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CONTRACTION OF CONVEX HYPERSURFACES BY THEIR AFFINE NORMAL

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Abstract

An affine-invariant evolution equation for convex hypersurfaces in Euclidean space is defined by assigning to each point a velocity equal to the affine normal vector. For an arbitrary compact, smooth, strictly convex initial hypersurface, it is shown that this deformation produces a unique, smooth family of convex hypersurfaces, which converge to a point in finite time. Furthermore, the hypersurfaces converge smoothly to an ellipsoid after rescaling about the final point to make the enclosed volume constant. The result leads to simple proofs of some affine-geometric isoperimetric inequalities.

1. Introduction

Consider a smooth, strictly convex hypersurface in Euclidean space $\mathbf{R}^{n+1}$ given by a smooth embedding $\varphi_0 : S^n \to \mathbf{R}^{n+1}$, where S^n is the unit sphere in $\mathbf{R}^{n+1}$. We consider the evolution of such an embedding to produce a family of embeddings $\varphi : S^n \times [0,T) \to \mathbf{R}^{n+1}$ satisfying the following equation:

$$(1.1)\qquad \begin{aligned} \frac{\partial}{\partial t}\varphi(z,t) &= \mathcal{N}(z,t) \\ \varphi(z,0) &= \varphi_0(z) \end{aligned}$$

for all z in S^n and t in $[0,T)$. Here $\mathcal{N}$ is the affine normal vector. This equation is the unique second order parabolic evolution equation for hypersurfaces in Euclidean space which is invariant under affine transformations of $\mathbf{R}^{n+1}$: Suppose $\varphi : S^n \otimes [0,T) \to \mathbf{R}^{n+1}$ is a solution of Equation (1.1), and $L : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is an affine transformation with the modulus of its determinant $|L| > 0$. Then the family of embeddings $\varphi_L : S^n \otimes [0, |L|^{\frac{2}{n+2}}T) \to \mathbf{R}^{n+1}$ given by $\varphi_L(z,t) = L \circ \varphi(z, |L|^{-\frac{2}{n+2}}t)$ is a solution to Equation (1.1) with initial condition $(\varphi_L)_0 = L \circ \varphi_0$.

Note that the round sphere in $\mathbf{R}^{n+1}$ is a homothetic solution of Equation (1.1): If we have $\varphi_0(z) = r_0 z$ as initial condition, then the solution

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is given by $\varphi_t(z) = r_t z$, where $r_t = \left(r_0^{\frac{2(n+1)}{n+2}} - \frac{2(n+1)}{n+2}t\right)^{\frac{n+2}{2(n+1)}}$. The affine invariance then gives the result that *every ellipsoid evolves homothetically* under Equation (1.1). More generally, the homothetic solutions of (1.1) are precisely those hypersurfaces for which the affine normal vector is proportional to the position vector. These are called *affine hyperspheres*, and have been studied extensively (see [10],[15]). In particular, elliptic affine hyperspheres are those for which the affine normals are converging; these are ellipsoids, and give contracting homothetic solutions of (1.1). Parabolic affine hyperspheres have parallel affine normals. These are paraboloids, and give rise to translating homothetic solutions. Hyperbolic affine hyperspheres have diverging affine normals, and give expanding homothetic solutions. These are in one-to-one correspondence, up to scaling, with the set of convex cones in $\mathbf{R}^{n+1}$, where a hypersphere corresponds to its asymptotic cone at infinity — hyperboloids are particular examples.

The Equation (1.1) can be rewritten in a more familiar form (for those familiar with curvature flows of hypersurfaces): After including an additional tangential diffeomorphism, the evolution equation becomes the following:

$$\begin{aligned} &\frac{\partial}{\partial t}\varphi(z,t) = -K(z,t)^{\frac{1}{n+2}}\nu(z,t) \\ &\varphi(z,p) = \varphi_0(z) \end{aligned} \tag{1.2}$$

where K is the Gauss curvature. This is therefore one of the Gauss curvature flows — a class that has been considered before by Chow [11]. It is proved in [11] that there exists a unique, smooth solution to Equation (1.2), which converges uniformly to a point as the final time is approached. Chow has also proved a Harnack inequality for this equation (see [13]), following a similar estimate for the mean curvature flow by Hamilton [25]. Entropy estimates, which are bounds on certain integral quantities, were proved by the author in [5] for a class of flows including Equation (1.2).

