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# POINTS IN PROJECTIVE SPACES AND APPLICATIONS

Ivan Cheltsov

## Abstract

We prove the factoriality of a nodal hypersurface in $\mathbb{P}^4$ of degree $d$ that has at most $2(d-1)^2/3$ singular points, and we prove the factoriality of a double cover of $\mathbb{P}^3$ branched over a nodal surface of degree $2r$ having less than $(2r-1)r$ singular points.

## 1. Introduction

Let $\Sigma$ be a finite subset in $\mathbb{P}^n$ and $\xi \in \mathbb{N}$, where $n \geqslant 2$. Then the points of the set $\Sigma$ impose independent linear conditions on homogeneous forms of degree $\xi$ if and only if for every point $P \in \Sigma$ there is a homogeneous form of degree $\xi$ that vanishes at every point of the set $\Sigma \setminus P$, and does not vanish at the point $P$. The latter is equivalent to the equality

$$h^1\Big(\mathcal{I}_\Sigma \otimes \mathcal{O}_{\mathbb{P}^n}(\xi)\Big) = 0,$$

where $\mathcal{I}_\Sigma$ is the ideal sheaf of the subset $\Sigma \subset \mathbb{P}^n$.

In this paper we prove the following result (see Section 2).

**Theorem 1.** *Suppose that there is a natural number $\lambda \geqslant 2$ such that at most $\lambda k$ points of the set $\Sigma$ lie on a curve in $\mathbb{P}^n$ of degree $k$. Then*

$$h^1\Big(\mathcal{I}_\Sigma \otimes \mathcal{O}_{\mathbb{P}^n}(\xi)\Big) = 0$$

*in the case when one of the following conditions holds:*

- $\xi = \lfloor 3\lambda/2 - 3 \rfloor$ *and* $|\Sigma| < \lambda \lceil \lambda/2 \rceil$;
- $\xi = \lfloor 3\mu - 3 \rfloor$, $|\Sigma| \leqslant \lambda\mu$ *and* $\lfloor 3\mu \rfloor - \mu - 2 \geqslant \lambda \geqslant \mu$ *for some* $\mu \in \mathbb{Q}$;
- $\xi = \lfloor n\mu \rfloor$, $|\Sigma| \leqslant \lambda\mu$ *and* $(n-1)\mu \geqslant \lambda$ *for some* $\mu \in \mathbb{Q}$.

Let us consider applications of Theorem 1.

**Definition 2.** An algebraic variety $X$ is factorial if $\mathrm{Cl}(X) = \mathrm{Pic}(X)$.

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We assume that all varieties are projective, normal, and defined over $\mathbb{C}$.
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Let $\pi\colon X \to \mathbb{P}^3$ be a double cover branched over a surface $S \subset \mathbb{P}^3$ of degree $2r \geqslant 4$ such that the only singularities of the surface $S$ are isolated ordinary double points. Then $X$ is a hypersurface

$$w^2 = f_{2r}\big(x, y, z, t\big) \subset \mathbb{P}\big(1, 1, 1, 1, r\big) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z, t, w\big]\Big),$$

where $\mathrm{wt}(x) = \cdots = \mathrm{wt}(t) = 1$, $\mathrm{wt}(w) = r$, and $f_{2r}(x, y, z, t)$ is a homogeneous polynomial of degree $2r$ such that $S \subset \mathbb{P}^3$ is given by

$$f_{2r}\big(x, y, z, t\big) = 0 \subset \mathbb{P}^3 \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z, t\big]\Big).$$

The following conditions are equivalent (see [**10**] and [**8**]):

- the threefold $X$ is factorial;
- the singularities of the threefold $X$ are $\mathbb{Q}$-factorial[1] ;
- the equality $\mathrm{rk}\, H_4(X, \mathbb{Z}) = 1$ holds;
- the ring
$$\mathbb{C}\big[x, y, z, t, w\big]\Big/\big\langle w^2 - f_{2r}\big(x, y, z, t\big)\big\rangle$$
is a unique factorization domain;
- the points of the set $\mathrm{Sing}(S)$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $3r - 4$.

**Theorem 3.** *Suppose that the inequality*

$$\Big|\mathrm{Sing}\big(S\big)\Big| < \big(2r - 1\big)r$$

*holds. Then the threefold $X$ is factorial.*

*Proof.* The subset $\mathrm{Sing}(S) \subset \mathbb{P}^3$ is a set-theoretic intersection of surfaces of degree $2r - 1$. Then $X$ is factorial by Theorem 1. q.e.d.

The assertion of Theorem 3 is proved in [**4**] in the case when $r = 3$.

**Example 4.** Suppose that the surface $S$ is given by an equation

$$g_r^2\big(x, y, z, t\big) = g_1\big(x, y, z, t\big)g_{2r-1}\big(x, y, z, t\big) \subset \mathbb{P}^3, \tag{5}$$

where $g_i$ is a general homogeneous polynomial of degree $i$. Then

$$\Big|\mathrm{Sing}\big(S\big)\Big| = \big(2r - 1\big)r,$$

and $S$ has at most ordinary double points. But $X$ is not factorial.

For $r = 3$, the threefold $X$ is non-rational if it is factorial (see [**4**]), but the threefold $X$ is rational if the surface $S$ is the Barth sextic (see [**1**]).

We prove the following generalization of Theorem 3 in Section 3.

[1] A variety is $\mathbb{Q}$-factorial if some non-zero integral multiple of every Weil divisor on it is a Cartier divisor. This property is not local in the analytic topology, because ordinary double points of threefolds are not locally analytically $\mathbb{Q}$-factorial.

**Theorem 6.** *Suppose that the inequality*

$$\left|\operatorname{Sing}\big(S\big)\right| \leqslant \big(2r-1\big)r+1$$

*holds. Then $X$ is not factorial $\iff$ $S$ can be defined by equation* 5.

The assertion of Theorem 6 is proved in [**11**] in the case when $r = 3$.

Let $V$ be a hypersurface in $\mathbb{P}^4$ of degree $d$ such that $V$ has at most isolated ordinary double points. Then $V$ can be given by the equation

$$f_d\big(x,y,z,t,u\big) = 0 \subset \mathbb{P}^4 \cong \operatorname{Proj}\Big(\mathbb{C}\big[x,y,z,t,u\big]\Big),$$

where $f_d(x,y,z,t,u)$ is a homogeneous polynomial of degree $d$.

The following conditions are equivalent (see [**10**] and [**8**]):

- the threefold $V$ is factorial;
- the threefold $V$ has $\mathbb{Q}$-factorial singularities;
- the equality $\operatorname{rk} H_4(V,\mathbb{Z}) = 1$ holds;
- the ring
$$\mathbb{C}\big[x,y,z,t,u\big]\Big/\big\langle f_d\big(x,y,z,t,u\big)\big\rangle$$
is a unique factorization domain;
- the points of the set $\operatorname{Sing}(V)$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^4$ of degree $2d-5$.

The threefold $V$ is not rational if it is factorial and $d = 4$ (see [**12**]), but general determinantal quartic threefolds are known to be rational.

**Conjecture 7.** Suppose that the inequality

$$\left|\operatorname{Sing}\big(V\big)\right| < \big(d-1\big)^2$$

holds. Then the threefold $V$ is factorial.

The assertion of Conjecture 7 is proved in [**3**] and [**5**] for $d \leqslant 7$.

**Example 8.** Suppose that $V$ is given by the equation

$$xg\big(x,y,z,t,u\big) + yf\big(x,y,z,t,u\big) = 0 \subset \mathbb{P}^4 \cong \operatorname{Proj}\Big(\mathbb{C}\big[x,y,z,t,u\big]\Big),$$

where $g$ and $f$ are general homogeneous forms of degree $d-1$. Then

$$\left|\operatorname{Sing}\big(V\big)\right| = \big(d-1\big)^2$$

and $V$ has at most ordinary double points. But $V$ is not factorial.

The threefold $V$ is factorial if $|\operatorname{Sing}(V)| \leqslant (d-1)^2/4$ by [**2**].

**Theorem 9.** *Suppose that the inequality*

$$\left|\operatorname{Sing}\big(V\big)\right| \leqslant \frac{2\big(d-1\big)^2}{3}$$

*holds. Then the threefold $V$ is factorial.*

*Proof.* The set $\mathrm{Sing}(V)$ is a set-theoretic intersection of hypersurfaces of degree $d-1$. Then $V$ is factorial for $d \geqslant 7$ by Theorem 1.

For $d \leqslant 6$, the threefold $V$ is factorial by Theorem 2 in [**9**]. q.e.d.

Let $Y$ be a complete intersection of hypersurfaces $F$ and $G$ in $\mathbb{P}^5$ of degree $m$ and $k$, respectively, such that $m \geqslant k$, and the complete intersection $Y$ has at most isolated ordinary double points.

**Example 10.** Let $F$ and $G$ be general hypersurfaces that contain a two-dimensional linear subspace in $\mathbb{P}^5$. Then

$$\Big|\mathrm{Sing}\big(Y\big)\Big| = \big(m+k-2\big)^2 - \big(m-1\big)\big(k-1\big)$$

and $Y$ has at most ordinary double points. But $Y$ is not factorial.

The threefold $Y$ is factorial if G is smooth and singular points of Y impose independent linear conditions on homogeneous forms of degree $2m+k-6$ (see [**8**]).

