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WILLMORE IMMERSIONS AND LOOP GROUPS

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Abstract

We propose a characterisation of Willmore immersions inspired from the works of R. Bryant on Willmore surfaces and J. Dorfmeister, F. Pedit, H.-Y. Wu on harmonic maps between a surface and a compact homogeneous manifold using moving frames and loop groups. We use that formulation in order to construct a Weierstrass type representation of all conformal Willmore immersions in terms of closed one-forms.

Let $\mathbb{R}^3$ be the Euclidean space and let us consider the set $\mathcal{D}$ of all compact, oriented surfaces without boundary which are immersed in $\mathbb{R}^3$ (the immersion being of class $\mathcal{C}^k$ for $k \geq 4$). For a surface $\mathcal{S} \in \mathcal{D}$ we consider the area 2-form dA induced by the first fundamental form of the immersion on $\mathcal{S}$ and the principal curvatures $k_1 \leq k_2$ computed using the first and the second fundamental forms. A point of $\mathcal{S}$ such that $k_1 = k_2$ is called an *umbilic point.* Let $H := (k_1 + k_2)/2$ be the mean curvature and $K := k_1 k_2$ the Gauss curvature. The quantity

$$\mathcal{W}(\mathcal{S}) := \int_{\mathcal{S}} H^2 dA$$

defines a functional on $\mathcal{D}$ called *Willmore functional.* A variant of $\mathcal{W}$ is

$$\tilde{\mathcal{W}}(\mathcal{S}) := \int_{\mathcal{S}} \frac{1}{4}(k_1 - k_2)^2 dA,$$

which differs from $\mathcal{W}(\mathcal{S})$ by $\mathcal{W}(\mathcal{S}) - \tilde{\mathcal{W}}(\mathcal{S}) = \int_{\mathcal{S}} K dA = 4\pi(1-g)$ where g is the genus of $\mathcal{S}$. Both functionals having the same critical points on $\mathcal{D}$ called *Willmore surfaces*; they are solutions of the equation

$$\Delta H + 2H(H^2 - K) = 0,$$

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(see [23]) where the Laplacian Δ is constructed by using the first fundamental form of the immersion. This problem was proposed by T. Willmore in the 1960's, but it was also considered by K. Voss during the 1950's. Later T. Willmore and K. Voss discovered in the book of W. Blaschke [1] that part of the theory was already known in the beginning of the century, in particular from the work of Thomsen and Shadow in 1923 [20]. For a general presentation of the problem, see for example the last chapter of [23].

Natural questions are: is there a Willmore surface for all genus? Are there surfaces minimizing the Willmore energy functional in each genus class? The simplest examples of Willmore surfaces are the round spheres. These are the only totally umbilic surfaces and they minimize the Willmore functional among surfaces of genus 0. Actually all Willmore surfaces of genus 0 have been characterized using a Weierstrass type representation by R. Bryant ([3] and [4]).

The next question is to understand the genus 1 case, i.e., Willmore tori. In 1965, T. Willmore constructed a torus of revolution and conjectured that this torus minimizes the Willmore functional among all tori. This conjecture is still unsolved despite some partial (positive) answer obtained by P. Li and S.T. Yau in [15] or S. Montiel and A. Ros in [16]. Recently, L. Simon proved that the minimum of that functional is achieved among tori [19], but it is unknown whether or not it coincides with Willmore's candidate.

A very important property of Willmore functional and Willmore surfaces is the invariance under the conformal transformations of $\mathbb{R}^3 \cup \{\infty\}$ (Möbius group). If T is a conformal transformation of $\mathbb{R}^3 \cup \{\infty\}$, then for all $\mathcal{S} \in \mathcal{D}$

$$\mathcal{W}(T(\mathcal{S})) = \mathcal{W}(\mathcal{S}).$$

The reason for this is that under T the quantity $\frac{1}{4}(k_1 - k_2)^2 dA$ is locally preserved. A corollary is that if $\mathcal{S}$ is a Willmore surface, then $T(\mathcal{S})$ is also a Willmore surface. This property has been observed by J.H. White in 1973 [22], but was known at the beginning of the century.

This symmetry implies that we need to enlarge the set of tori proposed by Willmore as candidates to be minimizing by adding all the images of these tori under conformal transformations. Actually the Euclidean structure on $\mathbb{R}^3$ that we used from the beginning to define the Willmore functional is not necessary and it suffices to use the conformal structure of S^3. In other words since the stereographic projection is a conformal diffeomorphism from $\mathbb{R}^3 \cup \{\infty\}$ to S^3, we may as well consider the problem on S^3 (or on the hyperbolic space $\mathbb{H}^3$).

Lastly one can remark that minimal surfaces in S^3, $\mathbb{R}^3$ or $\mathbb{H}^3$ are Willmore surfaces. Hence any conformal local diffeomorphism, say from S^3, $\mathbb{R}^3$ or $\mathbb{H}^3$ into $\mathbb{R}^3$, will map such a minimal surface into a Willmore surface, and then we get new examples of Willmore surfaces. For instance the Willmore torus corresponds to the minimal Clifford torus in S^3. Also all the family of minimal surfaces in $\mathbb{R}^3$ constructed by H. B. Lawson [14] provides examples of Willmore surfaces of arbitrary genus (they are also good candidates to be Willmore minimizers).

The conformal Gauss map

The importance of the classical Gauss map is well-known in the Euclidean geometry of surfaces. For Willmore surface this notion is not relevant anymore but has to be replaced by the *conformal Gauss map* which is an oriented sphere (or plane) in $\mathbb{R}^3$. Let us denote $\mathcal{Q} :=$ {oriented spheres of $\mathbb{R}^3$} $\cup$ { oriented planes of $\mathbb{R}^3$}. For any surface $\mathcal{S} \in \mathcal{D}$, $m \in \mathcal{S}$ we denote $S^2_{\gamma(m)}$ the unique element of $\mathcal{Q}$ such that $m \in S^2_{\gamma(m)}$, $S^2_{\gamma(m)}$ is tangent ot $\mathcal{S}$ at m, with the same orientation, $S^2_{\gamma(m)}$ and $\mathcal{S}$ have the same mean curvature at the point m. The map $m \longmapsto S^2_{\gamma(m)}$ is called conformal Gauss map. In [3] R. Bryant proved that the conformal Gauss map $\mathcal{S} \longrightarrow \mathcal{Q}$ is weakly conformal and that $\mathcal{S}$ is a Willmore surface if and only if its conformal Gauss map is harmonic. (But it was already in the book of W. Blaschke.)

The conformal transform

It is a construction which associates to each surface another one "which has the same conformal Gauss map". In particular if the first surface is Willmore, then its conformal transform is also Willmore. But we need to take care of a major difficulty: it works basically outside the umbilic points. Let us denote $\mathcal{U}$ the set of umbilic points of $\mathcal{S}$. The construction is the following: considering the family

$$\{S^2_{\gamma(m)} \in \mathcal{Q}/m \in \mathcal{S} \setminus \mathcal{U}\},$$

one can show (see [3]) that this family has two enveloppe surfaces: $\mathcal{S} \setminus \mathcal{U}$ and another which will be denoted $\hat{\mathcal{S}}$ and is precisely the conformal transform of $\mathcal{S}$.

All that can be translated in the framework of the Minkowski space $\mathbb{R}^{4,1}$. Points in $\mathbb{R}^3 \cup \{\infty\} \simeq S^3$ are identified with half lines contained in the light cone of the Minkowski space. Then the set $\mathcal{Q}$ may also be identified with the Minkowski sphere $S^{3,1} := \{y \in \mathbb{R}^{4,1}/|y|^2 = 1\}$, and the action of the group of conformal transformations of S^3 coincides with the action of the Lorentz group. This setting was introduced by R. Bryant in [3] and is reviewed in the first chapter of that paper.

Weierstrass representation for harmonic maps

Beside that is the recent work of J. Dorfmeister, F. Pedit, H.-Y. Wu [7] proving that any harmonic map from a simply connected surface into a homogeneous manifold which is the quotient of a *compact* Lie group by some subgroup can be constructed algebraically from holomorphic (or meromorphic) datas. This theory used strongly the loop group representation of such harmonic maps, and is based on loop group decompositions (sometimes known as Riemann-Hilbert problem or Birkhoff-Grothendieck Theorem) which formally generalizes to loop groups the Iwasawa decomposition for the complexification of compact Lie groups. This circle of ideas is familiar in the context of integrable systems (see [17] and [18]). Also the formulation of the harmonic map problem using these loop groups was already used succesfully by K. Uhlenbeck [21], F. Burstall, D. Ferus, F. Pedit, U. Pinkall [5] and other authors. (See also [10] for a basic exposition.)

The present paper combines the theories developped in [3] and [7]. The goal is to provide a Weierstrass type construction of all conformal Willmore immersions using holomorphic or meromorphic datas. As in [3] we will use a representation of a conformal Willmore immersion X of a simply connected domain U in $\mathbb{C}$ using a moving frame e in $\mathbb{R}^{4,1}$ lifting X and which encodes the tangent plane and the conformal Gauss map. When working outside the umbilic set $\mathcal{U}$ we can furthermore incorporate the conformal transform of X in that moving frame. Such a moving frame can be represented as a map F from U into the conformal group $SO(4,1)$. Its geometry can be described in an economic way by using the Maurer-Cartan form $\omega := F^{-1}.dF$, a 1-form with coefficients in the Lie algebra of the conformal group. Notice that this Maurer-Cartan form satisfies the structure equation $d\omega + \frac{1}{2}[\omega \wedge \omega] = 0$. We consider in Section 2.2 a family of deformations of ω of the form

$$\omega_\lambda = \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1''$$

(where $\alpha_1' + \alpha_0 + \alpha_1'' = \omega$) depending on a complex parameter $\lambda \in S^1 \subset \mathbb{C}^\star$. Then X is a Willmore immersion if and only if ω_λ still satisfies the structure equation $d\omega_\lambda + \frac{1}{2}[\omega_\lambda \wedge \omega_\lambda] = 0$ (Theorem 2, Section 2.2). Hence we can construct a family of moving frames F_λ depending on $\lambda \in S^1$ and such that $F_1 = F$. F_λ is called an *"extended conformal Willmore immersion"* (ECWI) and is the solution to

$$dF_\lambda = F_\lambda.\omega_\lambda \text{ on } U \text{ and } F_\lambda(p) = \mathbb{1},$$

where p is a fixed base point in U, and $\mathbb{1}$ is the identity.

We propose several ways to realize that in Theorem 2 of Chapter 2. One is obtained by applying the Dorfmeister, Pedit, Wu theory to the conformal Gauss map, a conformal harmonic map into $SO(4,1)/SO(3,1)$ (in this situation α_1' and α_1'' are respectively a (1,0)-form and a (0,1)-form with coefficients in the complexification of the Lie algebra of $SO(3,1)$). Another relies on the existence of a family of "roughly harmonic" maps Z associated to X with values into

$$SO(4,1)/SO(3) \times SO(1,1)$$

(the Grassmannian of three-dimensional spacelike subspaces).

Such a map Z is constructed as follows: to each z in U we associate a three-dimensional spacelike subspace of $\mathbb{R}^{4,1}$ orthogonal to the light line spanned by $X(z)$ and which contains the conformal Gauss map. Notice that such a map is not unique since for any z there is a two parameters family of choices for $Z(z)$, this is why we speak of a family of maps. "Roughly harmonic" means that for each map Z it corresponds to an ECWI F_λ uniquely up to gauge transformations $F_\lambda \longrightarrow F_\lambda.g$, where g is a map from U into $SO(3) \times SO(1,1)$, lifting X and Z. Moreover in the decomposition of ω_λ, α_1' and α_1'' are 1-forms with coefficients in the complexification of the Lie algebra of $SO(3) \times SO(1,1)$ (see Section 2.2), but now α_1' (respectively α_1'') is not necessarily of type (1,0) (respectively (0,1)). If Z and $\tilde{Z}$ are two roughly harmonic maps describing the same Willmore immersion X, then an ECWI F_λ lifting X and Z and an ECWI $\tilde{F}_\lambda$ lifting X and $\tilde{Z}$ are related by some special gauge transformation $F_\lambda = \tilde{F}_\lambda.\Psi_\lambda$ (Section 2.4). Given some roughly harmonic map Z, by such a gauge transformation we can construct *locally* a roughly harmonic map $\tilde{Z}$ which is really a harmonic map (see Lemma 3 in Section 2.4). But this is not possible *globally* in general.

We will choose the second representation using maps into $SO(4,1)/SO(3) \times SO(1,1)$ because it will be more suitable in the following for representation of Willmore surfaces. In particular one important difficulty is to work with umbilic points (Notice that in [2], M. Babich and A. Bobenko constructed a Willmore torus which contains a line of umbilic point.). Indeed on one hand we know how to recover a Willmore immersion from its conformal Gauss map if this conformal Gauss map is a spacelike immersion, which is true outside the umbilic set. But on the other hand we have to allow that conformal Gauss map to degenerate at some points where the tangent plane to its image shrinks to a light line (or sometimes to a point). It corresponds precisely to the umbilic set. The difficulty is to recover the (candidate to be) Willmore immersion from the conformal Gauss map. Singularities may appear. The second formulation avoids that difficulty because there the Weierstrass datas are in correspondance with the first derivatives of X.

Our results are the following. We call a Weierstrass data some couple $(ldz, mdz + \gamma l d\overline{z})$ where

l is a map into $\mathbb{C}^3$ which does not vanish such that ${}^tl.l = 0$,

m is a map into $\mathbb{C}^3$ such that ${}^tm.l = 0$,

γ is a map into $\mathbb{C}$,

$d(ldz) = d(mdz + \gamma l d\overline{z}) = 0.$

We allow these maps to have isolated point singularities, and denote S to be the singular set. Our main results are the following:

(i) *If X is a conformal Willmore immersion, Z a roughly harmonic map associated to X, and F_λ an extended conformal Willmore immersion which lifts X and Z, then we can associate algebraically to F_λ a Weierstrass data $(ldz, mdz + \gamma l d\overline{z})$. Moreover this Weierstrass data depends uniquely on X and Z* (Theorem 9).

Remark. The singular set S of $(ldz, mdz + \gamma l d\overline{z})$ depends on X and also on F_λ (more precisely F_λ is builded by integrating the equation $dF_\lambda = F_\lambda.\omega_\lambda$ with the initial condition $F_\lambda(p) = \mathbb{1}$ and the Weierstrass data depends also on the base point p.)

(ii) *Conversely given any data $(ldz, mdz + \gamma l d\overline{z})$ which satisfies the above condition and given a point z_0 in $U \setminus S$, we can construct by an algebraic algorithm a conformal Willmore immersion on a neighbourhood of z_0 in $U \setminus S$ such that its Weierstrass data is $(ldz, mdz + \gamma l d\overline{z})$.* (Theorems 8 and 9).

Remark. One sees here that (ii) is only a partial converse to (i) since: first we do not know a condition which would garantee that the moving frame F constructed using a Weierstrass data which is singular on S is smooth on S (notice however that in [6] a characterisation of meromorphic Weierstrass potentials leading to smooth constant mean surfaces in $\mathbb{R}^3$ is given); second the construction (ii) is only local. This kind of restriction is new in comparaison to [7] and is due to the fact that the Lie group of symmetries, $SO(4,1)$, is not compact. We do not know whether it is possible to remove this restriction but some progresses have been obtained very recently concerning the loop groups decompositions for noncompact real Lie groups, which could help for that question [12].

(iii) *Given a conformal Willmore immersion X and two roughly harmonic maps Z and $\tilde{Z}$ associated to X, two ECWI's F_λ and $\tilde{F}_\lambda$ which lift X and respectively Z and $\tilde{Z}$ are related by some special gauge transformation $F_\lambda = \tilde{F}_\lambda.\Psi_\lambda$* (Section 2.4). *Under this action the corresponding Weierstrass data is changed according to*

$$(\tilde{l}dz, \tilde{m}dz + \tilde{\gamma}\tilde{l}d\overline{z}) = (ldz, mdz + \gamma l d\overline{z} + d(\delta l))$$

for some map δ into $\mathbb{C}$ (Proposition 3, Section 4.3). *Notice that the above gauge transformation exists only* locally *in general.*

We see that if δ is a solution of $\frac{\partial\delta}{\partial\overline{z}} + \gamma = 0$, then the Weierstrass data for $\tilde{F}_\lambda$ is $(\tilde{l}dz, \tilde{m}dz)$, a pair of meromorphic forms. This is possible around each point but only locallly.

(iv) *By a Theorem of R. Bryant [3], we know that if the umbilic set $\mathcal{U}_X$ is different from U, then its complementary $U \setminus \mathcal{U}_X$ is open and dense in U. In this situation there exists a unique extended conformal Willmore immersion F_λ (up to gauge transformations $F_\lambda \longrightarrow F_\lambda.g$ for g into $SO(3) \times SO(1,1)$) on $U \setminus \mathcal{U}_X$ such that its Weierstrass data is of the form*

$$(ldz, \nu l dz)$$

for some meromorphic map $\nu : U \setminus \mathcal{U}_X \longrightarrow \mathbb{C}$ (see Section 5.1).

(v) *Special Willmore surfaces are minimal surfaces of S^3, $\mathbb{R}^3$ or $\mathbb{H}^3$. In the above representation (outside $\mathcal{U}_X$) such Willmore immersions are characterised by the condition that ν is a real constant* (Theorem 10):

if $\nu < 0$, $\mathcal{S}$ *is a minimal surface in* S^3;

if $\nu = 0$, $\mathcal{S}$ *is a minimal surface in* $\mathbb{R}^3$;

if $\nu > 0$, $\mathcal{S}$ *is a minimal surface in* $\mathbb{H}^3$.

Moreover in the case where $\nu = 0$, ldz *is the classical Weierstrass data for minimal surfaces in* $\mathbb{R}^3$ (Section 5.1).

This paper is organised as follows. The first chapter recalls some results and notation of Bryant's paper. The second chapter introduces new formulations of the problem using loop groups. The third chapter contains technical results on loop groups and in particular loop groups factorisation theorems (which are essentially adaptations of results in [7]). The fourth chapter presents the proof of Weierstrass representations. The fifth chapter gives some geometrical interpretations.

Other approachs concerning Willmore surfaces using the theory of Hamiltonian completely integrable systems have been developped, in particular by D. Ferus and F. Pedit [8], M. Babish and A. Bobenko [2] and B.G. Konopolchenko and I.A. Taimanov [13].

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1. Willmore immersions

In the following we review some important properties of Willmore surfaces : the existence of the conformal Gauss map and of the dual Willmore surface following R. Bryant in [3]. In order to apply the conformal invariance of the problem we will work in the Minkowski space $\mathbb{R}^{4,1}$ on which the Lorentz group $SO(4,1)$ acts linearly. The reason for that is that the group of conformal transformations of $\mathbb{R}^3 \cup \{\infty\}$, which we will denote $Conf(\mathbb{R}^3)$, coincides with the connected component of the identity $SO_0(4,1)$ of $SO(4,1)$. Also since $\mathbb{R}^3 \cup \{\infty\}$ and S^3 are conformally equivalent by stereographic projections, we may identify $Conf(\mathbb{R}^3) = Conf(S^3) = SO_0(4,1)$.

1.1. The Minkowski space

The five-dimensional Minkowski space $\mathbb{R}^{4,1}$ is the vectorial space $\mathbb{R}^5$ equipped with the Minkowski scalar product

$$B = (B_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(Here we do not use the standard representation.) We denote by $\mathfrak{G}$ the connected component of $\mathbb{1}$ in the group - isomorphic to the Lorentz group $SO(4,1)$ - of linear isometries of $\mathbb{R}^{4,1}$ preserving the volume element. The Lie algebra of $\mathfrak{G}$ will be denoted $\mathfrak{g}$.

