  $R$ . Let  $S^2$  be a sphere in  $\mathbb{R}^3$  with mean curvature equal to the mean curvature of  $M$ , which lies below  $E$  and which is disjoint from the solid cylinder  $\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq R^2\}$ . By vertically translating  $S^2$  upward across the  $(x_1, x_2)$ -plane and applying the maximum principle for  $CMC$  surfaces, we find that as  $S^2$  translates across  $E$ , the portions of the translated sphere that lie above  $E$  do not intersect  $M = \partial W$ . Thus, some vertical translate  $\widehat{S}^2$  of  $S^2$  lies inside  $W$ . Next, translate  $\widehat{S}^2$  inside  $W$  so that it touches  $\partial W$  a first time. The usual application of the maximum principle for  $CMC$  surfaces implies that  $M$  is a sphere, which is not possible, since  $M$  is not compact.

Note that if some point  $p \in W$  had distance at least  $\frac{1}{H}$  from  $\partial W$ , then  $\partial\mathbb{B}(p, \frac{1}{H})$  is a sphere of mean curvature  $H$  in  $W$ . The arguments in the previous paragraph show that no such sphere can exist, and this contradiction completes the proof of item 3.

The uniform local area estimates for  $M$  given in item 2 and the assumed bound on the second fundamental form of  $M$ , together with standard compactness arguments, imply that for any divergent sequence of points  $\{p_n\}_n$  in  $M$ , a subsequence of the translated surfaces  $M - p_n$  converges on compact sets of  $\mathbb{R}^3$  to a strongly Alexandrov-embedded  $CMC$  surface  $\mathbb{M}_\infty$  in  $\mathbb{R}^3$ . The component  $M_\infty$  of  $\mathbb{M}_\infty$  passing through the origin is a surface in  $\mathcal{T}(M)$  (if  $M_\infty$  is not embedded at the origin, then one obtains two elements in  $\mathcal{T}(M)$  depending on a choice of one of the two pointed components). Hence,  $\mathcal{T}(M)$  is nonempty.

Let  $\Sigma \in \mathcal{T}(M)$  and  $\Sigma' \in \mathcal{T}(\Sigma)$ . By definition of  $\mathcal{T}(\Sigma)$ , any compact domain of  $\Sigma'$  can be approximated arbitrarily well by translations of compact domains “at infinity” in  $\Sigma$ . In turn, by definition of  $\mathcal{T}(M)$ , these compact domains “at infinity” in  $\Sigma$  can be approximated arbitrarily well by translated compact domains “at infinity” on  $M$ . Hence, a standard diagonal argument implies that  $\Sigma' \in \mathcal{T}(M)$ . Thus,  $\mathcal{T}(M)$  is  $\mathcal{T}$ -invariant, which proves item 4.

Suppose now that  $\Sigma \in \mathcal{T}(M)$  is embedded at the origin. In this case, there exists an  $\varepsilon > 0$  depending only on the bound of the second fundamental form of  $M$ , so that there exists a disk  $D(\Sigma) \subset \Sigma \cap \overline{\mathbb{B}}(\varepsilon)$  with  $\partial D(\Sigma) \subset \partial\overline{\mathbb{B}}(\varepsilon)$ ,  $\vec{0} = (0, 0, 0) \in D(\Sigma)$  and such that  $D(\Sigma)$  is a graph with gradient at most 1 over its projection to the tangent plane  $T_{\vec{0}}D(\Sigma) \subset \mathbb{R}^3$ . Given another such  $\Sigma' \in \mathcal{T}(M)$ , define

$$d_{\mathcal{T}(M)}(\Sigma, \Sigma') = d_{\mathcal{H}}(D(\Sigma), D(\Sigma')),$$

where  $d_{\mathcal{H}}$  is the Hausdorff distance. If  $\vec{0}$  is not a point where  $\Sigma$  is embedded, then since we consider  $\Sigma$  to represent one of two different pointed

surfaces in  $\mathcal{T}(M)$ , we choose  $D(\Sigma)$  to be the disk in  $\Sigma \cap \mathbb{B}(\varepsilon)$  containing the chosen base point. With this modification, the above metric is well-defined on  $\mathcal{T}(M)$ .

Using the fact that the surfaces in  $\mathcal{T}(M)$  have uniform local area and curvature estimates (see item 2), we will now prove  $\mathcal{T}(M)$  is sequentially compact and hence compact. Let  $\{\Sigma_n\}_n$  be a sequence of surfaces in  $\mathcal{T}(M)$  and let  $\{D(\Sigma_n)\}_n$  be the related sequence of graphical disks defined in the previous paragraph. A standard compactness argument implies that a subsequence  $\{D(\Sigma_{n_i})\}_{n_i}$  of these disks converges to a graphical *CMC* disk  $D_\infty$ . Using item 2, it is straightforward to show that  $D_\infty$  lies on a complete, strongly Alexandrov-embedded surface  $\Sigma_\infty$  with the same constant mean curvature as  $M$ . Furthermore,  $\Sigma_\infty$  is a limit of compact domains  $\Delta_{n_i} \subset \Sigma_{n_i}$ . In turn, the  $\Delta_{n_i}$ 's are limits of translations of compact domains in  $M$ , where the translations diverge to infinity. Hence,  $\Sigma_\infty$  is in  $\mathcal{T}(M)$  and by definition of  $d_{\mathcal{T}(M)}$ , a subsequence of  $\{\Sigma_n\}_n$  converges to  $\Sigma_\infty$ . Thus,  $\mathcal{T}(M)$  is a compact metric space with respect to the metric  $d_{\mathcal{T}(M)}$ . We remark that this compactness argument can be easily modified to prove that the topology of  $\mathcal{T}(M)$  is independent of the sufficiently small radius  $\varepsilon$  used to define  $d_{\mathcal{T}(M)}$ . It follows that the topological structure on  $\mathcal{T}(M)$  is determined ( $\varepsilon$  chosen sufficiently small), and it is in this sense that the topological structure is natural. This completes the proof of item 5.

Suppose now that  $M \in \mathcal{T}(M)$ . Note that whenever  $X \in \mathcal{T}(M)$ , then the path connected set of translates  $\text{Trans}(X) = \{X - q \mid q \in X\}$  is a subset of  $\mathcal{T}(M)$ . In particular,  $\text{Trans}(M)$  is a subset of  $\mathcal{T}(M)$ . We claim that the closure of  $\text{Trans}(M)$  in  $\mathcal{T}(M)$  is equal to  $\mathcal{T}(M)$ . By definition of closure, the closure of  $\text{Trans}(M)$  is a subset of  $\mathcal{T}(M)$ . Using the definition of  $\mathcal{T}(M)$  and the metric space structure on  $\mathcal{T}(M)$ , it is straightforward to check that  $\mathcal{T}(M)$  is contained in the closure of  $\text{Trans}(M)$ ; hence,  $\overline{\text{Trans}(M)} = \mathcal{T}(M)$ . Since the closure of a path connected set in a topological space is always connected, we conclude that  $\mathcal{T}(M)$  is connected, which completes the proof of item 6.

We now prove item 7. Suppose  $\Delta$  is a nonempty, minimal  $\mathcal{T}$ -invariant set and  $\Sigma \in \Delta$ . By definition of  $\mathcal{T}$ -invariance,  $\mathcal{T}(\Sigma) \subset \Delta$ . By item 4,  $\mathcal{T}(\Sigma)$  is a nonempty  $\mathcal{T}$ -invariant set. By definition of minimal  $\mathcal{T}$ -invariant set,  $\mathcal{T}(\Sigma) = \Delta$ , which proves one of the desired implications. Suppose now that  $\Delta \subset \mathcal{T}(\Sigma)$  is nonempty and that whenever  $\Sigma \in \Delta$ ,  $\mathcal{T}(\Sigma) = \Delta$ ; it follows that  $\Delta$  is a  $\mathcal{T}$ -invariant set. If  $\Delta' \subset \Delta$  is a nonempty  $\mathcal{T}$ -invariant set, then there exists a  $\Sigma' \in \Delta'$ , and thus,  $\Delta = \mathcal{T}(\Sigma') \subset \Delta' \subset \Delta$ . Hence,  $\Delta' = \Delta$ , which means  $\Delta$  is a minimal  $\mathcal{T}$ -invariant set and item 7 is proved.

Now we prove item 8 through an application of Zorn's lemma. Suppose  $\Delta \subset \mathcal{T}(M)$  is a nonempty  $\mathcal{T}$ -invariant set and  $\Sigma \in \Delta$ . Using the definition of  $\mathcal{T}$ -invariance, an elementary argument proves  $\mathcal{T}(\Sigma)$  is a

nonempty  $\mathcal{T}$ -invariant set in  $\Delta$  which is a closed set of  $\mathcal{T}(M)$ ; essentially, this is because the set of limit points of a set in a topological space forms a closed set (also see the proofs of items 4 and 5 for this type of argument). Next, consider the set  $\Lambda$  of all nonempty  $\mathcal{T}$ -invariant subsets of  $\Delta$  which are closed sets in  $\mathcal{T}(M)$ , and as we just observed, this collection is nonempty. Also, observe that  $\Lambda$  has a partial ordering induced by inclusion  $\subset$ .

We first check that any linearly ordered set in  $\Lambda$  has a lower bound, and then apply Zorn's lemma to obtain a minimal element of  $\Lambda$  with respect to the partial ordering  $\subset$ . To do this, suppose  $\Lambda' \subset \Lambda$  is a nonempty linearly ordered subset and we will prove that the intersection  $\bigcap_{\Delta' \in \Lambda'} \Delta'$  is an element of  $\Lambda$ . In our case, this means that we only need to prove that such an intersection is nonempty, because the intersection of closed (respectively  $\mathcal{T}$ -invariant) sets in a topological space is a closed set (respectively  $\mathcal{T}$ -invariant) set. Since each element of  $\Lambda'$  is a closed set of  $\mathcal{T}(M)$  and the finite intersection property holds for the collection  $\Lambda'$ , then the compactness of  $\mathcal{T}(M)$  implies  $\bigcap_{\Delta' \in \Lambda'} \Delta' \neq \emptyset$ . Thus,  $\bigcap_{\Delta' \in \Lambda'} \Delta' \in \Lambda$  is a lower bound for  $\Lambda'$ . By Zorn's lemma applied to  $\Lambda$  under the partial ordering  $\subset$ ,  $\Delta$  contains a smallest, nonempty, closed  $\mathcal{T}$ -invariant set  $\Omega$ . We now check that  $\Omega$  is a nonempty, minimal  $\mathcal{T}$ -invariant subset of  $\Delta$ . If  $\Omega'$  is a nonempty  $\mathcal{T}$ -invariant subset of  $\Omega$ , then there exists a  $\Sigma' \in \Omega'$ . By our previous arguments,  $\mathcal{T}(\Sigma') \subset \Omega' \subset \Omega$  is a nonempty  $\mathcal{T}$ -invariant set in  $\Delta$  which is a closed set in  $\mathcal{T}(M)$ , i.e.,  $\mathcal{T}(\Sigma') \in \Lambda$ . Hence, by the minimality property of  $\Omega$  in  $\Lambda$ , we have  $\mathcal{T}(\Sigma') = \Omega' = \Omega$ . Thus,  $\Omega$  is a nonempty, minimal  $\mathcal{T}$ -invariant subset of  $\Delta$ , which proves item 8.

Let  $\Delta \subset \mathcal{T}(M)$  be a nonempty, minimal  $\mathcal{T}$ -invariant set and let  $\Sigma \in \Delta$ . By item 7,  $\mathcal{T}(\Sigma) = \Delta$ . Since  $\mathcal{T}(\Sigma)$  is a closed set in  $\mathcal{T}(M)$  and  $\mathcal{T}(M)$  is compact, then  $\Delta$  is compact. Since  $\Sigma \in \mathcal{T}(\Sigma) = \Delta$ , item 6 implies  $\Delta$  is also connected which completes the proof of item 9. q.e.d.

**Remark 2.4.** It turns out that any complete, connected, noncompact, embedded  $CMC$  surface  $M \subset \mathbb{R}^3$  with compact boundary and bounded second fundamental form is properly embedded in  $\mathbb{R}^3$ , has a fixed-sized regular neighborhood on its mean convex side, and so has cubical area growth; these properties of  $M$  follow from simple modifications of the proof of these properties in the case when  $M$  has empty boundary (see [20, 25]). For such an  $M$ , the space  $\mathcal{T}(M)$  also can be defined and consists of a nonempty set of strongly Alexandrov embedded  $CMC$  surfaces without boundary. We will use this remark in the next section where  $M$  is allowed to have compact boundary. Also we note that items 4–9 of the Dynamics Theorem make sense under small modifications and hold for properly embedded separating  $CMC$  hypersurfaces  $M$  with bounded second fundamental form in noncompact homogeneous  $n$ -manifolds  $N$ , where  $\mathcal{T}(M)$  is the set of connected properly immersed



surfaces that pass through a fixed base point of  $N$  and which are components of limits of  $M$  under a sequence of “translational” isometries of  $N$  which take a divergent sequence of points in  $M$  to the base point; see [22] for details.

### 3. The Minimal Element Theorem

In this section, we give applications of the Dynamics Theorem to the theory of complete embedded *CMC* surfaces  $M$  in  $\mathbb{R}^3$  with bounded second fundamental form and compact boundary. Let  $R$  be the radial distance to the origin in  $\mathbb{R}^3$ . We will obtain several results concerning the geometry of minimal elements in  $\mathcal{T}(M)$ , when the area growth of  $M$  is less than cubical in  $R$  or when the genus of the surfaces  $M \cap \mathbb{B}(R)$  grows slower than cubically in  $R$ . With this in mind, we now define some growth constants for the area and genus of  $M$  in  $\mathbb{R}^3$ .