**Theorem 11.** *Suppose that $G$ is smooth, and the inequalities*

$$\Big|\mathrm{Sing}\big(Y\big)\Big| \leqslant \big(m+k-2\big)\big(2m+k-6\big)/5$$

*and $m \geqslant 7$ hold. Then the threefold $Y$ is factorial.*

*Proof.* The set $\mathrm{Sing}(Y)$ is a set-theoretic intersection of hypersurfaces of degree $m+k-2$. Then $Y$ is factorial by Theorem 1. q.e.d.

Arguing as in the proof of Theorem 11, we obtain the following result.

**Theorem 12.** *Suppose that $G$ is smooth, and the inequalities*

$$\Big|\mathrm{Sing}\big(Y\big)\Big| \leqslant \big(2m+k-3\big)\big(m+k-2\big)/3$$

*and $m \geqslant k+6$ hold. Then the threefold $Y$ is factorial.*

Let $H$ be a smooth hypersurface in $\mathbb{P}^4$ of degree $d \geqslant 2$, and let

$$\eta\colon U \longrightarrow H$$

be a double cover branched over a surface $R \subset H$ such that

$$R \sim \mathcal{O}_{\mathbb{P}^4}\big(2r\big)\Big|_H$$

and $2r \geqslant d$. Suppose that $S$ has at most isolated ordinary double points.

**Theorem 13.** *Suppose that the inequalities*

$$\Big|\mathrm{Sing}\big(R\big)\Big| \leqslant \big(2r+d-2\big)r/2$$

*and $r \geqslant d+7$ hold. Then the threefold $U$ is factorial.*

*Proof.* The subset $\mathrm{Sing}(R) \subset \mathbb{P}^4$ is a set-theoretic intersection of hypersurfaces of degree $2r+d-2$. Then $U$ is factorial by Theorem 1, because it is factorial if the points of $\mathrm{Sing}(R)$ impose independent linear conditions on homogeneous forms of degree $3r+d-5$ (see [**8**]). q.e.d.

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## 2. Main result

Let $\Sigma$ be a finite subset in $\mathbb{P}^n$, where $n \geqslant 2$. Now we prove the following special case of Theorem 1, leaving the other cases to the reader.

**Proposition 14.** *Let $r \geqslant 2$ be a natural number. Suppose that*

$$\Big|\Sigma\Big| < \big(2r-1\big)r,$$

*and at most $(2r-1)k$ points in $\Sigma$ lie on a curve of degree $k$. Then*

$$h^1\Big(\mathcal{I}_\Sigma \otimes \mathcal{O}_{\mathbb{P}^n}\big(3r-4\big)\Big) = 0.$$

The following result is Corollary 4.3 in [**7**].

**Theorem 15.** *Let $\pi\colon Y \to \mathbb{P}^2$ be a blow up of points $P_1, \ldots, P_\delta \in \mathbb{P}^2$, and let $E_i$ be the $\pi$-exceptional divisor such that $\pi(E_i) = P_i$. Then*

$$\Big|\pi^*\Big(\mathcal{O}_{\mathbb{P}^2}\big(\xi\big)\Big) - \sum_{i=1}^{\delta} E_i\Big|$$

*does not have base points if at most $k(\xi+3-k)-2$ points in $\{P_1, \ldots, P_\delta\}$ lie on a curve of degree $k$ for every $k \leqslant (\xi+3)/2$, and the inequality*

$$\delta \leqslant \max\left\{\Big\lfloor\frac{\xi+3}{2}\Big\rfloor\left(\xi+3-\Big\lfloor\frac{\xi+3}{2}\Big\rfloor\right)-1, \Big\lfloor\frac{\xi+3}{2}\Big\rfloor^2\right\}$$

*holds, where $\xi$ is a natural number such that $\xi \geqslant 3$.*

Therefore, it follows from Theorem 15 that to prove Proposition 14, we may assume that $n = 3$ due to the following result.

**Lemma 16.** *Let $\Pi \subset \mathbb{P}^n$ be an $m$-dimensional linear subspace, and let*

$$\psi\colon \mathbb{P}^n \dashrightarrow \Pi \cong \mathbb{P}^m$$

*be a projection from a linear subspace $\Omega \subset \mathbb{P}^n$ such that*

- *the subspace $\Omega$ is sufficiently general and* $\dim(\Omega) = n - m - 1$*,*
- *there is a subset $\Lambda \subset \Sigma$ such that*

$$\Big|\Lambda\Big| \geqslant \lambda k + 1,$$

*but the set $\psi(\Lambda)$ is contained in an irreducible curve of degree $k$,*

*and $n > m \geqslant 2$. Let $\mathcal{M}$ be the linear system that contains all hypersurfaces in $\mathbb{P}^n$ of degree $k$ that pass through all points in $\Lambda$. Then*

$$\dim\Big(\mathrm{Bs}\big(\mathcal{M}\big)\Big) = 0,$$

*and either $m = 2$, or $k > \lambda$.*

*Proof.* Suppose that there is an irreducible curve $Z$ such that

$$Z \subset \mathrm{Bs}\big(\mathcal{M}\big),$$

and put $\Xi = Z \cap \Lambda$. We may assume that $\psi|_Z$ is a birational morphism, and

$$\psi\big(Z\big) \cap \psi\Big(\Lambda \setminus \Xi\Big) = \varnothing,$$

because $\Omega$ is general. Then $\deg(\psi(Z)) = \deg(Z)$.

Let $C$ be an irreducible curve in $\Pi$ of degree $k$ that contains $\psi(\Lambda)$, and let $W$ be the cone in $\mathbb{P}^n$ over the curve $C$ and with vertex $\Omega$. Then

$$W \in \mathcal{M},$$

which implies that $W$ contains the curve $Z$. Thus, we have

$$\psi\big(Z\big) = C,$$

which implies that $\Xi = \Lambda$ and $\deg(Z) = k$. But $|Z \cap \Sigma| \leqslant \lambda k$. We have

$$\dim\Big(\mathrm{Bs}\big(\mathcal{M}\big)\Big) = 0.$$

Suppose that $m > 2$ and $k \leqslant \lambda$. Let us show that the latter assumption leads to a contradiction. We may assume that $m = 3$ and $n = 4$, because $\psi$ as a composition of $n - m$ projections from points.

Let $\mathcal{Y}$ be the set of all irreducible reduced surfaces in $\mathbb{P}^4$ of degree $k$ that contains all points of the set $\Lambda$, and let $\Upsilon$ be a subset of $\mathbb{P}^4$ consisting of points that are contained in every surface of $\mathcal{Y}$. Then

$$\Lambda \subseteq \Upsilon,$$

but the previous arguments imply that $\Upsilon$ is a finite set.

Let $\mathcal{S}$ be the set of all surfaces in $\mathbb{P}^3$ of degree $k$ such that

$$S \in \mathcal{S} \iff \exists\ Y \in \mathcal{Y}\ \Big|\ \psi\big(Y\big) = S \text{ and } \psi\big|_Y \text{ is a birational morphism},$$

and let $\Psi$ be a subset of $\mathbb{P}^3$ consisting of points that are contained in every surface of the set $\mathcal{S}$. Then $\mathcal{S} \neq \varnothing$ and

$$\psi\big(\Lambda\big) \subseteq \psi\big(\Upsilon\big) \subseteq \Psi.$$

The generality of $\Omega$ implies that $\psi(\Upsilon) = \Psi$. Indeed, for every point

$$O \in \Pi \setminus \Psi$$

and for a general surface $Y \in \mathcal{Y}$, we may assume that the line passing through $O$ and $\Omega$ does not intersect $Y$, but $\psi|_Y$ is a birational morphism.

The set $\Psi$ is a set-theoretic intersection of surfaces in $\Pi$ of degree $k$, which implies that at most $\delta k$ points in $\Psi$ lie on a curve in $\Pi$ of degree $\delta$.

We see that at most $k^2$ points in $\Psi$ lie on a curve in $\Pi$ of degree $k$, but the set $\psi(\Lambda)$ contains at least $\lambda k + 1$ points that are contained in an irreducible curve in $\Pi$ of degree $k$, which is a contradiction. q.e.d.

We have a finite subset $\Sigma \subset \mathbb{P}^3$ and a natural number $r \geqslant 2$ such that

$$\big|\Sigma\big| < \big(2r-1\big)r,$$

and at most $(2r-1)k$ points in $\Sigma$ lie on a curve of degree $k$. Then

$$\big|\Sigma\big| < \big(2r-1\big)\big(r-\epsilon\big)$$

for some integer $\epsilon \geqslant 0$. Let us prove the following result.

**Proposition 17.** *The equality $h^1(\mathcal{I}_\Sigma \otimes \mathcal{O}_{\mathbb{P}^3}(3r-4-\epsilon)) = 0$ holds.*

Fix a point $P \in \Sigma$. To prove Proposition 17, it is enough to construct a surface[2] of degree $3r-4-\epsilon$ that contains $\Sigma\setminus P$ and does not contain $P$.

We assume that $r \geqslant 3$ and $\epsilon \leqslant r-3$, because the assertion of Proposition 17 follows from Theorem 2 in [**9**] and Theorem 15 otherwise.