We call a vector x respectively a space-like, light-like or time-like vector if respectively $\langle x,x\rangle > 0$, $\langle x,x\rangle = 0$ or $\langle x,x\rangle < 0$. We choose an orientation on $\mathbb{R}^{4,1}$ by requiring

$$dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 > 0.$$

The set $\{x \in \mathbb{R}^{4,1} \setminus \{0\} / \langle x,x\rangle \leq 0\}$ is divided into two connected components. We choose a time orientation by claiming that one of these is composed of positive vectors. The cone of light-like positive vectors is

$$\mathcal{C}^+ = \{x \in \mathbb{R}^{4,1} / ||x||^2 = 0, x \text{ is a positive vector }\}.$$

Consider the quotient $\mathcal{C}^+/\mathbb{R}_+^*$ which is the set of positive half light-lines. If x belongs to $\mathcal{C}^+$, we will denote by $[x]$ the half light-line spanned by x over $\mathbb{R}_+^*$. It is then simple to see that S^3 is diffeomorphic to $\mathcal{C}^+/\mathbb{R}_+^*$. Moreover the action of *Conf*(S^3) on S^3 corresponds to the action of $\mathfrak{G}$ on $\mathcal{C}^+$ through this diffeomorphism.

We choose a reference basis $\epsilon = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ of $\mathbb{R}^{4,1}$ such that

(a) ϵ is a direct basis, and

(b)
$$\langle \epsilon_i, \epsilon_j \rangle = B_{ij},$$
where the B_{ij} are the elements of the matrix B.

We assume also that

(c) ϵ_0 and ϵ_4 are positive light-like vectors.

We denote by ${}^t(x^0, x^1, x^2, x^3, x^4)$ the coordinates of some vector x in this basis. In these coordinates, the scalar product reads

$$\langle x,y\rangle = -x^0y^4 - x^4y^0 + \sum_{j=1}^{3} x^jy^j = {}^tx.B.y,$$

and a vector is positive if and only if $x^0 + x^4 > 0$.

We denote by $\mathcal{F}$ the set of all pseudo-orthonormal frames which satisfies (a), (b) and (c). It turns out that

$$\mathfrak{G} = \{g \in M(5, \mathbb{R}) / {}^t g.B.g = B, \det g = 1 \text{ and } g_0^0 > 0\},$$

$$\mathfrak{g} = \{\xi \in M(5, \mathbb{R}) / {}^t \xi.B + B.\xi = 0\},$$

and

$$\mathcal{F} = \{e = (e_0, e_1, e_2, e_3, e_4) / \exists! g \in \mathfrak{G} \text{ such that } e = \epsilon.g\}.$$

In the following $(\epsilon.g)_i$ will denote the vector number i of the basis $\epsilon.g$, i.e., $\epsilon_k g_i^k$. Then $[(\epsilon.g)_0]$ represents the point in S^3 spanned by $(\epsilon.g)_0$.

1.2. Conformal geometry of an immersion

Let us consider some immersion X of an open subset U of $\mathbb{C}$ into S^3. We may represent it by some smooth map e_0 from U to $\mathcal{C}^+$ such that $[e_0] = X$. Such a representation is of course not unique. We consider the following bundle over U,

$$\mathcal{F}_X^{(0)} = \{(z, e) \in U \times \mathcal{F} / e_0(z) \text{ spans } X(z)\}.$$

The group which acts on the right on this bundle is $\mathfrak{G}_0 = \{g \in \mathfrak{G} / (\epsilon.g)_0 = r^{-1}\epsilon_0, \text{for some } r \text{ in } \mathbb{R}_+^*\}$. Note that it is easy to construct global sections e of $\mathcal{F}_X^{(0)}$ since U is simply connected. For any such a section we consider the unique map F from U to $\mathfrak{G}$ such that $\epsilon.F = e$ and the Maurer-Cartan form

$$\omega = F^{-1}.dF. \tag{1}$$

Note that the elements ω_j^i of ω are 1-forms which satisfy the structure equation

$$d\omega_j^i + \omega_k^i \wedge \omega_j^k = 0, \tag{2}$$

or

$$d\omega + \frac{1}{2}[\omega \wedge \omega] = 0.$$

Coefficients ω_a^b satisfy also the relations

$$B_{ac}\omega_b^c + B_{bc}\omega_a^c = 0. \tag{3}$$

Following R. Bryant we will construct a series of subbundles by adding successively constraints on the first, second and third derivatives.

1.3. First order subbundle

We select here frames such that $\langle de_0, e_3\rangle = \omega_0^3 = 0$. We then obtain the bundle

$$\mathcal{F}_X^{(1)} = \{(z,e) \in \mathcal{F}_X^{(0)} / \omega_0^3 = 0 \text{ and } \omega_0^1 \wedge \omega_0^2 > 0\},$$

and each section (z, e) of this bundle is such that the tangent space to $e_0(U)$ at $e_0(z)$ is orthogonal to $e_3(z)$.

The bundle $\mathcal{F}_X^{(1)}$ is a principal bundle with a right action of the group

$$\mathfrak{G}^{(1)} = \{g \in \mathfrak{G}/g = \begin{pmatrix} r^{-1} & {}^t p.R & \frac{1}{2} r\, {}^t p.p \\ 0 & R & rp \\ 0 & 0 & r \end{pmatrix}, \text{with}$$

$$p = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix} \in \mathbb{R}^3, R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \theta \in \mathbb{R}, r \in \mathbb{R}_+^* \}.$$

1.4. Second order subbundle

Let us start with some smooth section $e = \epsilon.F$ of $\mathcal{F}_X^{(1)}$. Since $\omega_0^4 = 0$ by structure and since $\omega_0^3 = 0$ by construction, the equation $d\omega_0^3 + \sum_{j=0}^4 \omega_j^3 \wedge \omega_0^j = 0$ reduces to

$$\omega_1^3 \wedge \omega_0^1 + \omega_2^3 \wedge \omega_0^2 = 0. \tag{4}$$

From Cartan's lemma it follows that there exist smooth functions h_{11}, $h_{12} = h_{21}$ and h_{22} such that

$$\left\{ \begin{array}{lcl} \omega_1^3 & = & h_{11}\omega_0^1 + h_{12}\omega_0^2, \\ \omega_2^3 & = & h_{21}\omega_0^1 + h_{22}\omega_0^2. \end{array} \right. \tag{5}$$

We build a new section $\tilde{e} = \epsilon.F.g$ where

$$g = \begin{pmatrix} 1 & 0 & 0 & p^3 & \frac{1}{2}(p^3)^2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & p^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and $p^3 : U \longrightarrow \mathbb{R}$ (corresponding to the third component of p) is any smooth function. Then a computation shows that

$$\begin{pmatrix} \tilde{h}_{11} & \tilde{h}_{12} \\ \tilde{h}_{21} & \tilde{h}_{22} \end{pmatrix} = \begin{pmatrix} h_{11} - p^3 & h_{12} \\ h_{21} & h_{22} - p^3 \end{pmatrix},$$

where the $\tilde{h}_{ij}$ refer to the coefficients corresponding to $\tilde{e}$. It follows that if we choose $p^3 = \frac{1}{2}(h_{11} + h_{22})$, we obtain a new section such that $\tilde{h}_{11} + \tilde{h}_{22} = 0$. We define

$$\mathcal{F}_X^{(\gamma)} = \{(z, e) \in \mathcal{F}_U^{(1)} / h_{11} + h_{22} = 0\}.$$

We just proved the existence of a smooth section e of this bundle. The group which acts on the right on the fibers of $\mathcal{F}_X^{(\gamma)}$ is

$$\mathfrak{G}^{(\gamma)} = \{g \in \mathfrak{G} / g = \begin{pmatrix} r^{-1} & {}^t p.R & \frac{1}{2} r \, {}^t p.p \\ 0 & R & rp \\ 0 & 0 & r \end{pmatrix} \text{ with}$$

$$p = \begin{pmatrix} p^1 \\ p^2 \\ 0 \end{pmatrix} \in \mathbb{R}^2, R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, r \in \mathbb{R}_+^*, \theta \in \mathbb{R}\}.$$

A remarkable fact is that for any smooth $g' : U \longrightarrow \mathfrak{G}^{(\gamma)}$, the gauge transformation on $\mathcal{F}_X^{(\gamma)}$ given by $\epsilon.F \longmapsto \epsilon.F.g'$ does not change $e_3(z)$. Thus this exhibits a map $\gamma : U \longmapsto S^{3,1}$, where

$$S^{3,1} = \{y \in \mathbb{R}^{4,1} / ||y||^2 = 1\},$$

such that $\forall z \in U, \gamma(z) = e_3(z)$, for any e_3 corresponding to a section (z, e) of $\mathcal{F}_U^{(\gamma)}$. This γ is called the *conformal Gauss map*. Geometrically, γ represents the unique oriented sphere in S^3, which is tangent to $X(U)$ at $X(z)$, has the same mean curvature as $X(U)$ at $X(z)$ with the same orientation. The correspondance is given by the following. For any γ in $S^{3,1}$, the projection mapping $\Pi : \mathcal{C}^+ \longmapsto \mathcal{C}^+ / \mathbb{R}_+^* = S^3$ sends the three-dimensionnal subcone $\mathcal{C}^+ \cap \gamma^\perp$ of $\mathbb{R}^{4,1}$ onto a sphere S_γ^2 in S^3. An orientation can be addressed to this sphere according to the fact that γ and $-\gamma$ give the same sphere with opposite orientations.

A further property is that, provided that (h_{11}, h_{12}) does not vanish, γ is a conformal immersion (conformal means with respect to the complex structure induced on $X(U)$ by its immersion in S^3.) Indeed it follows from (5) that

$$\omega_1^3 - i\omega_2^3 = (h_{11} - ih_{12})(\omega_0^1 + i\omega_0^2). \tag{6}$$

Moreover we have

$$d\gamma = de_3 = e_0\omega_3^0 + e_1\omega_3^1 + e_2\omega_3^2,$$

which ensures that γ is conformal whenever X is so, since e_0 is an isotropic vector.

Note that the area element covered by γ is equal to

$$\omega_1^3 \wedge \omega_2^3 = -|k|^2\omega_0^1 \wedge \omega_0^2,$$

where

$$k = h_{11} - ih_{12}.$$

Thus it turns to be exactly minus the Willmore energy density.

We define the set of umbilic points to be

$$\mathcal{U}_X = \{z \in U / k(z) = 0\}.$$

This locus will have a dramatic importance in the following. The compliment of $\mathcal{U}_X$, $U \backslash \mathcal{U}_X$ will be denoted $\mathcal{N}_X$.

1.5. Further subbundles

In the case where $\mathcal{N}_X \neq \emptyset$, it is possible to impose new restrictions on a frame. Indeed we may define another second order bundle by

$$\mathcal{F}^{(2)}_{\mathcal{N}_X} = \{(z,e) \in \mathcal{F}^{(\gamma)}_X / z \in \mathcal{N}_X, k = 1\}.$$

$\mathcal{F}^{(2)}_{\mathcal{N}_X}$ is defined only over $\mathcal{N}_X$. The group associated to this bundle is

$$\mathfrak{G}^{(2)} = \{g \in \mathfrak{G} / g = \begin{pmatrix} 1 & {}^tp.R & \frac{1}{2}{}^tp.p \\ 0 & R & p \\ 0 & 0 & 1 \end{pmatrix} \text{ with}$$

$$p = \begin{pmatrix} p^1 \\ p^2 \\ 0 \end{pmatrix} \in \mathbb{R}^2, R = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}.$$

Lastly by the action of $\mathfrak{G}^{(2)}$ on $\mathcal{F}^{(2)}_{\mathcal{N}_X}$, it is possible to construct also a section of the following bundle :

$$\mathcal{F}^{(3)}_{\mathcal{N}_X} = \{(z,e) \in \mathcal{F}^{(2)}_{\mathcal{N}_X} / \omega^0_3 = 0\}.$$

Note that here the fiber of $\mathcal{F}^{(3)}_{\mathcal{N}_X}$ at a point z is just composed of two points, $e(z) = (e_0, \pm e_1, \pm e_2, e_3, e_4)(z)$. We remark that the last vector e_4 is the same in both. Thus we conclude that there exists a map

$$\hat{X} : \mathcal{N}_X \longmapsto \mathcal{C}^+/\mathbb{R}^*_+ = S^3,$$

such that $\hat{X}(z) = [e_4(z)]$. R. Bryant remarks that $\hat{X}$ is a parametrization of the conformal transform of the image of X.

1.6. The Euler-Lagrange equation of the Willmore functionnal

From the previous analysis we know that we may write the Willmore functional as

$$\mathcal{W}(X) = \int_U -\omega^1_3 \wedge \omega^2_3 \equiv \int_U \Omega_X,$$

which is precisely minus the area covered by γ. In [3], the Euler-Lagrange equation for an immersion $X : U \longrightarrow S^3$ - not necessarily conformal - of a critical point is derived.

To write it consider such an immersion X and a section e of the associated bundle $\mathcal{F}^{(\gamma)}_X$. We can set $\omega^0_3 := h_1\omega^1_0 + h_2\omega^2_0$. And one may prove, using Cartan lemma, that there exist smooth functions p_{11}, $p_{12} = p_{21}$ and p_{22} such that

$$(7) \qquad \begin{cases} dh_1 + 2\omega^0_0 h_1 = \omega^2_1 h_2 + h_{11}\omega^0_1 + h_{12}\omega^0_2 & \\ \qquad\qquad +p_{11}\omega^1_0 + p_{12}\omega^2_0 & (a) \\ dh_2 + 2\omega^0_0 h_2 = \omega^1_2 h_1 + h_{21}\omega^0_1 + h_{22}\omega^0_2 & \\ \qquad\qquad +p_{21}\omega^1_0 + p_{22}\omega^2_0. & (b) \end{cases}$$

Then X parametrizes a Willmore surface if and only if

$$(8) \qquad p_{11} + p_{22} = 0.$$

2. Formulation using a curvature free connection form

In this section we shall give a characterisation of Willmore immersions by introducing a family of connections which depend on complex parameters, with curvature zero. Such a construction is known for harmonic maps (see e.g. [21], [5], [7],...) and familiar in the theory of completely integrable Hamiltonian systems. Here we are guided by the similarities between Willmore surfaces and harmonic maps. An important result in this direction is the following due to R. Bryant.

Theorem 1 *[3]. If X is a conformal Willmore immersion, then its conformal Gauss map γ is harmonic conformal.*

In view of this we could study harmonic conformal maps from a surface onto $S^{3,1}$ and try to deduce all the conformal parametrisations of Willmore surfaces. One difficulty then is that the way to construct a Willmore conformal immersion from a given harmonic conformal map into $S^{3,1}$ becomes degenerate over umbilic points. We will here use another point of view which exploits the fact that the map which associates to each $z \in U$ the three-dimensional spacelike subspace of $\mathbb{R}^{4,1}$ spanned by (e_1, e_2, e_3) is -roughly - harmonic.

2.1. Notation and preliminary computations

In order to present the computations which are relatively heavy we need to introduce first some new notation for the rest of the paper.

The complexification of $\mathfrak{G}$ is denoted $\mathfrak{G}^{\mathbb{C}}$, and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ is the Lie algebra of $\mathfrak{G}^{\mathbb{C}}$. We introduce the matrices

$$(A_+, A_-, A_0, B_+, B_-, B_0, C_+, C_-, D_+, D_-),$$

which form a basis of $\mathfrak{g}^{\mathbb{C}}$ given by

$$A_+ = \begin{pmatrix} 0 & 0 & 0 \\ X_+ & 0 & 0 \\ 0 & {}^tX_+ & 0 \end{pmatrix}, \qquad A_- = \begin{pmatrix} 0 & 0 & 0 \\ X_- & 0 & 0 \\ 0 & {}^tX_- & 0 \end{pmatrix},$$
$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ X_0 & 0 & 0 \\ 0 & {}^tX_0 & 0 \end{pmatrix}, \tag{9}$$

$$B_+ = \begin{pmatrix} 0 & {}^tX_+ & 0 \\ 0 & 0 & X_+ \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_- = \begin{pmatrix} 0 & {}^tX_- & 0 \\ 0 & 0 & X_- \\ 0 & 0 & 0 \end{pmatrix},$$
$$B_0 = \begin{pmatrix} 0 & {}^tX_0 & 0 \\ 0 & 0 & X_0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{10}$$

(11)
$$C_+ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \qquad C_- = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

(12)
$$D_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad D_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here (X_+, X_-, X_0) is a basis of $\mathbb{C}^3$ given by

$$X_+ = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{13}$$

Below is the table of Lie brackets of the elements of this basis. It gives $[X, Y]$ in function of X and Y.

X \ Y	A_+	A_-	A_0	B_+	B_-	B_0	C_+	C_-	D_+	D_-
A_+	0	0	0	0	$-2C_+$	$-D_+$	$2A_+$	0	0	$-2A_0$
A_-	0	0	0	$-2C_-$	0	$-D_-$	0	$2A_-$	$-2A_0$	0
A_0	0	0	0	D_+	D_-	$-E$	A_0	A_0	A_+	A_-
B_+	0	$2C_-$	$-D_+$	0	0	0	0	$-2B_+$	0	$-2B_0$
B_-	$2C_+$	0	$-D_-$	0	0	0	$-2B_-$	0	$-2B_0$	0
B_0	D_+	D_-	E	0	0	0	$-B_0$	$-B_0$	B_+	B_-
C_+	$-2A_+$	0	$-A_0$	0	$2B_-$	B_0	0	0	$-D_+$	D_-
C_-	0	$-2A_-$	$-A_0$	$2B_+$	0	B_0	0	0	D_+	$-D_-$
D_+	0	$2A_0$	$-A_+$	0	$2B_0$	$-B_+$	D_+	$-D_-$	0	$C_+ - C_-$
D_-	$2A_0$	0	$-A_-$	$2B_0$	0	$-B_-$	$-D_-$	D_-	$C_- - C_+$	0

where $E = \frac{1}{2}(C_+ + C_-)$.

We let σ and τ be the two matrices:

$$\sigma = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \ \tau = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The automorphisms of $\mathfrak{g}^{\mathbb{C}}$ given by

$$Ad_\sigma : \xi \longmapsto \sigma\xi\sigma^{-1},$$

and

$$Ad_\tau : \xi \longmapsto \tau\xi\tau^{-1}$$

lead to the following decompositions of $\mathfrak{g}^{\mathbb{C}}$. First

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}, \text{ and } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{14}$$

where

$$\mathfrak{k} = \{\xi \in \mathfrak{g}/Ad_\sigma(\xi) = \xi\}, \ \mathfrak{p} = \{\xi \in g/Ad_\sigma(\xi) = -\xi\},$$

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \otimes \mathbb{C} \text{ and } \mathfrak{p}^{\mathbb{C}} = \mathfrak{p} \otimes \mathbb{C}.$$

Notice that $\mathfrak{k}$ is the Lie algebra of the subgroup $\mathfrak{K} = \{u \in \mathfrak{G}/\sigma u \sigma^{-1} = u\}$ and that $\mathfrak{k}^{\mathbb{C}}$ is spanned by (C_+, C_-, D_+, D_-) over $\mathbb{C}$. The homogeneous manifold $\mathfrak{G}/\mathfrak{K}$ coincides with the Grassmannian of three-dimensionnal space-like subspaces of $\mathbb{R}^{4,1}$, $Gr_3(\mathbb{R}^{4,1})$. Second

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{q}^{\mathbb{C}}, \text{ and } \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q}, \tag{15}$$

where

$$\mathfrak{l} = \{\xi \in \mathfrak{g}/Ad_\tau(\xi) = \xi\}, \ \mathfrak{q} = \{\xi \in \mathfrak{g}/Ad_\tau(\xi) = -\xi\},$$

$$\mathfrak{l}^{\mathbb{C}} = \mathfrak{l} \otimes \mathbb{C} \text{ and } \mathfrak{q}^{\mathbb{C}} = \mathfrak{q} \otimes \mathbb{C}.$$

Here $\mathfrak{l}$ is the Lie algebra of the subgroup $\mathfrak{L} = \{u \in \mathfrak{G}/\tau u \tau^{-1} = u\}$ and $\mathfrak{l}^{\mathbb{C}}$ is spanned by $(A_+, A_-, B_+, B_-, C_+, C_-)$ over $\mathbb{C}$. The homogenous manifold $\mathfrak{G}/\mathfrak{L}$ coincides with $S^{3,1}$.