For any  $p \in M$ , we denote by  $M(p, R)$  the connected component of  $M \cap \mathbb{B}(p, R)$  which contains  $p$ ; if  $M$  is not embedded at  $p$  and there are two immersed components  $M(p, R)$ ,  $M'(p, R)$  corresponding to two pointed immersions, then in what follows we will consider both of these components separately.

**Definition 3.1** (Growth Constants). For  $n = 1, 2, 3$ , we define

$$A_{\sup}(M, n) = \limsup \sup_{p \in M} (\text{Area}[M(p, R)] \cdot R^{-n}),$$

$$A_{\inf}(M, n) = \liminf \inf_{p \in M} (\text{Area}[M(p, R)] \cdot R^{-n}),$$

$$G_{\sup}(M, n) = \limsup \sup_{p \in M} (\text{Genus}[M(p, R)] \cdot R^{-n}),$$

$$G_{\inf}(M, n) = \liminf \inf_{p \in M} (\text{Genus}[M(p, R)] \cdot R^{-n}).$$

In the above definition, note that  $\sup_{p \in M} (\text{Area}[M(p, R)] \cdot R^{-n})$  and the other similar expressions are functions from  $(0, \infty)$  to  $\mathbb{R}$  and therefore they each have a lim sup or a lim inf, respectively.

By item 2 of Theorem 2.3 and Remark 2.4,  $A_{\sup}(M, 3)$  is a finite number. We now prove that  $G_{\sup}(M, 3)$  is also finite. Since  $M$  has bounded second fundamental form, it admits a triangulation  $T$  whose edges are geodesic arcs or smooth arcs in the boundary of  $M$  of lengths bounded between two small positive numbers, and so that the areas of 2-simplices in  $T$  also are bounded between two small positive numbers. Let  $T(M(p, R))$  be the set of simplices in  $T$  which intersect  $M(p, R)$ . Note that for  $R$  large, the number of edges in  $T(M(p, R))$  which intersect  $M(p, R)$  is less than some constant  $K$  times the area of  $M(p, R)$ , where  $K$  depends only on the second fundamental form of  $M$ . Hence, the number of generators of the first homology group  $H_1(T(M(p, R)), \mathbb{R})$  is less than  $K$  times the area of  $M(p, R)$ . Since there are at least

Genus $[M(p, R)]$  linearly independent simplicial homology classes in  $H_1(T(M(p, R)), \mathbb{R})$ , then

$$(2) \quad \text{Genus}[M(p, R)] \leq K \text{Area}[M(p, R)] \quad \text{for } R \text{ large.}$$

In particular, since  $A_{\text{sup}}(M, 3)$  is finite, equation (2) implies that  $G_{\text{sup}}(M, 3)$  is also finite.

**Definition 3.2.** Suppose that  $M \subset \mathbb{R}^3$  is a complete, noncompact, connected embedded *CMC* surface with compact boundary (possibly empty) and with bounded second fundamental form.

- 1) For any divergent sequence of points  $p_n \in M$ , a subsequence of the translated surfaces  $M - p_n$  converges to a properly immersed surface of the same constant mean curvature which bounds a smooth open subdomain on its mean convex side. Let  $\mathbb{T}(M)$  denote the collection of all such limit surfaces.
- 2) If there exists a constant  $C > 0$  such that for all  $p, q \in M$  with  $d_{\mathbb{R}^3}(p, q) \geq 1$ ,  $d_M(p, q) \leq C \cdot d_{\mathbb{R}^3}(p, q)$ , then we say that  $M$  is *chord-arc*. (Note that the triangle inequality implies that if  $M$  is chord-arc and  $p, q \in M$  with  $d_{\mathbb{R}^3}(p, q) < 1$ , then  $d_M(p, q) < 6C$ .)

We note that in the above definition and in Theorem 3.3 below, the embedded hypothesis on  $M$  can be replaced by the weaker hypothesis that  $M$  has a fixed-size one-sided neighborhood on its mean convex side (see Remark 2.4).

We now state the main theorem of this section. For the statement of this theorem, recall that a plane  $P \subset \mathbb{R}^3$  is a *plane of Alexandrov symmetry* for a surface  $M \subset \mathbb{R}^3$  if it is a plane of symmetry which separates  $M$  into two open components  $M^+$ ,  $M^-$ , each of which is a graph over a fixed subdomain of  $P$ .

**Theorem 3.3** (Minimal Element Theorem). *Let  $M \subset \mathbb{R}^3$  be a complete, noncompact, connected embedded CMC surface with possibly empty compact boundary and bounded second fundamental form. Then the following statements hold.*

- 1) *If  $\Sigma \in \mathcal{T}(M)$  is a minimal element, then either every surface in  $\mathbb{T}(\Sigma)$  is the translation of a fixed Delaunay surface or every surface in  $\mathbb{T}(\Sigma)$  has one end. In particular, if  $\Sigma \in \mathcal{T}(M)$  is a minimal element, then every surface in  $\mathbb{T}(\Sigma)$  is connected and  $\mathcal{T}(\Sigma) = \mathbb{T}(\Sigma)$ .*
- 2) *Minimal elements of  $\mathcal{T}(M)$  are chord-arc.*
- 3) *Let  $\Sigma$  be a minimal element of  $\mathcal{T}(M)$ . For all  $D, \varepsilon > 0$ , there exists a  $d_{\varepsilon, D} > 0$  such that the following statement holds. For every compact domain  $X \subset \Sigma$  with extrinsic diameter less than  $D$  and for each  $q \in \Sigma$ , there exists a smooth compact, domain  $X_{q, \varepsilon} \subset \Sigma$  and a translation,  $\tau: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , such that*

$$d_{\Sigma}(q, X_{q, \varepsilon}) < d_{\varepsilon, D} \quad \text{and} \quad d_{\mathcal{H}}(X, \tau(X_{q, \varepsilon})) < \varepsilon,$$

where  $d_\Sigma$  is the distance function on  $\Sigma$  and  $d_{\mathcal{H}}$  is the Hausdorff distance on compact sets in  $\mathbb{R}^3$ . Furthermore, if  $X$  is connected, then  $X_{q,\varepsilon}$  can be chosen to be connected.

- 4) If  $M$  has empty boundary and lies in the halfspace  $\{x_3 \geq 0\}$ , then some minimal element of  $\mathcal{T}(M)$  has the  $(x_1, x_2)$ -plane as a plane of Alexandrov symmetry.
- 5) If  $E$  is an end representative<sup>5</sup> of  $M$  such that  $\mathbb{R}^3 - E$  contains balls of arbitrarily large radius, then  $\mathcal{T}(M)$  contains a surface with a plane of Alexandrov symmetry.
- 6) The following statements are equivalent:
  - a)  $A_{\text{inf}}(M, 3) = 0$ .
  - b)  $G_{\text{inf}}(M, 3) = 0$ .
  - c)  $\mathcal{T}(M)$  contains a minimal element with a plane of Alexandrov symmetry.
  - d)  $A_{\text{inf}}(M, 2)$  is finite.
  - e)  $G_{\text{inf}}(M, 2)$  is finite.
- 7) If  $M$  has an infinite number of ends, then there exists a minimal element in  $\mathcal{T}(M)$  with a plane of Alexandrov symmetry.
- 8) If  $\mathcal{T}(M)$  does not contain an element with a plane of Alexandrov symmetry, then the following statements hold.
  - a) There exists a constant  $F$  such that for every end representative  $E$  of a surface in  $\mathbb{T}(M)$ , there exists a positive number  $R(E)$  such that

$$[\mathbb{R}^3 - \mathbb{B}(R(E))] \subset \{x \in \mathbb{R}^3 \mid d_{\mathbb{R}^3}(x, E) < F\}.$$

In particular, if  $E_1$  and  $E_2$  are end representatives of a surface in  $\mathbb{T}(M)$ , then for  $R$  sufficiently large, for any  $x \in E_1 - \mathbb{B}(R)$ ,  $d_{\mathbb{R}^3}(x, E_2 - \mathbb{B}(R)) < F$ .

- b) There is a uniform upper bound on the number of ends of any element in  $\mathbb{T}(M)$ . In particular, there is a uniform upper bound on the number of components of any element in  $\mathbb{T}(M)$ .
- 9) Suppose  $\Sigma$  is a minimal element of  $\mathcal{T}(M)$ . Then the following statements are equivalent.
  - a)  $A_{\text{inf}}(\Sigma, 2) = 0$ .
  - b)  $G_{\text{inf}}(\Sigma, 2) = 0$ .
  - c)  $\Sigma$  is a Delaunay surface.
  - d)  $A_{\text{inf}}(\Sigma, 1)$  is finite.
  - e)  $G_{\text{inf}}(\Sigma, 1)$  is finite.

The following corollary gives some immediate consequences of Theorem 3.3. The proof of this corollary appears after the proof of Theorem 3.3.

---

<sup>5</sup>A proper noncompact domain  $E \subset M$  is called an *end representative* for  $M$  if it is connected and has compact boundary.

**Corollary 3.4.** *Let  $M \subset \mathbb{R}^3$  be a complete, noncompact, connected, embedded CMC surface with compact boundary and bounded second fundamental form. Then the following statements hold.*

- 1)  $A_{\text{sup}}(M, 3) = 0 \implies G_{\text{sup}}(M, 3) = 0 \implies$   
 $\implies$  Every minimal element in  $\mathcal{T}(M)$  has a plane of Alexandrov symmetry.
- 2)  $A_{\text{sup}}(M, 2) = 0 \implies G_{\text{sup}}(M, 2) = 0 \implies$   
 $\implies$  Every minimal element in  $\mathcal{T}(M)$  is a Delaunay surface.

We make the following conjecture related to the Minimal Element Theorem. Note that item 9 of Theorem 3.3 implies that the conjecture holds for  $n = 1$ .

**Conjecture 3.5.** *Suppose that  $M \subset \mathbb{R}^3$  satisfies the hypotheses of Theorem 3.3. Then for any minimal element  $\Sigma \in \mathcal{T}(M)$  and for  $n = 1, 2, \text{ or } 3$ ,*

$$\lim_{R \rightarrow \infty} \text{Area}[\Sigma \cap \mathbb{B}(R)] \cdot R^{-n} \quad \text{and} \quad \lim_{R \rightarrow \infty} \text{Genus}[\Sigma \cap \mathbb{B}(R)] \cdot R^{-n}$$

*exist (possibly infinite). Furthermore,*

$$A_{\text{inf}}(\Sigma, n) = A_{\text{sup}}(\Sigma, n) = \lim_{R \rightarrow \infty} \text{Area}[\Sigma \cap \mathbb{B}(R)] \cdot R^{-n}$$

$$G_{\text{inf}}(\Sigma, n) = G_{\text{sup}}(\Sigma, n) = \lim_{R \rightarrow \infty} \text{Genus}[\Sigma \cap \mathbb{B}(R)] \cdot R^{-n}.$$

*Proof of Theorem 3.3.* We postpone the proofs of items 1, 2, 3 to after the proofs of the items 4–9 of the theorem.

Assume that  $M$  has empty boundary and  $M \subset \{x_3 \geq 0\}$ . Using techniques similar to the ones discussed by Ros and Rosenberg in [27], we now prove that some element of  $\mathcal{T}(M)$  has a horizontal plane of Alexandrov symmetry, that is, item 4. Let  $W_M$  be the smooth open domain in  $\mathbb{R}^3 - M$  on the mean convex side of  $M$ . Note that  $W_M \subset \{x_3 \geq 0\}$ . After a vertical translation of  $M$ , assume that  $M$  is not contained in a smaller halfspace of  $\{x_3 \geq 0\}$ . Since  $M$  has a fixed-size regular neighborhood on its mean convex side and  $M$  has bounded second fundamental form, then for any generic and sufficiently small  $\varepsilon > 0$ ,  $M_\varepsilon = M \cap \{x_3 \leq \varepsilon\}$  is a nonempty graph of small gradient over its projection to  $P_0 = \{x_3 = 0\}$ ; we let  $P_t = \{x_3 = t\}$ . Note that the mean curvature vector to  $M_\varepsilon$  is upward pointing. In what follows,  $R_{P_t}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  denotes reflection in  $P_t$ , while  $\Pi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  denotes orthogonal projection onto  $P_0$ .

For any  $t > 0$ , consider the new surface with boundary,  $\widehat{M}_t$ , obtained by reflecting  $M_t = M \cap \{x_3 \leq t\}$  across the plane  $P_t$ , i.e.,  $\widehat{M}_t = R_{P_t}(M_t)$ . Let  $T = \sup\{t \in (0, \infty) \mid \text{for } t' < t, \text{ the surface } M_{t'} \text{ is a graph over its projection to } P_0, \widehat{M}_{t'} \cap M = \partial \widehat{M}_{t'} = \partial M_{t'} \text{ and the infimum of the angles that the tangent spaces to } M \text{ along } \partial M_{t'} \text{ make with vertical planes is}$

bounded away from zero}. Recall that by height estimates for *CMC* graphs with zero boundary values [27],  $\varepsilon < T \leq \frac{1}{H}$ , where  $H$  is the mean curvature of  $M$ .

If there is a point  $p \in \partial M_T$  such that the tangent plane  $T_p M$  is vertical, then the classical Alexandrov reflection principle implies that the plane  $P_T$  is a plane of Alexandrov symmetry. Next, suppose that the angles that the tangent spaces to  $M_T$  make with  $(0, 0, 1)$  along  $\partial M_T$  are not bounded away from zero. In this case, let  $p_n \in \partial M_T$  be a sequence of points such that the tangent planes  $T_{p_n} M$  converge to the vertical (the dot products of the normal vectors to the planes with  $(0, 0, 1)$  are going to zero) and let  $\Sigma \in \mathcal{T}(M)$  be a related limit of the translated surfaces  $M - p_n$ . One can check that  $\Sigma \cap \{x_3 < 0\}$  is a graph over  $P_0$  and that its tangent plane at the origin is vertical. Now the usual application of the boundary Hopf maximum principle at the origin, or equivalently, the Alexandrov reflection argument, implies  $P_0$  is a plane of Alexandrov symmetry for  $\Sigma$ .