**Lemma 18.** *Suppose that there is a hyperplane $\Pi \subset \mathbb{P}^3$ that contains the set $\Sigma$. Then there is a surface of degree $3r-4-\epsilon$ that contains every point of the set $\Sigma \setminus P$ and does not contain the point $P$.*

*Proof.* Suppose that $|\Sigma \setminus P| > \lfloor (3r-1-\epsilon)/2 \rfloor^2$. Then

$$\big(2r-1\big)\big(r-\epsilon\big) - 2 \geqslant \big|\Sigma \setminus P\big| \geqslant \Big\lfloor \frac{3r-1-\epsilon}{2} \Big\rfloor^2 + 1 \geqslant \frac{(3r-2-\epsilon)^4}{4} + 1,$$

which implies that $(r-4)^2+2\epsilon r+\epsilon^2 \leqslant 0$. We have $r=4$ and $\epsilon=0$. Then

$$\big|\Sigma \setminus P\big| \leqslant \Big\lfloor \frac{3r-1-\epsilon}{2} \Big\rfloor \left(3r-1-\epsilon - \Big\lfloor \frac{3r-1-\epsilon}{2} \Big\rfloor\right).$$

Thus, in every possible case, the number $|\Sigma \setminus P|$ does not exceed

$$\max\left(\Big\lfloor \frac{3r-1-\epsilon}{2} \Big\rfloor \left(3r-1-\epsilon - \Big\lfloor \frac{3r-1-\epsilon}{2} \Big\rfloor\right),\ \Big\lfloor \frac{3r-1-\epsilon}{2} \Big\rfloor^2\right).$$

At most $3r-4-\epsilon$ points of $\Sigma\setminus P$ lie on a line, because $3r-4-\epsilon \geqslant 2r-1$.

Let us prove that at most $k(3r-1-\epsilon-k)-2$ points in $\Sigma \setminus P$ can lie on a curve of degree $k \leqslant (3r-1-\epsilon)/2$. It is enough to show that

$$k\big(3r-1-\epsilon-k\big) - 2 \geqslant k\big(2r-1\big)$$

for all $k \leqslant (3r-1-\epsilon)/2$. We must prove this only for $k > 1$ such that

$$k\big(3r-1-\epsilon-k\big) - 2 < \big|\Sigma \setminus P\big| \leqslant \big(2r-1\big)\big(r-\epsilon\big) - 2,$$

because otherwise the condition that at most $k(3r-1-k)-2$ points in the set $\Sigma \setminus P$ can lie on a curve of degree $k$ is vacuous.

We may assume that $k < r-\epsilon$. But

$$k(3r-1-\epsilon-k) - 2 \geqslant k(2r-1) \iff r > k-\epsilon,$$

which immediately implies that at most $k(3r-1-\epsilon-k)-2$ points in the set $\Sigma \setminus P$ can lie on a curve of degree $k$.

[2]For simplicity we consider homogeneous forms on $\mathbb{P}^n$ as hypersurfaces.

It follows from Theorem 15 that there is a curve

$$C \subset \Pi \cong \mathbb{P}^2$$

of degree $3r-4-\epsilon$ that contains $\Sigma \setminus P$ and does not contain $P \in \Sigma$.

A general cone in $\mathbb{P}^3$ over the curve $C$ is the required surface. q.e.d.

Fix a general hyperplane $\Pi \subset \mathbb{P}^3$. Let $\psi\colon \mathbb{P}^3 \dashrightarrow \Pi$ be a projection from a sufficiently general point $O \in \mathbb{P}^3$. Put $\Sigma' = \psi(\Sigma)$ and $P' = \psi(P)$.

**Lemma 19.** *Suppose that at most $(2r-1)k$ points in $\Sigma'$ lie on a curve of degree $k$. Then there is a surface in $\mathbb{P}^3$ of degree $3r-4-\epsilon$ that contains all points of the set $\Sigma \setminus P$ but does not contain the point $P \in \Sigma$.*

*Proof.* Arguing as in the proof of Lemma 18, we obtain a curve

$$C \subset \Pi \cong \mathbb{P}^2$$

of degree $3r-4-\epsilon$ that contains $\Sigma' \setminus P'$ and does not pass through $P'$.

Let $Y$ be the cone in $\mathbb{P}^3$ over $C$ whose vertex is $O$. Then $Y$ is a surface of degree $3r-4-\epsilon$ that contains all points of the set $\Sigma \setminus P$ but does not contain the point $P \in \Sigma$. q.e.d.

To conclude the proof of Proposition 14, we may assume that there is a natural number $k$ such that at least $(2r-1)k+1$ points of $\Sigma'$ lie on a curve of degree $k$, where $k$ is the smallest number of such property.

**Lemma 20.** *The inequality $k \geqslant 3$ holds.*

*Proof.* The inequality $k \geqslant 2$ holds by Lemma 16, which implies $r \geqslant 3$.

Suppose that there is a subset $\Phi \subseteq \Sigma$ such that

$$\big|\Phi\big| > 2\big(2r-1\big),$$

but $\psi(\Phi)$ is contained in a conic $C \subset \Pi$. Then $C$ is irreducible.

Let $\mathcal{D}$ be a linear system of quadrics in $\mathbb{P}^3$ containing $\Phi$. Then

$$\dim\Big(\mathrm{Bs}\big(\mathcal{D}\big)\Big) = 0$$

by Lemma 16. Let $W$ be a cone in $\mathbb{P}^3$ over $C$ with the vertex $\Omega$. Then

$$8 = D_1 \cdot D_2 \cdot W \geqslant \sum_{\omega \in \Phi} \mathrm{mult}_{\omega}(D_1)\mathrm{mult}_{\omega}(D_2) \geqslant |\,\Phi| > 2\big(2r-1\big) \geqslant 8,$$

where $D_1$ and $D_2$ are general divisors in $\mathcal{D}$. q.e.d.

Therefore, there is a subset $\Lambda_k^1 \subseteq \Sigma$ such that

$$\big|\Lambda_k^1\big| > \big(2r-1\big)k,$$

but the subset $\psi(\Lambda_k^1) \subset \Pi \cong \mathbb{P}^2$ is contained in an irreducible curve of degree $k \geqslant 3$. Similarly, we obtain a disjoint union

$$\bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i,$$

where $\Lambda_j^i$ is a subset in $\Sigma$ such that

$$\big|\Lambda_j^i\big| > \big(2r-1\big)j,$$

the subset $\psi(\Lambda_j^i)$ is contained in an irreducible reduced curve of degree $j$, and at most $(2r-1)\zeta$ points of the subset

$$\psi\Big(\Sigma \setminus \Big(\bigcup_{j=k}^{l}\bigcup_{i=1}^{c_j}\Lambda_j^i\Big)\Big) \subsetneq \Sigma' \subset \Pi \cong \mathbb{P}^2$$

lie on a curve in $\Pi$ of degree $\zeta$. Put $\Lambda = \cup_{j=k}^{l}\cup_{i=1}^{c_j}\Lambda_j^i$.

Let $\Xi_j^i$ be the base locus of the linear system of surfaces of degree $j$ that pass through the set $\Lambda_j^i$. Then $\Xi_j^i$ is a finite set by Lemma 16, and

$$\big|\Sigma \setminus \Lambda\big| < \big(2r-1\big)\big(r-\epsilon\big) - 1 - \sum_{i=k}^{l} c_i\big(2r-1\big)i. \tag{21}$$

**Corollary 22.** *The inequality* $\sum_{i=k}^{l} ic_i \leqslant r-\epsilon-1$ *holds.*

We have $\Lambda_j^i \subseteq \Xi_j^i$. But the set $\Xi_j^i$ imposes independent linear conditions on homogeneous forms of degree $3(j-1)$ by the following result.

**Lemma 23.** *Let* $\mathcal{M}$ *be a linear subsystem in* $|\mathcal{O}_{\mathbb{P}^n}(\lambda)|$ *such that*

$$\dim\Big(\mathrm{Bs}\big(\mathcal{M}\big)\Big) = 0,$$

*where* $\lambda \geqslant 2$. *Then the points in* $\mathrm{Bs}(\mathcal{M})$ *impose independent linear conditions on homogeneous forms on* $\mathbb{P}^n$ *of degree* $n(\lambda-1)$.

*Proof.* See Lemma 22 in [**2**] or Theorem 3 in [**6**]. q.e.d.

Put $\Xi = \cup_{j=k}^{l}\cup_{i=1}^{c_j}\Xi_j^i$. Then $\Lambda \subseteq \Xi$.

**Lemma 24.** *Suppose that* $\Sigma$ *is contained in* $\Xi$. *Then there is a surface of degree* $3r-4-\epsilon$ *that contains* $\Sigma \setminus P$ *and does not contain* $P \in \Sigma$.

*Proof.* For every $\Xi_j^i$ containing $P$ there is a surface of degree $3(j-1)$ that contains the set $\Xi_j^i \setminus P$ and does not contain $P$ by Lemma 23.

For every $\Xi_j^i$ not containing $P$ there is a surface of degree $j$ that contains $\Xi_j^i$ and does not contain $P$ by the definition of the set $\Xi_j^i$.

We have $j < 3(j-1)$, because $k \geqslant 2$. For every $\Xi_j^i$ there is a surface

$$F_i^j \subset \mathbb{P}^3$$

of degree $3(j-1)$ that contains the set $\Xi_j^i \setminus (\Xi_j^i \cap P)$ and does not contain the point $P$. The union $\cup_{j=k}^{l}\cup_{i=1}^{c_j} F_j^i$ is a surface of degree

$$\sum_{i=k}^{l} 3(i-1)c_i \leqslant \sum_{i=k}^{l} 3ic_i - 3c_k \leqslant 3r-6-3\epsilon \leqslant 3r-4-\epsilon$$

that contains the set $\Sigma \setminus P$ and does not contain the point $P$. q.e.d.