Now let $X : U \longrightarrow S^3$ be some smooth immersion, and $e = \epsilon.F$ be a smooth section of the corresponding bundle $\mathcal{F}_X^{(1)}$. The pull-back of the Maurer-Cartan form is $\omega = F^{-1}.dF$. We will sometime also denote $\omega = e^{-1}.de$. We can decompose ω in our basis of $\mathfrak{g}^{\mathbb{C}}$ as:

$$\begin{aligned}\omega = a^+A_+ + a^-A_- + b^+B_+ + b^-B_- + (b^0 + \overline{b^0})B_0 \\ +c^+C_+ + c^-C_- + d^+D_+ + d^-D_- \ ,\end{aligned} \tag{16}$$

where

$$a^+ = \frac{1}{2}(\omega_0^1 + i\omega_0^2), \tag{17}$$

$$a^- = \frac{1}{2}(\omega_0^1 - i\omega_0^2), \tag{18}$$

$$b^+ = \frac{1}{2}(\omega_1^0 + i\omega_2^0), \tag{19}$$

$$b^- = \frac{1}{2}(\omega_1^0 - i\omega_2^0), \tag{20}$$

$$b^0 = (h_1 - ih_2)a^+ = 2ha^+, \tag{21}$$

$h = \frac{1}{2}(h_1 - ih_2)$ being defined by

$$\omega_3^0 = h_1\omega_0^1 + h_2\omega_0^2, \tag{22}$$

$$c^+ = \frac{1}{2}(\omega_0^0 + i\omega_2^1), \tag{23}$$

$$c^- = \frac{1}{2}(\omega_0^0 - i\omega_2^1), \tag{24}$$

$$d^+ = \frac{1}{2}(\omega_1^3 + i\omega_2^3), \tag{25}$$

$$d^- = \frac{1}{2}(\omega_1^3 - i\omega_2^3). \tag{26}$$

Notice that there is no component on A_0 in ω. This reflects the condition that $e = \epsilon.F$ is a section of $\mathcal{F}_X^{(1)}$. Moreover using (5) and denoting $H := \frac{1}{2}(h_{11} + h_{22})$ and $k := \frac{1}{2}(h_{11} - h_{22}) - ih_{12}$, we have

$$d^- = k\ a^+ + Ha^- \text{ and } d^+ = \overline{k}a^- + Ha^+, \tag{27}$$

and $a^- = \overline{a^+}$, $b^- = \overline{b^+}$, $c^- = \overline{c^+}$, $d^- = \overline{d^+}$, $\omega_3^0 = b^0 + \overline{b^0}$.

We now list the structure equations $d\omega_j^i + \omega_k^i \wedge \omega_j^k = 0$ in the new notation. For $\omega_j^i = \omega_0^0$ and ω_2^1 we get

$$dc^+ - 2a^+ \wedge b^- - d^- \wedge d^+ = 0, \tag{28}$$

for $\omega_1^3 - i\omega_2^3$ and using (35) we get

$$d(d^-) + (c^+ - c^-) \wedge d^- + b^0 \wedge a^- = 0, \tag{29}$$

for $\omega_0^3 = 0$ we get

$$a^+ \wedge d^- + a^- \wedge d^+ = 0, \tag{30}$$

for ω_3^0 as we will compute in the next Lemma we have, denoting $P := \frac{1}{2}(p_{11} + p_{22})$,

$$\begin{aligned} &db^0 + (c^+ + c^-) \wedge b^0 - 2b^+ \wedge d^- \\ &\quad = 2H(b^- \wedge a^+ - b^+ \wedge a^-) + 2Pa^- \wedge a^+, \end{aligned} \tag{31}$$

for $\omega_0^1 + i\omega_0^2$ we get

$$da^+ + 2a^+ \wedge c^+ = 0, \tag{32}$$

for $\omega_1^0 + i\omega_2^0$ we get

$$db^+ + 2c^- \wedge b^+ + (b^0 + \overline{b^0}) \wedge d^+ = 0. \tag{33}$$

A special attention is devoted to the proof of (31) as follows:

Lemma 1. *The relation (31) holds.*

Proof. Let us use the complex notation $h = \frac{1}{2}(h_1 - ih_2)$ rewrite equation (7)

$$dh + (2\omega_0^0 + i\omega_2^1)h =$$

$$\frac{1}{2}k(\omega_1^0+i\omega_2^0)+\frac{1}{2}q(\omega_0^1+i\omega_0^2)+\frac{1}{4}(h_{11}+h_{22})(\omega_1^0-i\omega_2^0)+\frac{1}{4}(p_{11}+p_{22})(\omega_0^1-i\omega_0^2),$$

where we set

$$q = \frac{1}{2}(p_{11} - p_{22}) - ip_{12}.$$

This is equivalent to

$$dh + (3c^+ + c^-)h = kb^+ + qa^+ + Hb^- + Pa^-. \tag{34}$$

Hence by (32) we obtain

$$\begin{aligned} d(ha^+) &= dh \wedge a^+ + hda^+ \\ &= [kb^+ + qa^+ + Hb^- + Pa^- - (3c^+ + c^-)h] \wedge a^+ + 2hc^+ \wedge a^+ \\ &= b^+ \wedge (d^- - Ha^-) + Hb^- \wedge a^+ + Pa^- \wedge a^+ - h(c^+ + c^-) \wedge a^+ \\ &= b^+ \wedge d^- + H(b^- \wedge a^+ - b^+ \wedge a^-) \\ &\quad +Pa^- \wedge a^+ - (c^+ + c^-) \wedge (ha^+), \end{aligned}$$

where we have used (27). This implies by (21) that

$$db^0 + (c^+ + c^-) \wedge b^0 = 2b^+ \wedge d^- + 2H(b^- \wedge a^+ - b^+ \wedge a^-) + 2Pa^- \wedge a^+,$$

which is precisely (31). q.e.d.

2.2. A reformulation of the Willmore equation

The previous computations will lead us to show that we can associate to each Willmore immersion a family of connections depending on complex parameters with zero curvature. More precisely given an immersion X of U (not necessarily Willmore for the moment), and a map F from U to $\mathfrak{G}$ such that $e = \epsilon.F$ is a section of $\mathcal{F}_X^{(1)}$, we set for any λ, μ in $\mathbb{C}^*$,

$$\begin{aligned} \omega_{\lambda,\mu} =& \lambda^{-1}\mu^{-1}b^0 B_0 + \lambda^{-1}(a^+A_+ + b^+B_+) + \mu^{-1}d^-D_- \\ & + c^+C_+ + c^-C_- + \lambda\mu\overline{b^0}B_0 \\ & + \lambda(a^-A_- + b^-B_-) + \mu d^+D_+. \end{aligned} \tag{35}$$

We define also

$$\begin{aligned} \omega_\lambda = \omega_{\lambda,1} =& \lambda^{-1}(a^+A_+ + b^+B_+ + b^0B_0) \\ & + c^+C_+ + c^-C_- + d^+D_+ + d^-D_- \\ & + \lambda(a^-A_- + b^-B_- + \overline{b^0}B_0) \\ :=& \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1'', \end{aligned} \tag{36}$$

$$\begin{aligned} \omega_\mu = \omega_{1,\mu} =& \mu^{-1}(b^0B_0 + d^-D_-) \\ & + a^+A_+ + a^-A_- + b^+B_+ + b^-B_- \\ & + c^+C_+ + c^-C_- + \mu(\overline{b^0}B_0 + d^+D_+) \\ :=& \mu^{-1}\beta_1' + \beta_0 + \mu\beta_1'', \end{aligned} \tag{37}$$

and recall that $\omega = F^{-1}.dF = \omega_{1,1}$. We have the following.

Theorem 2. *The following four assertions are equivalent.*

a) $e = \epsilon.F$ is a section of $\mathcal{F}_X^{(\gamma)}$, i.e., $H = \frac{1}{2}(h_{11} + h_{22}) = 0$ and X is a Willmore immersion, i.e., $P = \frac{1}{2}(p_{11} + p_{22}) = 0$.

b) For any λ, μ in $\mathbb{C}^*$, $\omega_{\lambda,\mu}$ has zero curvature, i.e.,

$$d\omega_{\lambda,\mu} + \frac{1}{2}[\omega_{\lambda,\mu} \wedge \omega_{\lambda,\mu}] = 0. \tag{38}$$

c) For any λ *in* $\mathbb{C}^*$, ω_λ *has curvature zero, i.e.,*

$$d\omega_\lambda + \frac{1}{2}[\omega_\lambda \wedge \omega_\lambda] = 0. \tag{39}$$

d) For any μ *in* $\mathbb{C}^*$, ω_μ *has curvature zero, i.e.,*

$$d\omega_\mu + \frac{1}{2}[\omega_\mu \wedge \omega_\mu] = 0. \tag{40}$$

Proof. It is a consequence of the following direct computation (true for any general immersion) which uses all the structure relations of the previous section and Lemma 1. For any λ, μ in $\mathbb{C}^*$,

$$\begin{aligned}
&d\omega_{\lambda,\mu} + \frac{1}{2}[\omega_{\lambda,\mu} \wedge \omega_{\lambda,\mu}] \\
&\quad = -(\lambda^{-1}\mu^{-1}a^+ \wedge d^- + \lambda\mu a^- \wedge d^+)A_0 \\
&\qquad + (\lambda\mu^2 - \lambda^{-1})\overline{b^0} \wedge d^+ B_+ + (\lambda^{-1}\mu^{-2} - \lambda)b^0 \wedge d^- B_- \\
&\qquad + (\lambda^{-1}\mu^{-1} - \lambda\mu)[2H(b^- \wedge a^+ - b^+ \wedge a^-) + 2Pa^- \wedge a^+]B_0.
\end{aligned}$$

Using relations (35) and (41) we have

$$\begin{aligned}
d\omega_{\lambda,\mu}+&\frac{1}{2}[\omega_{\lambda,\mu} \wedge \omega_{\lambda,\mu}] \\
&= -(\lambda^{-1}\mu^{-1} - \lambda\mu)Ha^+ \wedge a^- A_0 \\
&\quad + (\lambda\mu^2 - \lambda^{-1})2\overline{h}Ha^- \wedge a^+ B_+ \\
&\quad + (\lambda^{-1}\mu^{-2} - \lambda)2hHa^+ \wedge a^- B_- \\
&\quad + (\lambda^{-1}\mu^{-1} - \lambda\mu)[2H(b^- \wedge a^+ - b^+ \wedge a^-) \\
&\quad + 2Pa^- \wedge a^+]B_0.
\end{aligned} \tag{41}$$

Hence, if one of the three assertions b), c) and d) occurs, then the cancelation of the coefficient of A_0 forces $H = 0$ and

$$d\omega_{\lambda,\mu} + \frac{1}{2}[\omega_{\lambda,\mu} \wedge \omega_{\lambda,\mu}] = (\lambda^{-1}\mu^{-1} - \lambda\mu)(p_{11} + p_{22})a^- \wedge a^+ B_0,$$

so we must have also $p_{11} + p_{22} = 0$ and a) is proven. Conversely the relation (41) shows that a) implies the three assertions b), c) and d).

q.e.d.

In the following we will assume that X is a conformal immersion. It follows then from the condition $\omega_0^1 \wedge \omega_0^2 > 0$ that a^+ is a $(1,0)$-form, i.e., that

$$a^+(\frac{\partial}{\partial \overline{z}}) = 0.$$

The above theorem is familiar with the theory of harmonic maps between a surface and a homogeneous manifold of compact type developped using integrable systems. In particular let us recall how one may apply the construction of harmonic maps given by J. Dorfmeister, F. Pedit and H.-Y. Wu [7] to the conformal Gauss map γ to get a characterisation of conformal Willmore immersions.

One studies the harmonic map $\gamma : U \longrightarrow S^{3,1}$ through its lift $e : U \longrightarrow \mathcal{F}$. We decompose

$$e^{-1}.de = \beta_0 + \beta_1,$$

where β_0 is a $\mathfrak{k}$-valued 1-form and β_1 is a $\mathfrak{q}$-valued 1-form. Furthermore we split β_1 into $\beta_1' + \beta_1''$ where β_1' is the (1,0) part of β_1 and β_1'' is the (0,1) part of β_1.($\beta_1' = \beta_1(\frac{\partial}{\partial z})dz$, $\beta_1'' = \beta_1(\frac{\partial}{\partial \overline{z}})d\overline{z}$).

Then the following result is a consequence of Theorem 1 and a straightforward adaptation of [7] to noncompact homogeneous manifolds

Theorem 3 *[7]. X is a conformal Willmore immersion (or γ is a harmonic conformal map) and $e = \epsilon.F$ is a section of $\mathcal{F}_X^{(\gamma)}$ if and only if X is conformal and the 1-form*

$$\beta_\mu = \mu^{-1}\beta_1' + \beta_0 + \mu\beta_1''$$

solves the zero-curvature condition

$$d\beta_\mu + \frac{1}{2}[\beta_\mu \wedge \beta_\mu] = 0$$

for any $\mu \in \mathbb{C}^*$.

Notice that obviously $\beta_\mu = \omega_\mu$ and hence we deduce that the equivalence between a) and d) in Theorem 2 was already contained in Theorem 3.

2.3. Introducing loop groups

We now focus on equation c) of Theorem 2 and its exploitation using loop groups. For that purpose, we introduce the following notation. For any Lie group $\mathfrak{A}$, $L\mathfrak{A}$ is the set of maps from the unit circle S^1 (i.e., loops) with values into $\mathfrak{A}$. We choose measurable maps which are bounded in the H^s topology, for $s > \frac{1}{2}$ (other choices are possible, see [17] and [7]). To be more precise, we set

$$S^1 = \{\lambda \in \mathbb{C}/|\lambda| = 1\}.$$

We adopt the convention that any map defined on S^1 will be denoted with the subscript $_\circ$: an object like $g_\circ$ is a map defined on the circle and its value at some $\lambda \in S^1$ is g_λ. This is a way to avoid confusion between elements in finite dimensional Lie groups and Lie algebras with elements in corresponding loop groups and algebras. We assume that $\mathfrak{A}$ is some subgroup of a matrix group $GL(n,\mathbb{C})$, and we define for $s > \frac{1}{2}$

$$L\mathfrak{A} = H^s(S^1,\mathfrak{A}) = \{g_\circ : S^1 \longrightarrow \mathfrak{A}/g_\lambda = \sum_{k\in\mathbb{Z}}(g_\circ)_k\lambda^k \\ \text{with } \sum_{k\in\mathbb{Z}}(1+k^2)^s|(g_\circ)_k|^2 < +\infty\},$$

equipped with the norm $||g_\circ|| = ||g_\circ||_{H^s} = [\sum_{k\in\mathbb{Z}}(1+k^2)^s|(g_\circ)_k|^2]^{\frac{1}{2}}$.

We define the product of two elements $a_\circ, b_\circ$ in $L\mathfrak{A}$ by the rule

$$a_\circ.b_\circ : \lambda \longmapsto a_\lambda.b_\lambda.$$

Note that $L\mathfrak{A}$ is a group in the sense that if $a_\circ, b_\circ \in L\mathfrak{A}$, then $a_\circ.b_\circ \in L\mathfrak{A}$. Moreover there exists some constant C such that

$$||a_\circ.b_\circ|| \leq C||a_\circ||.||b_\circ||, \qquad \forall a_\circ, b_\circ \in L\mathfrak{A}. \tag{42}$$

Similarly if $\mathfrak{a}$ is a Lie algebra and we assume for simplicity that $\mathfrak{a}$ is contained in a matrix algebra $M(n,\mathbb{C})$ then we will denote

$$L\mathfrak{a} = H^s(S^1,\mathfrak{a}).$$

This is an infinite dimensional vectorial space which is a Banach space when equipped with the norm $||\xi_\circ|| := ||\xi_\circ||_{H^s}$.

We also define a bracket on $L\mathfrak{a}$ by the rule

$$[\xi_\circ, \eta_\circ] : \lambda \longmapsto [\xi_\lambda, \eta_\lambda].$$

It is then easy to check that if $\xi_\circ, \eta_\circ \in L\mathfrak{a}$, then $[\xi_\circ, \eta_\circ] \in L\mathfrak{a}$, $||[\xi_\circ, \eta_\circ]|| \leq C||\xi_\circ||.||\eta_\circ||$ and $L\mathfrak{a}$ endowed with this bracket is a Lie algebra. Moreover if $\mathfrak{a}$ is the Lie algebra of $\mathfrak{A}$, then $L\mathfrak{a}$ is the Lie algebra of $L\mathfrak{A}$.

Twisted loop groups

We need also to consider

$$L\mathfrak{G}_\sigma = \{g_\circ \in L\mathfrak{G}/\sigma g_\lambda \sigma^{-1} = g_{-\lambda}\},$$

$$L\mathfrak{G}^{\mathbb{C}}_\sigma = \{g_\circ \in L\mathfrak{G}^{\mathbb{C}} / \sigma g_\lambda \sigma^{-1} = g_{-\lambda}\},$$
$$L\mathfrak{g}_\sigma = \{\xi_\circ \in L\mathfrak{g} / \sigma \xi_\lambda \sigma^{-1} = \xi_{-\lambda}\},$$
$$L\mathfrak{g}^{\mathbb{C}}_\sigma = \{\xi_\circ \in L\mathfrak{g}^{\mathbb{C}} / \sigma \xi_\lambda \sigma^{-1} = \xi_{-\lambda}\}.$$

Notice that $L\mathfrak{g}_\sigma$ is the Lie algebra of $L\mathfrak{G}_\sigma$ and $L\mathfrak{g}^{\mathbb{C}}_\sigma$ is the Lie algebra of $L\mathfrak{G}^{\mathbb{C}}_\sigma$. Moreover, each $\xi_\circ \in L\mathfrak{g}^{\mathbb{C}}$ can be decomposed as a Fourier series

$$\xi_\lambda = \sum_{k\in\mathbb{Z}} (\xi_\circ)_k \lambda^k,$$

where $(\xi_\circ)_k \in \mathfrak{g}^{\mathbb{C}}$, and the twisting condition $\sigma\xi_\lambda\sigma^{-1} = \xi_{-\lambda}$ is equivalent to

$$(\xi_\circ)_k \in \mathfrak{k}^{\mathbb{C}} \text{ if } k \text{ is even,}$$
$$(\xi_\circ)_k \in \mathfrak{p}^{\mathbb{C}} \text{ if } k \text{ is odd.}$$

We may interpret the family of 1-forms ω_λ from Theorem 2, on U with coefficients in $\mathfrak{g}^{\mathbb{C}}$ and depending on the complex parameter $\lambda \in \mathbb{C}^\star$ as a 1-form on U with coefficients in $L\mathfrak{g}^{\mathbb{C}}$ (actually in $L\mathfrak{g}$). Then it follows from Frobenius Theorem that (39) is the necessary and sufficient condition for the local existence of a map $F_\circ$ from U to $L\mathfrak{G}$ such that

$$dF_\circ = F_\circ . \omega_\circ. \tag{43}$$

In the following we will assume that U is simply connected. Then the existence of $F_\circ$ will be global. We choose a base point $p \in U$ and impose the condition that $F_\circ(p) = \mathbb{1}$. Inspired by the terminology of [7] we will call $F_\circ$ an extended conformal Willmore immersion (ECWI). Clearly the set of conformal Willmore immersions of a domain U is not in bijection with the set of all maps from U to $L\mathfrak{G}$. Thus we need to characterize the ECWI's among maps from U to $L\mathfrak{G}$.