The main result of this paper is following:

Theorem 1.3. *Let $\varphi_0 : S^n \to \mathbf{R}^{n+1}$ be a smooth, strictly convex embedding. Then there is a unique, smooth solution of Equation* (1.1) *which converges to a point in finite time. After rescaling about the final point to make the enclosed volume constant, the solution converges in C^∞ to an ellipsoid.*

This should be compared with various other results which are known for evolving convex hypersurfaces: The model result is due to Huisken

[28], who proved that the flow by mean curvature contracts compact convex hypersurfaces in $\mathbf{R}^{n+1}, n \geq 2$ to points, and makes them spherical in the process. The analogous result for curves ($n = 1$) was proved by Gage and Hamilton (see [18], [19], [20]), and extended to the non-convex case by Grayson [22]. Huisken's methods have been applied to certain other flows (see [11], [12], [2]), all of which have speeds that are homogeneous of degree one in the principal curvatures. Some results have been obtained for flows in which the speed has a different degree of homogeneity (see [17], [33], [11], [1], [3], [5], [21], [29], [34], [35]). For positive degrees of homogeneity, it can be shown for a wide variety of flows that the solutions converge to points in finite time. However, little is known about the asymptotic shape of the solutions. It was proved in [5] that for small positive degrees of homogeneity, the rescaled solutions do not always converge to spheres; it seems reasonable to conjecture that in general the solutions of such equations do not converge at all after rescaling, but degenerate onto lines or hyperplanes. The result proved here therefore represents a critical case — we expect that the Gauss curvature flows with larger exponents will always converge to spheres, while those with smaller exponents will diverge for generic initial data.

The paper is arranged as follows: In section 2, the required notation is introduced, and various preliminary results are summarised. In sections 3 and 4, we obtain estimates on the cubic ground form and the affine curvature. The first of these is related to an estimate proved by Calabi in [10] for affine hyperspheres; the second is a more powerful version of the Harnack estimate previously obtained by Chow [13]. Section 5 provides uniform estimates for the solutions after rescaling, and after suitable affine transformations. In section 6, we complete the proof of Theorem (1.3) by deducing the convergence of the solutions after rescaling. We conclude in section 7 by using the main result to prove some affine-geometric isoperimetric inequalities.

2. Notation and preliminary results

Let M^n be a compact strictly convex hypersurface in $\mathbf{R}^{n+1}$. We can describe M in terms of its support function $s : S^n \to \mathbf{R}$ defined by $s(z) = \sup\{\langle z, y\rangle : y \in M^n\}$: If M is smooth, then it is given as the image of the embedding $\bar{\varphi} : S^n \to \mathbf{R}^{n+1}$ defined by $\bar{\varphi}(z) = s(z)z + \bar{\nabla} s(z)$, where $\bar{\nabla}$ is the connection on S^n coming from the standard metric $\bar{g}$. The curvature of M is given by the identity $\mathcal{W}^{-1} = \bar{g}^* \operatorname{Hess}_{\bar{\nabla}} s + \operatorname{Id} s$, where $\mathcal{W}$ is the Weingarten map of M. For convenience we define for any function f on S^n the bilinear form $A[f] = \operatorname{Hess}_{\bar{\nabla}} f + \bar{g} f$. This gives in particular $A[s](u,v) = \bar{g}(\mathcal{W}^{-1}(u),v)$. Full details of these formulae and other useful expressions involving the support function may be found in [3].

The affine differential geometry of hypersurfaces is the study of those properties of hypersurfaces which are invariant under special affine transformations of Euclidean space. A remarkable fact is that the affine metric, defined by $\tilde{g} = K^{-1/(n+2)}A[s]$ is such an invariant. If M is strictly convex, then $\tilde{g}$ is indeed a metric, and we deduce the existence of an affine-invariant connection $\tilde{\nabla}$, and higher invariants.