Suppose now that the tangent planes of  $M$  along  $\partial M_T$  are bounded away from the vertical. In this case,  $P_T$  is not a plane of Alexandrov symmetry. So, by the usual application of the Alexandrov reflection principle, we conclude that  $\widehat{M}_T \cap M = \partial \widehat{M}_T = \partial M_T$ . By definition of  $T$ , there exist  $\delta_n > 0$ ,  $\delta_n \rightarrow 0$ , such that  $F_n = \widehat{M}_{T+\delta_n} \cap M$  is not contained in  $\partial M_{T+\delta_n}$ . We first show that not only is  $\Pi(F_n)$  contained in the interior of  $\Pi(M_T)$ , but for some  $\eta > 0$ , it stays at a distance at least  $\eta$  from  $\Pi(\partial M_T)$  for  $\delta_n$  sufficiently small. In fact, since we are assuming that the tangent planes of  $M$  along  $\partial M_T$  are bounded away by a fixed positive angle from the vertical, if  $\delta$  is small enough, the tangent planes of  $M$  along  $\partial M_{T+\delta}$  are also bounded away by a fixed positive angle from the vertical. Thus, the previous statement on the existence of an  $\eta > 0$  is a consequence of the existence of a fixed-size one-sided regular neighborhood for  $M$  in  $W_M$ .

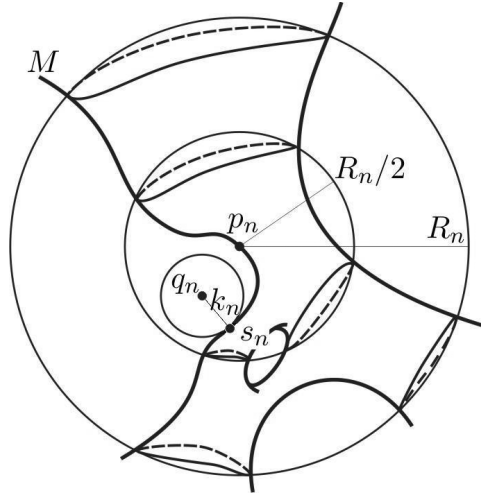
The discussion in the previous paragraph implies that there exists a sequence of points  $p_n \in M_T$  which stay at a distance at least  $\eta$  from  $\partial M_T$  and such that the distance from  $R_T(p_n)$  and  $M - M_T$  is going to zero. The fact that  $p_n$  stays at a distance at least  $\eta$  from  $\partial M_T$  implies that for  $n$  large there exists an  $\varepsilon > 0$  such that  $R_T(\mathbb{B}(p_n, \varepsilon) \cap M)$  is disjoint from  $M$  and it is a graph over  $\Pi(\mathbb{B}(p_n, \varepsilon) \cap M)$ . Consider the element  $\Sigma \in \mathcal{T}(M)$  obtained as a limit of the translated surfaces  $M - \Pi(p_n)$  and let  $\lim_{n \rightarrow \infty} p_n = p = (0, 0, S) \in \Sigma$ . From the way  $\Sigma$  is obtained,  $p$  is a positive distance from  $\partial \Sigma_T$ . Moreover,  $R_T(p) \in \Sigma - \Sigma_T$  and  $\widehat{\Sigma}_T$  is tangent to  $\Sigma - \Sigma_T$  and lies on its mean convex side. The maximum principle implies that  $P_T$  is a plane of Alexandrov symmetry which contradicts the assumption that tangent planes of  $M$  along  $\partial M_T$  are bounded away by a fixed positive angle from the vertical. This completes the proof that there exists a surface  $\Sigma \in \mathcal{T}(M)$  with the  $(x_1, x_2)$ -plane as a plane of

Alexandrov symmetry. It then follows from item 8 of Theorem 2.3 that the nonempty  $\mathcal{T}$ -invariant set  $\mathcal{T}(\Sigma) \subset \mathcal{T}(M)$  contains minimal element of  $\mathcal{T}(M)$  with the  $(x_1, x_2)$ -plane as a plane of Alexandrov symmetry, which proves item 4.

We now prove item 5 holds. Assume now that  $M$  has possibly nonempty compact boundary and there exists a sequence of open balls  $\mathbb{B}(q_n, n) \subset \mathbb{R}^3 - M$ . Note that these balls can be chosen so that they are at distance at least  $n$  from the boundary of  $M$  and so that there exists a sequence of points  $p_n \in \partial\mathbb{B}(q_n, n) \cap M$  diverging in  $\mathbb{R}^3$ . After choosing a subsequence, we may assume that the translated balls  $\mathbb{B}(q_n, n) - p_n$  converge to an open halfspace  $K$  of  $\mathbb{R}^3$  and a subsequence of the translated surfaces  $M - p_n$  gives rise to an element  $M_\infty \in \mathcal{T}(M)$  with  $M_\infty$  contained in the halfspace  $\mathbb{R}^3 - K$  and  $\partial M_\infty = \emptyset$ . By the previous discussion when  $M$  has empty boundary (item 4),  $\mathcal{T}(M_\infty) \subset \mathcal{T}(M)$  contains a minimal element with a plane of Alexandrov symmetry. This completes the proof of item 5.

We now prove item 6 in the theorem. First, observe that 6d  $\implies$  6a and that 6e  $\implies$  6b. Also, equation (2) implies that 6a  $\implies$  6b and that 6d  $\implies$  6e. We now prove that 6c  $\implies$  6d. Suppose that  $\mathcal{T}(M)$  contains a minimal element  $\Sigma$  which has a plane of Alexandrov symmetry and let  $W_\Sigma$  denote the embedded three-manifold on the mean convex side of  $\Sigma$ . In this case  $W_\Sigma$  is contained in a slab, and by item 2 of Theorem 2.3, the area growth of  $\Sigma$  is comparable to the volume growth of  $W_\Sigma$ . Note that the volume of  $W_\Sigma$  grows at most like the volume of the slab which contains it, and so, the volume growth of  $W_\Sigma$  and the area growth of  $\Sigma$  is at most quadratic in  $R$ . By the definitions of  $\mathcal{T}(M)$  and  $A_{\text{inf}}(M, 2)$ , we see that  $A_{\text{inf}}(M, 2)$  is finite which implies 6d.

In order to complete the proof of item 6, it suffices to show 6b  $\implies$  6c. However, since the proof of 6b  $\implies$  6c uses the fact that 6a  $\implies$  6c, we first show that 6a  $\implies$  6c. Assume that  $A_{\text{inf}}(M, 3) = 0$  and we will prove that  $\mathcal{T}(M)$  contains a surface  $\Sigma$  which lies in a halfspace of  $\mathbb{R}^3$ . Since  $A_{\text{inf}}(M, 3) = 0$ , we can find a sequence of points  $\{p_n\}_n \subset M$  and positive numbers  $R_n, R_n \rightarrow \infty$ , such that the connected component  $M(p_n, R_n)$  of  $M \cap \overline{\mathbb{B}}(p_n, R_n)$  containing  $p_n$  has area less than  $\frac{1}{n}R_n^3$ . Since  $M$  has bounded second fundamental form, there exists an  $\varepsilon > 0$  such that for any  $q \in \mathbb{R}^3$ , if  $\mathbb{B}(q, r) \cap M \neq \emptyset$ , then  $\text{Area}(\mathbb{B}(q, r+1) \cap M) \geq \varepsilon$ . Using this observation, together with the inequality  $\text{Area}(M \cap \overline{\mathbb{B}}(p_n, R_n)) \leq \frac{1}{n}R_n^3$  and the equality  $\text{Volume}(\mathbb{B}(p_n, R_n)) = \frac{4\pi}{3}R_n^3$ , we can find a sequence of points  $q_n \in \mathbb{B}(p_n, R_n)$ , numbers  $k_n$  with  $k_n \rightarrow \infty$ , such that  $\mathbb{B}(q_n, k_n) \subset [\mathbb{B}(p_n, \frac{R_n}{2}) - M(p_n, R_n)]$  and such that there are points  $s_n \in \partial\mathbb{B}(q_n, k_n) \cap M(p_n, R_n)$  with  $|s_n| \rightarrow \infty$  (see Figure 1). Let  $\Sigma \in \mathcal{T}(M)$  be a limit surface arising from the sequence of translated surfaces  $M(p_n, R_n) - s_n$ . Note that  $\Sigma$  is disjoint from an open halfspace obtained from a limit of a subsequence of the translated balls  $\mathbb{B}(q_n, k_n) - s_n$ . Since



**Figure 1.** Finding large balls in the complement of  $M(p_n, R_n)$

$\Sigma$  lies in a halfspace of  $\mathbb{R}^3$ , item 4 in the theorem implies  $\mathcal{T}(M)$  contains a minimal element with a plane of Alexandrov symmetry. The existence of this minimal element proves that 6a  $\implies$  6c.

We now prove that 6b  $\implies$  6c, and this will complete the proof of item 6. Assume that  $G_{\text{inf}}(M, 3) = 0$ . Since  $G_{\text{inf}}(M, 3) = 0$ , there exists a sequence of points  $p_n \in M$  and  $R_n \rightarrow \infty$ , such that the genus of  $M(p_n, R_n) \subset \mathbb{B}(p_n, R_n)$  is less than  $\frac{1}{n}R_n^3$ . Using the fact that the genus of disjoint surfaces is additive, a simple geometric argument, which is similar to the argument that proved 6a  $\implies$  6c, shows that we can find a sequence of points  $q_n \in \mathbb{B}(p_n, R_n)$  diverging in  $\mathbb{R}^3$  and numbers  $k_n$ , with  $k_n \rightarrow \infty$ , such that one of the following statements holds.

- 1)  $\text{Genus}(M(q_n, k_n)) = 0$ .
- 2)  $\mathbb{B}(q_n, k_n) \subset [\mathbb{B}(p_n, \frac{R_n}{2}) - M(p_n, R_n)]$  and, as  $n$  varies, there exist points  $s_n \in \partial\mathbb{B}(q_n, k_n) \cap M(p_n, R_n)$  diverging in  $\mathbb{R}^3$ .

If statement 2 holds, then our previous arguments imply that  $\mathcal{T}(M)$  contains a surface  $\Sigma$  which lies in a halfspace of  $\mathbb{R}^3$  and that  $\mathcal{T}(M)$  contains a minimal element with a plane of Alexandrov symmetry. Thus, we may assume statement 1 holds.

Since statement 1 holds, then the sequence of translated surfaces  $M - q_n$  yields a limit surface  $\Sigma \in \mathcal{T}(M)$  of genus zero. If  $\Sigma$  has a finite number of ends, then  $\Sigma$  has an annular end  $E$ . By the main theorem in [18],  $E$  is contained in a solid cylinder in  $\mathbb{R}^3$ . Under a sequence of translations of  $E$ , we obtain a limit surface  $D \in \mathcal{T}(\Sigma)$  which is contained in a solid cylinder. By item 4, there is a minimal element  $D' \in \mathcal{T}(D) \subset \mathcal{T}(M)$  which has a plane of Alexandrov symmetry; this conclusion also follows from the main result in [13].

Suppose now  $\Sigma$  has genus zero and an infinite number of ends. For each  $n \in \mathbb{N}$ , there exists numbers,  $T_n$  with  $T_n \rightarrow \infty$ , such that the number  $k(n)$  of noncompact components,

$$\{\Sigma_1(T_n), \Sigma_2(T_n), \dots, \Sigma_{k(n)}(T_n)\},$$

in  $\Sigma - \mathbb{B}(T_n)$  is at least  $n$ . Fix points  $p_i(n) \in \Sigma_i(T_n) \cap \partial\mathbb{B}(2T_n)$ , for each  $i \in \{1, 2, \dots, k(n)\}$ . Note that  $\sum_{i=1}^{k(n)} \text{Area}(\Sigma(p_i(n), T_n)) \leq \text{Area}(\Sigma \cap \mathbb{B}(3T_n))$ . Since  $\Sigma$  has no boundary, then  $\text{Area}(\Sigma \cap \mathbb{B}(3T_n)) \leq \frac{4}{3}\pi c_2(3T_n)^3$  (see item 2 of Theorem 2.3). Therefore, we obtain that for all  $n$ , there exists an  $i$ , such that

$$\text{Area}(\Sigma(p_i(n), T_n)) \leq \frac{c}{n}T_n^3,$$

for a fixed constant  $c$ . By definition of  $A_{\text{inf}}(\Sigma, 3)$ , we conclude that  $A_{\text{inf}}(\Sigma, 3) = 0$ . Since we have shown that 6a  $\implies$  6c,  $\mathcal{T}(\Sigma)$  contains a minimal element  $\Sigma'$  with a plane of Alexandrov symmetry. Since  $\mathcal{T}(\Sigma) \subset \mathcal{T}(M)$ ,  $\mathcal{T}(M)$  contains a minimal element with a plane of Alexandrov symmetry. Thus 6b  $\implies$  6c, which completes the proof of item 6.

We next prove item 7. Assume that  $M$  has an infinite number of ends. If  $M$  has empty boundary, then by the arguments in the previous paragraph,  $A_{\text{inf}}(M, 3) = 0$  and thus  $\mathcal{T}(M)$  contains a minimal element with a plane of Alexandrov symmetry. By Remark 2.4, if  $M$  has nonempty, compact boundary, then it has a fixed-size regular neighborhood on its mean convex side, which is sufficient for item 2 of Theorem 2.3 to hold and then to apply the arguments in the previous paragraph. This proves that item 7 holds.