We recall that for any conformal extended Willmore immersions $F_\circ$,

$$F_\lambda^{-1}.dF_\lambda = \omega_\lambda = \alpha'_1\lambda^{-1} + \alpha_0 + \alpha''_1\lambda,$$

where

$$\begin{aligned} \alpha'_1 &= a^+A_+ + b^+B_+ + b^0B_0, \\ \alpha_0 &= c^+C_+ + c^-C_- + d^+D_+ + d^-D_-, \\ \alpha''_1 &= a^-A_- + b^-B_- + \overline{b^0}B_0 = \overline{\alpha'_1}. \end{aligned}$$

Let us collect some observations:

(i) $F_\circ \in L\mathfrak{G}$, i.e., it is a map with values into $L\mathfrak{G}^{\mathbb{C}}$ which checks the reality condition

$$F_{\overline{\lambda^{-1}}} = \overline{F_\lambda}, \tag{44}$$

(ii)

$$F_\lambda^{-1}.dF_\lambda \text{ is a linear combination of } \lambda^{-1}, \lambda^0 \text{ and } \lambda. \tag{45}$$

(iii)

$$\begin{array}{l} \alpha_1' \text{ and } \alpha_1'' \text{ have coefficients in } \mathfrak{p}^{\mathbb{C}} \\ \alpha_0 \text{ has coefficients in } \mathfrak{k}, \end{array} \tag{46}$$

(iv) The special structure of α_1' may be written

$$\alpha_1' = \begin{pmatrix} 0 & {}^t(\eta dz + \zeta d\overline{z}) & 0 \\ \xi dz & 0 & \eta dz + \zeta d\overline{z} \\ 0 & {}^t\xi dz & 0 \end{pmatrix}, \tag{47}$$

where ξ, η, ζ are smooth maps from U to $\mathbb{C}^3$ such that

$$ {}^t\xi.\xi = {}^t\xi.\eta = 0, \tag{48}$$

$$\xi \text{ never vanishes,} \tag{49}$$

$$\exists \beta : U \longrightarrow \mathbb{C}, \zeta = \beta\xi. \tag{50}$$

Indeed we have in this particular case

$$\begin{array}{rcl} \xi dz &=& a^+ X_+, \\ \zeta d\overline{z} &=& b^+(\frac{\partial}{\partial \overline{z}}) X_+ d\overline{z}, \\ \eta dz &=& b^+(\frac{\partial}{\partial z}) X_+ dz + b^0 X_0. \end{array}$$

Lastly properties (44), (45), (46) can be expressed in a more compact way. For that purpose we introduce the notation

$$L^+\mathfrak{G}^{\mathbb{C}} = \{g_\circ \in L\mathfrak{G}^{\mathbb{C}} / g_\circ \text{ admits a holomorphic extension inside the disk } |\lambda| < 1\},$$

$$L^+\mathfrak{g}^{\mathbb{C}} = \{g_\circ \in L\mathfrak{g}^{\mathbb{C}} / g_\circ \text{ admits a holomorphic extension inside the disk } |\lambda| < 1\}.$$

Lemma 2. *Conditions (44), (45), (46) are equivalent to the following conditions*

$$\forall z \in U, \qquad F_\circ(z) \in L\mathfrak{G}_\sigma, \tag{51}$$

$$\lambda \longmapsto \lambda F_\lambda^{-1}.dF_\lambda \in L^+\mathfrak{g}^{\mathbb{C}}. \tag{52}$$

Proof. Indeed on one hand the implication (44), (45), (46) $\Rightarrow$ (51), (52) is obvious. Conversely if we assume (52) we then have

$$F_\lambda^{-1}.dF_\lambda = \sum_{k\geq -1} \theta_k \lambda^k,$$

and condition (51) implies two facts: a reality condition, $\theta_{-k} = \overline{\theta_k}$ from which we deduce (44) and

$$F_\lambda^{-1}.dF_\lambda = \theta_{-1}\lambda^{-1} + \theta_0 + \overline{\theta_{-1}}\lambda,$$

- and hence (45) follows - and lastly the twisting condition

$$\sigma.F_\lambda^{-1}.dF_\lambda.\sigma^{-1} = F_{-\lambda}^{-1}.dF_{-\lambda},$$

which implies (46). q.e.d.

Let us denote

$$\mathcal{E} = \{F_\circ : U \longrightarrow L\mathfrak{G}_\sigma / F_\circ(p) = 1\!\!1, F_\circ \text{ satisfies} \\ (47), (48), (49), (50), (51), (52)\}$$

and

$$\mathcal{W} = \{X : U \longrightarrow S^3 / X \text{ is a conformal Willmore immersion} \\ \text{such that } X(p) = X_0\}.$$

We define the mapping $\mathcal{P}$ which associates to each $F_\circ \in \mathcal{E}$ the map $X = \mathcal{P}(F_\circ) = [(\epsilon.F_1)_0]$. The following result asserts that $\mathcal{P}$ maps $\mathcal{E}$ into $\mathcal{W}$ and is surjective. In other words, $\mathcal{E}$ is the set of ECWI's.

Theorem 4. *Starting with any $X \in \mathcal{W}$ we can construct an ECWI $F_\circ \in \mathcal{E}$ such that $[(\epsilon.F_1)_0] = X$, and conversely given any $F_\circ \in \mathcal{E}$ the map $[(\epsilon.F_1)_0]$ is a conformal Willmore immersion.*

Proof. The construction of some ECWI $F_\circ$ from any given conformal Willmore immersion X is contained in the previous discussion. Let us prove the converse. We consider some $F_\circ \in \mathcal{E}$ and set

$$X = [(\epsilon.F_1)_0].$$

Notice that conditions (47), (48), (49) garantee that X is always a conformal immersion. We now need to prove that its image is a Willmore surface.

By Theorem 2 it suffices to construct a ECWI $\tilde{e}_\circ = \epsilon.\tilde{F}_\circ$ lifting X such that $\tilde{F}_\lambda^{-1}.d\tilde{F}_\lambda$ has the form (36) to prove that X is a conformal Willmore immersion.

We will construct such a map by looking for

$$\tilde{F}_\circ = F_\circ.g,$$

where $g : U \longrightarrow \mathfrak{K}$. Hence we need to find maps $r : U \longrightarrow]0, +\infty[$ and $R : U \longrightarrow SO(3)$ to produce

$$g = \begin{pmatrix} r^{-1} & 0 & 0 \\ 0 & R^{-1} & 0 \\ 0 & 0 & r \end{pmatrix}.$$

According to (48) and (49) there exists a unique g such that

$$R.\xi = rX_+ ,$$

and this condition dictates our choice for g. We then have

$$\begin{aligned} \tilde{F}_\lambda^{-1}.d\tilde{F}_\lambda &= g^{-1}.F_\lambda^{-1}.dF_\lambda.g + g^{-1}.dg \\ &= \lambda^{-1}\tilde{\alpha}_1' + \tilde{\alpha}_0 + \lambda\tilde{\alpha}_1'', \end{aligned}$$

with $\tilde{\alpha}_0$ taking values into $\mathfrak{k}$, $\tilde{\alpha}_1'$ and $\tilde{\alpha}_1''$ taking values into $\mathfrak{p}^{\mathbb{C}}$, $\tilde{\alpha}_1'' = \overline{\tilde{\alpha}_1'}$ and

$$\tilde{\alpha}_1' = g^{-1}.\alpha_1'.g = A_+dz + r^2\beta B_+d\overline{z} + r\begin{pmatrix} 0 & {}^t(R.\eta) & 0 \\ 0 & 0 & R.\eta \\ 0 & 0 & 0 \end{pmatrix} dz.$$

Remark that since $R \in SO(3)$ and because of (48),

$${}^tX_+.(R.\eta) = {}^t(r^{-1}R.\xi).(R.\eta) = 0,$$

and thus $\exists u, v \in \mathbb{C}$ such that

$$R.\eta = uX_+ + vX_0,$$

so that

$$\tilde{\alpha}_1' = A_+dz + (r^2\beta d\overline{z} + rudz)B_+ + rvB_0dz.$$

We then see that this expression for $\tilde{F}_\lambda^{-1}.d\tilde{F}_\lambda$ is similar to (36). A straightforward verification shows that $X = [(\epsilon.\tilde{F}_1)_0]$ and hence Theorem 4 is proved. q.e.d.

Remark 1. The proof of Theorem 4 consists essentially in proving that any extended conformal Willmore immersion $e_\circ = \epsilon.F_\circ$ can be deformed into another one $\tilde{e}_\circ = \epsilon.\tilde{F}_\circ$ by a gauge transformation

$$\tilde{F}_\circ(z) = F_\circ(z).g(z), \tag{53}$$

where $g : U \longrightarrow \mathfrak{K}$ and such that $\tilde{F}_\lambda^{-1}.d\tilde{F}_\lambda$ has the form (36). We will say that $F_\circ$ is in the *normalized form* if property (36) holds. Actually we proved also that it is possible to impose the condition $a^+ = dz$, through such a gauge transformation (which is then uniquely defined). Moreover all the ECWI's which are in the same gauge orbit correspond to the same conformal Willmore immersion.

2.4. Gauge transformations

We now want to study other gauge transformations acting on $\mathcal{E}$ which preserve the fibers of $\mathcal{P}$. These gauge transformations extend to loop groups the action of $\mathfrak{G}^{(2)}$ on $\mathcal{F}^{(\gamma)}$ which was described in Section 1.5.

For any smooth function $f : U \longrightarrow \mathbb{C}$ we define the maps

$$\Psi_\circ : U \longrightarrow L\mathfrak{G}_\sigma, \qquad \psi_\circ : U \longrightarrow L^+\mathfrak{G}^{\mathbb{C}}$$

and $\psi_\circ^\star : U \longrightarrow L^-\mathfrak{G}^{\mathbb{C}}$ by

$$\begin{aligned} \psi_\lambda^\star &= \exp\left(\lambda^{-1}\tfrac{f}{2}B_+\right) = 1\!\!1 + \lambda^{-1}\tfrac{f}{2}B_+, \\ \psi_\lambda &= \exp\left(\lambda\tfrac{\overline{f}}{2}B_-\right) = 1\!\!1 + \lambda\tfrac{\overline{f}}{2}B_-, \end{aligned} \tag{54}$$

(notice that $B_+^2 = 0$)

$$\Psi_\lambda(z) = \psi_\lambda(z).\psi_\lambda^\star(z) = \psi_\lambda^\star(z).\psi_\lambda(z). \tag{55}$$

We then have the following.

Proposition 1. *Let $e_\circ = \epsilon.F_\circ$ be an ECWI in the reduced form. Then*

$$\tilde{e}_\circ = \epsilon.\tilde{F}_\circ = \epsilon.F_\circ.\Psi_\circ^{-1}$$

is also an ECWI in the normalized form, such that $\mathcal{P}(\tilde{e}_\circ) = \mathcal{P}(e_\circ)$. Moreover we have

$$\tilde{F}_\lambda^{-1}.d\tilde{F}_\lambda = \lambda^{-1}\tilde{\alpha}_1' + \tilde{\alpha}_0 + \lambda\tilde{\alpha}_1'', \tag{56}$$

with

$$\tilde{\alpha}_0 = \tilde{c}^+ C_+ + \tilde{c}^- C_- + \tilde{d}^+ D_+ + \tilde{d}^- C_-, \tag{57}$$

$$\tilde{\alpha}'_1 = \tilde{a}^+ A_+ + \tilde{b}^+ B_+ + \tilde{b}^0 B_0 = \overline{\tilde{\alpha}''_1}, \tag{58}$$

and

$$\left\{ \begin{array}{lcl} \tilde{a}^+ & = & a^+ \ , \\ \tilde{b}^+ & = & b^+ - \frac{1}{2}(df + 2fc^- + f^2 a^-), \\ \tilde{b}^0 & = & b^0 - fd^-, \end{array} \right. \tag{59}$$

$$\left\{ \begin{array}{lcl} \tilde{c}^+ & = & c^+ + \overline{f} a^+ \ , \\ \tilde{c}^- & = & c^- + f a^- \ , \\ \tilde{d}^+ & = & d^+ \ , \\ \tilde{d}^- & = & d^-. \end{array} \right. \tag{60}$$

Proof. It consists essentially in a computation of $\tilde{F}_\circ^{-1}.d\tilde{F}_\circ$. We have first

$$\tilde{F}_\circ^{-1}.d\tilde{F}_\circ = \Psi_\circ.(F_\circ^{-1}.dF_\circ).\Psi_\circ^{-1} - d\Psi_\circ.\Psi_\circ^{-1}. \tag{61}$$

Using the fact that

$$\begin{array}{lcl} (\psi_\circ^\star)^{-1}.\alpha'_1.\psi_\circ^\star & = & \alpha'_1, \\ (\psi_\circ)^{-1}.\alpha''_1.\psi_\circ & = & \alpha''_1, \end{array}$$

and $\psi_\circ.\psi_\circ^\star = \psi_\circ^\star.\psi_\circ$ we get

$$\begin{array}{lcl} \tilde{F}_\lambda^{-1}.d\tilde{F}_\lambda & = & \lambda^{-1}\psi_\lambda.\alpha'_1.\psi_\lambda^{-1} + \Psi_\lambda.\alpha_0.\Psi_\lambda^{-1} + \lambda\psi_\lambda^\star.\alpha''_1.(\psi_\lambda^\star)^{-1} \\ & & -d\psi_\lambda.\psi_\lambda^{-1} - d\psi_\lambda^\star.(\psi_\lambda^\star)^{-1}. \end{array} \tag{62}$$

We then conclude that

$$\begin{aligned} \tilde{F}_\lambda^{-1}.d\tilde{F}_\lambda =& \lambda^{-1}[\alpha'_1 - \frac{1}{2}(df + 2c^- f + f^2 a^-)B_+ - fd^- B_0] \\ &+ [\alpha_0 + \overline{f} a^+ C_+ + f a^- C_-] \\ &+ \lambda[\alpha''_1 - \frac{1}{2}(d\overline{f} + 2c^+\overline{f} + \overline{f}^2 a^+)B_- - \overline{f} d^+ B_0], \end{aligned}$$

which leads to the result. q.e.d.

The above proposition will be crucial in the following since it will help us to show that locally any conformal Willmore immersion can be represented by an ECWI in a reduced form such that α'_1 is a 1-form of type (1,0), i.e., $\alpha'_1(\frac{\partial}{\partial \overline{z}}) = 0$. In other words we can reduce our problem locally to a situation very similar to the harmonic map problem.

Lemma 3. *Let $e_\circ$ be in $\mathcal{E}$. Then for every point $z_0 \in U$ there exists a neighbourhood U_{z_0} of z_0 and a gauge transformation of $e_\circ$ on U_{z_0} onto $\tilde{e}_\circ = e_\circ.\Psi_\circ$ where $\Psi_\circ$ is defined by (54) and (55) and such that*

$$\tilde{\alpha}_1'(\frac{\partial}{\partial \overline{z}}) = 0. \tag{63}$$

Moreover we can assume that $\Psi_\circ(z_0) = \psi_\circ(z_0) = \psi_\circ^\star(z_0) = \mathbb{1}$. We will call any ECWI satisfying (63) a harmonic *ECWI.*

Proof. First we can assume without loss of generality that $e_\circ$ is in the normalized form (see Theorem 4 and Remark 1). We need to find a function f from a neighbourhood of z_0 into $\mathbb{C}$ solving equation (63). Use of equations (58) and (59), which relate α_1' to $\tilde{\alpha}_1'$, gives the necessary and sufficient condition

$$0 = \tilde{b}^+(\frac{\partial}{\partial \overline{z}}) = b^+(\frac{\partial}{\partial \overline{z}}) - \frac{1}{2}(\frac{\partial f}{\partial \overline{z}} + 2fc^-(\frac{\partial}{\partial \overline{z}}) + f^2 a^-(\frac{\partial}{\partial \overline{z}})), \tag{64}$$

which is a quadratic Ricatti equation with smooth coefficients. It is known that such an equation admits smooth local solutions but no global solutions in general. Notice that solutions to equation (64) can be obtained by first solving the linear system

$$\frac{\partial \chi}{\partial \overline{z}} = M.\chi, \tag{65}$$

where

$$\chi = \begin{pmatrix} u \\ v \end{pmatrix} \text{ and } M = \begin{pmatrix} -c^-(\frac{\partial}{\partial \overline{z}}) & 2b^+(\frac{\partial}{\partial \overline{z}}) \\ a^-(\frac{\partial}{\partial \overline{z}}) & c^-(\frac{\partial}{\partial \overline{z}}) \end{pmatrix},$$

and then by setting $f = u/v$. To solve (65) locally, consider the linear operator $T(\chi)(z) = \int_{\mathbb{C}} \frac{M(\zeta).\chi(\zeta)}{\pi(z-\zeta)}\beta(\zeta)d\zeta^1 d\zeta^2$, where β is some cut-off function, and choose a holomorphic function H with values into $\mathbb{C}^2$. Then any χ satisfying $\chi - T(\chi) = H$ solves (65). Thus it suffices to construct solutions to that latter equation, and this is easily done using a fixed point argument.

Remark 2. A particular solution to equation (63) may be obtained on the nonumbilic set $\mathcal{N}_X$ (notice that if $\mathcal{N}_X \neq \emptyset$, then a Theorem of Bryant [3] ensures that $\mathcal{N}_X$ is a dense open subset of U). Indeed on $\mathcal{N}_X$ we have $d^- \neq 0 \Leftrightarrow k \neq 0$, and formula (73) proves that by choosing $f = 2h/k$ in the gauge transformation $\Psi_\circ$ we obtain a new ECWI $\tilde{F}_\circ$ such that $\tilde{b}^0 = 0$. Then from relation (45) it follows that $\tilde{b}^+ \wedge \tilde{d}^- = 0$ which implies $\tilde{b}^+(\frac{\partial}{\partial \overline{z}}) = 0$. The condition $\tilde{b}^0 = 0$ is precisely equivalent to the condition $\tilde{\omega}_3^0 = 0$ which is one of the conditions characterizing $\mathcal{F}^{(3)}_{\mathcal{N}_X}$. It corresponds to the choice $f = -p^1 - ip^2$ where p^1 and p^2 are the components of the $\mathfrak{G}^{(2)}$-valued gauge transformation used to select a section of $\mathcal{F}^{(3)}_{\mathcal{N}_X}$ starting from a section of $\mathcal{F}^{(2)}_{\mathcal{N}_X}$.