The proof of Lemma 24 implies that there is surface of degree

$$\sum_{i=k}^{l} 3(i-1)c_i$$

containing $(\Xi\cap\Sigma)\setminus(\Xi\cap P)$ and not containing $P$, and a surface of degree

$$\sum_{i=k}^{l} ic_i$$

containing $\Xi\cap\Sigma$ and not containing any point of the set $\Sigma\setminus(\Xi\cap\Sigma)$.

**Lemma 25.** *Let $\Lambda$ and $\Delta$ be disjoint finite subsets in $\mathbb{P}^n$ such that*

- *there is a hypersurface in $\mathbb{P}^n$ of degree $\zeta$ that contains all points in the set $\Lambda$ and does not contain any point in the set $\Delta$,*
- *the points of the sets $\Lambda$ and $\Delta$ impose independent linear conditions on hypersurfaces in $\mathbb{P}^n$ of degree $\xi$ and $\xi-\zeta$, respectively,*

*where $\xi\geqslant\zeta$ are natural numbers. Then the points of the set $\Lambda\cup\Delta$ impose independent linear conditions on hypersurfaces in $\mathbb{P}^n$ of degree $\xi$.*

*Proof.* Let $Q$ be a point in $\Lambda\cup\Delta$. To conclude the proof we must find a hypersurface of degree $\xi$ that passes through the set $(\Lambda\cup\Delta)\setminus Q$ and does not contain the point $Q$. We may assume that $Q\in\Lambda$.

Let $F$ be the homogenous form of degree $\xi$ that vanishes at every point of the set $\Lambda\setminus Q$ and does not vanish at the point $Q$. Put

$$\Delta=\Big\{Q_1,\ldots,Q_\delta\Big\},$$

where $Q_i$ is a point. There is a homogeneous form $G_i$ of degree $\xi$ that vanishes at every point in $(\Lambda\cup\Delta)\setminus Q_i$ and does not vanish at $Q_i$. Then

$$F\big(Q_i\big)+\mu_i G_i\big(Q_i\big)=0$$

for some $\mu_i\in\mathbb{C}$, because $g_i(Q_i)\neq 0$. Then the homogenous form

$$F+\sum_{i=1}^{\delta}\mu_i G_i$$

vanishes on set $(\Lambda\cup\Delta)\setminus Q$ and does not vanish at the point $Q$. q.e.d.

Put $d=3r-4-\epsilon-\sum_{i=k}^{l} ic_i$ and

$$\bar{\Sigma}=\psi\Big(\Sigma\setminus\big(\Xi\cap\Sigma\big)\Big).$$

To prove Proposition 17, we may assume that $\emptyset\neq\bar{\Sigma}\subsetneq\Sigma'$.

It follows from Lemma 25 that to prove Proposition 17 it is enough to show that $\bar{\Sigma}\subset\Pi$ and $d$ satisfy the hypotheses of Theorem 15.

**Lemma 26.** *The inequality $|\bar{\Sigma}|\leqslant\lfloor(d+3)/2\rfloor^2$ holds.*

*Proof.* Suppose that the inequality $|\bar{\Sigma}| \geqslant \lfloor (d+3)/2 \rfloor^2 + 1$ holds. Then

$$\big(2r-1\big)\Big(r-\epsilon-\sum_{i=k}^{l} c_i i\Big) - 2 \geqslant \big|\bar{\Sigma}\big| \geqslant \frac{\Big(3r-2-\epsilon-\sum_{i=k}^{l} ic_i\Big)^2}{4} + 1$$

by Corollary 22. Put $\Delta = \epsilon + \sum_{i=k}^{l} c_i i$. Then $\Delta \geqslant k \geqslant 3$ and

$$4\big(2r-1\big)\big(r-\Delta\big) - 12 \geqslant \big(3r-2-\Delta\big)^2,$$

which implies that $0 < r^2 - 8r + 16 + 2r\Delta + \Delta^2 \leqslant 0$. q.e.d.

The inequality $d \geqslant 3$ holds by Corollary 22, because $r \geqslant 3$.

**Lemma 27.** *Suppose that at least $d+1$ points in the set $\bar{\Sigma}$ are contained in a line. Then there is a surface in $\mathbb{P}^3$ of degree $3r-4-\epsilon$ that contains all points of the set $\Sigma \setminus P$ and does not contains the point $P \in \Sigma$.*

*Proof.* We have $|\bar{\Sigma}| \geqslant d+1$. It follows from inequality 21 that

$$3r-3-\epsilon-\sum_{i=k}^{l} ic_i < \big(2r-1\big)\big(r-\epsilon\big) - 1 - \sum_{i=k}^{l} c_i\big(2r-1\big)i,$$

which gives $\sum_{i=k}^{l} ic_i \neq r-\epsilon-1$. Now it follows from Corollary 22 that

$$\sum_{i=k}^{l} ic_i \leqslant r-\epsilon-2,$$

but $2r-1 \geqslant 3r-3-\epsilon-\sum_{i=k}^{l} ic_i$. Then $\sum_{i=k}^{l} ic_i = r-\epsilon-2$ and $d = 2r-2$.

We have a surface of degree $\sum_{i=k}^{l} 3(i-1)c_i \leqslant 3r-4-\epsilon$ that contains

$$\Big(\Xi \cap \Sigma\Big) \setminus \Big(\Xi \cap P\Big)$$

and does not contain $P$. But we have a surface of degree $r-\epsilon-2$ that contains $\Xi \cap \Sigma$ and does not contain any point of the set $\Sigma \setminus (\Xi \cap \Sigma)$.

The set $\Sigma \setminus (\Xi \cap \Sigma)$ contains at most $4r-4$ points, at most $2r-1$ points of the set $\Sigma$ lie on a line. It follows from Theorem 2 in [**9**] that the set

$$\Sigma \setminus \Big(\Xi \cap \Sigma\Big)$$

imposes independent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $2r-2$. Applying Lemma 25, we complete the proof. q.e.d.

So, we may assume that at most $d$ points in $\bar{\Sigma}$ lie on a line.

**Lemma 28.** *For every $t \leqslant (d+3)/2$, at most*

$$t\big(d+3-t\big) - 2$$

*points in $\bar{\Sigma}$ lie on a curve of degree $t$ in $\Pi \cong \mathbb{P}^2$.*

*Proof.* At most $(2r-1)t$ of the points in $\bar{\Sigma}$ lie on a curve of degree $t$, which implies that to conclude the proof it is enough to show that

$$t\big(d+3-t\big)-2 \geqslant \big(2r-1\big)t$$

for every $t \leqslant (d+3)/2$ such that $t>1$ and $t(d+3-t)-2<|\bar{\Sigma}|$. But

$$t(d+3-t)-2 \geqslant t(2r-1) \iff r-\epsilon-\sum_{i=k}^{l} ic_i > t,$$

because $t>1$. Thus, we may assume that $t(d+3-t)-2<|\bar{\Sigma}|$ and

$$r-\epsilon-\sum_{i=k}^{l} ic_i \leqslant t \leqslant \frac{d+3}{2}.$$

Let $g(x)=x(d+3-x)-2$. Then

$$g\big(t\big) \geqslant g\Big(r-\epsilon-\sum_{i=k}^{l} ic_i\Big),$$

because $g(x)$ is increasing for $x<(d+3)/2$. Therefore, we have

$$\big(2r-1\big)\Big(r-\epsilon-\sum_{i=k}^{l} ic_i\Big)-2 \geqslant \big|\bar{\Sigma}\big| > g(t) \geqslant \Big(r-\epsilon-\sum_{i=k}^{l} ic_i\Big)\big(2r-1\big)-2,$$

because inequality 21 holds. q.e.d.

We can apply Theorem 15 to the blow up of the plane $\Pi$ at the points of the set $\bar{\Sigma}$ and to the integer $d$. Then applying Lemma 25, we obtain a surface in $\mathbb{P}^3$ of degree $3r-4-\epsilon$ containing $\Sigma \setminus P$ and not containing $P$.

The assertion of Proposition 17 is completely proved, which implies the assertion of Proposition 14. The proof of Theorem 1 is similar.

## 3. Auxiliary result

Now we prove Theorem 6. Let $\pi\colon X \to \mathbb{P}^3$ be a double cover branched over a surface $S$ of degree $2r \geqslant 4$ with isolated ordinary double points.

**Lemma 29.** *Let $F$ be a hypersurface in $\mathbb{P}^n$ of degree $d$ that has isolated singularities, and let $C$ be a curve in $\mathbb{P}^n$ of degree $k$. Then*

- *the inequality $|\operatorname{Supp}(C) \cap \operatorname{Sing}(F)| \leqslant k(d-1)$ holds,*
- *the equality $|\operatorname{Supp}(C) \cap \operatorname{Sing}(F)| = k(d-1)$ implies that*

$$\operatorname{Sing}\big(C\big) \cap \operatorname{Sing}\big(F\big) = \varnothing.$$

*Proof.* Let $f(x_0,\ldots,x_n)$ be the homogeneous form of degree $d$ such that $f(x_0,\ldots,x_n)=0$ defines $F \subset \mathbb{P}^n$, where $(x_0:\ldots:x_n)$ are homogeneous coordinates on $\mathbb{P}^n$. Put

$$\mathcal{D}=\left|\sum_{i=0}^{n} \lambda_i \frac{\partial f}{\partial x_i}=0\right| \subset \big|\mathcal{O}_{\mathbb{P}^n}(d-1)\big|,$$

where $\lambda_0, \ldots, \lambda_n$ are complex numbers. Then

$$\mathrm{Bs}\big(\mathcal{D}\big) = \mathrm{Sing}\big(F\big),$$

which implies that the curve $C$ intersects a generic member of the linear system $\mathcal{D}$ at most $(d-1)k$ times, which implies the assertion. q.e.d.