We next prove item 8a. Arguing by contrapositive, suppose that the conclusion of item 8a fails to hold and we will prove that  $\mathcal{T}(M)$  contains an element with a plane of Alexandrov symmetry. Since the conclusion of 8a fails to hold, there exists a sequence of surfaces  $\Sigma(n) \in \mathbb{T}(M)$  with end representatives  $E(n)$ , and positive numbers  $F(n) \rightarrow \infty$  as  $n \rightarrow \infty$  such that for any  $R(n) > 0$ , there exist balls  $B_n$  of radius  $F(n)$  such that

$$B_n \subset [\mathbb{R}^3 - (\mathbb{B}(R(n)) \cup E(n))].$$

Choose  $R(n) > F(n)$  sufficiently large so that  $\partial E(n) \subset \mathbb{B}(\frac{R(n)}{2})$ . After rotating  $B_n$  around an axis passing through the origin, we obtain a new ball  $K_n \subset \mathbb{R}^3 - (\mathbb{B}(R(n)) \cup E(n))$  of radius  $F(n)$  such that  $\partial K_n$  intersects  $E(n)$  at a point  $p_n$  of extrinsic distance at least  $\frac{R(n)}{2}$  from  $\partial E(n)$ . After choosing a subsequence, suppose that  $E(n) - p_n$  converges to a surface  $\Sigma_\infty \in \mathbb{T}(M)$  which lies in a halfspace of  $\mathbb{R}^3$  which is a limit of some subsequence of the translated balls  $K_n - p_n$ . By item 4,  $\mathcal{T}(\Sigma_\infty) \subset \mathcal{T}(M)$  contains a surface with a plane of Alexandrov symmetry, which completes the proof of item 8a.

The proof of item 8b is a modification of the proof of item 7. In fact, if  $\Sigma_n \in \mathbb{T}(M)$  is a sequence of surfaces with at least  $n$  ends,  $n$  going



to infinity, then  $A_{\text{inf}}(M, 3) = 0$ , which implies that  $\mathcal{T}(M)$  contains a minimal element with a plane of Alexandrov symmetry.

We now prove that item 9 holds. First, observe that 9d  $\implies$  9a and that 9e  $\implies$  9b. Also, equation (2) implies that 9a  $\implies$  9b and that 9d  $\implies$  9e. An argument similar to the proof of 6c  $\implies$  6d shows that 9c  $\implies$  9d. In order to complete the proof of item 9, it suffices to show 9b  $\implies$  9c. Let  $\Sigma$  be a minimal element of  $\mathcal{T}(M)$  satisfying 9b. By item 6, there exists a minimal element  $\Sigma' \in \mathcal{T}(\Sigma)$  with a plane of Alexandrov symmetry. By minimality of  $\Sigma$ ,  $\Sigma \in \mathcal{T}(\Sigma')$ , and so  $\Sigma$  also has a plane  $P$  of Alexandrov symmetry (the same plane as  $\Sigma'$  up to some translation). In particular,  $\Sigma$  lies in a fixed slab in  $\mathbb{R}^3$ .

After a possible rotation of  $\Sigma$ , assume that  $P = \{x_3 = 0\}$  and so  $\Sigma \subset \{-a \leq x_3 \leq a\}$  for some  $a > 0$ . Since  $G_{\text{inf}}(\Sigma, 2) = 0$ , there exists a sequence of points  $p_n = (x_1(n), x_2(n), 0) \in \Sigma$ , numbers  $R_n$  with  $R_n \rightarrow \infty$ , such that  $\text{Genus}(\Sigma(p_n, R_n)) < \frac{1}{n}R_n^2$ . Similar to the proof of 6b  $\implies$  6c, the fact that  $G_{\text{inf}}(\Sigma, 2) = 0$  implies that we can find a sequence of points  $q_n \in \mathbb{B}(p_n, R_n) \cap P$  diverging in  $\mathbb{R}^3$  and numbers  $k_n$ , with  $k_n \rightarrow \infty$ , such that one of the following statements holds.

- 1)  $\text{Genus}(\Sigma(q_n, k_n)) = 0$ .
- 2)  $\mathbb{B}(q_n, k_n) \subset [\Sigma(p_n, \frac{R_n}{2}) - \Sigma(p_n, R_n)]$  and, as  $n$  varies, there exist points  $s_n \in \partial\mathbb{B}(q_n, k_n) \cap \Sigma(p_n, R_n) \cap P$  diverging in  $\mathbb{R}^3$ .

We will consider the two cases above separately. If statement 1 holds, then a subsequence of the translated surfaces  $\Sigma - q_n$  yields a limit surface  $\Sigma_\infty \in \mathcal{T}(\Sigma)$  of genus zero with  $P$  as a plane of Alexandrov symmetry. If  $\Sigma_\infty$  has a finite number of ends, then it has an annular end. In this case, the end is asymptotic to a Delaunay surface [13]. Therefore,  $\mathcal{T}(\Sigma)$  contains a Delaunay surface  $\Sigma'$  and since  $\Sigma$  is a minimal element,  $\Sigma \in \mathcal{T}(\Sigma')$  which implies  $\Sigma$  itself is a Delaunay surface. Suppose  $\Sigma_\infty$  has an infinite number of ends. Note that  $\Sigma_\infty$  lies in a slab which implies that  $\text{Area}(\Sigma_\infty \cap \mathbb{B}(R)) \leq C_2 R^2$  for some constant  $C_2$ . In this case, a modification of the end of the proof that 6b  $\implies$  6c shows that for each  $n \in \mathbb{N}$ , there exist numbers  $T_n$  with  $T_n \rightarrow \infty$  such that the number  $k(n)$  of components  $\{\Sigma_1(T_n), \Sigma_2(T_n), \dots, \Sigma_{k(n)}(T_n)\}$  in  $\Sigma_\infty - \mathbb{B}(T_n)$  is at least  $n$  and, after possibly reindexing, there is a point  $p_1(n) \in \Sigma_1(T_n) \cap \partial\mathbb{B}(2T_n)$ , a constant  $C$  such that  $\text{Area}(\Sigma_1(p_1(n), T_n)) \leq \frac{C}{n}T_n^2$ . This implies that one can find diverging points  $q_n \in \mathbb{B}(p_1(n), T_n) \cap P$  and numbers  $r_n \rightarrow \infty$  such that  $\mathbb{B}(q_n, r_n) \subset [\mathbb{B}(p_1(n), \frac{T_n}{2}) - \Sigma_1(p_1(n), T_n)]$  and there are points  $s_n \in \partial\mathbb{B}(q_n, r_n) \cap \Sigma_1(p_1(n), T_n)$  such that  $|s_n| \rightarrow \infty$ . It follows that a subsequence of the surfaces  $\Sigma_1(p_1(n), T_n) - s_n$  converges to a surface  $\Sigma_\infty$  which lies in halfspace whose boundary plane is a vertical plane. Item 6 of Theorem 2.3 implies that  $\mathcal{T}(\Sigma_\infty)$  contains a surface  $\Sigma'$  with the plane  $P$  as a plane of Alexandrov symmetry as well as a vertical plane of Alexandrov symmetry. Therefore,  $\Sigma'$  is cylindrically bounded

and so it is a Delaunay surface [13]. Since  $\Sigma \in \mathcal{T}(\Sigma')$ ,  $\Sigma$  is a Delaunay surface [13].

We now consider the case where statement 2 holds. A modification of the proof of the case where statement 1 holds (this time translating  $\Sigma$  by the points  $-s_n$  instead by the points  $-q_n$ ) then demonstrates that there is a  $\Sigma' \in \mathcal{T}(\Sigma)$  with both the plane  $P$  and a vertical plane as planes of Alexandrov symmetry. As before, we conclude that  $\Sigma$  is a Delaunay surface. This completes the proof of item 9.

We now prove item 1. Let  $\Sigma$  be a minimal element in  $\mathcal{T}(M)$ . If  $\Sigma$  has a plane of Alexandrov symmetry and  $\mathbb{T}(\Sigma)$  contains a surface  $\Sigma'$  with more than one end, then Theorem 4.1, which does not depend on the proof of this item, implies that  $\Sigma'$  has at least one annular end, from which it follows that  $\mathcal{T}(\Sigma)$  contains a Delaunay surface  $D$ . Since  $\Sigma$  and  $D$  are minimal elements of  $\mathcal{T}(\Sigma)$ , then  $\Sigma \in \mathcal{T}(\Sigma) = \mathcal{T}(D)$ , and so  $\Sigma$  is a translation of  $D$ . Since  $\Sigma$  is a Delaunay surface (a translation of  $D$ ), then clearly every surface in  $\mathbb{T}(\Sigma)$  is also a translation of a Delaunay surface, which proves item 1 under the additional hypothesis that  $\Sigma$  has a plane of Alexandrov symmetry.

Thus, arguing by contradiction, suppose that  $\Sigma$  fails to have a plane of Alexandrov symmetry and  $\mathbb{T}(\Sigma)$  has a surface with more than one end. Since  $\Sigma$  is a minimal element, then  $\Sigma \in \mathcal{T}(\tilde{\Sigma})$  for any  $\tilde{\Sigma} \in \mathcal{T}(\Sigma)$ , and so no element of  $\mathcal{T}(\Sigma)$  has a plane of Alexandrov symmetry. By item 8b, there is a bound on the number of ends of any surface in  $\mathbb{T}(\Sigma)$ . Let  $\Sigma' \in \mathbb{T}(\Sigma)$  be a surface with the largest possible number  $n \geq 2$  of ends and let  $\{E_1, E_2, \dots, E_n\}$  be pairwise disjoint end representatives for its  $n$  ends. By item 8a, the ends  $E_1, E_2, \dots, E_n$  are uniformly close to each other. It now follows from the definition of  $\mathbb{T}(\Sigma')$  that every element of  $\mathbb{T}(\Sigma')$  must have at least  $n$  components, each such component arising from a limit of translations of each of the ends  $E_1, E_2, \dots, E_n$ .

By our choice of  $n$ , we find that every surface in  $\mathbb{T}(\Sigma') \subset \mathbb{T}(\Sigma)$  has exactly  $n$  components. From the minimality of  $\Sigma$ ,  $\Sigma$  must be a component of some element  $\Sigma'' \in \mathbb{T}(\Sigma')$ . But then our previous arguments imply  $\mathbb{T}(\Sigma'')$  contains a surface  $\Delta$  with  $n - 1$  ends coming from translational limits of the components of  $\Sigma''$  different from  $\Sigma$  and at least two additional components (in fact  $n$  components) arising from translational limits of  $\Sigma \subset \Sigma''$ . Hence,  $\mathbb{T}(\Sigma'') \subset \mathbb{T}(\Sigma')$  contains a surface  $\Delta$  with at least  $n + 1$  components, which contradicts the definition of  $n$ . This contradiction completes the proof of item 1.

We are now in a position to prove item 2 of the theorem. The first step in this proof is the following assertion.

**Assertion 3.6.** *Suppose  $\Sigma \in \mathbb{T}(M) \cup \{M\}$  and every element in  $\mathbb{T}(\Sigma)$  is connected. There exists a function  $f: [1, \infty) \rightarrow [1, \infty)$  such that for*

every  $\Omega \in \mathcal{T}(\Sigma)$  and for all points  $p, q \in \Omega$  with  $1 \leq d_{\mathbb{R}^3}(p, q) \leq R$ , then

$$d_{\Omega}(p, q) \leq f(R)d_{\mathbb{R}^3}(p, q).$$

Furthermore, if no element in  $\mathbb{T}(\Sigma)$  has a plane of Alexandrov symmetry, then  $\Sigma$  is chord-arc.

*Proof.* Suppose  $\Sigma \in \mathbb{T}(M) \cup \{M\}$  and every surface in  $\mathbb{T}(\Sigma)$  is connected. If there fails to exist the desired function  $f$ , then there exists a positive number  $R$ , a sequence of surfaces  $\Omega(n) \in \mathbb{T}(\Sigma)$  and points  $p_n, q_n \in \Omega(n)$  such that for  $n \in \mathbb{N}$ ,

$$1 \leq d_{\mathbb{R}^3}(p_n, q_n) \leq R \quad \text{and} \quad n \cdot d_{\mathbb{R}^3}(p_n, q_n) \leq d_{\Omega(n)}(p_n, q_n).$$

Since by hypothesis every surface in  $\mathbb{T}(\Sigma)$  is connected,  $\mathbb{T}(\Sigma) = \mathcal{T}(\Sigma)$ . As  $\mathcal{T}(\Sigma)$  is sequentially compact and  $\mathcal{T}(\Sigma) = \mathbb{T}(\Sigma)$ , the sequence of surfaces  $\Omega(n) - p_n \in \mathcal{T}(\Sigma)$  can be chosen to converge to a  $\Sigma_{\infty} \in \mathbb{T}(\Sigma) = \mathcal{T}(\Sigma)$  and the points  $q_n - p_n$  converge to a point  $q \in \Sigma_{\infty}$ . Clearly  $\Sigma_{\infty}$  is not connected because it has a component passing through  $q$  and another component passing through the origin (the intrinsic distance between  $\vec{0} \in \Omega(n) - p_n$  and  $q_n - p_n \in \Omega(n) - p_n$  is at least  $n$ ). But by assumption, every surface in  $\mathbb{T}(\Sigma)$  is connected. This contradiction proves the existence of the desired function  $f$ .