3. Lie groups and loop groups decompositions

In order to exploit the previous description of conformal Willmore immersions and to give Weierstrass type representations we need further results concerning loop groups. Basic facts about all the theory involved are exposed in the book of A. Pressley and G. Segal [17]. The results which follow are corollaries or adaptations of the results in [7]. We first recall the Iwasawa decomposition for finite dimensional Lie groups which we use and prove after two kinds of loop groups factorisation. A major difficulty here is that $\mathfrak{G}$ is a noncompact Lie group and we do not know whether all the results in [7] extend to the noncompact case (we learned recently that some work by P. Kellersch [12] brings progresses concerning loop groups Iwasawa decompositions). For that reason one of these results will be replaced by a local version.

A first ingredient is that there exists a solvable subgroup $\mathfrak{B}$ of $\mathfrak{K}^{\mathbb{C}}$ such that $\mathfrak{K}^{\mathbb{C}} = \mathfrak{K}.\mathfrak{B}$. This is the Iwasawa decomposition of $\mathfrak{K}^{\mathbb{C}}$. To be more precise

$$\mathfrak{B} = \{g \in \mathfrak{K}^{\mathbb{C}} / g = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix}, \theta \in \mathbb{R}, R \in \mathfrak{C}\}.$$

Here $\mathfrak{C}$ is a solvable subgroup of $SO(3)^{\mathbb{C}}$ such that the Iwasawa decomposition $SO(3)^{\mathbb{C}} = SO(3).\mathfrak{C}$ holds (see [11]). One may choose for instance $\mathfrak{C} = \{R \in SO(3)^{\mathbb{C}} / R.X_+ = \nu X_+, \text{ for some } \nu \in (0, +\infty)\}$. We will denote $\mathfrak{c}$ the Lie algebra of $\mathfrak{C}$, and $\mathfrak{b}$ the Lie algebra of $\mathfrak{B}$. We have the following result.

Lemma 4. *The product mapping $\mathfrak{K} \times \mathfrak{B} \longrightarrow \mathfrak{K}^{\mathbb{C}}$ is a diffeomorphism. This means in particular that for any $g \in \mathfrak{K}^{\mathbb{C}}$, $\exists!(a, b) \in \mathfrak{K} \times \mathfrak{B}$ such that*

$$g = a.b.$$

Proof. Notice that $\mathfrak{K}^{\mathbb{C}}$ is diffeomorphic to $SO(3)^{\mathbb{C}} \times \mathbb{C}^\star$ and that by this diffeomorphism, $\mathfrak{K}$ corresponds to $SO(3) \times \mathbb{R}^\star_+$ and $\mathfrak{B}$ corresponds to $\mathfrak{C}\times S^1$. Hence it suffices to prove the Iwasawa decomposition for each factor

$$\begin{array}{rcl} SO(3)^{\mathbb{C}} & = & SO(3).\mathfrak{C}, \\ \mathbb{C}^\star & = & \mathbb{R}_+^\star.S^1. \end{array}$$

The second decomposition is obvious. For the first one let us assume that $\mathfrak{C} = \{R \in SO(3)^{\mathbb{C}}/R.X_+ = \nu X_+,\text{ for some } \nu \in (0,+\infty)\}$. Let $g \in SO(3)^{\mathbb{C}}$ and observe that $Y := g.X_+$ is an isotropic vector of $\mathbb{C}^3$ which is different from 0, i.e., of the form $Y_1 - iY_2 \neq 0$ where

$$|Y_1|^2 - |Y_2|^2 = \langle Y_1, Y_2\rangle = 0.$$

Thus there exist a unique $R \in SO(3)$ and a unique $t \in \mathbb{R}_+^*$ such that $Y = tR.X_+$. We pose $a = R$ and $b = R^{-1}.g$ and it is clear that $g = a.b$ and $b \in \mathfrak{C}$. This proves the existence and the uniqueness of the decomposition. The diffeomorphism property is easily obtained.

q.e.d.

We now study decompositions of loop groups. Let us introduce some further terminology. We recall that $L^+\mathfrak{G}_\sigma^{\mathbb{C}}$ (respectively $L^-\mathfrak{G}_\sigma^{\mathbb{C}}$) is the subgroup of $L\mathfrak{G}_\sigma^{\mathbb{C}}$ of loops which admit a holomorphic extension inside the disk $|\lambda| < 1$ (repectively inside $\{\lambda \in \mathbb{C}\cup\{\infty\}/|\lambda| > 1\}$). If $\mathfrak{A}$ is any subgroup of $\mathfrak{G}^{\mathbb{C}}$ with Lie algebra $\mathfrak{a}$, we denote

$$L^+_{\mathfrak{A}}\mathfrak{G}_\sigma^{\mathbb{C}} = \{g_\circ \in L^+\mathfrak{G}_\sigma^{\mathbb{C}}/g_0 \in \mathfrak{A}\},$$

$$L^-_{\mathfrak{A}}\mathfrak{G}_\sigma^{\mathbb{C}} = \{g_\circ \in L^-\mathfrak{G}_\sigma^{\mathbb{C}}/g_\infty \in \mathfrak{A}\}.$$

We also use the notation $L^+_{\mathfrak{a}}\mathfrak{g}_\sigma^{\mathbb{C}}$ and $L^-_{\mathfrak{a}}\mathfrak{g}_\sigma^{\mathbb{C}}$ for the corresponding Lie algebras. Lastly we denote $L^+_\star\mathfrak{G}_\sigma^{\mathbb{C}} = L^+_{\{\mathbb{1}\}}\mathfrak{G}_\sigma^{\mathbb{C}}$ and $L^-_\star\mathfrak{G}_\sigma^{\mathbb{C}} = L^-_{\{\mathbb{1}\}}\mathfrak{G}_\sigma^{\mathbb{C}}$.

We consider the product mappings

$$P_1 : L\mathfrak{G}_\sigma \times L^+_{\mathfrak{B}}\mathfrak{G}_\sigma^{\mathbb{C}} \longrightarrow L\mathfrak{G}_\sigma^{\mathbb{C}},$$

and

$$P_2 : L^-_\star\mathfrak{G}_\sigma^{\mathbb{C}} \times L^+\mathfrak{G}_\sigma^{\mathbb{C}} \longrightarrow L\mathfrak{G}_\sigma^{\mathbb{C}}.$$

We do not know whether the first product mapping P_1 is a diffeomorphism (it would be true if $\mathfrak{G}$ were compact as proved in [7]). We are able however to prove that it is true locally. The situation is different for P_2. It is proven in [7] that if $\mathfrak{G}$ were a compact Lie group, then P_2 would be a diffeomorphism into an open dense subset of the identity component of $L\mathfrak{G}_\sigma^{\mathbb{C}}$ called *big cell*. We will show that this result applies also here because $\mathfrak{G}^{\mathbb{C}}$ coincides with the compactification of a compact Lie group. Before stating these results we need the following preliminary result.

Lemma 5. *There exist a neighbourhood $\mathcal{W}_0$ of 0 in $LM(5,\mathbb{C})$ and a neighbourhood $\mathcal{W}_{1\!\!1}$ of $1\!\!1$ in $LGL(5,\mathbb{C})$ such that the exponential mapping exp:$\mathcal{W}_0 \longrightarrow \mathcal{W}_{1\!\!1}$, $a \longmapsto e^a$ is an analytical diffeomorphism. We will denote log:$\mathcal{W}_{1\!\!1} \longrightarrow \mathcal{W}_0$ the inverse diffeomorphism.*

Moreover the restriction of exp to any Lie subalgebra of $LM(5,\mathbb{C})$ is a diffeomorphism into the corresponding Lie subgroup of $LGL(5,\mathbb{C})$.

Proof. Let us first recall (42). This inequality implies that all algebraic operations in $LGL(5,\mathbb{C})$ are smooth analytical (and in particular continuous). Thus the result follows from the inverse mapping theorem because the differential of exp at 0 is the identity map. q.e.d.

Theorem 5. *Let $g^0_\circ \in L\mathfrak{G}^{\mathbb{C}}_\sigma$ such that $\exists! a^0_\circ, b^0_\circ \in L\mathfrak{G}_\sigma \times L^+_{\mathfrak{B}}\mathfrak{G}^{\mathbb{C}}_\sigma$ with $g^0_\circ = a^0_\circ.b^0_\circ$. Then there exist a neighbourhood $\mathcal{V}_{g^0_\circ}$ of $g^0_\circ$ in $L\mathfrak{G}^{\mathbb{C}}_\sigma$ and neighbourhoods*

$$_{a^0_\circ}\mathcal{W} = \{a^0_\circ.exp\alpha_\circ/\alpha_\circ \in \mathcal{W}_0 \cap L\mathfrak{g}_\sigma\},$$

$$\mathcal{W}_{b^0_\circ} = \{exp\beta_\circ.b^0_\circ/\beta_\circ \in \mathcal{W}_0 \cap L^+_{\mathfrak{b}}\mathfrak{g}^{\mathbb{C}}_\sigma\}$$

such that the product mapping $_{a^0_\circ}\mathcal{W} \times \mathcal{W}_{b^0_\circ} \longrightarrow \mathcal{V}_{g^0_\circ}$ is a diffeomorphism.

Proof. As a preliminary we remark the linear decomposition

$$L\mathfrak{g}^{\mathbb{C}}_\sigma = L\mathfrak{g}_\sigma \oplus L^+_{\mathfrak{b}}\mathfrak{g}^{\mathbb{C}}_\sigma,$$

for $\forall \xi_\circ = \sum_{k\in\mathbb{Z}}(\xi_\circ)_k\lambda^k \in L\mathfrak{g}^{\mathbb{C}}_\sigma$ we may split $\xi_\circ = \eta_\circ + \phi_\circ$ with

$$\begin{array}{rcll} \eta_\lambda &=& (\xi_0)_{\mathfrak{k}} + \sum_{k<0}\xi_k\lambda^k + \overline{\xi_k}\lambda^{-k} & \in L\mathfrak{g}_\sigma, \\ \phi_\lambda &=& (\xi_0)_{\mathfrak{b}} + \sum_{k>0}(\xi_k - \overline{\xi_{-k}})\lambda^k & \in L^+_{\mathfrak{b}}\mathfrak{g}^{\mathbb{C}}_\sigma, \end{array}$$

where $(\xi_0)_{\mathfrak{k}}$ and $(\xi_0)_{\mathfrak{b}}$ are the components of ξ_0 according to the decomposition $\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{b}$. This defines a linear diffeomorphism

$$\begin{array}{rccc} S: & L\mathfrak{g}_\sigma \times L^+_{\mathfrak{b}}\mathfrak{g}^{\mathbb{C}}_\sigma & \longrightarrow & L\mathfrak{g}^{\mathbb{C}}_\sigma, \\ & (\eta_\circ, \phi_\circ) & \longmapsto & \eta_\circ + \phi_\circ. \end{array}$$

Let us consider the mapping

$$\begin{array}{rccc} \tilde{S}: & (L\mathfrak{g}^{\mathbb{C}}_\sigma) \cap \mathcal{W}_0 & \longrightarrow & (L\mathfrak{g}^{\mathbb{C}}_\sigma) \cap \mathcal{W}_0, \\ & \xi_\circ & \longmapsto & \log(\exp\eta_\circ.\exp\phi_\circ), \end{array}$$

where $S(\eta_\circ, \phi_\circ) = \xi_\circ$. According to Lemma 4, this map exists and is smooth analytical if $\mathcal{W}_0$ is a sufficiently small neighbourhood of 0. Moreover the differential of $\tilde{S}$ at 0 is the identity map. Hence we can apply the inverse mapping theorem to deduce that $\tilde{S}$ is a local diffeomorphism onto its image. Using Lemma 5, one checks easily that this statement is equivalent to the assertion of Theorem 5 in the case $g_0 = 1\!\!1$.

If we deal with an arbitrary $g_\circ^0$, we need to solve the equation

$$g_\circ = a_\circ^0.\exp\alpha_\circ.\exp\beta_\circ.b_\circ^0,$$

with $\alpha_\circ \in L\mathfrak{g}_\sigma$, $\beta_\circ \in L^+_{\mathfrak{b}}\mathfrak{g}^{\mathbb{C}}_\sigma$ close to 0 and $g_\circ$ close to $g_\circ^0$. Thanks to Lemma 5, this is equivalent to

$$\log((a_\circ^0)^{-1}.g_\circ.(b_\circ^0)^{-1}) = \log(\exp\alpha_\circ.\exp\beta_\circ).$$

We then deduce the result from the fact that there exists a neighbourhood $\mathcal{V}_{g_\circ^0}$ of $g_\circ^0$ in $L\mathfrak{G}^{\mathbb{C}}_\sigma$ such that

$$\begin{array}{ccc} \mathcal{V}_{g_\circ^0} & \longrightarrow & L\mathfrak{g}^{\mathbb{C}}_\sigma, \\ g_\circ & \longmapsto & \log((a_\circ^0)^{-1}.g_\circ.(b_\circ^0)^{-1}) \end{array}$$

is a local diffeomorphism and using also the diffeomorphism $\tilde{S}$. q.e.d.

Theorem 6. *For any $g_\circ^0$ in $L^+\mathfrak{G}^{\mathbb{C}}_\sigma$ there exist a neighbourhood $\mathcal{V}^-_{\star,1\!\!1}$ of $1\!\!1$ in $L^-_\star\mathfrak{G}^{\mathbb{C}}_\sigma$ and a neighbourhood $\mathcal{V}^+_{g_\circ^0}$ of $g_\circ^0$ in $L^+\mathfrak{G}^{\mathbb{C}}_\sigma$ such that the product mapping*

$$\begin{array}{ccc} \mathcal{V}^-_{\star,1\!\!1} \times \mathcal{V}^+_{g_\circ^0} & \longrightarrow & L\mathfrak{G}^{\mathbb{C}}_\sigma, \\ (a_\circ, b_\circ) & \longmapsto & a_\circ.b_\circ \end{array}$$

is a diffeomorphism into its image which is a neighbourhood of $g_\circ^0$ in $L\mathfrak{G}^{\mathbb{C}}_\sigma$.

One may prove this theorem by the same strategy as the one used for Theorem 5, but the result is also a consequence of the following more general result.

Theorem 7. *There exists an open subset of $L\mathfrak{G}^{\mathbb{C}}_\sigma$ called big cell, such that the product mapping*

$$\begin{array}{ccc} L^-_\star\mathfrak{G}^{\mathbb{C}}_\sigma \times L^+\mathfrak{G}^{\mathbb{C}}_\sigma & \longrightarrow & L\mathfrak{G}^{\mathbb{C}}_\sigma, \\ (a_\circ, b_\circ) & \longmapsto & a_\circ.b_\circ \end{array}$$

is a diffeomorphism onto the big cell.

Proof. This theorem is actually a straightforward corollary of Theorem 2.3 in [7]. Let us denote

$$\tilde{Q} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1\!\!1 & 0 \\ i\frac{\sqrt{2}}{2} & 0 & i\frac{\sqrt{2}}{2} \end{pmatrix}.$$

The group $\tilde{\mathfrak{G}} = \tilde{Q}^{-1}.\mathrm{SO}(5).\tilde{Q}$ is a compact subgroup of $\mathfrak{G}^{\mathbb{C}}$. Hence applying Theorem 2.3 in [7] we know that the product mapping

$$\begin{array}{ccc} L_{\star}^{-}\tilde{\mathfrak{G}}_{\sigma}^{\mathbb{C}} \times L^{+}\tilde{\mathfrak{G}}_{\sigma}^{\mathbb{C}} & \longrightarrow & L\tilde{\mathfrak{G}}_{\sigma}^{\mathbb{C}}, \\ (a_{\circ}, b_{\circ}) & \longmapsto & a_{\circ}.b_{\circ} \end{array}$$

is a diffeomorphism onto the big cell. But we also remark that the complexcification of $\tilde{\mathfrak{G}}$ coincides with $\mathfrak{G}^{\mathbb{C}}$. Hence $L\mathfrak{G}_{\sigma}^{\mathbb{C}} = L\tilde{\mathfrak{G}}_{\sigma}^{\mathbb{C}}$, and our result follows. (Notice that the subgroup $\tilde{\mathfrak{K}} = \{u \in \tilde{\mathfrak{G}}/\sigma.u.\sigma^{-1} = u\}$ analogous to $\mathfrak{K}$ in $\tilde{\mathfrak{G}}$ is diffeomorphic to $SO(3) \times SO(2)$). q.e.d.

The above result has a nice geometrical interpretation in terms of the Grassmannian model (see [17]). For the convenience of the reader, we survey the basic facts about that theory in the Appendix of this paper.

We also need the following result which is proved in [7].

Lemma 6. *[7] If $h_{\circ} : \mathbb{C} \supset U \longrightarrow LGL(n, \mathbb{C})$ is a holomorphic curve, then*

(i) either $h_{\circ}$ never meets the big cell (which corresponds to the case described in the Appendix where $\pi \circ (h_{\circ}(z).H_{+}^{(n)}) \in \mathbb{P}(\mathcal{K})$ for all $z \in U$),

(ii) either there exists a subset S of U composed of isolated points such that $h_{\circ}(z)$ is contained in the big cell for any $z \in U \setminus S$.

In case (ii), Theorem 7 implies that there exists a unique pair of holomorphic maps $g_{\circ}^{-} : U \setminus S \longrightarrow L_{\star}^{-}\mathfrak{G}_{\sigma}^{\mathbb{C}}$ and $g_{\circ}^{+} : U \setminus S \longrightarrow L^{+}\mathfrak{G}_{\sigma}^{\mathbb{C}}$ such that $g_{\circ} = g_{\circ}^{-}.g_{\circ}^{+}$ on $U \setminus S$.

Moreover the behaviour of $g_{\circ}^{-}$ and $g_{\circ}^{+}$ is precised by

Lemma 7 [7]. *Under the above hypotheses, $g_{\circ}^{-}$ and $g_{\circ}^{+}$ extend meromorphically across U, i.e., admit poles of finite order at each singular point in S.*

4. Weierstrass representations

We now exploit the results of the two previous sections in order to construct an (abstract) algorithm for constructing all conformal Willmore immersions from holomorphic (or meromorphic) datas.

4.1. Holomorphic potentials

Let us first give a "constructive" result. We define below the set of holomorphic potentials $\mathcal{P}$.

Definition 1. Let V be an open subset of $\mathbb{C}$. $\mathcal{P}_V$ denotes the set of holomorphic 1-forms $\mu_\circ$ on V (i.e., closed forms of type (1,0)) with coefficients in $\lambda^{-1}L^+\mathfrak{g}^{\mathbb{C}} \cap L\mathfrak{g}^{\mathbb{C}}_\sigma$ and such that the lower degree term of $\mu_\circ$ has the expression

$$(\mu_\circ)_{-1} = \begin{pmatrix} 0 & {}^t m & o \\ l & 0 & m \\ 0 & {}^t l & 0 \end{pmatrix} dz, \tag{66}$$

with the conditions that $l, m : V \longrightarrow \mathbb{C}^3$, ${}^t l.l = {}^t l.m = 0$ and l does not vanish.