The affine normal is defined by $\mathcal{N} = \frac{1}{n}\tilde{\Delta}\varphi$, where $\tilde{\Delta}$ is the Laplace-Beltrami operator of the metric $\tilde{g}$ and φ is the inclusion map of M into $\mathbf{R}^{n+1}$. Having defined this, we have natural normal and tangential projections at each point of the hypersurface, and we define the affine curvature $\mathcal{A}$ and the cubic ground form $\mathcal{C}$ by the following equations:

$$
\begin{aligned}
&D_{T\varphi(u)}T\varphi(v) = \tilde{g}(u,v)\mathcal{N} + T\varphi\Big(\tilde{\nabla}_u v + \mathcal{C}(u,v)\Big) \\
&D_u\mathcal{N} = -T\varphi(\mathcal{A}(u))
\end{aligned} \tag{2.1}
$$

for every u and v in TM, where $T\varphi$ is the tangent mapping of φ. Thus $\mathcal{A}$ is in $TM \otimes T^*M$, and $\mathcal{C}$ is in $T^*M \otimes T^*M \otimes TM$.

By differentiating these equations we obtain a series of structure conditions: First, the affine curvature $\mathcal{A}$ is symmetric and satisfies a kind of Codazzi equation:

$$
\begin{aligned}
&\tilde{g}(\mathcal{A}(u),v) = \tilde{g}(u,\mathcal{A}(v)) \\
&\tilde{\nabla}\mathcal{A}(w,u) + \mathcal{C}(\mathcal{A}(u),w) = \tilde{\nabla}\mathcal{A}(u,w) + \mathcal{C}(\mathcal{A}(w),u).
\end{aligned} \tag{2.2}
$$

The cubic ground form is totally symmetric, trace-free, and also satisfies a kind of Codazzi identity:

(2.3)
$$
\begin{aligned}
&\mathcal{C}(u,v) = \mathcal{C}(v,u); \quad \tilde{g}(\mathcal{C}(u,v),w) = \tilde{g}(\mathcal{C}(w,v),u); \quad \mathrm{tr}_{\tilde{g}}\,\mathcal{C} = 0; \\
&\tilde{\nabla}\mathcal{C}(u,v,w) - \tilde{\nabla}\mathcal{C}(v,u,w) = \frac{1}{2}\tilde{g}(v,w)\mathcal{A}(u) + \frac{1}{2}\tilde{g}(\mathcal{A}(u),w)v \\
&\qquad\qquad - \frac{1}{2}\tilde{g}(u,w)\mathcal{A}(v) - \frac{1}{2}\tilde{g}(\mathcal{A}(v),w)u.
\end{aligned}
$$

In view of the symmetries of $\mathcal{A}$ and $\mathcal{C}$, we will write $\mathcal{A}(u,v) = \tilde{g}(\mathcal{A}(u),v)$ and $\mathcal{C}(u,v,w) = \tilde{g}(\mathcal{C}(u,v),w)$. Finally, the curvature $\tilde{R}$ of the metric $\tilde{g}$

has the following expression:

$$\tilde{R}(u,v)w = \frac{1}{2}\tilde{g}(u,w)\mathcal{A}(v) - \frac{1}{2}\tilde{g}(v,w)\mathcal{A}(u) + \frac{1}{2}\mathcal{A}(u,w)v - \frac{1}{2}\mathcal{A}(v,w)u + \mathcal{C}(u,\mathcal{C}(v,w)) - \mathcal{C}(v,\mathcal{C}(u,w)) \tag{2.4}$$

These identities hold for all u, v and w in TM. A complete derivation of these structure equations may be found in [10] or [15].

The following expressions give the affine invariants in terms of the support function, using the metric $\bar{g}$ and connection $\bar{\nabla}$ on S^n:

(2.5)

$$\begin{aligned}
\tilde{g}(u,v) &= \frac{A[s](u,v)}{K^{\frac{1}{n+2}}}\\
\mathcal{N}(z) &= -K^{\frac{1}{n+2}}z - \bar{\nabla}(K^{\frac{1}{n+2}})\\
\mathcal{A}(u) &= A[s]^{-1}\circ A[K^{\frac{1}{n+2}}]\\
\mathcal{C}(u,v,w) &= \frac{1}{2}K^{-\frac{1}{n+2}}\bar{\nabla}A[s](u,v,w) - \frac{1}{2}A[s](u,w)\bar{\nabla}K^{-\frac{1}{n+2}}(v)\\
&\quad - \frac{1}{2}A[s](v,w)\bar{\nabla}K^{-\frac{1}{n+2}}(u) - \frac{1}{2}A[s](u,v)\bar{\nabla}K^{-\frac{1}{n+2}}(w)
\end{aligned}$$