**Lemma 30.** *Let $\Pi \subset \mathbb{P}^3$ be a hyperplane, and let $C \subset \Pi$ be a reduced curve of degree $r$. Suppose that the equality*

$$\mathrm{Supp}\big(C\big) \cap \mathrm{Sing}\big(S\big) = \big(2r-1\big)r$$

*holds. Then $S$ can be defined by equation* 5.

*Proof.* Put

$$S\Big|_{\Pi} = \sum_{i=1}^{\alpha} m_i C_i,$$

where $C_i$ is an irreducible reduced curve, and $m_i$ is a natural number.

We assume that $C_i \neq C_j$ for $i \neq j$, and $C = \sum_{i=1}^{\beta} C_i$, where $\beta \leqslant \alpha$.

It follows from Lemma 29 and from the equalities

$$\sum_{i=1}^{\beta} \mathrm{deg}(C_i) = r = \frac{\sum_{i=1}^{\alpha} m_i \mathrm{deg}(C_i)}{2} \tag{31}$$

that $C_i \cap \mathrm{Sing}(S) = (2r-1)\mathrm{deg}(C_i)$ if $i \leqslant \beta$, and

$$\mathrm{Sing}\big(C\big) \cap \mathrm{Sing}\big(S\big) = \varnothing.$$

Suppose that $m_\gamma = 1$ for some $\gamma \leqslant \beta$. Then

$$C_\gamma \cap \mathrm{Sing}(S) = \big(2r-1\big)\mathrm{deg}\big(C_\gamma\big),$$

but the curve $S|_\Pi = \sum_{i=1}^{\alpha} m_i C_i$ must be singular at every singular point of the surface $S$ that is contained in $C_\gamma$. Thus, we have

$$\mathrm{Sing}\big(S\big) \cap \mathrm{Supp}\big(C_\gamma\big) \subseteq \bigcup_{i \neq \gamma} C_i \cap C_\gamma,$$

but $|\, C_i \cap C_\gamma| \leqslant (C_i \cdot C_\gamma)_\Pi = \mathrm{deg}(C_i)\mathrm{deg}(C_\gamma)$ for $i \neq \gamma$. Hence, we have

$$\sum_{i \neq \gamma} \mathrm{deg}(C_i)\mathrm{deg}(C_\gamma) \geqslant \big(2r-1\big)\mathrm{deg}(C_\gamma),$$

but on the plane $\Pi$ we have the equalities

$$\big(2r - \mathrm{deg}(C_\gamma)\big)\mathrm{deg}(C_\gamma) = \Big(S\Big|_{\Pi} - C_\gamma\Big) \cdot C_\gamma = \sum_{i \neq \gamma} m_i \mathrm{deg}(C_i)\mathrm{deg}(C_\gamma),$$

which implies that $\mathrm{deg}(C_\gamma) = 1$ and $m_i = 1$ for every $i$.

Now, equalities 31 imply that $\beta < \alpha$, but every singular point of the surface $S$ that is contained in the curve $C$ must lie in the set

$$C \cap \bigcup_{i=\beta+1}^{\alpha} C_i$$

that consists of at most $r^2$ points, which is a contradiction.

Thus, we see that $m_i \geqslant 2$ for every $i \leqslant \beta$. Therefore, it follows from the equalities 31 that $\alpha = \beta$ and $m_i = 2$ for every $i$.

Let $f(x,y,z,w)$ be the homogeneous form of degree $2r$ such that

$$f\big(x,y,z,w\big) = 0$$

defines the surface $S \subset \mathbb{P}^3$, where $(x:y:z:w)$ are homogeneous coordinates on $\mathbb{P}^3$. We may assume that $\Pi$ is given by $x = 0$. Then

$$f\big(0,y,z,w\big) = g_r^2\big(y,z,w\big),$$

where $g_r(y,z,w)$ is a form of degree $r$ such that $C$ is given by

$$x = g_r\big(y,z,w\big) = 0,$$

which implies that $S$ can be defined by equation 5. q.e.d.

It follows from Lemma 29 that at most $(2r-1)k$ singular points of the surface $S$ can lie on a curve in $\mathbb{P}^3$ of degree $k$.

**Lemma 32.** *Let $C$ be an irreducible reduced curve in $\mathbb{P}^3$ of degree $k$ that is not contained in a hyperplane. Then*

$$\Big|C \cap \mathrm{Sing}\big(S\big)\Big| \leqslant \big(2r-1\big)k - 2.$$

*Proof.* Suppose that the curve $C$ contains at least $(2r-1)k-1$ singular points of the surface $S$. Then $C \subset S$, because otherwise we have

$$2rk = \deg\big(C\big)\deg\big(S\big) \leqslant 2\big(2r-1\big)k - 2 = 4rk - 2k - 2,$$

which leads to $2k(r-1) \leqslant 2$. But $r \geqslant 2$ and $k \geqslant 3$.

Let $O$ be a sufficiently general point of the curve $C$, and let

$$\psi \colon \mathbb{P}^3 \dashrightarrow \Pi$$

be a projection from $O$, where $\Pi$ is a general plane in $\mathbb{P}^3$. Then

$$\psi\Big|_C \colon C \dashrightarrow \psi\big(C\big)$$

is a birational morphism, because $C$ is not a plane curve.

Put $Z = \psi(C)$. Then $Z$ has degree $k-1$.

Let $Y$ be a cone in $\mathbb{P}^3$ over $Z$ with the vertex $O$. Then $C \subset Y$.

The point $O$ is not contained in a hyperplane in $\mathbb{P}^3$ that is tangent to the surface $S$ at some point of the curve $C$, because $C$ is not contained in a hyperplane. Then $Y$ does not tangent $S$ along the curve $C$. Put

$$S\Big|_Y = C + R,$$

where $R$ is a curve of degree $2rk - k - 2r$. The generality in the choice of the point $O$ implies that $R$ does not contain rulings of the cone $Y$.

Let $\alpha : \bar{Z} \to Z$ be the normalization of $Z$. Then the diagram

$$\begin{array}{ccc} \bar{Y} & \xrightarrow{\ \beta\ } & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \psi|_{Y}} \\ \bar{Z} & \xrightarrow[\ \alpha\ ]{} & Z \end{array}$$

commutes, where $\beta$ is a birational morphism, the surface $\bar{Y}$ is smooth, and $\pi$ is a $\mathbb{P}^1$-bundle. Let $L$ be a general fiber of $\pi$, and $E$ be a section of the $\mathbb{P}^1$-bundle $\pi$ such that $\beta(E) = O$. Then $E^2 = -k + 1$ on $\bar{Y}$.

Let $Q$ be an arbitrary point of the set

$$\mathrm{Sing}\big(S\big) \cap C,$$

and let $\bar{C}$ and $\bar{R}$ be proper transforms of the curves $C$ and $R$ on the surface $\bar{Y}$, respectively. Then there is a point $\bar{Q} \in \bar{Y}$ such that

$$\bar{Q} \in \mathrm{Supp}\Big(\bar{C} \cdot \bar{R}\Big)$$

and $\beta(\bar{Q}) = Q$. But we have

$$\bar{R} \equiv \big(2r - 2\big)E + \big(2rk - k - 2r\big)L$$

and $\bar{C} \equiv E + kL$. Therefore, we have

$$\big(2r - 1\big)k - 2 = \bar{C} \cdot \bar{R} \geqslant \big(2r - 1\big)k - 1,$$

which is a contradiction. q.e.d.

Now we prove Theorem 6 by reductio ad absurdum, where we assume that $r \geqslant 4$, because the case $r = 3$ is done in [**11**].

Put $\Sigma = \mathrm{Sing}(S)$, and suppose that the following conditions hold:

- the inequalities $|\Sigma| \leqslant (2r - 1)r + 1$ and $r \geqslant 3$ hold;
- the surface $S$ can not be defined by equation 5;
- the threefold $X$ is not factorial.

There is a point $P \in \Sigma$ such that every surface in $\mathbb{P}^3$ of degree $3r - 4$ that pass through the set $\Sigma \setminus P$ contains the point $P$ as well.

**Lemma 33.** *Let* $\Pi$ *be a hyperplane in* $\mathbb{P}^3$. *Then* $|\Pi \cap \Sigma| \leqslant 2r$.

*Proof.* Suppose that the inequality $|\Pi \cap \Sigma| > 2r$ holds. Let us show that this assumption leads to a contradiction.

Let $\Gamma$ be the subset of the set $\Sigma$ that consists of all points that are not contained in the plane $\Pi$. Then $\Gamma$ contains at most

$$\big(2r - 1\big)\big(r - 1\big) - 1$$

points, which impose independent linear conditions on homogeneous forms of degree $3r - 5$ by Proposition 17.

Suppose that $P \notin \Pi$. There is a surface $F \subset \mathbb{P}^3$ of degree $3r-5$ that contains the set $\Gamma \setminus P$ and does not contain the point $P$. Then

$$F \cup \Pi \subset \mathbb{P}^3$$

is the surface of degree $3r-4$ that contains the set $\Sigma \setminus P$ and does not contain the point $P$, which is impossible. Therefore, we have $P \in \Pi$.

Arguing as in the proof of Lemma 29, we see that

$$\big|\Pi \cap \Sigma\big| \leqslant \big(2r-1\big)r,$$

because $S|_{\Pi}$ is singular in every point of the set $\Pi \cap \Sigma$.