Theorem 8. *Assume that U is simply connected. Let $\mu_\circ \in \mathcal{P}_U$ and $z_0 \in U$. Then*

1. *(Integration) There exists a unique holomorphic map $g_\circ : U \longrightarrow L\mathfrak{G}^{\mathbb{C}}_\sigma$ such that*
$$\begin{aligned} dg_\circ &= \mu_\circ . g_\circ \text{ on } U, \\ g_\circ(z_0) &= 1\!\!1. \end{aligned}$$
2. *(Local decomposition of $g_\circ$ around z_0) There exists a neighbourhood V_0 of z_0 in U on which two maps $F_\circ : V_0 \longrightarrow L\mathfrak{G}_\sigma$ and $h_\circ : V_0 \longrightarrow L^+_{\mathfrak{B}}\mathfrak{G}^{\mathbb{C}}_\sigma$ are defined such that*

$$g_\circ(z) = F_\circ(z).h_\circ(z), \forall z \in V_0. \tag{67}$$

3. *$F_\circ$ is a harmonic ECWI and thus $z \longmapsto [(\epsilon.F_1(z))_0]$ is a conformal Willmore immersion.*

Proof. We first state that $\mu_\circ$ is a curvature free connection form. This is a consequence of the equations

$$d\mu_\circ = [\mu_\circ \wedge \mu_\circ] = 0.$$

Hence Claim 1 follows from Frobenius' Theorem.

We now remark that $g_\circ$ is close to $1\!\!1$ around z_0. Hence applying Theorem 5 with $g_0 = a_0 = b_0 = 1\!\!1$ we deduce the existence of a neighbourhood V_0 of z_0 such that Claim 2 holds.

Let us now prove the last Claim. From (67) we deduce that $F_\circ = g_\circ.h_\circ^{-1}$ so that

$$\begin{array}{rcl} F_\circ^{-1}.dF_\circ & = & h_\circ.g_\circ^{-1}.dg_\circ.h_\circ^{-1} - dh_\circ.h_\circ^{-1} \\ & = & h_\circ.\mu_\circ.h_\circ^{-1} - dh_\circ.h_\circ^{-1}. \end{array}$$

It turns out that $F_\circ^{-1}.dF_\circ \in \lambda^{-1}L^+\mathfrak{g}^{\mathbb{C}} \cap L\mathfrak{g}_\sigma$ and therefore

$$F_\lambda^{-1}.dF_\lambda = \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1''.$$

Moreover we know that $\alpha_1' = h_0.(\mu_\circ)_{-1}.h_0^{-1}$ and $h_0 \in \mathfrak{B}$, which means that there exist maps $r : V_0 \longrightarrow S^1$ and $R : V_0 \longrightarrow \mathfrak{C}$ such that $h_0 = \begin{pmatrix} r & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & r^{-1} \end{pmatrix}$. Hence (66) leads to

$$\alpha_1' = \begin{pmatrix} 0 & {}^t(rR.m) & 0 \\ r^{-1}R.l & 0 & rR.m \\ 0 & {}^t(r^{-1}R.l) & 0 \end{pmatrix} dz.$$

Since $R \in SO(3)^{\mathbb{C}}$, ${}^t(r^{-1}R.l).(r^{-1}R.l) = r^{-2t}l.l = 0$ and ${}^t(r^{-1}R.l).(rR.m) =^t l.m = 0$. It proves $F_\circ \in \mathcal{E}$, and the last Claim by Theorem 4. q.e.d.

We now will study the converse algorithm, i.e., the "analysis" of some conformal Willmore immersion. Let us first establish a local result.

Proposition 2. *Let X be a conformal Willmore immersion on $U \subset \mathbb{C}$. Then for any $z_0 \in U$ there exists a neighbourhood V_0 of z_0 in U, on which we can construct some potential $\mu_\circ \in \mathcal{P}_{V_0}$ such that X derivates from $\mu_\circ$ according the above Theorem.*

In other words there exists a potential $\mu_\circ \in \mathcal{P}_{V_0}$ such that the solution $g_\circ : V_0 \longrightarrow L\mathfrak{G}_\sigma^{\mathbb{C}}$ of the equation $dg_\circ = \mu_\circ.g_\circ$ is holomorphic and can be decomposed as

$$g_\circ = \Phi_\circ.b_\circ \text{ on } V_0,$$

where $\Phi_\circ$ is an ECWI lifting X (i.e., $[(\epsilon.\Phi_1)_0] = X$) and $b_\circ : V_0 \longrightarrow L_{\mathfrak{B}}^+\mathfrak{G}_\sigma^{\mathbb{C}}$.

Proof. Let $F_\circ$ be an ECWI lifting X. We assume without loss of generality that $F_\circ$ is in the normalized form (Theorem 4). A first step is to use Lemma 2 in order to perform a gauge transformation of $F_\circ$ on a neighbourhhod of z_0 in such a way that $F_\circ$ becomes a harmonic ECWI, i.e., $\alpha_1'(\frac{\partial}{\partial \bar{z}}) = 0$ on this neighbourhood. Then we need to find two maps

$g_\circ$ and $h_\circ$ defined on that neighbourhood of z_0, into $L\mathfrak{G}_\sigma^{\mathbb{C}}$ and $L^+\mathfrak{G}_\sigma^{\mathbb{C}}$ respectively such that $g_\circ = F_\circ.h_\circ$ and $g_\circ$ is as holomorphic function of z. This latter condition means that

$$g_\circ^{-1}.dg_\circ = h_\circ^{-1}.F_\circ^{-1}.dF_\circ.h_\circ + h_\circ^{-1}.dh_\circ$$

is a (1,0)-form, and leads to the equation

$$h_\lambda^{-1}.(\alpha_0(\frac{\partial}{\partial \overline{z}}) + \lambda\alpha_1''(\frac{\partial}{\partial \overline{z}})).h_\lambda + h_\lambda^{-1}.\frac{\partial h_\lambda}{\partial \overline{z}} = 0,$$

which is equivalent to

$$\frac{\partial h_\lambda}{\partial \overline{z}}.h_\lambda^{-1} = -(\alpha_0(\frac{\partial}{\partial \overline{z}}) + \lambda\alpha_1''(\frac{\partial}{\partial \overline{z}})). \tag{68}$$

In the Appendix of [7], the existence of a solution $h_\circ$ to (68), assuming the condition $h_\circ(z_0) = 1\!\!1$ on a neighbourhood V_0 of z_0 is proven. Notice that in general this solution takes values into $L^+\mathfrak{G}_\sigma^{\mathbb{C}}$ and we would need the further condition that $h_0 \in \mathfrak{B}$ (and hence to substitute for $h_\circ$ a solution into $L_{\mathfrak{B}}^+\mathfrak{G}_\sigma^{\mathbb{C}}$). For that purpose it suffices to decompose $h_\circ$ as $h_\circ = H.b_\circ$ where H is a map into $\mathfrak{K}$ and $b_\circ$ is a maps into $L_{\mathfrak{B}}^+\mathfrak{G}_\sigma^{\mathbb{C}}$. This decomposition exists and is unique because of the decomposition $L^+\mathfrak{G}_\sigma^{\mathbb{C}} = \mathfrak{K}.L_{\mathfrak{B}}^+\mathfrak{G}_\sigma^{\mathbb{C}}$, a consequence of the Iwasawa decomposition $\mathfrak{K}^{\mathbb{C}} = \mathfrak{K}.\mathfrak{B}$ stated in Lemma 3. Thus we rewrite things as

$$g_\circ = F_\circ.H.b_\circ = \Phi_\circ.b_\circ,$$

where $\Phi_\circ = F_\circ.H$ is an ECWI lifting the same conformal Willmore immersion X as $F_\circ$.

Lastly we need to check that $\mu_\circ := g_\circ^{-1}.dg_\circ$ belongs to $\mathcal{P}_{V_0}$. We already know that $\mu_\circ$ is a holomorphic 1-form, and we easily verify that its coefficients are in $\lambda^{-1}L^+\mathfrak{G}^{\mathbb{C}} \cap L\mathfrak{G}_\sigma^{\mathbb{C}}$. But we have to check that the lower degree term is right; it is $(\mu_\circ)_{-1} = h_0^{-1}.\alpha_1'.h_0$. We leave to the reader the verification that it is of the type defined in Definition 1, the proof being the same as that of Theorem 4 or Theorem 8 and based on the fact that $h_0 \in \mathfrak{K}^{\mathbb{C}}$. q.e.d.

4.2. Weierstrass datas

As in [7] we will now see that it is possible to improve the above Proposition by showing that it suffices to look for potentials of the type $\mu_\circ = \lambda^{-1}(\mu_\circ)_{-1}$ where $(\mu_\circ)_{-1}$ is a 1-form with coefficients in $\mathfrak{p}^{\mathbb{C}}$. The price to pay however is to allow $(\mu_\circ)_{-1}$ to be meromorphic instead of holomorphic. A second improvement is that, because of the uniqueness of such a meromorphic potential, we are able to produce a global result. But here also we need to enlarge the class of potentials to those of the same type as above but where $(\mu_\circ)_{-1}$ is a closed form with isolated singularities but not necessary of the type (1,0).

Theorem 9. *Let U be a simply connected domain of $\mathbb{C}$ and $X : U \longrightarrow S^3$ a conformal Willmore immersion. Consider any ECWI $e_\circ = \epsilon.F_\circ : U \longrightarrow L\mathfrak{G}_\sigma$ lifting X, and denote the coefficient of λ^{-1} in $F_\circ^{-1}.dF_\circ$ as*

$$\alpha_1' = \begin{pmatrix} 0 & {}^t(\eta dz + \beta\xi d\overline{z}) & 0 \\ \xi dz & 0 & \eta dz + \beta\xi d\overline{z} \\ 0 & {}^t l dz & 0 \end{pmatrix}.$$

(Notice that if we assume that $F_\circ$ is in the normalized form, then $\xi dz = a^+A_+$ and $\eta dz + \beta\xi d\overline{z} = b^+B_+ + b^0B_0$.) Then there exists a subset S of U composed of isolated points such that on $U \setminus S$, $F_\circ$ decomposes uniquely as

$$F_\circ = F_\circ^-.F_\circ^+, \tag{69}$$

where $F_\circ^- : U \setminus S \longrightarrow L_\star^-\mathfrak{G}_\sigma^{\mathbb{C}}$, $F_\circ^+ : U \setminus S \longrightarrow L^+\mathfrak{G}_\sigma^{\mathbb{C}}$, and $F_\circ^-$ derivates from a linear potential $\mu_\circ$ such that

$$\begin{aligned} \mu_\lambda =& (F_\lambda^-)^{-1}.dF_\lambda^- \\ =& \lambda^{-1} \begin{pmatrix} 0 & {}^t(mdz + \gamma l d\overline{z}) & 0 \\ ldz & 0 & mdz + \gamma l d\overline{z} \\ 0 & {}^t l dz & 0 \end{pmatrix}, \end{aligned} \tag{70}$$

where l, m are maps from $U \setminus S$ to $\mathbb{C}^3$, and γ is a map from $U \setminus S$ to $\mathbb{C}$ such that

$$l \text{ does not vanish,} \tag{71}$$

$${}^t l.l = {}^t l.m = 0, \tag{72}$$

$$d(ldz) = d(mdz + \gamma l d\overline{z}) = 0, \tag{73}$$

(hence l is holomorphic and m is a solution of $\frac{\partial m}{\partial \overline{z}} = \frac{\partial(\gamma l)}{\partial z}$). More precisely, $l = r^{-1}R.\xi$, $m = rR.\eta$, $\gamma = r^2\beta$ for some $r \in \mathbb{C}^\star$, $R \in SO(3)^{\mathbb{C}}$.

We call $(ldz, mdz + \gamma l d\overline{z})$ *the Weierstrass data of* $F_\circ$. *Conversely if we choose maps* l, m *from* $U \setminus S$ *to* $\mathbb{C}^3$ *and* γ *from* $U \setminus S$ *to* $\mathbb{C}$ *satisfying (71), (72) and (73) and if* $a \in U \setminus S$, *there exists a neighbourhood of* a *in* $U \setminus S$ *on which we can construct an ECWI* $F_\circ$, *and the Weierstrass data of which is* $(ldz, mdz + \gamma l d\overline{z})$.

Proof. We divide the demonstration into 6 steps. Let

$$\begin{aligned} S &= \{z \in U / F_\circ(z) \text{ does not belong to the big cell }\} \\ &= \{z \in U / \pi(F_\circ(z).H_+^{(n)}) \in \mathbb{P}(\mathcal{K})\} \text{ (see the Appendix)}, \end{aligned}$$

and assume from the beginning that $F_\circ$ is the normalized form. The main difficulty is to prove that S is composed of isolated points; this will be stated in Step 4.

Step 1. We apply Lemma 3 around each point $a \in U$. It ensures that there exists an open neighbourhood V'_a of a in U and maps ${}_a\psi_\circ$, ${}_a\psi_\circ^\star$ and ${}_a\Psi_\circ$ from V'_a respectively to $L_\star^+\mathfrak{G}_\sigma^{\mathbb{C}}$, $L_\star^-\mathfrak{G}_\sigma^{\mathbb{C}}$ and $L\mathfrak{G}_\sigma$ satisfying ${}_a\psi_\circ(a) = {}_a\psi_\circ^\star(a) = {}_a\Psi_\circ(a) = 1\!\!1$ and ${}_a\Psi_\circ = {}_a\psi_\circ . {}_a\psi_\circ^\star$ such that the map

$${}_aF_\circ = F_\circ . {}_a\Psi_\circ^{-1}$$

is a harmonic ECWI, i.e., the map has the property that ${}_a\alpha'_1(\frac{\partial}{\partial \overline{z}}) = 0$, where we use the decomposition ${}_aF_\circ^{-1}.d_aF_\circ = \lambda^{-1}{}_a\alpha'_1 + {}_a\alpha_0 + \lambda_a\alpha''_1$.

Step 2. We use Proposition 2 to decompose ${}_aF_\circ$. It follows that there exists a neighbourhood V_a of a, which is a subset of V'_a, and there exist maps ${}_ag_\circ : V_a \longrightarrow L\mathfrak{G}_\sigma^{\mathbb{C}}$, ${}_ah_\circ : V_a \longrightarrow L_{\mathfrak{B}}^+\mathfrak{G}_\sigma^{\mathbb{C}}$ such that

$${}_aF_\circ = {}_ag_\circ . {}_ah_\circ,$$

and ${}_ag_\circ$ is holomorphic. That equation implies (using Theorem 7) that ${}_aF_\circ$ belongs to the big cell if and only if ${}_ag_\circ$ does also so, because ${}_ah_\circ \in L_{\mathfrak{B}}^+\mathfrak{G}_\sigma^{\mathbb{C}}$. Using a more geometrical language (see the Appendix) the mapping $z \longmapsto {}_aF_\circ(z).H_+^{(n)} = {}_ag_\circ(z).H_+^{(n)}$ is a holomorphic map with values into $Gr(H^{(n)})$.

Hence by Lemma 6 either ${}_aF_\circ$ never meets the big cell, or ${}_aF_\circ$ takes its values into the big cell outside isolated points. We will denote S_a the subset of points z in V_a such that ${}_aF_\circ(z)$ does not belong to the big cell.

Step 3. We will show that $S_a = S \cap V_a$. Let us assume first that for some $z \in V_a$, ${}_aF_\circ(z)$ belongs to the big cell. Then by Theorem 7 there exist ${}_aF_\circ^-(z) \in L_\star^-\mathfrak{G}_\sigma^{\mathbb{C}}$ and ${}_aF_\circ^+(z) \in L^+\mathfrak{G}_\sigma^{\mathbb{C}}$ such that ${}_aF_\circ(z) = {}_aF_\circ^-(z).{}_aF_\circ^+(z)$.

Notice that obviously ${}_aF_\circ^+(z)$ belongs to $L^+\mathfrak{G}_\sigma^{\mathbb{C}}$ and that ${}_a\psi_\circ^\star$ belongs to a neighbourhood of $1\!\!1$ in $L_\star^-\mathfrak{G}_\sigma^{\mathbb{C}}$ as small as we want (since we can choose V_a to be so). Thus we can apply Theorem 6 with ${}_aF_\circ^+(z).{}_a\psi_\circ^\star(z)$. We deduce that $\exists !_a\tilde{\psi}_\circ^\star(z) \in L_\star^-\mathfrak{G}_\sigma^{\mathbb{C}}$, $\exists !_a\tilde{F}_\circ^+(z) \in L^+\mathfrak{G}_\sigma^{\mathbb{C}}$ such that

$$ {}_aF_\circ^+(z).{}_a\psi_\circ^\star(z) = {}_a\tilde{\psi}_\circ^\star(z).{}_a\tilde{F}_\circ^+(z), \tag{74} $$

and deduce from (74) that on V_a,

$$ \begin{aligned} F_\circ(z) &= {}_aF_\circ(z).{}_a\Psi_\circ(z) \\ &= {}_aF_\circ^-(z).{}_aF_\circ^+(z).{}_a\psi_\circ^\star(z).{}_a\psi_\circ(z) \\ &= {}_aF_\circ^-(z).{}_a\tilde{\psi}_\circ^\star(z).{}_a\tilde{F}_\circ^+(z).{}_a\psi_\circ(z) \\ &= F_\circ^-(z).F_\circ^+(z), \end{aligned} $$

where $F_\circ^-(z) = {}_aF_\circ^-(z).{}_a\tilde{\psi}_\circ^\star(z) \in L_\star^-\mathfrak{G}_\sigma^{\mathbb{C}}$ and $F_\circ^+(z) = {}_a\tilde{F}_\circ^+(z).{}_a\psi_\circ(z) \in L^+\mathfrak{G}_\sigma^{\mathbb{C}}$.

Thus we conclude that $F_\circ(z)$ belongs to the big cell. Conversely a similar argument by applying Theorem 6 with $F_\circ^+.({}_a\psi_\circ^\star)^{-1}$ shows that if $F_\circ(z)$ belongs to the big cell, then ${}_aF_\circ(z)$ does also so. Hence $S_a = S \cap V_a$.

Step 4. From the covering $U \subset \cup_{a\in U} V_a$ we extract some locally finite covering $U \subset \cup_{a\in A} V_a$ for some subset A of U. We denote by A_1 the set of points $a \in A$ such that $S \cap V_a = S_a$ is composed of isolated points, and by A_2 the set of $a \in A$ such that $S \cap V_a = V_a$. Steps 2 and 3 implies basically that $A = A_1 \bigsqcup A_2$. Now we let $S_1 = \cup_{a\in A_1} S_a$ and $S_2 = \cup_{a\in A_2} S_a = \cup_{a\in A_2} V_a$.

We remark that $U \setminus S_1$ is connected and that S_2 is an open and closed subset of $U \setminus S_1$. Hence either $S_2 = \emptyset$ or $S_2 = U \setminus S_1$, which would imply that $S = U$. But notice that for some $z_0 \in U, F_\circ(z_0) = 1\!\!1$ belongs obviously to the big cell. This excludes the second alternative. Thus S is composed of isolated points.