The evolution Equation (1.1) can be written as an evolution equation for the support function s by using the expression for $\mathcal{N}$ given above:

$$\frac{\partial}{\partial t}s(z,t) = -K(z,t)^{\frac{1}{n+2}} \tag{2.6}$$

The evolution equations for the curvature $A[s]$ and other quantities follow from this:

$$\frac{\partial}{\partial t}A[s] = -A[K^{\frac{1}{n+2}}] = -K^{\frac{1}{n+2}}\mathcal{A} \tag{2.7}$$

$$\frac{\partial}{\partial t}K^{\frac{1}{n+2}} = \frac{1}{n+2}K^{\frac{1}{n+2}}\mathcal{H} \tag{2.8}$$

where $\mathcal{H} = \operatorname{tr}\mathcal{A}$ is the affine mean curvature.

An entropy estimate holds for Equation (1.1), as proved in [5]:

Theorem 2.9. *Let φ be a strictly convex, smooth solution to Equation* (1.1). *Then the following estimate holds*:

$$\frac{\partial}{\partial t}\left(V^{-\frac{n}{n+2}}\int_{S^n} K^{-\frac{n+1}{n+2}}d\mu\right) \geq 0$$

where the inequality is strict unless φ is a homothetic solution of the equation. Here V is the enclosed volume of the hypersurface $\varphi(S^n)$.

Note that the entropy integral is just the volume of M with respect to the affine metric, normalised by the enclosed volume. Thus Theorem (2.9) says that the affine isoperimetric ratio improves.

3. The estimate on the cubic ground form

We begin by computing evolution equations for some affine-geometric quantities:

Lemma 3.1. *Under Equation* (1.1), *the following evolution equations hold*:

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{g}(u,v) &= -\frac{1}{n+2}\mathcal{H}\tilde{g}(u,v) - \mathcal{A}(u,v)\\
\frac{\partial}{\partial t}\mathcal{C}(u,v,w) &= -\frac{1}{2}\left(\tilde{\nabla}\mathcal{A}(u,v,w) - \mathcal{C}(\mathcal{A}(u),v,w)\right) - \frac{\mathcal{H}}{n+2}\mathcal{C}(u,v,w)\\
&\quad - \frac{1}{2}\left(\mathcal{C}(u,v,\mathcal{A}(w)) + \mathcal{C}(u,\mathcal{A}(v),w) + \mathcal{C}(\mathcal{A}(u),v,w)\right)\\
&\quad + \frac{1}{2(n+2)}\left(\tilde{g}(u,v)\tilde{\nabla}_w\mathcal{H} + \tilde{g}(u,w)\tilde{\nabla}_v\mathcal{H} + \tilde{g}(v,w)\tilde{\nabla}_u\mathcal{H}\right).
\end{aligned}$$

These expressions are easily derived from the formulae (2.5) and the evolution Equations (2.7) and (2.8).

The next step is to express the evolution of $\mathcal{C}$ in a parabolic form, with an elliptic operator as the leading term. For this reason we require the following result, which gives an expression for the Laplace-Beltrami operator of the cubic ground form. This is a generalisation of an estimate obtained by Calabi [10] for affine hyperspheres.

Theorem 3.2. *The following identity holds*:

$$\begin{aligned}
\tilde{\Delta}\mathcal{C}(u,v,w) &= -\frac{n+2}{2}\left(\tilde{\nabla}\mathcal{A}(u,v,w) - \mathcal{C}(\mathcal{A}(u),v,w)\right) + \frac{\mathcal{H}}{2}\mathcal{C}(u,v,w)\\
&\quad + \frac{1}{2}\left(\tilde{g}(u,v)\tilde{\nabla}_w\mathcal{H} + \tilde{g}(u,w)\tilde{\nabla}_v\mathcal{H} + \tilde{g}(v,w)\tilde{\nabla}_u\mathcal{H}\right)\\
&\quad + \mathcal{P}(u,\mathcal{C}(v,w)) + \mathcal{P}(w,\mathcal{C}(u,v)) + \mathcal{P}(v,\mathcal{C}(u,w))\\
&\quad - 2\mathcal{Q}(u,v,w).
\end{aligned}$$

Here $\mathcal{P}(u,v) = \mathrm{tr}(\mathcal{C}(u)\circ\mathcal{C}(v))$, *and* $\mathcal{Q}(u,v,w) = \mathrm{tr}(\mathcal{C}(u)\circ\mathcal{C}(v)\circ\mathcal{C}(w))$.