It follows from Lemma 30 that $\Pi \cap \Sigma$ is not contained in a curve of degree $r$ if $|\Pi \cap \Sigma| = (2r-1)r$. Arguing as in the proof of Lemma 18, we see that there is a surface of degree $3r-4$ that contains the set

$$\Big(\Pi \cap \Sigma\Big) \setminus P$$

and does not contain $P$, which concludes the proof by Lemma 25. q.e.d.

The inequality $|\Sigma| \geqslant (2r-1)r$ holds by Proposition 14.

**Lemma 34.** *Let $L_1 \neq L_2$ be lines in $\mathbb{P}^3$. Then*

$$\big|\big(L_1 \cup L_2\big) \cap \Sigma\big| < 4r-2.$$

*Proof.* Suppose that $|(L_1 \cup L_2) \cap \Sigma| \geqslant 4r-2$. Then

$$\big|L_1 \cap \Sigma\big| = \big|L_1 \cap \Sigma\big| = 2r-1$$

by Lemma 29. Then $L_1 \cap L_2 = \varnothing$ by Lemma 33.

Fix two points $Q_1$ and $Q_2$ in the set

$$\Sigma \setminus \Big(\big(L_1 \cup L_2\big) \cap \Sigma\Big)$$

different from $P$ such that $Q_1 \neq Q_2$. Let $\Pi_i$ be a hyperplane in $\mathbb{P}^3$ that contains $L_i$ and $Q_i$. Then $|\Pi_i \cap \Sigma| = 2r$ by Lemma 33.

Suppose that $P \notin \Pi_1 \cup \Pi_2$. There is a surface $F \subset \mathbb{P}^3$ of degree $3r-6$ that does not contain the point $P$ and contains all points of the set

$$\Big(\Sigma \setminus \Big(\Sigma \cap \big(\Pi_1 \cup \Pi_2\big)\Big)\Big) \setminus P$$

by Proposition 17. Hence, the union

$$F \cup \Pi_1 \cup \Pi_2$$

is a surface in $\mathbb{P}^3$ of degree $3r-4$ that contains $\Sigma \setminus P$ and does not contain $P$, which is impossible. Therefore, we have $P \in \Pi_1 \cup \Pi_2$.

The set $\Sigma \cap (\Pi_1 \cup \Pi_2)$ consists of $4r$ points by Lemma 33. The points in

$$\Sigma \cap \Big(\Pi_1 \cup \Pi_2\Big)$$

impose independent linear conditions on homogeneous forms $\mathbb{P}^3$ of degree $3r-4$ by Theorem 2 in [**9**]. On the other hand, the inequality

$$\Big|\Sigma \setminus \Big(\Sigma \cap \big(\Pi_1 \cup \Pi_2\big)\Big)\Big| < \big(2r-1\big)\big(r-2\big)$$

holds. Then the points in $\Sigma \setminus (\Sigma \cap (\Pi_1 \cup \Pi_2))$ impose independent linear conditions homogeneous forms of degree $3r-6$ by Proposition 17, which leads to a contradiction by applying Lemma 25. q.e.d.

**Lemma 35.** *Let $C$ be a curve in $\mathbb{P}^3$ of degree $k \geqslant 2$. Then*

$$\big|C \cap \Sigma\big| < (2r-1)k.$$

*Proof.* Suppose that $|C \cap \Sigma| \geqslant (2r-1)k$. Then

$$\big|C \cap \Sigma\big| = (2r-1)k$$

by Lemma 29, and $C$ is not contained in a hyperplane by Lemma 33.

The curve $C$ must be reducible by Lemma 32. Put

$$C = \sum_{i=1}^{\alpha} C_i,$$

where $\alpha \geqslant 2$ and $C_i$ is an irreducible curve. Then

$$k = \sum_{i=1}^{\alpha} d_i,$$

where $d_i = \deg(C_i)$. Then $|C_i \cap \Sigma| = (2r-1)d_i$ by Lemma 29.

The curve $C_i$ is contained in a hyperplane in $\mathbb{P}^3$ by Lemma 32. Then

$$d_1 = d_2 = \cdots = d_\alpha = 1$$

and $\alpha = k \neq 1$ by Lemma 33, which contradicts Lemma 34. q.e.d.

**Lemma 36.** *Let $L$ be a line in $\mathbb{P}^3$. Then $|L \cap \Sigma| \leqslant 2r-2$.*

*Proof.* Suppose that the inequality $|L \cap \Sigma| \geqslant 2r-1$ holds. Then

$$\big|L \cap \Sigma\big| = 2r-1$$

by Lemma 29. Let $\Phi$ be a hyperplane in $\mathbb{P}^3$ such that $\Phi$ passes through the line $L$, and $\Phi$ contains a point of the set $\Sigma \setminus (L \cap \Sigma)$. Then

$$\big|\Phi \cap \Sigma\big| = 2r$$

by Lemma 33. Put $\Delta = \Sigma \setminus (\Phi \cap \Sigma)$. Then $|\Delta| \leqslant (2r-1)(r-1)$.

The points in $\Delta$ impose dependent linear conditions on homogeneous forms of degree $3r-5$, because otherwise the points in $\Sigma$ impose independent linear conditions on forms of degree $3r-4$ by Lemma 25.

Therefore, we see that there is a point $Q \in \Delta$ such that every surface of degree $3r-5$ containing $\Delta \setminus Q$ must pass through $Q$. Then

$$\big|\Delta\big| = \big(2r-1\big)\big(r-1\big)$$

and $|\Sigma| = (2r-1)r+1$ by Proposition 17.

Fix sufficiently general hyperplane $\Pi \subset \mathbb{P}^3$ and a point $O \in \mathbb{P}^3$. Let

$$\psi\colon \mathbb{P}^3 \dashrightarrow \Pi$$

be a projection from $O$. Put $\Delta' = \psi(\Delta)$ and $Q' = \psi(Q)$.

At most $2r-2$ points in $\Delta'$ lie on a line by Lemmas 16 and 34.

Suppose that at most $(2r-1)k$ points in the set $\Delta'$ lie on any curve of degree $k$ for every $k$, and there is a curve $Z \subset \Pi$ of degree $r-1$ that contains the whole set $\Delta'$. Then

$$h^1\Big(\mathcal{I}_\Delta \otimes \mathcal{O}_{\mathbb{P}^3}\big(3r-5\big)\Big) = 0$$

by Lemmas 16, 23 and 35 in the case when $Z$ is irreducible. So, we have

$$Z = \sum_{i=1}^{\alpha} Z_i,$$

where $\alpha \geqslant 2$, and $Z_i$ is an irreducible curve of degree $d_i$. Then

$$\big|Z_i \cap \Delta'\big| = (2r-1)d_i,$$

because $r = \sum_{i=1}^{\alpha} d_i$. Then every point of the set $\Delta'$ is contained in one irreducible component of the curve $Z$. We have $d_i \neq 1$ for every $i$.

Let $Z_\beta$ be the unique component of the curve $Z$ such that $Q' \in Z_\beta$, and let $\Gamma \subset \Delta$ be a subset such that

$$\psi\big(\Gamma\big) = \Delta' \cap Z_\beta \subset \Pi \cong \mathbb{P}^2,$$

which implies that $Q \in \Gamma$. There is a surface $F_\beta \subset \mathbb{P}^3$ of degree $3(d_\beta - 1)$ that contains $\Gamma \setminus Q$ and does not contain $Q$ by Lemmas 16, 23 and 35.

Let $Y_i$ be a cone over $Z_i$ whose vertex is the point $O$. Then

$$F_\beta \cup \bigcup_{i \neq \beta} Y_i$$

is a surface of degree $3d_i - 3 + \sum_{i\neq\beta} d_i = 2d_i + r - 4$ containing $\Delta \setminus Q$ and not containing $Q$, which is impossible, because $2d_i + r - 4 \leqslant 3r-5$.

Hence, we proved that

- either at least $(2r-1)k+1$ points in $\Delta'$ lie on a curve of degree $k$;
- or there is no curve of degree $r-1$ that contains the set $\Delta'$.

Suppose that at most $(2r-1)k$ points of the set $\Delta'$ lie on every curve of degree $k$ for every natural $k$. Then it follows from Theorem 15 that there is a curve in $\Pi$ of degree $3r-5$ that contains $\Delta' \setminus Q'$ and does not contain the point $Q'$, which is a contradiction.

So, at least $(2r-1)k+1$ points in $\Delta'$ lie on some curve in $\Pi$ of degree $k$, where $k \geqslant 3$ by Lemma 20. Thus, the proof of Proposition 17 implies the existence of a subset $\Xi \subseteq \Delta$ such that

- at most $(2r-1)k$ points in $\psi(\Delta \setminus \Xi)$ lie on a curve of degree $k$,
- there is a surface in $\mathbb{P}^3$ of degree $\mu \leqslant r-2$ that contains all points of the set $\Xi$ and does not contain any point of the set $\Delta \setminus \Xi$,

- the inequality $|\Delta \setminus \Xi| \leqslant (2r-1)(r-1-\mu)-1$ holds and

$$h^1\Big(\mathcal{I}_{\Xi} \otimes \mathcal{O}_{\mathbb{P}^3}\big(3r-5\big)\Big)=0.$$

Put $\bar{\Delta}=\psi(\Delta\setminus\Xi)$ and $d=3r-5-\mu$. The points of $\bar{\Delta}$ impose dependent linear conditions on homogeneous forms of degree $d$ by Lemma 25, which implies that there is a point $\bar{Q}\in\bar{\Delta}$ such that $\bar{\Delta}\setminus\bar{Q}$ and $d$ do not satisfy one of the hypotheses of Theorem 15.