Suppose now that  $\mathcal{T}(\Sigma)$  contains no element with a plane of Alexandrov symmetry and let  $f$  be a function satisfying the first statement in the assertion. Since  $\Sigma$  is an end representative of  $\Sigma$  itself, item 5 of the theorem implies that there exists an  $R_0 > 0$  such that every ball in  $\mathbb{R}^3$  of radius at least  $R_0$  intersects  $\Sigma$  in some point. Let  $k$  be a positive integer greater than  $R_0 + 1$ . Fix any two points  $p, q \in \Sigma$  of extrinsic distance at least  $4k$ . Let  $v = \frac{q-p}{|q-p|}$ ,  $p_0 = p$  and  $p_{i+1} = p_i + 2kv$ , where  $i \in \{0, 1, \dots, n-1\}$  and  $q \in \overline{\mathbb{B}(p_n, k)}$ . By our choice of  $k$ , an open ball of radius  $k-1$  always intersects  $\Sigma$  at some point. For each  $0 < i < n$ , let  $s_i \in \Sigma \cap \mathbb{B}(p_i, k-1)$ ; we choose  $s_0 = p$  and  $s_n = q$ . Since for each  $i < n$ ,  $d_{\mathbb{R}^3}(s_i, s_{i+1}) \leq 4k$  and  $d_{\mathbb{R}^3}(s_i, s_{i+1}) \geq 1$ , then  $d_{\Sigma}(s_i, s_{i+1}) \leq f(4k)4k$ . Using the triangle inequality and  $2(n-1)k \leq d_{\mathbb{R}^3}(p, q)$ , we obtain

$$\begin{aligned} d_{\Sigma}(p, q) &\leq \sum_{i=0}^{n-1} d_{\Sigma}(s_i, s_{i+1}) \\ &\leq nf(4k)4k \leq 2f(4k)(d_{\mathbb{R}^3}(p, q) + 2k) < 4f(4k)d_{\mathbb{R}^3}(p, q). \end{aligned}$$

Thus,  $\Sigma$  is chord-arc with constant  $4f(4k)$ , which completes the proof of the assertion. q.e.d.

We now return to the proof of item 2. Let  $\Sigma \in \mathcal{T}(M)$  be a minimal element. By the last statement in item 1, the minimal element  $\Sigma$  satisfies  $\mathcal{T}(\Sigma) = \mathbb{T}(\Sigma)$ , and so every surface in  $\mathbb{T}(\Sigma)$  is connected. Thus, by Assertion 3.6, if  $\Sigma$  fails to have a plane  $P$  of Alexandrov symmetry, then  $\Sigma$  is chord-arc. Suppose now that  $\Sigma$  has a plane  $P$  of Alexandrov symmetry. If  $\Sigma$  were to fail to be chord-arc, then the proof of item 9

shows that either  $\Sigma$  is a Delaunay surface or else there exists an  $R_0 > 0$  such that every ball  $B$  in  $\mathbb{R}^3$  of radius  $R_0$  and centered at a point of  $P$  must intersect  $\Sigma$ . In the first case,  $\Sigma$  is a Delaunay surface, which is clearly chord-arc. In the second case, the existence of points in  $B \cap \Sigma$  allows one to modify the proof of Assertion 3.6 to show that  $\Sigma$  is chord-arc. Thus, item 2 of the theorem is proved.

In order to prove item 3, we need the following lemma.

**Lemma 3.7.** *Let  $\Sigma$  be a minimal element in  $\mathcal{T}(M)$ . For all  $D, \varepsilon > 0$ , there exists a  $d_{\varepsilon, D} > 0$  such that the following statement holds. For any  $B_{\Sigma}(p, D) \subset \Sigma$  and for all  $q \in \Sigma$ , there exists  $q' \in \Sigma$  such that  $B_{\Sigma}(q', D) \subset B_{\Sigma}(q, d_{\varepsilon, D})$  and  $d_{\mathcal{H}}(B_{\Sigma}(p, D) - p, B_{\Sigma}(q', D) - q') < \varepsilon$ . Here  $B_{\Sigma}(p, R)$  denotes the intrinsic ball of radius  $R$  centered at  $p$ .*

*Proof.* Arguing by contradiction, suppose that the claim in the lemma is false. Then there exist  $D, \varepsilon > 0$  such that the following holds. For all  $n \in \mathbb{N}$ , there exist intrinsic balls  $B_{\Sigma}(p_n, D) \subset \Sigma$  and  $q_n \in \Sigma$  such that for any  $B_{\Sigma}(q', D) \subset B_{\Sigma}(q_n, n)$ , then  $d_{\mathcal{H}}(B_{\Sigma}(p_n, D) - p_n, B_{\Sigma}(q', D) - q') > \varepsilon$ . In what follows, we further simplify the notation and we let  $B_{\Sigma}(p)$  denote  $B_{\Sigma}(p, D)$ . After going to a subsequence, we can assume that the set of translated surfaces,  $\Sigma - p_n$ , converges  $C^2$  to a complete, strongly Alexandrov-embedded, CMC surface  $\Sigma_{\infty}$  passing through the origin  $\vec{0}$ . By item 1,  $\Sigma_{\infty}$  is connected and we consider it to be pointed so that  $B_{\Sigma}(p_n) - p_n$  converges to  $B_{\Sigma_{\infty}}(\vec{0})$ . Also, we can assume that  $B_{\Sigma}(q_n, n) - q_n$  converges to a complete, connected, pointed, strongly Alexandrov-embedded CMC surface  $\Sigma'_{\infty}$ . The previous discussion implies that for any  $z \in \Sigma'_{\infty}$ , there exists a sequence  $B_{\Sigma}(z_n) \subset B_{\Sigma}(q_n, n)$ , such that

$$(3) \quad d_{\mathcal{H}}(B_{\Sigma}(z_n) - z_n, B_{\Sigma'_{\infty}}(z) - z) < \frac{\varepsilon}{4} \quad \text{for } n \text{ large.}$$

Furthermore, we can also assume that

$$(4) \quad d_{\mathcal{H}}(B_{\Sigma}(p_n) - p_n, B_{\Sigma_{\infty}}(\vec{0})) < \frac{\varepsilon}{4},$$

and since  $B_{\Sigma}(z_n) \subset B_{\Sigma}(q_n, n)$ , then

$$(5) \quad d_{\mathcal{H}}(B_{\Sigma}(p_n) - p_n, B_{\Sigma}(z_n) - z_n) > \varepsilon.$$

Recall that since  $\Sigma$  is a minimal element, item 7 in Theorem 2.3 implies that

$$\Sigma, \Sigma_{\infty}, \Sigma'_{\infty} \in \mathcal{T}(\Sigma) = \mathcal{T}(\Sigma_{\infty}) = \mathcal{T}(\Sigma'_{\infty}).$$

In order to obtain a contradiction, it suffices to show that there exists an  $\alpha > 0$  such that

$$d_{\mathcal{H}}(B_{\Sigma'_{\infty}}(z) - z, B_{\Sigma_{\infty}}(\vec{0})) > \alpha$$

for any  $z \in \Sigma'_{\infty}$  because this inequality clearly implies that  $\Sigma_{\infty} \notin \mathcal{T}(\Sigma'_{\infty})$ . Fix  $z \in \Sigma'_{\infty}$  and let  $z_n$  and  $p_n$  be as given by equations (3) and (4).

In what follows, we are going to start with equation (5), apply the triangle inequality for the Hausdorff distance between compact sets, then apply the triangle inequality and equation (3), and finally apply (4). For  $n$  large,

$$\begin{aligned} \varepsilon &< d_{\mathcal{H}}(B_{\Sigma}(p_n) - p_n, B_{\Sigma}(z_n) - z_n) \leq \\ &\leq d_{\mathcal{H}}(B_{\Sigma}(p_n) - p_n, B_{\Sigma'_\infty}(z) - z) + d_{\mathcal{H}}(B_{\Sigma'_\infty}(z) - z, B_{\Sigma}(z_n) - z_n) < \\ &< d_{\mathcal{H}}(B_{\Sigma}(p_n) - p_n, B_{\Sigma_\infty}(\vec{0})) + d_{\mathcal{H}}(B_{\Sigma_\infty}(\vec{0}), B_{\Sigma'_\infty}(z) - z) + \frac{\varepsilon}{4} < \\ &< \frac{\varepsilon}{2} + d_{\mathcal{H}}(B_{\Sigma'_\infty}(z) - z, B_{\Sigma_\infty}(\vec{0})). \end{aligned}$$

This inequality implies  $d_{\mathcal{H}}(B_{\Sigma'_\infty}(z) - z, B_{\Sigma_\infty}(\vec{0})) > \frac{\varepsilon}{2}$ , which, after setting  $\alpha = \frac{\varepsilon}{2}$ , completes the proof of the lemma. q.e.d.

Notice that if  $X \subset \Sigma$  is a compact domain of intrinsic diameter less than  $D$ , then for a point  $p \in \Sigma$ ,  $X \subset B_{\Sigma}(p, 2D)$ . The next lemma is a consequence of Lemma 3.7 and the following observation regarding the Hausdorff distance: Given three compact sets  $A, B, X \subset \Sigma$  with  $X \subset A$ , if  $d_{\mathcal{H}}(A, B) < \varepsilon$ , then there exists  $X' \subset B$  such that  $d_{\mathcal{H}}(X, X') < \varepsilon$ .

**Lemma 3.8.** *Let  $\Sigma$  be a minimal element of  $\mathcal{T}(M)$ . For all  $D, \varepsilon > 0$ , there exists a  $d_{\varepsilon, D} > 0$  such that the following statement holds. For every smooth, connected compact domain  $X \subset \Sigma$  with intrinsic diameter less than  $D$  and for each  $q \in \Sigma$ , there exists a smooth compact, connected domain  $X_{q, \varepsilon} \subset \Sigma$  and a translation,  $i: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , such that*

$$d_{\Sigma}(q, X_{q, \varepsilon}) < d_{\varepsilon, D} \quad \text{and} \quad d_{\mathcal{H}}(X, i(X_{q, \varepsilon})) < \varepsilon,$$

where  $d_{\Sigma}$  is distance function on  $\Sigma$ .

In order to finish the proof of item 3, we remark that item 2 implies intrinsic and extrinsic distances are comparable when the intrinsic distance between the points is at least one. Thus, the above lemma implies the first statement in item 3. The second statement is an immediate consequence of the first statement, which completes the proof.

Theorem 3.3 is now proved. q.e.d.

*Proof of Corollary 3.4.* We first prove item 1 of the corollary. By equation (2),  $A_{\text{sup}}(M, 3) = 0$  implies  $G_{\text{sup}}(M, 3) = 0$ . On the other hand, if  $G_{\text{sup}}(M, 3) = 0$ , then for any  $\Sigma \in \mathcal{T}(M)$ ,  $G_{\text{sup}}(\Sigma, 3) = 0$ . In particular, for any minimal element  $\Sigma \in \mathcal{T}(M)$ ,  $G_{\text{inf}}(\Sigma, 3) = 0$ . By item 6 of Theorem 3.3,  $\mathcal{T}(\Sigma)$  contains a minimal element  $\Sigma'$  with a plane of Alexandrov symmetry. Since  $\Sigma$  is a minimal element,  $\Sigma \in \mathcal{T}(\Sigma')$  and therefore has a plane of Alexandrov symmetry. This proves that item 1 holds.

The proof of item 2 follows from arguments similar to the ones in the proof of item 1, using item 9 of Theorem 3.3 instead of item 6. q.e.d.

**Remark 3.9.** In regards to item 4 of Theorem 3.3, it has been conjectured by Meeks [18] that if  $M$  is a properly embedded  $CMC$  surface in  $\mathbb{R}^3$  which lies in the halfspace  $\{x_3 \geq 0\}$ , then it has a horizontal plane of Alexandrov symmetry. This conjecture holds when  $M$  has finite topology [13]

**Remark 3.10.** In [22], we give a natural generalization of Theorems 2.3 and 3.3 to the more general case of separating  $CMC$  hypersurfaces  $M$  with bounded second fundamental form in an  $n$ -dimensional noncompact homogeneous manifold  $N$ . In that paper, we obtain some interesting applications of this generalization to the classical setting where  $N$  is  $\mathbb{R}^n$  or hyperbolic  $n$ -space,  $\mathbb{H}^n$ , which are similar to the applications given in Theorem 3.3.

**Remark 3.11.** In [26], we prove that if  $M \subset \mathbb{R}^3$  is a strongly Alexandrov-embedded  $CMC$  surface with bounded second fundamental form and  $\mathcal{T}(M)$  contains a Delaunay surface, then  $M$  is rigid<sup>6</sup>. In [29], Smyth and Tinaglia show that if  $M$  contains a surface with a plane of Alexandrov symmetry, then  $M$  is locally rigid<sup>7</sup>. In relation to these rigidity results note that Theorem 3.3 gives several different constraints on the geometry or the topology of  $M$  that guarantee the existence of a Delaunay surface or a surface with a plane of Alexandrov symmetry in  $\mathcal{T}(M)$ . The first author conjectures that *the helicoid is the only complete, embedded, constant mean curvature surface in  $\mathbb{R}^3$  which admits more than one noncongruent, isometric, constant mean curvature immersion into  $\mathbb{R}^3$  with the same constant mean curvature*. Since intrinsic isometries of the helicoid extend to ambient isometries, this conjecture would imply that *an intrinsic isometry of a complete, embedded, constant mean curvature surface in  $\mathbb{R}^3$  extends to an ambient isometry of  $\mathbb{R}^3$* .

#### 4. Embedded $CMC$ surfaces with a plane of Alexandrov symmetry and more than one end

In this section we prove the following topological result that uses techniques from the proof of Theorem 3.3. In the next theorem the hypothesis that the surface  $M$  be embedded can be replaced by the weaker condition that it is embedded in the complement of its Alexandrov plane of symmetry.