Step 5. We now work on $U \setminus S$. Using Theorem 7 we know that there exist maps $F_\circ^- : U \setminus S \longrightarrow L_\star^-\mathfrak{G}_\sigma^{\mathbb{C}}$ and $F_\circ^+ : U \setminus S \longrightarrow L^+\mathfrak{G}_\sigma^{\mathbb{C}}$ such that

$$ F_\circ = F_\circ^-.F_\circ^+. $$

This implies that $F_\circ^- = F_\circ.(F_\circ^+)^{-1}$, and

$$ \mu_\circ := (F_\circ^-)^{-1}.dF_\circ^- = F_\circ^+.F_\circ^{-1}.dF_\circ.(F_\circ^+)^{-1} - dF_\circ^+.(F_\circ^+)^{-1} \tag{75} $$

has coefficients in $\lambda^{-1}L^+\mathfrak{g}^{\mathbb{C}} \cap L^-_\star \mathfrak{g}^{\mathbb{C}}_\sigma$. Hence

$$\mu_\lambda = \lambda^{-1}(\mu_\circ)_{-1},$$

and necessarily $d\mu_\circ + \frac{1}{2}[\mu_\circ \wedge \mu_\circ] = 0$, which implies

$$d(\mu_\circ)_{-1} = [(\mu_\circ)_{-1} \wedge (\mu_\circ)_{-1}] = 0. \tag{76}$$

Step 6. We analyze $(\mu_\circ)_{-1}$, and deduce from (75) that $(\mu_\circ)_{-1} = F_0^+.\alpha_1'.(F_0^+)^{-1}$. Let us denote $F_0^+ = \begin{pmatrix} r & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & r^{-1} \end{pmatrix}$ where $r : U \setminus S \longrightarrow \mathbb{C}^\star$, $R : U \setminus S \longrightarrow SO(3)^{\mathbb{C}}$. Then

$$(\mu_\circ)_{-1} = \begin{pmatrix} 0 & {}^t(rR.\eta dz + r\beta R.\xi d\overline{z}) & 0 \\ r^{-1}R.\xi dz & 0 & rR.\eta dz + r\beta R.\xi d\overline{z} \\ 0 & {}^t(r^{-1}R.\xi dz) & 0 \end{pmatrix}.$$

Letting $l = r^{-1}R.\xi$, $m = rR.\eta$ and $\gamma l = r\beta R.\xi$ we obtain the conclusions (70), (71) and (72) of the Theorem by conditions (61), (62), (63) and (64) on ξ, η, γ. Moreover equation (76) implies (73).

The proof of the (local) converse of that result is left to the reader: it is very similar to the proof of Theorem 8. q.e.d.

The effect of a gauge transformation on the Weierstrass data

To conclude this Chapter we analyze the effect of a gauge transformations (as in Proposition 1) on the Weierstrass data $\mu_\circ$ of a Willmore immersion. Let $F_\circ$ be an ECWI in the normalized form, i.e., $\alpha_1' = a^+A_+ + b^+B_+ + b^0B_0$ and $F_\circ'$ be a second one related to $F_\circ$ by the relation

$$F_\circ' = F_\circ.\Psi_\circ,$$

where $\Psi_\circ = \psi_\circ^\star.\psi_\circ$, $\psi_\lambda^\star = \lambda^{-1}\frac{f}{2}B_+ + \mathbb{1}$ and $\psi_\lambda = \mathbb{1} + \lambda\frac{\overline{f}}{2}B_-$. Assume that $\psi_\circ^\star$ is close to $\mathbb{1}$.

On $U \setminus S$ (S is composed of isolated points) we have the decomposition $F_\circ = F_\circ^-.F_\circ^+$ with $F_\circ^- \in L^-_\star \mathfrak{G}^{\mathbb{C}}_\sigma$, $F_\circ^+ \in L^+\mathfrak{G}^{\mathbb{C}}_\sigma$. We repeat the arguments of the Step 3 in the proof of Theorem 9. Using the fact that $\psi_\circ^\star$ is close to $\mathbb{1}$ and Lemma 7, on some neighbourhood of a given point we have

$$F_\circ^+.\psi_\circ^\star = \tilde{\psi}_\circ^\star.\tilde{F}_\circ^+,$$

where a straighforward computation shows that

$$\tilde{\psi}^\star_\circ = \lambda^{-1}\frac{f}{2}F_0^+.B_+.(\tilde{F}_0^+)^{-1} + 1\!\!1 \in L^-_\star\mathfrak{G}^{\mathbb{C}}_\sigma,$$

$\tilde{F}^+_\circ \in L^+\mathfrak{G}^{\mathbb{C}}_\sigma$. Hence on $U \setminus S$,

$$F'_\circ = F_\circ.\Psi_\circ = {F'}^-_\circ.{F'}^+_\circ,$$

with ${F'}^-_\circ = F^-_\circ.\tilde{\psi}^\star_\circ \in L^-_\star\mathfrak{G}^{\mathbb{C}}_\sigma$, ${F'}^+_\circ \in L^+\mathfrak{G}^{\mathbb{C}}_\sigma$. We want to compare $\mu'_\circ = ({F'}^-_\circ)^{-1}.d{F'}^-_\circ$ and $\mu_\circ = (F^-_\circ)^{-1}.dF^-_\circ$, in function of f, which determinates $\Psi_\circ$. For that purpose let us first express $\tilde{\psi}^\star_\circ$ in more details. Remark that we need to compute $F_0^+.B_+.(\tilde{F}_0^+)^{-1}$. We know that

$$F_0^+ = \begin{pmatrix} r & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & r^{-1} \end{pmatrix}; \qquad \tilde{F}_0^+ = \begin{pmatrix} \tilde{r} & 0 & 0 \\ 0 & \tilde{R} & 0 \\ 0 & 0 & \tilde{r}^{-1} \end{pmatrix},$$

for some $r, \tilde{r} \in \mathbb{C}^\star$, $R, \tilde{R} \in SO(3)^{\mathbb{C}}$. Hence

$$F_0^+.B_+.(\tilde{F}_0^+)^{-1} = \begin{pmatrix} 0 & {}^t(r\tilde{R}.X_+) & 0 \\ 0 & 0 & \tilde{r}R.X_+ \\ 0 & 0 & 0 \end{pmatrix}. \tag{77}$$

Since this element belongs to $\mathfrak{K}^{\mathbb{C}}$, it is necessary to have $r\tilde{R}.X_+ = \tilde{r}R.X_+$. Beside (77) we also have

$$F_0^+.B_+.(F_0^+)^{-1} = \begin{pmatrix} 0 & {}^t(rR.X_+) & 0 \\ 0 & 0 & rR.X_+ \\ 0 & 0 & 0 \end{pmatrix}. \tag{78}$$

From (77) and (78) it follows that $F_0^+.B_+.(\tilde{F}_0^+)^{-1} = \frac{\tilde{r}}{r}F_0^+.B_+.(F_0^+)^{-1}$. Thus

$$\tilde{\psi}^\star_\lambda = \lambda^{-1}\frac{f\tilde{r}}{2r}F_0^+.B_+.(F_0^+)^{-1} + 1\!\!1. \tag{79}$$

Now we go back to

$$\mu'_\circ = (\tilde{\psi}^\star_\lambda)^{-1}.\mu_\circ.\tilde{\psi}^\star_\lambda + (\tilde{\psi}^\star_\lambda)^{-1}.d\tilde{\psi}^\star_\lambda.$$

(79) and the relation $\mu_\lambda = \lambda^{-1}F_0^+.\alpha'_1.(F_0^+)^{-1}$ lead to

$$\begin{aligned}(\tilde{\psi}^\star_\lambda)^{-1}.\mu_\lambda.\tilde{\psi}^\star_\lambda = &- \lambda^{-3}(\frac{f\tilde{r}}{2r})^2F_0^+.B_+.\alpha'_1.B_+.(F_0^+)^{-1} \\ &+ \lambda^{-2}\frac{f\tilde{r}}{2r}F_0^+[\alpha'_1, B_+](F_0^+)^{-1} + \mu_\lambda.\end{aligned}$$

Using the fact that B_+ commutes with A_+, B_+, B_0 and $B_+^2 = 0$, we conclude $(\tilde{\psi}^\star_\lambda)^{-1}.\mu_\lambda.\tilde{\psi}^\star_\lambda = \mu_\lambda$, and easily obtain

$$\mu'_\lambda = \mu_\lambda + \lambda^{-1} d\left(\frac{f\tilde{r}}{2r} F_0^+.B_+.(F_0^+)^{-1}\right). \tag{80}$$

Remember that $l = r^{-1}a^+(\frac{\partial}{\partial z})R.X_+$, so that $rR.X_+ = \frac{r^2 l}{a^+(\frac{\partial}{\partial z})}$, the substitution of which in (78) gives

$$F_0^+.B_+.(F_0^+)^{-1} = \frac{r^2}{a^+(\frac{\partial}{\partial z})}\begin{pmatrix} 0 & {}^t l & 0 \\ 0 & 0 & l \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus we can rewrite (80) as

$$\mu'_\lambda = \mu_\lambda + \lambda^{-1} d\left(\delta \begin{pmatrix} 0 & {}^t l & 0 \\ 0 & 0 & l \\ 0 & 0 & 0 \end{pmatrix}\right),$$

where $\delta = \frac{fr\tilde{r}}{2a^+(\frac{\partial}{\partial z})}$.

Hence we obtain

Proposition 3. *For a gauge transformation of the type $F'_\circ = F_\circ.\Psi_\circ$ the corresponding Weierstrass datas $(ldz, mdz + \gamma l d\overline{z})$ associated to $F_\circ$ and $(l'dz, m'dz + \gamma' l' d\overline{z})$ associated to $F'_\circ$ are related by:*

$$\begin{cases} l' &= l, \\ m' &= m + \dfrac{\partial \delta}{\partial z} l + \delta \dfrac{\partial l}{\partial z}, \\ \gamma' &= \gamma + \dfrac{\partial \delta}{\partial \overline{z}}. \end{cases}$$

5. Some illustrations of the theory

5.1. The umbilic set and Bryant's quartic differential revisited

We first expose some results which were proved by R. Bryant in [3]. In the following $X : U \longrightarrow S^3$ is a conformal Willmore immersion.

First, a theorem in [3] states the following alternative: either the umbilic set $\mathcal{U}_X$ is equal to U, or $\mathcal{U}_X$ is a closed subset of U, the interior of which is empty. Recall that $\mathcal{U}_X$ is characterized by the equation $k = 0$ or equivalently $d^- = 0$.

Second, there exists a quartic differential $\mathcal{Q}_X$ on U, associated to X, which is holomorphic (see [3]). The general definition of $\mathcal{Q}_X$ is relatively complicated. However we may characterize it without using the complete definition:

1. if $\mathcal{U}_X = U$, then $\mathcal{Q}_X = 0$;
2. if $\mathcal{U}_X \neq U$, then $\mathcal{N}_X$ (the complementary set of $\mathcal{U}_X$) is a dense open subset of U, and we can determinate $\mathcal{Q}_X$ by its value on $\mathcal{N}_X$ as follows.

Indeed on $\mathcal{N}_X$ it is possible (using the gauge action of $\mathfrak{G}^{(2)}$) to construct a frame F lifting X (for instance a section of $\mathcal{F}^{(3)}_{\mathcal{N}_X}$) such that $\omega^0_3 = 0 \Leftrightarrow b^0 = 0 \Leftrightarrow h = 0$ (see Remark 2). Using such a frame we have an expression for $\mathcal{Q}_X$:

$$\mathcal{Q}_X = (h_{11} - ih_{12})(p_{11} - ip_{12})(\omega^1_0 + i\omega^2_0)^4 = 16kq(a^+)^4.$$

Since $ka^+ = d^-$, and also relation (48) together with the condition $h = 0$ implies $kb^+ + qa^+ = 0$, it follows that

$$\mathcal{Q}_X = -16(d^-)^2 b^+ a^+.$$

The reader may verify that this quartic form is of type (4,0) and is closed by using (43), (46), (47) and $h = 0$.

Using the above concepts R. Bryant gave a general classification of Willmore immersions as follows.

1. $\mathcal{U}_X = U$. Then X parametrizes a round sphere in S^3. Indeed in this case we have $d^- = 0$ which by (43) implies that $b^0 \wedge a^- = 0$. But this equation means that $h = 0$ because of (35). Hence $e_3 = \gamma$ is a constant map and X lies in the sphere determined by this constant.

2. $\mathcal{U}_X \neq U$. We can construct a section $e = \epsilon.F$ of the bundle $\mathcal{F}^{(3)}_{\mathcal{N}_X}$ over the dense open subset $\mathcal{N}_X$. In particular we have $b^0 = 0$. Moreover d^- does not vanish on $\mathcal{N}_X$.

(i) $\mathcal{Q}_X = 0$. This implies that $(d^-)^2 a^+ b^+ = 0$ on $\mathcal{N}_X$, so that $b^+ = 0$ on $\mathcal{N}_X$. Hence if $F_\circ$ is the ECWI associated to F, we have

$$F_\lambda^{-1}.dF_\lambda = \lambda^{-1}a^+A_+ + (c^+C_+ + c^-C_- + d^+D_+ + d^-D_-) + \lambda a^-A_-.$$

Let $\mu_\circ$ be the Weierstrass data associated to $F_\circ$. It follows from Theorem 9 that $\mu_\circ$ should be of the form

$$\mu_\lambda = \lambda^{-1} \begin{pmatrix} 0 & 0 & 0 \\ l & 0 & 0 \\ 0 & {}^t l & 0 \end{pmatrix} dz$$

with $l \neq 0$, $(l,l) = 0$. Since μ_λ is spanned by A_+,A_-,A_0 and these 3 matrices commute, the integration of $\mu_\circ$ is relatively easy, We let $L : U \longrightarrow \mathbb{C}^3$ be defined by

$$L(z) = \int_p^z l(v)dv.$$

Then the solution of the system $dF_\circ^- = F_\circ^-.\mu_\circ$ and $F_\circ^-(p) = \mathbb{1}$ is precisely

$$F_\lambda^- = \begin{pmatrix} 1 & 0 & 0 \\ \lambda^{-1}L & \mathbb{1} & 0 \\ \frac{1}{2}\lambda^{-2}\, {}^tL.L & \lambda^{-1}\, {}^tL & 1 \end{pmatrix}.$$

We let

$$F_\lambda^+ = \begin{pmatrix} 1 & 0 & 0 \\ \lambda\overline{L} & \mathbb{1} & 0 \\ \frac{1}{2}(\lambda^2\, {}^t\overline{L}.\overline{L}) & \lambda^t\, \overline{L} & 1 \end{pmatrix},$$

and we remark that $F_\lambda^+ \in L^+\mathfrak{G}_\sigma^{\mathbb{C}}$ is the complex conjugate of F_λ^- and that $F_\circ^+$ and $F_\circ^-$ commute. Hence their product $F_\circ^-.F_\circ^+$ takes values into $L\mathfrak{G}_\sigma$ and should coincide with $F_\circ.g$, where g is some constant in $\mathfrak{G}$. We will assume for simplicity that this constant is $\mathbb{1}$. A computation shows that

$$F_\lambda = F_\lambda^-.F_\lambda^+ = \begin{pmatrix} 1 & 0 & 0 \\ \lambda^{-1}L + \lambda\overline{L} & \mathbb{1} & 0 \\ \frac{1}{2}(\lambda^{-2t}L.L) + \lambda^2\ \frac{{}^t\overline{L}.\overline{L}}{2} & {}^t(\lambda^{-1}L + \lambda\overline{L}) & 1 \end{pmatrix}.$$

Thus $e_0 = \begin{pmatrix} 1 \\ L + \overline{L} \\ \frac{1}{2}\ {}^t(L+\overline{L}).(L+\overline{L}) \end{pmatrix}$. We recover the Weierstrass representation of R. Bryant, related to the classical Weierstrass representation used for minimal surfaces in $\mathbb{R}^3$: by a suitable stereographic projection $S^3 \longrightarrow \mathbb{R}^3$ the Willmore immersion is given by $L + \overline{L}$.

(ii) $\mathcal{Q}_X \neq 0$. Then the analysis is of course much more complicated since the Weierstrass data is such that $mdz + \gamma ld\bar{z}$ does not vanish. Assuming that $b^0 = 0$ we have for the ECWI $F_\circ$ associated to F:

$$\begin{aligned} F_\lambda^{-1}.dF_\lambda =& \lambda^{-1}(a^+A_+ + b^+B_+) \\ &+ (c^+C_+ + c^-C_- + d^+D_+ + d^-D_-) \\ &+ \lambda(a^-A_- + b^-B_+-) \end{aligned}$$

and $b^+(\frac{\partial}{\partial \bar{z}}) = 0$ (see Remark 2). From Theorem 9 we deduce that on $\mathcal{N}_X$

$$\mu_\lambda = \lambda^{-1}\begin{pmatrix} 0 & {}^t(\nu l) & 0 \\ l & 0 & \nu l \\ 0 & {}^t l & 0 \end{pmatrix} dz,$$

for $l \neq 0$, $(l,l) = 0$ and $\nu \in \mathbb{C}$. Of course in this situation ν is not identically equal to 0.

Some particular classes of Willmore surfaces are obtained from minimal surfaces in three-dimensional spaces of constant curvature (the sphere S^3, the Euclidean space $\mathbb{R}^3$, the hyperbolic space $\mathbb{H}^3$). It relies on the fact that all these spaces are locally conformally equivalent, and any minimal surface in such a space is a Willmore surface. We have already seen above the Willmore surfaces which are obtained from minimal surfaces on $\mathbb{R}^3$. Let us see the general case.

Any three-dimensional space of constant curvature $\mathcal{M}$ can be embedded isometrically as the intersection of the half lightcone $\mathcal{C}^+$ in $\mathbb{R}^{4,1}$ with some affine hyperplane $H_{c,t} = \{v \in \mathbb{R}^{4,1}/\langle c, v\rangle = t\}$, where c is some fixed vector different from 0, and t is a positive real number. If c is timelike then $\mathcal{M}$ is isometric to a sphere, if c is in the light cone $\mathcal{M}$ is isometric to $\mathbb{R}^3$, and if c is spacelike $\mathcal{M}$ is isometric to the hyperbolic ball. Let X be a conformal Willmore immersion, and F be a section of $\mathcal{F}_X^{(1)}$. Let us assume that X can be obtained from a minimal surface in some space $\mathcal{C}^+ \cap H_{c,t}$. Then first of all it is possible to choose F in such a way that e_0 lies in $\mathcal{C}^+ \cap H_{c,t}$. The relation $\langle e_0, c\rangle = t$ implies $d\langle e_0, c\rangle = 0$ from which we get $\langle e_1, c\rangle = \langle e_2, c\rangle = 0$, i.e., $e_1, e_2 \in c^\perp$. We may then choose e_3 such that (e_1, e_2, e_3) is an orthonormal basis of $c^\perp \cap e_0^\perp$, the tangent plane to $\mathcal{C}^+ \cap H_{c,t}$ at e_0.

Let us exploit the fact that e_0 is a conformal harmonic map into $\mathcal{C}^+ \cap H_{c,t}$. We remark that $\mathcal{C}^+ \cap H_{c,t} \simeq \mathfrak{G}_c/\mathfrak{K}_c$ where $\mathfrak{G}_c := \{g \in \mathfrak{G}/g(c) = g\}$ and $\mathfrak{K}_c = \mathfrak{G}_c \cap \mathfrak{K}$ (here if $c = \sum_{i=0}^4 e_i c^i$, we pose $g(c) = \sum_{i=0}^4 e_i c^i$ for

$e = \epsilon.g$). Then we use the theory of [7] to split $\omega = F^{-1}.dF$ according to the decomposition of the Lie algebra of $\mathfrak{G}_c$, $\mathfrak{g}_c = \mathfrak{k}_c \oplus \mathfrak{p}_c$ as $\omega = \alpha_0 + \alpha_1$ with $\alpha_0 \in \mathfrak{k}_c$ and $\alpha_1 \in \mathfrak{p}_c$. Furthermore we denote $\alpha_1' = \alpha_1(\frac{\partial}{\partial z})dz$ and $\alpha_1'' = \alpha_1(\frac{\partial}{\partial \overline{z}})d\overline{z}$. Now the Maurer-Cartan form

$$\omega_\lambda = \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1''$$

is a curvature free connection for all $\lambda \in S^1$. But this formulation coincides with the Willmore formulation given in Theorem 2. A first consequence is that F is a section of $\mathcal{F}_X^{(\gamma)}$ (i.e., e_3 is the conformal Gauss map of X).