We have $d\geqslant 3$, because $r\geqslant 4$. The proof of Lemma 26 gives

$$\big|\bar{\Delta}\setminus\bar{Q}\big|\leqslant\left\lfloor\frac{d+3}{2}\right\rfloor^2,$$

which implies that at least $t(d+3-t)-1$ points of the finite set $\bar{\Delta}\setminus\bar{Q}$ lie on a curve of degree $t$ for some natural number $t$ such that $t\leqslant(d+3)/2$.

Suppose that $t=1$. Then at least $d+1$ points of $\bar{\Delta}$ lie on a line, but at most $2r-2$ points of $\Delta'$ lie on a line by Lemmas 16 and 34, which implies that $d=2r-3$ and $|\bar{\Delta}|=2r-2$. Then the points in $\bar{\Delta}$ impose dependent linear conditions on homogeneous forms of degree $d$, which is impossible. Therefore, we see that $t\geqslant 2$.

At least $t(d+3-t)-1$ points in $\bar{\Delta}\setminus\bar{Q}$ lie on a curve of degree $t$. Then

$$t\big(d+3-t\big)-1\leqslant\big|\bar{\Delta}\setminus\bar{Q}\big|\leqslant\big(2r-1\big)\big(r-1\big)-2-\mu\big(2r-1\big)i,$$

but $t(d+3-t)-1\leqslant(2r-1)t$, because at most $(2r-1)t$ points in $\bar{\Delta}$ lie on a curve of degree $t$. Hence, we have $t\geqslant r-1-\mu$, which gives

$$\big(2r-1\big)\Big(r-1-\mu\Big)-2\geqslant\big|\bar{\Delta}\setminus\bar{Q}\big|\geqslant t(d+3-t)-1\geqslant\Big(r-1-\mu\Big)\big(2r-1\big)-1,$$

which is a contradiction. q.e.d.

**Corollary 37.** *Let $C$ be any curve in $\mathbb{P}^3$ of degree $k$. Then*

$$\big|C\cap\Sigma\big|<\big(2r-1\big)k.$$

Fix a hyperplane $\Pi\subset\mathbb{P}^3$ and a general point $O\in\mathbb{P}^3$. Let

$$\psi\colon\mathbb{P}^3\dashrightarrow\Pi\subset\mathbb{P}^3$$

be a projection from $O$. Put $\Sigma'=\psi(\Sigma)$ and $P'=\psi(P)$.

**Lemma 38.** *Let $C$ be an irreducible curve in $\Pi$ of degree $r$. Then*

$$\big|\,C\cap\Sigma'\big|<\big(2r-1\big)r.$$

*Proof.* Suppose that $|C\cap\Sigma'|\geqslant(2r-1)r$. Let $\Psi$ be a subset in $\Sigma$ that contains all points mapped to the curve $C$ by the projection $\psi$. Then

$$\big|\Psi\big|\geqslant\big(2r-1\big)r,$$

but less than $(2r-1)r$ points in $\Sigma$ lie on a curve of degree $r$.

Let $\mathcal{H}$ be a linear system of surfaces in $\mathbb{P}^3$ of degree $r$ that pass through the set $\Psi$, and let $\Phi$ be the base locus of $\mathcal{H}$. Then

$$\dim\big(\Phi\big)=0$$

is finite by Lemma 16. Put $\Upsilon = \Sigma \cap \Phi$. The points in $\Upsilon$ impose independent linear conditions on homogeneous forms of degree $3r-3$ by Lemma 23.

Let $\Gamma$ be a subset in $\Upsilon$ such that $\Upsilon \setminus \Gamma$ consists of $4r-6$ points. Then

$$\big|\Gamma\big| \leqslant 2r^2 - 5r - 5 \leqslant \frac{(r+2)(r+1)r}{6} - 1,$$

because $r \geqslant 4$. Therefore, there is a surface $F \subset \mathbb{P}^3$ of degree $r-1$ that contains all points of the set $\Gamma$.

Let $\Theta$ be a subset of the set $\Upsilon$ such that $\Theta$ consists of all points that are contained in the surface $F$. Then $\Theta$ imposes independent linear conditions on homogeneous forms of degree $3r-4$ by Theorem 3 in [**6**].

Put $\Delta = \Upsilon \setminus \Theta$. Using Theorem 2 in [**9**], we easily see that the points of the set $\Delta$ impose independent linear conditions on homogeneous forms of degree $2r-3$ by Lemmas 33 and 36. Then

$$h^1\Big(\mathcal{I}_{\Upsilon} \otimes \mathcal{O}_{\mathbb{P}^3}\big(3r-4\big)\Big) = 0$$

by Lemma 25, which also follows from Theorem 3 in [**6**].

We have $|\Sigma \setminus \Upsilon| \leqslant 1$. Thus, the points in $\Sigma$ impose independent linear conditions on homogeneous forms of degree $3r-4$ by Lemma 25. q.e.d.

**Lemma 39.** *There is a curve $Z \subset \Pi$ of degree $k$ such that*

$$\big|Z \cap \Sigma'\big| \geqslant (2r-1)k + 1.$$

*Proof.* Suppose that at most $(2r-1)k$ points of the set $\Sigma'$ lie on a curve of degree $k$ for every integer $k \geqslant 1$. Let us derive a contradiction.

The finite subset $\Sigma' \setminus P' \subset \Pi$ and the natural number $3r-4$ do not satisfy at least one of the hypotheses of Theorem 15. But

$$\big|\Sigma' \setminus P'\big| \leqslant \max\left(\left\lfloor\frac{3r-1}{2}\right\rfloor\left(3r-1-\left\lfloor\frac{3r-1}{2}\right\rfloor\right), \left\lfloor\frac{3r-1}{2}\right\rfloor^2\right),$$

and at most $2r-1 \leqslant 3r-4$ points in $\Sigma' \setminus P'$ lie on a line by Lemma 16.

We see that at least

$$k\big(3r-1-k\big) - 1$$

points in $\Sigma' \setminus P'$ lie on a curve of degree $k$ such that $2 \leqslant k \leqslant (3r-1)/2$, which implies that $k = r$, because at most $k(2r-1)$ points in $\Sigma'$ lie on a curve of degree $k$, and $|\Sigma' \setminus P'| \leqslant (2r-1)r$.

Thus, there is a curve $C \subset \Pi$ of degree $r$ such that

$$\Big|\mathrm{Supp}\big(C\big) \cap \Big(\Sigma' \setminus P'\Big)\Big| \geqslant (2r-1)r - 1,$$

which implies that $P' \in C$, because otherwise there is a curve in $\Pi$ of degree $3r-4$ that contains $\Sigma' \setminus P'$ and does not contain $P'$. Then

$$\big|\mathrm{Supp}\big(C\big) \cap \Sigma'\big| \geqslant (2r-1)r,$$

which implies that $C$ is reducible by Lemma 38. Put

$$C = \sum_{i=1}^{\alpha} C_i,$$

where $C_i$ is an irreducible curve of degree $d_i \geqslant 1$ and $\alpha \geqslant 2$. Then

$$(2r-1)r \leqslant \big|C \cap \Sigma'\big| \leqslant \sum_{i=1}^{\alpha} \big|C_i \cap \Sigma'\big| \leqslant \sum_{i=1}^{\alpha}(2r-1)\deg\big(C_i\big) = (2r-1)r,$$

which implies that $C_i$ contains $(2r-1)d_i$ points of the set $\Sigma$, and every point of the set $\Sigma$ is contained in at most one curve $C_i$.

Let $C_\upsilon$ be the component of $C$ that contains $P'$, and let $\Upsilon$ be a subset of the set $\Sigma$ that contains all points of the set $\Sigma$ that are mapped to the curve $C_\upsilon$ by the projection $\psi$. Then

$$\big|\Upsilon\big| = \big(2r-1\big)d_\upsilon,$$

but less than $(2r-1)d_\upsilon$ points of the set $\Sigma$ lie on a curve of degree $d_\upsilon$.

The points in $\Upsilon$ impose independent linear conditions on the homogeneous forms of degree $3(d_\upsilon - 1)$ by Lemmas 16 and 23.

There is a surface $F \subset \mathbb{P}^3$ of degree such that

$$\Upsilon \setminus P \subset F \not\ni P$$

and $\deg(F) = 3(d_\upsilon - 1)$. Let $Y_i$ be a cone in $\mathbb{P}^3$ over the curve $C_i$ whose vertex is the point $O$. Then the surface

$$F \cup \bigcup_{i \neq \upsilon} Y_i \in \big|\mathcal{O}_{\mathbb{P}^3}\big(2d_\upsilon - 3 + r\big)\big|$$

contains the set $\Sigma \setminus P$ and does not contain the point $P$. But

$$2d_\upsilon - 3 + r \leqslant 3r - 4,$$

which is a contradiction. q.e.d.

Arguing as in the proof of Theorem 1, we construct a disjoint union

$$\bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i \subseteq \Sigma$$

such that $|\Lambda_j^i| > (2r-1)j$, the subset $\psi(\Lambda_j^i)$ is contained in an irreducible curve of degree $j$, and at most $(2r-1)t$ points of the subset

$$\psi\Big(\Sigma \setminus \Big(\bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i\Big)\Big) \subsetneq \Sigma' \subset \Pi \cong \mathbb{P}^2$$

lie on a curve in $\Pi$ of degree $t$. Then $r > k \geqslant 3$ by Lemmas 20 and 38.