**Theorem 4.1.** *Suppose  $M$  is a not necessarily connected, complete embedded  $CMC$  surface with bounded second fundamental form, possibly*

<sup>6</sup> $M$  is *rigid* in the sense that the inclusion map is the unique isometric immersion in  $\mathbb{R}^3$  up to rigid motion with the same constant mean curvature.

<sup>7</sup> $M$  is *locally rigid* if any one-parameter family of isometric immersions  $M_t$  of  $M$ ,  $t \in [0, \varepsilon)$ ,  $M_0 = M$ , with same mean curvature as  $M$  is obtained by a family of rigid motions of  $M$ .

empty compact boundary, a plane of Alexandrov symmetry, at least  $n$  ends, and every component of  $M$  is noncompact. If  $n$  is at least two, then  $M$  has at least  $n$  annular ends. Furthermore, if  $M$  has empty boundary and more than one component, then each component of  $M$  is a Delaunay surface.

The following corollary is an immediate consequence of the above theorem and the result of Meeks [18] that a connected, noncompact, properly embedded CMC surface with one end must have infinite genus.

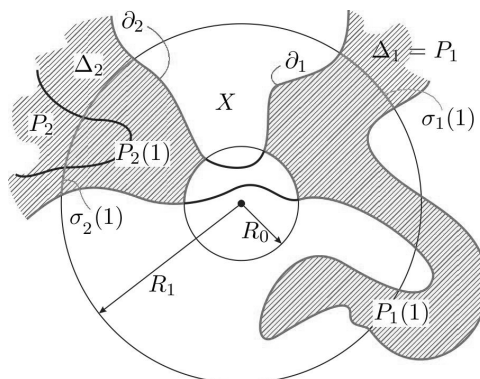
**Corollary 4.2.** *Suppose  $M$  is a connected, noncompact, complete embedded CMC surface with bounded second fundamental form and a plane of Alexandrov symmetry. Then  $M$  has finite topology if and only if  $M$  has a finite number of ends greater than one.*

In regards to Theorem 4.1 when  $n = \infty$ , we note that there exist connected surfaces of genus zero satisfying the hypotheses of the theorem which are singly periodic and have an infinite number of annular ends. It is important to notice that the hypothesis in Theorem 4.1 that  $M$  has bounded second fundamental form is essential; otherwise, there are counterexamples (see Remark 4.7).

*Proof.* We first describe some of the notation that we will use in the proof of the theorem. We will assume that  $M$  has a plane  $P$  of Alexandrov symmetry and  $P$  is the  $(x_1, x_2)$ -plane. We let  $\mathbb{S}^1(R) = \partial(P \cap \mathbb{B}(R))$ . Assume that  $M$  is a bigraph over a domain  $\Delta \subset P$  and  $R_0$  is chosen sufficiently large, so that  $\partial M \subset \mathbb{B}(R_0)$  and  $\Delta - \mathbb{B}(R_0)$  contains  $n$  noncompact components  $\Delta_1, \Delta_2, \dots, \Delta_n$ . Let  $M_1, M_2 \subset M$  denote the bigraphs with boundary over the respective regions  $\Delta_1, \Delta_2$ . Let  $X$  be the component in  $P - (\Delta_1 \cup \Delta_2)$  with exactly two boundary curves  $\partial_1, \partial_2$ , each a proper noncompact curve in  $P$  and such that  $\partial_1 \subset \partial\Delta_1, \partial_2 \subset \partial\Delta_2$ . The curve  $\partial_1$  separates  $P$  into two closed, noncompact, simply connected domains  $P_1, P_2$ , where  $\Delta_1 \subset P_1$  and  $\Delta_2 \subset P_2$ .

Now choose an increasing unbounded sequence of numbers  $\{R_n\}_{n \in \mathbb{N}}$  with  $R_1 > R_0$  chosen large enough so that for  $i = 1, 2$ , there exists a unique component of  $P_i \cap \mathbb{B}(R_1)$  which intersects  $P_i \cap \mathbb{S}^1(R_0)$  and so has  $P_i \cap \mathbb{S}^1(R_0)$  in its boundary; we will also assume that the circles  $\mathbb{S}^1(R_n)$  are transverse to  $\partial\Delta_1 \cup \partial\Delta_2$  for each  $n$ . By elementary separation properties, for  $i = 1, 2$ , there exists a unique component  $\sigma_i(n)$  of  $P_i \cap \mathbb{S}^1(R_n)$  which separates  $P_i$  into two components, exactly one of whose closure is a compact disk  $P_i(n)$  with  $P_i \cap \mathbb{S}^1(R_0)$  in its boundary; note that the collection of domains  $\{P_i(n)\}_n$  forms a compact exhaustion of  $P_i$ . See Figure 2.

Since  $\sigma_1(n)$  is disjoint from  $\sigma_2(n)$  and each of these sets is a connected arc in  $\mathbb{S}^1(R_n)$ , then, after possibly replacing the sequence  $\{R_n\}_{n \in \mathbb{N}}$  by a subsequence and possibly reindexing  $P_1, P_2$ , for each  $n \in \mathbb{N}$ , the arc

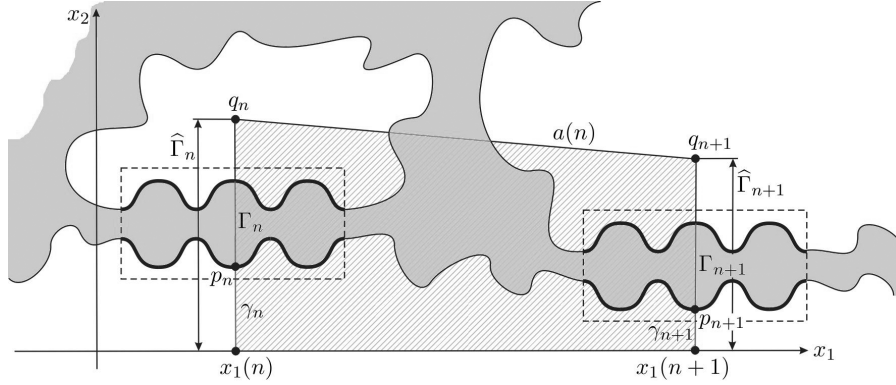


**Figure 2.**  $P_1(1)$  is the shaded region on the right-hand side containing  $\sigma_1(1)$  and an arc of  $\partial_1$  in its boundary. This figure illustrates the possibility that  $\Delta_1$  may equal  $P_1$  while  $\Delta_2$  may be strictly contained in  $P_2$ .

$\sigma_1(n)$  is contained in a closed halfspace  $K_n$  of  $\mathbb{R}^3$  with boundary plane  $\partial K_n$  being a vertical plane passing through the origin  $\vec{0}$  of  $\mathbb{R}^3$ . Let  $\Delta_1(n) = \Delta_1 \cap P_1(n)$  and let  $\overline{M}_1(n) \subset M_1$  be the compact bigraph over  $\Delta_1(n)$ . Let  $\widehat{K}_n$  be the closed halfspace in  $\mathbb{R}^3$  with  $K_n \subset \widehat{K}_n$  and such that the boundary plane  $\partial \widehat{K}_n$  is a distance  $\frac{2}{H} + R_0$  from  $\partial K_n$ , where  $H$  is the mean curvature of  $M$ . Note that  $\partial \overline{M}_1(n)$  is contained in the union of the solid cylinder over  $\overline{\mathbb{B}}(R_0)$  and the halfspace  $K_n$ . Thus, the distance from  $\partial \overline{M}(n)$  to  $\partial \widehat{K}_n$  is at least  $\frac{2}{H}$ . By the Alexandrov reflection principle and the  $\frac{1}{H}$  height estimate for  $CMC$  graphs with zero boundary values and constant mean curvature  $H$ , we find that  $\overline{M}_1(n) \subset \widehat{K}_n$ . After choosing a subsequence, the halfspaces  $\widehat{K}_n$  converge on compact sets of  $\mathbb{R}^3$  to a closed halfspace  $K$ . Since for all  $n \in \mathbb{N}$ ,  $\overline{M}_1(n) \subset \overline{M}_1(n+1)$  and  $\bigcup_{n=1}^{\infty} \overline{M}_1(n) = M_1$ , one finds that  $M_1 \subset K$ . After a translation in the  $(x_1, x_2)$ -plane and a rotation of  $M_1$  around the  $x_3$ -axis, we may assume that the new surface, which we will also denote by  $M_1$ , lies in  $\{(x_1, x_2, x_3) \mid x_2 > 0\}$  and it is a bigraph over a region  $\Delta_1 \subset \{(x_1, x_2, 0) \mid x_2 > 0\}$ . A straightforward application of the Alexandrov reflection principle and height estimates for  $CMC$  graphs shows that, after an additional translation in the  $(x_1, x_2)$ -plane and a rotation around the  $x_3$ -axis,  $\Delta_1$  also can be assumed to contain a divergent sequence of points  $p_n = (x_1(n), x_2(n), 0) \in \partial \Delta_1$  such that  $\frac{x_2(n)}{x_1(n)} \rightarrow 0$  as  $n$  approaches infinity. See Figure 3.

**Assertion 4.3.** *The points  $p_n$  can be chosen to satisfy the following additional properties.*





**Figure 3.** Choosing the points  $p_n$  and related data. The shaded trapezoidal region is  $T(n)$ .

- 1) The vertical line segments  $\gamma_n$  joining  $p_n$  to  $(x_1(n), 0, 0)$  intersect  $\Delta_1$  only at  $p_n$  and  $\frac{x_1(n+1)}{x_1(n)} > n$ .
- 2) The surfaces  $M_1 - p_n$  converge to a surface in  $\mathbb{T}(M)$  with a related component in  $\mathcal{T}(M)$  being a Delaunay surface  $F$  with  $P$  as a plane of Alexandrov symmetry and axis parallel to the  $x_1$ -axis.

*Proof.* The proof that the points  $p_n$  can be chosen to satisfy statement 1 is clear. To prove that they can also be chosen to satisfy statement 2 can be seen as follows. Let  $S_n \subset P$  be the circle passing through the points  $p_n$  and  $(\frac{x_1(n)}{10}, 0, 0)$  with center on the line  $\{(x_1(n), s, 0) \mid s < x_2(n)\}$  and let  $E_n$  denote the closed disk with boundary  $S_n$ . Consider the family of translated disks  $E_n(t) = E_n - (0, t, 0)$  and let  $t_0$  be the largest  $t \geq 0$  such that  $E_n(t)$  intersects  $\Delta_1$  at some point and let  $D_n = E_n(t_0)$ . By construction and after possibly replacing by a subsequence, points in  $D_n \cap \Delta_1$  satisfy the first statement in the assertion as well as the previous property that the ratio of their  $x_2$ -coordinates to their  $x_1$ -coordinates limit to zero as  $n \rightarrow \infty$ . Next, replace the previous point  $p_n$  by any point of  $\partial D_n \cap M_1$ , to obtain a new sequence of points which we also denote by  $p_n$ . A subsequence of certain compact regions of the translated surfaces  $M - p_n$  converges to a strongly Alexandrov-embedded surface  $M_\infty \in \mathcal{T}(M)$  which has  $P$  as a plane of Alexandrov symmetry and which lies in the halfspace  $x_2 \geq 0$ . It follows from item 4 of Theorem 3.3 (and its proof) that  $\mathcal{T}(\Sigma)$  contains a Delaunay surface  $D$  with axis being a bounded distance from the  $x_1$ -axis and which arises from a limit of translates of  $M_\infty$ . It is now clear how to choose the desired points described in the assertion, which again we denote by  $p_n$ , so that certain compact regions of the translated surfaces  $M - p_n$  converge to the desired Delaunay surface  $F$ . This completes the proof of the assertion. q.e.d.

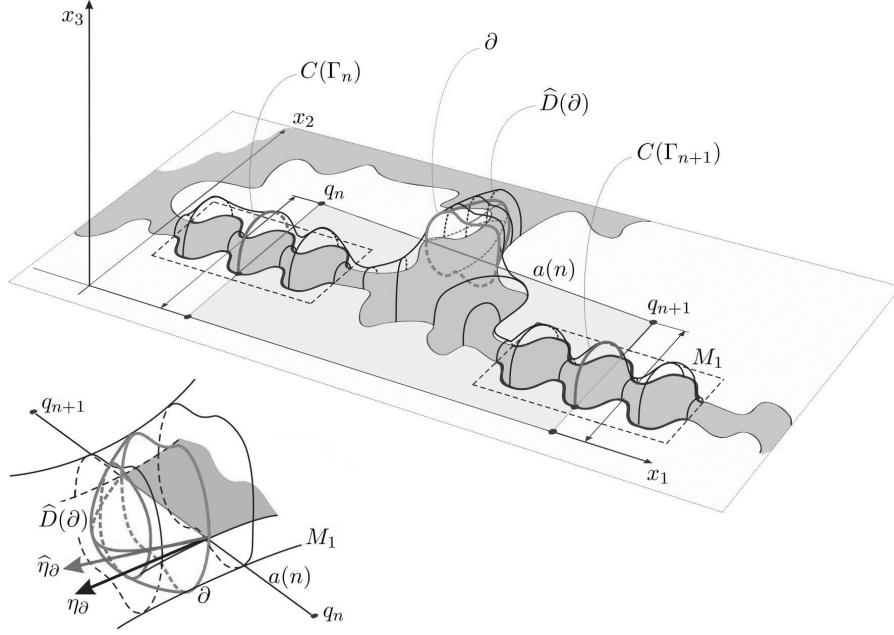
As a reference for the discussion which follows, we refer the reader to Figure 3. By Assertion 4.3, we may assume that around each point  $p_n$ , the surface  $M_1$  is closely approximated by a translation of a fixed large compact region of the Delaunay surface  $F$ . Without loss of generality, we may assume that the entire line containing  $\gamma_n$  is disjoint from the self-intersection set of  $\partial\Delta_1$ . Let  $\Gamma_n$  be the largest compact extension of  $\gamma_n$  so that  $\Gamma_n - \gamma_n \subset \Delta_1$ , and let  $\widehat{\Gamma}_n$  be a line segment extension of  $\Gamma_n$  near the endpoint of  $\Gamma_n$  with positive  $x_2$ -coordinate so that  $\widehat{\Gamma}_n \cap \Delta_1 = \Gamma_n \cap \Delta_1$  and so that the length of  $\widehat{\Gamma}_n - \Gamma_n$  is less than  $\frac{1}{n}$ . Let  $q_n$  denote the endpoint of  $\widehat{\Gamma}_n$  which is different from the point  $p_n$ . Without loss of generality, we may assume that the line segments  $a(n)$  in  $P$  joining  $q_n, q_{n+1}$  are transverse to  $\partial\Delta_1$  and intersect  $\Delta_1$  in a finite collection of compact intervals. If we denote by  $v(n)$  the upward-pointing unit vector in the  $(x_1, x_2)$ -plane perpendicular to  $a(n)$ , then the vectors  $v(n)$  converge to  $(0, 1, 0)$  as  $n$  goes to infinity.