Let us characterize more precisely the Lie algebra of the subgroup $\mathfrak{G}_c$. Since for some base point $p \in U$ we have $F(p) = 1\!\!1$, it implies that ϵ_1, ϵ_2 and ϵ_3 are contained in $c^\perp$. Hence $c = \epsilon_1 c^1 + \epsilon_4 c^4$. Notice that $c^4 = -t$ does not vanish. Thus we have

$$\mathfrak{g}_c = \{\xi \in \mathfrak{g} / \xi_0^i c^0 + \xi_4^i c^4 = 0, \text{ for all } i\},$$

and therefore $\omega_0^i c^0 + \omega_4^i c^4 = 0$ or equivalentely

$$\omega_0^0 = 0, \ b^+ = -\frac{c^0}{c^4}a^+ , \qquad b^0 = 0.$$

Lastly we can build the meromorphic potential of F_λ using our theory or the theory of [7] with the subgroup $\mathfrak{G}_c$. Because of the uniqueness of the Weierstrass data both contructions coincide and we deduce that the meromorphic potential takes its values into $\mathfrak{p}_c^{\mathbb{C}}$. Thus we conclude

Theorem 10. *A conformal Willmore immersion X derivates from a minimal surfaces in S^3, $\mathbb{R}^3$ or $\mathbb{H}^3$ if and only if there exists an ECWI $F_\circ$ lifting X, the Weierstrass data of which is $(ldz, \nu ldz)$ where ν is a real constant ($\nu = -\frac{c^0}{c^4}$). Moreover the case $\nu > 0$ corresponds to a minimal surface in $\mathbb{H}^3$, the case $\nu = 0$ to a minimal surface in $\mathbb{R}^3$ and the case $\nu < 0$ to a minimal surface in S^3.*

5.2. The S^1 action

As pointed out in [21] in the case of harmonic mappings, the loop group formulation reveals in a straightforward way an action of the circle on the set of conformal Willmore immersions of a simply connected domain U. This is simply done by observing that for each $\lambda \in S^1$, $[(\epsilon.F_\lambda)_0]$ is a conformal Willmore immersion which is in general different from $[(\epsilon.F_1)_0]$. If $X = [(\epsilon.F_1)_0]$ we will denote $\lambda\sharp X = [(\epsilon.F_\lambda)_0]$.

This action is not trivial in general. In the case where $\mathcal{Q}_X = 0$ and X is not totally umbilic, one sees easily in the light of the previous section that X is basically a minimal surface of $\mathbb{R}^3$ and the circle action coincides with the classical one. In the case where $\mathcal{Q}_X \neq 0$, one observes that

$$\mathcal{Q}_{\lambda\sharp X} = \lambda^{-2}\mathcal{Q}_X.$$

According to Theorem 2, one may construct an action of the torus $S^1 \times S^1$ on the set of conformal Willmore immersions by the following. Let us denote $F_{\lambda,\mu} : U \longrightarrow \mathfrak{G}$ the solution of

$$\begin{array}{rcl} dF_{\lambda,\mu} &=& F_{\lambda,\mu}.\omega_{\lambda,\mu} \quad \text{on } U, \\ F_{\lambda,\mu}(p) &=& 1\!\!1, \end{array}$$

for all $\lambda, \mu \in S^1$. If $X = [(\epsilon.F_{1,1})_0]$ we denote $(\lambda,\mu)\sharp X = [(\epsilon.F_{\lambda,\mu})_0]$. One may believe that this action generates a two-parameters family of nonisometric Willmore immersions. This is not true for the following reason. If we denote

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 & 0 \\ 0 & \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ for } \theta \in \mathbb{R},$$

then we observe that $R_{-\theta}.\omega_{\lambda,\mu}.R_\theta = \omega_{\lambda e^{i\theta},\mu e^{-i\theta}}$, which implies that $F_{\lambda e^{i\theta},\mu e^{-i\theta}} = R_{-\theta}.F_{\lambda,\mu}.R_\theta$. Thus for $\mu = e^{i\theta}$, $(\lambda,\mu)\sharp X = R_{-\theta}.(\lambda\mu\sharp X)$ and the torus action reduces to a trivial circle action combined with the other one, which is not trivial.

5.3. The Willmore torus

As an example let us study briefly the Willmore torus in the light of our theory. A conformal parametrisation of this tori is given by the following biperiodic mapping from $\mathbb{R}^2$ to S^3, $X(x,y) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos x \\ \sin x \\ \cos y \\ \sin y \end{pmatrix}$.

A section of $\mathcal{F}_X$ is given by $e = \epsilon.F^0.F$ where

$$F^0.F = \frac{1}{2}\begin{pmatrix} 1+\frac{1}{\sqrt{2}}\cos x & -\sqrt{2}\sin x & 0 & \cos x & 1-\frac{1}{\sqrt{2}}\cos x \\ \sin x & 2\cos x & 0 & \sqrt{2}\sin x & -\sin x \\ \cos y & 0 & -2\sin y & -\sqrt{2}\cos y & -\cos y \\ \sin y & 0 & 2\cos y & -\sqrt{2}\sin y & -\sin y \\ 1-\frac{1}{\sqrt{2}}\cos x & \sqrt{2}\sin x & 0 & -\cos x & 1+\frac{1}{\sqrt{2}}\cos x \end{pmatrix}.$$

We remark that here $e_4(x,y) = e_0(x+\pi, y+\pi)$ which means that the conformal dual of $X(x,y)$ is $X(x+\pi, y+\pi)$. Assuming the base point condition $F(0,0) = \mathbb{1}$ we have

$$F := \frac{1}{4}\begin{pmatrix} 2+\cos x+\cos y & -2\sin x & -2\sin y & \sqrt{2}(\cos x-\cos y) & 2-\cos x-\cos y \\ 2\sin x & 4\cos x & 0 & 2\sqrt{2}\sin x & -2\sin x \\ 2\sin y & 0 & 4\cos y & -2\sqrt{2}\sin y & -2\sin y \\ \sqrt{2}(\cos x-\cos y) & -2\sqrt{2}\sin x & 2\sqrt{2}\sin y & 2\cos x+2\cos y & -\sqrt{2}(\cos x-\cos y) \\ 2-\cos x-\cos y & 2\sin x & 2\sin y & -\sqrt{2}(\cos x-\cos y) & 2+\cos x+\cos y \end{pmatrix}.$$

The ECWI $F_\circ$ associated to F satisfies

$$F_\lambda^{-1}.dF_\lambda = \lambda^{-1}\frac{1}{4}(A_+ - B_+)dz - \frac{\sqrt{2}}{4}(D_-dz + D_+d\overline{z}) + \lambda\frac{1}{4}(A_- - B_-)d\overline{z}.$$

We notice that

$$F_\lambda^{-1}.dF_\lambda = g'_\lambda dz + g''_\lambda d\overline{z}$$

where g'_λ is a constant in $L^-\mathfrak{g}_\sigma^{\mathbb{C}}$, and g''_λ is a constant in $L^+\mathfrak{g}_\sigma^{\mathbb{C}}$. Moreover $[g'_\lambda, g''_\lambda] = 0$. Hence we deduce that

$$F_\lambda(x,y) = e^{zg'_\lambda}.e^{\overline{z}g''_\lambda}.$$

From this equation and $e^{zg'_\lambda} \in L^-\mathfrak{G}_\sigma^{\mathbb{C}}$, it follows that the $L_\star^-\mathfrak{G}_\sigma^{\mathbb{C}}$ component of $F_\lambda(x,y)$ is $F_\lambda^-(x,y) = e^{zg'_\lambda}.e^{-zg'_\infty}$ and thus

$$\begin{aligned}\mu_\lambda(x,y) =& e^{zg'_\infty}.g'_\lambda.e^{-zg'_\infty}dz + e^{zg'_\infty}.de^{-zg'_\infty} \\ =& \lambda^{-1}\frac{1}{4}e^{zg'_\infty}\begin{pmatrix} 0 & -1 & i & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ -i & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 & 0 \end{pmatrix}.e^{-zg'_\infty}dz.\end{aligned}$$

Since

$$e^{zg'_\infty} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1-\frac{z^2}{16} & -i\frac{z^2}{16} & z\frac{\sqrt{2}}{4} & 0 \\ 0 & -i\frac{z^2}{16} & 1+\frac{z^2}{16} & iz\frac{\sqrt{2}}{4} & 0 \\ 0 & -z\frac{\sqrt{2}}{4} & -iz\frac{\sqrt{2}}{4} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

we obtain

$$\mu_\lambda = \lambda^{-1}\frac{1}{4}\begin{pmatrix} 0 & -1+\frac{z^2}{8} & i(1+\frac{z^2}{8}) & -\frac{z}{\sqrt{2}} & 0 \\ 1-\frac{z^2}{8} & 0 & 0 & 0 & -1+\frac{z^2}{8} \\ -i(1+\frac{z^2}{8}) & 0 & 0 & 0 & i(1+\frac{z^2}{8}) \\ \frac{z}{\sqrt{2}} & 0 & 0 & 0 & -\frac{z}{\sqrt{2}} \\ 0 & 1-\frac{z^2}{8} & -i(1+\frac{z^2}{8}) & \frac{z}{\sqrt{2}} & 0 \end{pmatrix}dz,$$

from which we get $l = \frac{1}{4}\begin{pmatrix} 1-\frac{z^2}{8} \\ -i(1+\frac{z^2}{8}) \\ \frac{z}{\sqrt{2}} \end{pmatrix}$ and $\nu = -1$. We remark that the "associated" minimal surface in $\mathbb{R}^3$ (with l as classical Weierstrass data) is the Enneper surface.

Lastly we may also write $F_\lambda^{-1}.dF_\lambda = h_\lambda^1 dx + h_\lambda^2 dy$ where h_λ^1 and h_λ^2 are constants in $L\mathfrak{G}_\sigma$ such that $[h_\lambda^1, h_\lambda^2] = 0$, so that $F_\lambda = e^{xh_\lambda^1}.e^{yh_\lambda^1}$. The eigenvalues of h_λ^1 and h_λ^2 are the same and are $\{0, \pm i\sin\frac{\theta}{2}, \pm i\cos\frac{\theta}{2}\}$ for $\lambda = e^{i\theta}$. Thus $\lambda\sharp X$ will be periodic (and hence gives rise to an immersion of a torus) if and only if $\mathrm{tg}\frac{\theta}{2} \in \mathbb{Q}$.

6. Appendix: the Grassmannian model

We refer to [17] for a complete exposition. We let $H^{(n)} = L^2(S^1, \mathbb{C}^n)$. An element of $H^{(n)}$ will be denoted by $v_\circ$, and its value at some $\lambda \in S^1$ by v_λ. We remark that $LGL(n,\mathbb{C})$ acts on $H^{(n)}$ by the following rule: for $g_\circ \in LGL(n,\mathbb{C})$, $g_\circ.v_\circ$ takes the value $g_\lambda.v_\lambda$ at λ. This defines a continuous endomorphism on $H^{(n)}$ since the loops are bounded. Each element $v_\circ$ in $H^{(n)}$ can be written as a Fourier series $v_\lambda = \sum_{k\in\mathbb{Z}}(v_\circ)_k\lambda^k$, where $(v_\circ)_k \in \mathbb{C}^n$. This leads to the decomposition

$$H^{(n)} = H_+^{(n)} \oplus H_-^{(n)},$$

where

$$H_+^{(n)} = \{\sum_{k\geq 0}(v_\circ)_k\lambda^k \in H^{(n)}\}$$

and

$$H_-^{(n)} = \{\sum_{k<0}(v_\circ)_k\lambda^k \in H^{(n)}\}$$

(see [17, 6.2]). We denote $pr_+ : H^{(n)} \longrightarrow H_+^{(n)}$ and $pr_- : H^{(n)} \longrightarrow H_-^{(n)}$ the associated projections. The Grassmannian $Gr(H^{(n)})$ is the set of vectorial subspaces of $H^{(n)}$, which are comparable to $H^{(n)}$ in the sense that

$$W \in Gr(H^{(n)}) \Longleftrightarrow \begin{cases} pr_- : W \longrightarrow H_-^{(n)} \text{ is a Hilbert-Schmidt operator} \\ pr_+ : W \longrightarrow H_+^{(n)} \text{ is a Fredholm operator} \end{cases}$$

(see [17, 7.1]). The loop group $LGL(n,\mathbb{C})$ acts on $Gr(H^{(n)})$ by the following: for each $g_\circ \in LGL(n,\mathbb{C})$ and each $W \in Gr(H^{(n)})$ we let

$g_\circ.W = \{g_\circ.v_\circ / v_\circ \in W\}$. Notice that beside the fact that an element $g_\circ$ in $LGL(n,\mathbb{C})$ should be bounded in the $L^\infty(S^1)$ topology in order to act continuously on $H^{(n)}$ (see [17, 6.1]), one needs also to verify that when acting on $Gr(H^{(n)})$ such an element should respect the above conditions on $pr_{-|W}$ and $pr_{+|W}$. This is true if and only if $g_\circ$ is bounded in the $H^{\frac{1}{2}}(S^1)$ topology (see [17, 6.3]). Thus both conditions ($g_\circ \in L^\infty(S^1)$ and $g_\circ \in H^{\frac{1}{2}}(S^1)$) are true if we assume that $g_\circ \in H^s(S^1)$, for $s > \frac{1}{2}$.

For the study of $H^{(n)}$ and its Grassmannian, it is convenient to make the identification $H^{(n)} \simeq H := L^2(S^1,\mathbb{C})$: we choose some orthogonal basis $(\epsilon_1, ..., \epsilon_n)$ of $\mathbb{C}^n$, and substitute each $v_\lambda \in H^{(n)}$ by $\langle v_{\lambda^n}, \epsilon_1 + \overline{\lambda}\epsilon_2 + ... + \overline{\lambda^{n-1}}\epsilon_n, \rangle$ (see [17, 6.5]). Important properties follow.

a) $Gr(H)$ is a complex analytical manifold. Around each W there is a canonically defined neighbourhood U_W in $Gr(H)$ which is the set of all subspaces which are graphs of Hilbert-Schmidt operators from W to $W^\perp$. On U_W the data of the unique Hilbert-Schmidt operator associated to each subspace constitutes a local chart (see [17, 7.1]).

b) Although each subspace W in $Gr(H)$ is infinite dimensional, it is possible to define the notion of the *virtual dimension* of W, which amounts roughly to the dimension of W with respect to H_+. For that purpose we recall that $pr_{+|W}$ is a Fredholm operator and hence has an index

$$ind(pr_{+|W}) = dim(\ker pr_{+|W}) - dim(\mathrm{coker}\ pr_{+|W}) \ \in \mathbb{Z}.$$

We call this integer the virtual dimension of W, and denote $Gr(H)_d$ the subset of $Gr(H)$ of subspaces with a virtual dimension equal to d. These sets are exactly the connected components of $Gr(H)$ (see [17, 7.1]).

c) On each connected component $Gr(H)_d$ we define admissible basis as families $\{w_k\}_{k\geq -d}$ with the following properties:

i) The linear operator $w : \lambda^{-d}H_+ \longrightarrow W$, $\lambda^k \longmapsto w_k$ is a continuous isomorphism.

ii) If pr_d denotes the orthogonal projection onto $\lambda^{-d}H_+$, then the map $pr_d \circ w : \lambda^{-d}H_+ \longmapsto \lambda^{-d}H_+$ is an operator with a determinant.

For the sake of simplicity we denote w such a basis (see [17, 7.5]).

d) Special subspaces in $Gr(H)$ are obtained by the following way. We let Σ be the set of all subsets S of $\mathbb{Z}$ such that $S \setminus \mathbb{N}$ and $\mathbb{N} \setminus S$ are finite sets. The virtual dimension of S is by definition

$$\text{virt. dim.}S = \text{Card}(S \setminus \mathbb{N}) - \text{Card}(\mathbb{N} \setminus S) \ \in \mathbb{Z}.$$

To each $S \in \Sigma$ it corresponds an element H_S of $Gr(H)$ which is the subspace of H admiting $\{\lambda^s / s \in S\}$ as a Hilbertian basis. The virtual dimension of H_S is equal to virt. dim.S. Let $U_S := U_{H_S}$. Then the collection of all U_S constitutes a covering of $Gr(H)$ (see [17, 7.1]).

e) Suppose that virt. dim.W = virt. dim.S. Let pr_S be the orthogonal projection onto H_S. Then for every admissible basis w of W the map $pr_S \circ w : \lambda^{-d} H_+ \longrightarrow H_S$ admits a determinant. We denote $\pi_S(w)$ the determinant of $pr_S \circ w$. If virt. dim.$W \neq$ virt. dim.S, we let $\pi_S(w) = 0$. The collection of $\{\pi_S(w)/S \in \Sigma\}$ characterizes completely W. Moreover if w' is another admissible basis of W, then there exists some $\gamma \in \mathbb{C}^\star$ such that $\pi_S(w') = \gamma \pi_S(w), \forall S \in \Sigma$. Thus it constitutes projective coordinates. Let $\mathcal{H} = \ell^2(\Sigma)$ and $\mathbb{P}(\mathcal{H})$ be the set of complex vectorial lines in $\mathcal{H}$. We then have a smooth map $\pi : Gr(H) \longrightarrow \mathbb{P}(\mathcal{H})$ which associates to each W the collection of all $\pi_S(w)$'s for some admissible basis w and up to some complex multiplicative factor. We denote $\pi_S(W)$ the projective coordinates (see [17, 7.5]).

Lemma 8 *(see [17, 7.5]). The map $\pi : Gr(H^{(n)}) \longrightarrow \mathbb{P}(\mathcal{H})$ is a holomorphic embedding.*

Moreover we have

Lemma 9 *(see [17, 7.5]). For all $S \in \Sigma$, $W \in U_S \Leftrightarrow \pi_S(W) \neq 0$.*

For instance $U_{\mathbb{N}}$ is the subset of elements $W \in Gr(H)$ such that $\pi_{\mathbb{N}}(W) \neq 0$. Using more geometrical terms, if we denote $\mathcal{K}$ to be the closed hyperplane of $\mathcal{H}$ of equation $\pi_{\mathbb{N}} = 0$, then the projective space $\mathbb{P}(\mathcal{K})$ is a closed projective hyperplane of $\mathbb{P}(\mathcal{H})$, and $\pi^{-1}(\mathbb{P}(\mathcal{K}))$ is a complex analytical submanifold of codimension one in $Gr(H)$. Further, $U_{\mathbb{N}}$ is precisely the complementary of that submanifold in $Gr(H)_0$.

Let us now go back to $H^{(n)}$ and loop groups. A characterisation of the *big cell* in $LGL(n, \mathbb{C})$ is that it is the same as

$$\{g_\circ \in LGL(n, \mathbb{C}) / g_\circ . H_+^{(n)} \in U_{\mathbb{N}}\}$$

(see [17, 8.4]). Moreover for any Lie subgroup $\mathfrak{A}^{\mathbb{C}}$ it is clear that the corresponding big cell is obtained by the intersection of the above set with $L\mathfrak{A}^{\mathbb{C}}$.

Since the map $g_\circ \longmapsto g_\circ.H_+^{(n)}$ is holomorphic, it follows that the big cell in $L\mathfrak{A}^{\mathbb{C}}$ is the complementary of some closed complex analytical hypersurface in $Gr(H^{(n)})$. This illustrates Theorem 7 and Lemma 6.

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