Put $\Lambda = \cup_{j=k}^{l} \cup_{i=1}^{c_j} \Lambda_j^i$. Let $\Xi_j^i$ be the base locus of the linear system of surfaces in $\mathbb{P}^3$ of degree $j$ that pass through $\Lambda_j^i$. Then

$$(40)\quad \big|\Sigma\setminus\Lambda\big| \leqslant (2r-1)r+1-\sum_{i=k}^{l} c_i\Big((2r-1)i+1\Big) \leqslant (2r-1)\Big(r-\sum_{i=k}^{l} ic_i\Big),$$

which implies that $\sum_{i=k}^{l} ic_i \leqslant r$. The set $\Xi_j^i$ is finite by Lemma 16.

**Remark 41.** We have $\sum_{i=k}^{l} ic_i \leqslant r-1$, because the equality

$$\sum_{i=k}^{l} ic_i = r$$

and inequalities 40 imply that $k=l=r$, but $k<r$ by Lemma 38.

It follows from Lemma 23 that the points of $\Xi_j^i$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $3(j-1)$.

Put $\Xi = \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Xi_j^i$. Then

$$(42)\qquad \big|\Sigma \setminus \Big(\Xi\cap\Sigma\Big)\big| \leqslant (2r-1)r - \sum_{i=k}^{l} c_i(2r-1)i.$$

Therefore, we can find surfaces $F$ and $G$ in $\mathbb{P}^3$ of degree $\sum_{i=k}^{l} 3(i-1)c_i$ and $\sum_{i=k}^{l} ic_i$, respectively, such that

$$\Big(\Xi\cap\Sigma\Big)\setminus P \subset F \not\ni P,$$

the surface $G$ contains the set $\Xi\cap\Sigma$, and the surface $G$ does not contain any point in $\Sigma\setminus(\Xi\cap\Sigma)$. In particular, we have $\Sigma \not\subseteq \Xi$, because

$$\sum_{i=k}^{l} 3(i-1)c_i \leqslant \sum_{i=k}^{l} 3ic_i - 3c_k \leqslant 3r-6 < 3r-4.$$

Put $\bar{\Sigma} = \psi(\Sigma\setminus(\Xi\cap\Sigma))$ and $d = 3r-4-\sum_{i=k}^{l} ic_i$.

It follows from Lemma 25 that there is a point $\bar{Q}\in\bar{\Sigma}$ such that every curve in $\Pi$ of degree $d$ that contains the set $\bar{\Sigma}\setminus\bar{Q}$ must pass through the point $\bar{Q}$ as well. Therefore, we can not apply Theorem 15 to the points of the subset $\bar{\Sigma}\setminus\bar{Q}\subset\Pi$ and the natural number $d$.

The proof of Lemma 26 implies that the inequality

$$\big|\bar{\Sigma}\setminus\bar{Q}\big| \leqslant (2r-1)\Big(r-\sum_{i=k}^{l} c_i i\Big) - 1 \leqslant \Big\lfloor \frac{d+3}{2}\Big\rfloor^2$$

holds, but $d = 3r-4-\sum_{i=k}^{l} ic_i \geqslant 2r-3 \geqslant 3$, because $\sum_{i=k}^{l} ic_i \leqslant r-1$, which implies that at least $t(d+3-t)-1$ points of the set $\bar{\Sigma}\setminus\bar{Q}$ lie on a curve in $\Pi$ of degree $t\leqslant (d+3)/2$.

**Lemma 43.** *The inequality $t\neq 1$ holds.*

*Proof.* Suppose that $t=1$. Then at least $d+1$ points in $\bar{\Sigma}\setminus\bar{Q}$ lie on a line, which implies that $d+1\leqslant 2r-2$ by Lemmas 16 and 36.

The inequality $d+1\leqslant 2r-2$ gives $\sum_{i=k}^{l} ic_i=r-1$ and $d=2r-3$. It follows from inequality 42 that

$$\Big|\Sigma\setminus\Big(\Xi\cap\Sigma\Big)\Big|\leqslant 2r-1,$$

which implies that the set $\Sigma\setminus(\Xi\cap\Sigma)$ imposes independent linear conditions on the homogeneous forms of degree $2r-3$ by Theorem 2 in [**9**], which is impossible by Lemma 25. q.e.d.

There is a curve $C\subset\Pi$ of degree $t\geqslant 2$ that contains at least

$$t\big(d+3-t\big)-1$$

points of the set $\bar{\Sigma}\setminus\bar{Q}$, which implies that

$$t\big(d+3-t\big)-1\leqslant\big|\bar{\Sigma}\setminus\bar{Q}\big|$$

and $t(d+3-t)-1\leqslant(2r-1)t$. Therefore, we see that

$$t\geqslant r-\sum_{i=k}^{l} ic_i,$$

because $t\geqslant 2$. It follows from inequalities 40 that

$$\begin{aligned}\big(2r-1\big)\left(r-\sum_{i=k}^{l} ic_i\right)-1\geqslant\big|\bar{\Sigma}\setminus\bar{Q}\big|\geqslant t\big(d+3-t\big)-1\\ \geqslant\left(r-\sum_{i=k}^{l} ic_i\right)\big(2r-1\big)-1,\end{aligned}$$

which implies that $t=r-\sum_{i=k}^{l} ic_i$, the curve $C$ contains $\bar{\Sigma}\setminus\bar{Q}$, and inequalities 40 are actually equalities. We have $\Sigma\cap\Xi=\Lambda$ and

$$\begin{aligned}\big|\Sigma\setminus\Lambda\big|&=\big(2r-1\big)r+1-\sum_{i=k}^{l} c_i\Big(\big(2r-1\big)i+1\Big)\\&=\big(2r-1\big)\left(r-\sum_{i=k}^{l} ic_i\right),\end{aligned}$$

which implies that $l=k$, $c_k=1$, $d=3r-4-k$ and $\sum_{i=k}^{l} ic_i=k$.

**Lemma 44.** *The curve $C$ contains the set $\bar{\Sigma}$.*

*Proof.* Suppose that $\bar{\Sigma}\not\subset C$. Then $\bar{Q}\notin C$, which implies that there is a curve in $\Pi$ of degree $r-k$ that contains the set $\bar{\Sigma}\setminus\bar{Q}$ but does not contain the point $\bar{Q}$. The latter is impossible, because $d\geqslant r-k$. q.e.d.

We have $\deg(C) = r - k$ and $\psi(\Sigma \setminus \Lambda) \subset C$. The equality

$$\Big|\psi\big(\Sigma \setminus \Lambda\big)\Big| = (r-k)(2r-1)$$

holds. But there is an irreducible curve $Z \subset \Pi$ of degree $k$ that contains all points of the set $\psi(\Lambda)$, which consists of $k(2r-1)+1$. Then

$$\big|\Sigma\big| = \big|\Sigma \setminus \Lambda\big| + \big|\Lambda\big| = \big(r-k\big)\big(2r-1\big) + k\big(2r-1\big) + 1 = \big(2r-1\big)r + 1.$$

**Lemma 45.** *The curve $C$ is reducible.*

*Proof.* Suppose that $C$ is irreducible. Then $\Sigma \setminus \Lambda$ imposes independent linear conditions on forms of degree $3(r-k-1)$ by Lemmas 16, 23, and 35, but the points in $\Lambda$ impose independent linear conditions on forms of degree $3(k-1)$ by Lemmas 16 and 23. Then $\Sigma$ imposes independent linear conditions on forms of degree $3r-4$ by Lemma 25. q.e.d.

Put $C = \sum_{i=1}^{\alpha} C_i$, where $C_i$ is an irreducible curve of degree $d_i$. Then

$$r - k = \sum_{i=1}^{\alpha} d_i,$$

the curve $C_i$ contains $(2r-1)d_i$ points of the set $\bar{\Sigma}$, and every point of the set $\bar{\Sigma}$ is contained in a single irreducible component of the curve $C$.

**Lemma 46.** *The curve $Z$ contains the point $P'$.*

*Proof.* Suppose that $P' \notin Z$. Let $C_v$ be a component of $C$ such that

$$P' \in C_v,$$

and let $\Upsilon$ be a subset of the set $\Sigma$ that contains all points that are mapped to the curve $C_v$ by the projection $\psi$. Then $|\Upsilon| = (2r-1)d_v$.

The set $\Upsilon$ imposes independent linear conditions on the homogeneous forms of degree $3(d_v - 1)$ by Lemmas 16, 23 and 35. There is a surface

$$F \subset \mathbb{P}^3$$

of degree $3(d_v - 1)$ that contains $\Upsilon \setminus P$ and does not contain $P$.

Let $Y_i$ and $Y$ be the cones in $\mathbb{P}^3$ over the curves $C_i$ and $Z$, respectively, whose vertex is the point $O$. Then the union

$$F \cup Y \cup \bigcup_{i \neq v} Y_i$$

is a surface of degree $2d_v - 3 + r \leqslant 3r - 4$ that contains the set $\Sigma \setminus P$ and does not contain the point $P$, which is a contradiction. q.e.d.

The proof of Lemma 46 implies that the set $\Sigma \setminus \Lambda$ imposes independent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $3r - 4 - k$, but we already know that the set $\Lambda$ imposes independent linear conditions on homogeneous forms of degree $3(k-1)$ by Lemmas 16 and 23.

Applying Lemma 25, we obtain a contradiction.

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School of Mathematics
University of Edinburgh
Edinburgh EH9 3JZ, UK
*E-mail address*: I.Cheltsov@ed.ac.uk