As a reference for the discussion which follows, we refer the reader to Figure 4. Now fix some large  $n$  and consider the compact region  $T(n) \subset P$  bounded by the line segments  $\widehat{\Gamma}_n, \widehat{\Gamma}_{n+1}, a(n)$  and the line segment joining  $(x_1(n), 0, 0)$  to  $(x_1(n+1), 0, 0)$ . Consider  $T(n)$  to lie in  $\mathbb{R}^2$  and let  $T(n) \times \mathbb{R} \subset \mathbb{R}^3$  be the related convex domain in  $\mathbb{R}^3$ . Let  $M_1(n)$  be the component of  $M_1 \cap (T(n) \times \mathbb{R})$  which contains the point  $p_n$ . Note that  $M_1(n)$  is compact with boundary consisting of an almost-circle  $C(\Gamma_n)$  which is a bigraph over an arc in  $\Gamma_n$ , possibly also an almost circle  $C(\Gamma_{n+1})$  which is a bigraph over an interval in  $\Gamma_{n+1}$ , and a collection of bigraph components over a collection of intervals  $I_n$  in the line segment  $a(n)$ .

We denote by  $\alpha(n)$  the collection of boundary curves of  $M_1(n)$ . Let  $\alpha_2(n)$  be the subcollection of curves in  $\alpha(n)$  which intersect either  $\Gamma_n$  or  $\Gamma_{n+1}$ , that is,  $\alpha_2(n) = \{C(\Gamma_n), C(\Gamma_{n+1})\}$  or  $\alpha_2(n) = \{C(\Gamma_n)\}$ . Clearly, the collection of boundary curves of  $M_1(n)$  which are bigraphs over the collection of intervals  $I_n = \Delta_1 \cap a(n)$  is  $\alpha(n) - \alpha_2(n)$ . Let  $\alpha_3(n)$  be the subcollection of curves in  $\alpha(n) - \alpha_2(n)$  which bound a compact domain  $\Delta(\alpha) \subset M_1 - \partial M_1$ , and let  $\alpha_4(n) = \alpha(n) - (\alpha_2(n) \cup \alpha_3(n))$ . Note that in Figure 4,  $\alpha_2(n) = \{C(\Gamma_n), C(\Gamma_{n+1})\}$ ,  $\alpha_3(n)$  is empty and  $\alpha_4(n)$  consists of the single gray curve  $\partial$ .

**Assertion 4.4.** *For  $n$  sufficiently large, every boundary curve  $\partial$  of  $M_1(n)$  which is a graph over an interval in  $I_n$ , bounds a compact domain  $\Delta(\partial) \subset M_1 - \partial M_1$ ; in other words,  $\alpha_4(n)$  is empty.*

*Proof.* For any  $\alpha \in \alpha(n)$ , let  $\eta_\alpha$  denote the outward-pointing conormal to  $\alpha \subset \partial M_1(n)$  and let  $D(\alpha)$  be the planar disk bounded by  $\alpha$ . Consider a boundary component  $\partial \in \alpha_4(n)$ . By the “blowing a bubble” argument presented in [12], there exists another disk  $\widehat{D}(\partial)$  on the mean convex side of  $M_1$  of the same constant mean curvature



**Figure 4.** Blowing a bubble  $\widehat{D}(\partial)$  on the mean convex side of  $M_1$ .

as  $M_1$ ,  $\partial\widehat{D}(\partial) = \partial D(\partial)$ . Moreover,  $\widehat{D}(\partial)$  is a graph over  $D(\partial)$  and  $\widehat{D}(\partial) \cap (T(n) \times \mathbb{R}) = \partial\widehat{D}(\partial) = \partial$ . Let  $\widehat{\eta}_\partial$  denote the inward pointing conormal to  $\partial\widehat{D}(\partial)$ . The graphical disk  $\widehat{D}(\partial)$  is constructed so that  $\langle \eta_\partial - \widehat{\eta}_\partial, v(n) \rangle \geq 0$ , see Figure 4.

The piecewise smooth surface  $M_1(n) \cup (\bigcup_{\alpha \in \alpha_2(n) \cup \alpha_3(n)} D(\alpha)) \cup (\bigcup_{\alpha \in \alpha_4(n)} \widehat{D}(\alpha))$  is the boundary of a compact region  $W(n) \subset \mathbb{R}^3$ . An application of the divergence theorem given in [13] to the vector field  $v(n)$ , considered to be a constant vector field in  $\mathbb{R}^3$  in the region  $W(n)$ , gives rise to the following equation:

$$(6) \quad \sum_{\alpha \in \alpha_2(n) \cup \alpha_3(n)} \left[ \int_{\alpha} \langle \eta_\alpha, v(n) \rangle - 2H \int_{D(\alpha)} \langle v(n), N(n) \rangle \right] + \\ + \sum_{\partial \in \alpha_4(n)} \left[ \int_{\partial} \langle \eta_\partial, v(n) \rangle - 2H \int_{\widehat{D}(\partial)} \langle v(n), N(n) \rangle \right] = 0,$$

where  $H$  is the mean curvature of  $M$  and  $N(n)$  is the outward pointing conormal to  $W(n)$ . Note that  $\sum_{\alpha \in \alpha_2(n)} \left[ \int_{\alpha} \langle \eta_\alpha, v \rangle - 2H \int_{D(\alpha)} \langle v(n), N \rangle \right] = \varepsilon(n)$  converges to zero as  $n \rightarrow \infty$  because  $v(n)$  converges to  $(0, 1, 0)$

and the curves  $C(\Gamma_n), C(\Gamma_{n+1})$  converge to curves on Delaunay surfaces whose axes are perpendicular to  $(0, 1, 0)$ . Also note that this application of the divergence theorem in [13] implies that for  $\alpha \in \alpha_3(n)$ ,  $\int_\alpha \langle \eta_\alpha, v(n) \rangle - 2H \int_{D(\alpha)} \langle v(n), N(n) \rangle = 0$ . Thus, equation (6) reduces to the equation

$$(7) \quad \varepsilon(n) + \sum_{\partial \in \alpha_4(n)} \left[ \int_{\partial} \langle \eta_{\partial}, v(n) \rangle - 2H \int_{\widehat{D}(\partial)} \langle v(n), N(n) \rangle \right] = 0.$$

On the other hand, for each  $\partial \in \alpha_4(n)$ ,

$$(8) \quad \int_{\partial} \langle \eta_{\partial}, v(n) \rangle - 2H \int_{\widehat{D}(\partial)} \langle v(n), N(n) \rangle = \int_{\partial} \langle \eta_{\partial} - \widehat{\eta}_{\partial}, v(n) \rangle \geq 0$$

and the length of each  $\partial \in \alpha_4$  is uniformly bounded from below. Since  $\varepsilon(n)$  is going to zero as  $n$  goes to infinity, equations (7) and (8) imply that for  $n$  large, the conormals  $\eta_{\partial}$  and  $\widehat{\eta}_{\partial}$  are approaching each other uniformly (see the lower left-hand corner of Figure 4). Note that the intrinsic distance of any point on the graphs  $\widehat{D}(\partial)$  to  $\partial$  is uniformly bounded (independent of  $\partial$  and  $n$ )<sup>8</sup>. The Harnack inequality, the above remark, the facts that  $\widehat{D}(\partial)$  is simply-connected and the second fundamental form of  $M$  is bounded, imply that there exists  $\delta > 0$  such that if  $\int_{\partial} \langle \eta_{\partial} - \widehat{\eta}_{\partial}, v(n) \rangle < \delta$ , then there is a disk  $\Delta(\partial) \subset M_1 - M_1(n)$  which can be expressed as a small graph over  $\widehat{D}(\partial)$ . The existence of  $\Delta(\partial)$  contradicts that  $\partial \in \alpha_4(n)$ , which means  $\alpha_4(n) = \emptyset$  for  $n$  sufficiently large. This contradiction proves the assertion. q.e.d.

We now apply Assertion 4.4 to prove the following key partial result in the proof of Theorem 4.1.

**Assertion 4.5.**  *$M_1$  has at least one annular end.*

*Proof.* By Assertion 4.4, for some fixed  $n$  chosen sufficiently large, every boundary curve  $\alpha$  of  $M(n)$  in the collection  $\alpha(n) - \alpha_2(n)$  bounds a compact domain  $\Delta(\alpha) \subset M_1 - \partial M_1$ . By the Alexandrov reflection principle and height estimates for *CMC* graphs, we find that the surface  $\widehat{M}(n) = M(n) \cup \bigcup_{\alpha \in \alpha(n) - \alpha_2(n)} \Delta(\alpha)$  must have two almost circles in its boundary arising from  $\alpha_2(n)$ . Let  $\Sigma(k) = \bigcup_{j \leq k} \widehat{M}(n + j)$ . Note that by the Alexandrov reflection argument and height estimates for *CMC* graphs with zero boundary values, there exist half-cylinders  $C(n, k)$  in  $\mathbb{R}^3$  which contain  $\Sigma(k)$  and have fixed radii  $\frac{4}{H}$ . Hence, there is a limit half-cylinder  $C(n) \subset \mathbb{R}^3$  that contains  $\Sigma(\infty) = \bigcup_{k \in \mathbb{N}} \Sigma(k) \subset M$ . By the main result in [13],  $\Sigma(\infty)$  is asymptotic to a Delaunay surface, which proves the assertion. q.e.d.

---

<sup>8</sup>This uniform intrinsic distance estimate holds since *CMC* graphs are strongly stable (existence of a positive Jacobi function) and there are no strongly stable, complete *CMC* surfaces in  $\mathbb{R}^3$ ; see theorem 2 in [27] for a proof of this result.

It follows from the discussion at the beginning of the proof of Theorem 4.1 and Assertion 4.5 that if  $M$  has at least  $n$  ends,  $n > 1$ , then it has at least  $n - 1$  annular ends. It remains to prove that if  $M_1, M_2$  are given as in the beginning of the proof of Theorem 4.1 with  $M_1$  having an annular end, then  $M_2$  has an annular end as well. To see this, note that the annular end  $E_1 \subset M_1$  is asymptotic to the end  $F$  of a Delaunay surface and so after a rotation of  $M$ ,  $M_1$  is a graph over a domain  $\Delta_1$  which contains the axis of  $F$ , which we can assume to be the positive  $x_1$ -axis. Now translate  $M_2$  in the direction  $(-1, 0, 0)$  sufficiently far so that its compact boundary has negative  $x_1$ -coordinates less than  $-\frac{2}{H}$ ; call the translated surface  $M'_2$  and let  $\Delta'_2 \subset P$  be the domain over which  $M'_2$  is a bigraph. If for some  $n \in \mathbb{N}$  the line  $L_n = \{(n, t, 0) \mid t \in \mathbb{R}\}$  is disjoint from  $M'_2$ , then  $M'_2$  is contained in a halfplane of  $P$  and our previous arguments imply  $M'_2$  has an annular end. Thus without loss of generality, we may assume that every line  $L_n$  intersects  $\partial\Delta'_2$  a first time at some point  $s_n$  with positive  $x_2$ -coordinate.

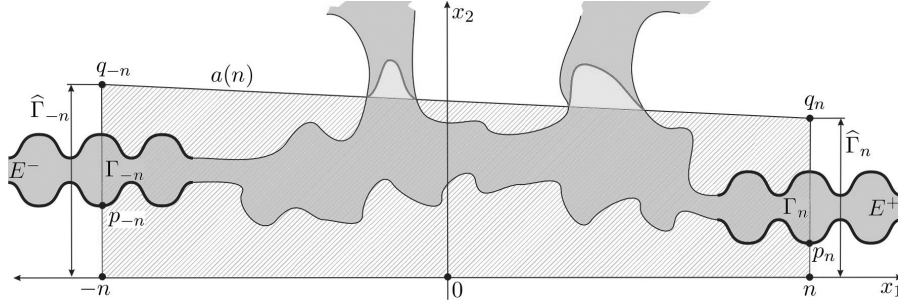
For  $\theta \in (0, \frac{\pi}{2}]$ , let  $r(\theta)$  be the ray with base point the origin and angle  $\theta$  and let  $W(\theta)$  be the closed convex wedge in  $P$  bounded by  $r(\theta)$  and the positive  $x_1$ -axis. Let  $\theta_0$  be the infimum of the set of  $\theta \in (0, \frac{\pi}{2}]$  such that  $W(\theta) \cap \{s_n\}_{n \in \mathbb{N}}$  is an infinite set. Because of our previous placement of  $\partial M'_2$ , a simple application of the Alexandrov reflection principle and height estimates for CMC graphs with zero boundary values implies that some further translate  $M''_2$  of  $M'_2$  in the direction  $(-1, 0, 0)$  must be disjoint from  $r(\theta_0)$ . Finally, after a clockwise rotation  $\widehat{M}_2$  of  $M''_2$  by angle  $\theta_0$ , our previous arguments prove the existence of an annular end of  $\widehat{M}_2$  of bounded distance from the positive  $x_1$ -axis. Thus, we conclude that  $M_2$  also has an annular end which completes the proof of the first statement in Theorem 4.1.

We next prove the last statement of the theorem. Suppose  $M \subset \mathbb{R}^3$  is a complete, properly embedded CMC surface with bounded second fundamental form and with the  $(x_1, x_2)$ -plane  $P$  as a plane of Alexandrov symmetry. Suppose  $M$  contains two noncompact components  $M_1, M_2$  and that we will prove that each of these surfaces is a Delaunay surface.

Consider  $M_1$  and  $M_2$  to be two disjoint end representatives of  $M$  defined as bigraphs over two disjoint connected domains  $\Delta_1, \Delta_2$  in  $P$ , respectively.

**Assertion 4.6.** *After possibly reindexing  $\Delta_1, \Delta_2$  and applying a rigid of  $\mathbb{R}^3$  preserving the plane  $P$ , then  $\Delta_1 \subset \{x_2 \geq 0\}$ .*

*Proof.* By what we have proved so far, we know that  $M_2$  has an annular end  $E$  which is asymptotic to the end  $D(E)$  of a Delaunay surface. Let  $r_E \subset \Delta_2$  be a ray contained in the axis of  $D(E)$ . After a rigid motion of  $M$  preserving  $P$ , assume  $r_E$  is a ray based at the origin of  $P$ . The arguments used to prove the first statement of the theorem



**Figure 5.** A picture of  $M_1$  with two bubbles blown on its mean convex side.

show that there are two disjoint annular ends  $F, G$  of  $M_1$  such that for  $R$  large the arc  $\alpha(F, G, R)$  in  $S^1(R) - \Delta_1$  which intersects  $r_E$ , has one of its endpoints in  $\Delta_1 \cap F$  and its other endpoint in  $\Delta_1 \cap G$ . Let  $D(F), D(G)$  be ends of Delaunay surfaces to which  $F, G$  are asymptotic. Let  $r_F, r_G \subset \Delta_1$  be rays contained in the axes of  $D(F), D(G)$ , respectively. Let  $\gamma_1$  be a properly embedded arc in  $\Delta_1$  consisting of  $r_F, r_G$  and a compact arc joining their endpoints. Let  $\gamma'_2$  be the proper arc in  $P - \mathbb{B}(R)$  consisting of  $\alpha(F, G, R)$ , a boundary arc in  $(E \cap \partial\Delta_1) - \mathbb{B}(R)$  and a boundary arc in  $(G \cap \partial\Delta_1) - \mathbb{B}(R)$ . After a small perturbation of  $\gamma'_2$  we obtain a proper arc  $\gamma_2$  contained in  $P - \Delta_1$  which intersects  $r_E$ . Note that  $\gamma_1$  is contained in a halfplane and since  $\gamma_2$  lies at a bounded distance from  $\gamma_1$ , the halfplane can be chosen to contain both  $\gamma_1$  and  $\gamma_2$ . After a rigid motion, we may assume that this halfplane is  $\{x_2 \geq 0\}$ . Since the region bounded by  $\gamma_1$  and  $\gamma_2$  is a strip by construction, by elementary separation arguments, either  $\gamma_1$  lies between  $\{x_2 = 0\}$  and  $\gamma_2$  or  $\gamma_2$  lies between  $\{x_2 = 0\}$  and  $\gamma_1$ . If  $\gamma_1$  lies between  $\{x_2 = 0\}$  and  $\gamma_2$ , then  $\Delta_2$  lies in the halfspace, otherwise  $\Delta_1$  does. After possibly reindexing, this completes the proof. q.e.d.

In the discussion which follows, we refer the reader to Figure 5. By the previous assertion, we may assume  $\Delta_1 \subset \{x_2 \geq 0\}$ . Previous arguments imply that after a rigid motion of  $M$ , we can further assume that  $M_1$  contains an annular end  $E^+$  with the property that for  $n \in \mathbb{N}$  sufficiently large, the line segments  $\{(n, t, 0) \mid t > 0\}$  intersect  $\Delta_1$  for a first time in a point  $p_n \in E_1$ . Furthermore,  $E^+$  is asymptotic to the end  $D^+$  of a Delaunay surface. Also we can assume that the half axis of revolution of  $D^+$  lies in  $P$  and is a bounded distance from the positive  $x_1$ -axis.

By the Alexandrov reflection principle and height estimates for *CMC* graphs with zero boundary values,  $\Delta_1$  must not be contained in a convex wedge of  $P$  with angle less than  $\pi$ . Therefore, for  $n \in \mathbb{N}$  sufficiently large the line segments  $\{(-n, t, 0) \mid t > 0\}$  intersect a second annular end  $\Delta_1$

in points  $p_{-n} \in E^-$  for a first time. In this case the annular end  $E^-$  is asymptotic to the end  $D^-$  of another Delaunay surface and the half axis of  $D^-$  in  $P$  is a bounded distance from the negative  $x_1$ -axis.

Similar to our previous arguments, we define for each  $n \in \mathbb{Z}$  with  $|n|$  sufficiently large, curves  $\gamma_n, \Gamma_n, \widehat{\Gamma}_n$  and points  $q_n$  as we did before (see Figures 3 and 5). For each  $n \in \mathbb{N}$  sufficiently large, we define the line segment  $a(n) \subset P$  whose endpoints are the points  $q_{-n}, q_n$ . Now define for any sufficiently large  $n$  the compact region  $T(n) \subset P$  bounded by the line segments  $\widehat{\Gamma}_{-n}, \widehat{\Gamma}_n, a(n)$  and the line segment joining  $(-n, 0, 0)$  to  $(n, 0, 0)$ , and let  $T(n) \times \mathbb{R} \subset \mathbb{R}^3$  be the related convex domain in  $\mathbb{R}^3$ . Let  $M_1(n)$  be the component of  $M_1 \cap (T(n) \times \mathbb{R})$  which contains the point  $p_{-n}$ . Note that  $M_1(n)$  is compact with boundary consisting of an almost-circle  $C(\Gamma_{-n})$  which is a bigraph over an arc on  $\Gamma_{-n}$ , and an almost-circle  $C(\Gamma_n)$  which is a bigraph over an arc on  $\Gamma_n$  and a collection of bigraph components over a collection of intervals  $I_n$  in the line segment  $a(n)$ .

As in previous arguments, an assertion similar to Assertion 4.4 holds in the new setting. With this slightly modified assertion, one finds that the almost-circles  $C(\Gamma_{-n})$  and  $C(\Gamma_n)$  bound a compact domain  $\widehat{M}_1(n) \subset M_1$ . A slight modification of the proof of Assertion 4.5 implies  $M_1$  is cylindrically bounded, and so by the main theorem in [13],  $M_1$  is a Delaunay surface. Note that the axis of  $M_1$  is an infinite line in  $\Delta_1$  and so  $\Delta_2$  also lies in a halfplane of  $P$ . The arguments above prove that  $M_2$  is also a Delaunay surface, which completes the proof of the theorem. q.e.d.

**Remark 4.7.** Using techniques in [10, 16], for every integer  $n > 1$ , it is possible to construct a surface  $M_n$  with empty boundary and  $n$  ends, none of which are annular, which satisfies the hypotheses of the surface  $M$  in the statement of Theorem 4.1 except for the bounded second fundamental form hypothesis. Hence, the hypothesis in the theorem that  $M$  has bounded second fundamental form is a necessary one in order for the conclusion of the theorem to hold.

## References

- [1] A.D. Alexandrov, *Uniqueness theorems for surfaces in the large I*, Vestnik Leningrad Univ. Math., **11**(19):5–17, 1956, MR 0150706.
- [2] B. Daniel & L. Hauswirth, *Half-space theorem, embedded minimal annuli and minimal graphs in the Heisenberg group*, Proc. Lond. Math. Soc. (3), **98**(2):445–470, 2009, Zbl pre05530823.
- [3] C. Delaunay, *Sur la surface de revolution dont la courbure moyenne est constante*, J. Math. Pures Appl., **16**:309–321, 1841.
- [4] K. Große Brauckmann, *New surfaces of constant mean curvature*, Math. Z., **214**:527–565, 1993, MR 1248112, Zbl 0806.53005.

- [5] I. Fernandez & P. Mira, *Holomorphic quadratic differentials and the Berstein problem in Heisenberg space*, to appear in Trans. Amer. Math. Soc., arxiv:0705.1436v2.
- [6] D. Fischer-Colbrie, *On complete minimal surfaces with finite Morse index in 3-manifolds*, Invent. Math., **82**:121–132, 1985, MR 0808112, Zbl 0573.53038.
- [7] M. Freedman, *An unknotting result for complete minimal surfaces in  $\mathbb{R}^3$* , Invent. Math, **109**(1):41–46, 1992, MR 1168364, Zbl 0773.53003.
- [8] C. Frohman & W.H. Meeks III, *The topological uniqueness of complete one-ended minimal surfaces and Heegaard surfaces in  $\mathbb{R}^3$* , J. Amer. Math. Soc., **10**(3):495–512, 1997, MR 1443545, Zbl 0886.57015.
- [9] L. Hauswirth, H. Rosenberg & J. Spruck, *On complete mean curvature  $\frac{1}{2}$  surfaces in  $\mathbb{H}^2 \times \mathbb{R}$* , Comm. Anal. Geom., **16**(5):989–1005, 2008, MR 2471365, Zbl pre05543232.
- [10] N. Kapouleas, *Complete constant mean curvature surfaces in Euclidean three space*, Ann. of Math., **131**:239–330, 1990, MR 1043269, Zbl 0699.53007.
- [11] H. Karcher, *The triply periodic minimal surfaces of Alan Schoen and their constant mean curvature companions*, Manuscripta Math., **64**:291–357, 1989, MR 1003093, Zbl 0687.53010.
- [12] N. Korevaar & R. Kusner, *The global structure of constant mean curvature surfaces*, Invent. Math., **114**:311–332, 1993, MR 1240641, Zbl 0803.53040.
- [13] N. Korevaar, R. Kusner & B. Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geom., **30**:465–503, 1989, MR 1010168, Zbl 0726.53007.
- [14] R. Kusner, *Global geometry of extremal surfaces in three-space*, PhD thesis, University of California, Berkeley, 1988.
- [15] H.B. Lawson, *Complete minimal surfaces in  $S^3$* , Ann. of Math., **92**:335–374, 1970, MR 0270280, Zbl 0205.52001.
- [16] R. Mazzeo & F. Pacard, *Constant mean curvature surfaces with Delaunay ends*, Comm. Anal. Geom., **9**(1):169–237, 2001, MR 1807955, Zbl 1005.53006.
- [17] R. Mazzeo, F. Pacard & D. Pollack, *Connected sums of constant mean curvature surfaces in euclidean 3-space*, J. Reine Angew. Math., **536**:115–165, 2001, MR 1837428, Zbl 0972.53010.
- [18] W.H. Meeks III, *The topology and geometry of embedded surfaces of constant mean curvature*, J. of Differential Geom., **27**:539–552, 1988, MR 0940118, Zbl 0617.53007.
- [19] W.H. Meeks III, J. Pérez & A. Ros, *Local removable singularity theorems for minimal and h-laminations*, preprint, available at <http://www.ugr.es/local/jperez/papers/papers.htm>.
- [20] W.H. Meeks III & H. Rosenberg, *Maximum principles at infinity*, J. Differential Geometry, **79**(1):141–165, 2008, MR 2401421, Zbl pre05285650.
- [21] W.H. Meeks III & G. Tinaglia, *Chord-arc properties for constant mean curvature disks*, work in progress.
- [22] ———, *The CMC dynamics theorem in homogeneous n-manifolds*, work in progress.
- [23] ———, *CMC surfaces in locally homogeneous three-manifolds*, work in progress.
- [24] ———, *Curvature estimates for constant mean curvature surfaces*, work in progress.



- [25] ———, *Properness results for constant mean curvature surfaces*, work in progress.
- [26] ———, *The rigidity of constant mean curvature surfaces*, preprint, arXiv:0801.3409.
- [27] A. Ros & H. Rosenberg, *Properly embedded surfaces with constant mean curvature*, preprint.
- [28] R. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom., **18**:791–809, 1983, MR 0730928, Zbl 0575.53037.
- [29] B. Smyth & G. Tinaglia, *The number of constant mean curvature isometric immersions of a surface with topology*, preprint, arXiv:0811.1231.
- [30] G. Tinaglia, *Multi-valued graphs in embedded constant mean curvature disks*, Trans. Amer. Math. Soc., **359**:143–164, 2007, MR 2247886, Zbl 1115.53010.
- [31] G. Tinaglia, *Structure theorems for embedded disks with mean curvature bounded in  $L^p$* , Comm. Anal. Geom., **16**(4):819–836, 2008, MR 2471371.

MATHEMATICS DEPARTMENT  
UNIVERSITY OF MASSACHUSETTS  
AMHERST, MA 01003

*E-mail address:* profmeeks@gmail.com

DEPARTMENT OF MATHEMATICS  
KING'S COLLEGE LONDON  
STRAND, LONDON, WC2R 2LS, U.K.

*E-mail address:* giuseppe.tinaglia@kcl.ac.uk