Proof. We begin with the identity

$$\mathrm{tr}\,\tilde{\nabla}_u\tilde{\nabla}_v\mathcal{C}(w) = 0$$

which follows since $\mathcal{C}$ is trace-free. Now apply the Codazzi Equation (2.3) for the cubic ground form:

$$0 = \operatorname{tr} \tilde{\nabla}_u \tilde{\nabla} \mathcal{C}(v, w) - \frac{1}{2}\tilde{g}(v, w)\tilde{\nabla}_u \mathcal{H} + \frac{n}{2}\tilde{\nabla}_u \mathcal{A}(v, w).$$

The derivatives in the leading term can now be commuted, introducing a curvature term:

$$\begin{aligned} 0 = &\operatorname{tr} \tilde{\nabla}\tilde{\nabla}_u \mathcal{C}(v, w) + \tilde{\operatorname{Ric}}(u, \mathcal{C}(v, w)) \\ &- \frac{1}{2}\tilde{g}(v, w)\tilde{\nabla}_u \mathcal{H} + \frac{n}{2}\tilde{\nabla}_u \mathcal{A}(v, w). \end{aligned}$$

where $\tilde{\operatorname{Ric}}$ is the Ricci tensor of the metric $\tilde{g}$. Now we can apply the Codazzi identity for $\mathcal{C}$ again to turn the leading term into a Laplace-Beltrami operator:

$$\begin{aligned} 0 = &\tilde{\Delta}\mathcal{C}(u, v, w) + \tilde{\operatorname{Ric}}(u, \mathcal{C}(v, w)) + \frac{n}{2}\tilde{\nabla}_u \mathcal{A}(v, w) \\ &+ \frac{1}{2}\tilde{\nabla}_u \mathcal{A}(v, w) + \frac{1}{2}\tilde{\nabla}_w \mathcal{A}(u, v) - \frac{1}{2}\tilde{g}(v, w)\tilde{\nabla}_u \mathcal{H} \\ &- \frac{1}{2}\tilde{g}(u, w)\operatorname{tr}\tilde{\nabla}\mathcal{A}(v) - \frac{1}{2}\tilde{g}(u, v)\operatorname{tr}\tilde{\nabla}\mathcal{A}(w). \end{aligned}$$

The result now follows by substituting the expression (2.4) for the curvature, and using the affine curvature Codazzi identity (2.2).

Lemma 3.3. *The cubic ground form $\mathcal{C}$ satisfies the following evolution equation:*

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{C}(u, v, w) = &\frac{1}{n+2}\tilde{\Delta}\mathcal{C}(u, v, w) - \frac{3\mathcal{H}}{2(n+2)}\mathcal{C}(u, v, w) + \frac{2}{n+2}\mathcal{Q}(u, v, w) \\ &- \frac{1}{2}(\mathcal{C}(\mathcal{A}(u), v, w) + \mathcal{C}(u, \mathcal{A}(v), w) + \mathcal{C}(u, v, \mathcal{A}(w))) \\ &- \frac{1}{n+2}(\mathcal{P}(u, \mathcal{C}(v, w)) + (\mathcal{P}(v, \mathcal{C}(u, w)) + (\mathcal{P}(w, \mathcal{C}(u, v))). \end{aligned}$$

This result becomes particularly simple if we choose vectors u, v and w which evolve in time so as to preserve the metric:

$$\frac{\partial}{\partial t}u = \frac{1}{2(n+2)}\mathcal{H}u + \frac{1}{2}\mathcal{A}(u), \tag{3.4}$$

with similar equations for v and w. This is a useful method of removing that part of the evolution which comes from the time-dependence of the metric; such methods were developed by Hamilton in [24] and [26].

The resulting remarkably simple evolution equation for the cubic ground form is the following:

(3.5)

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{C}(u,v,w) &= \frac{1}{n+2}\tilde{\Delta}\mathcal{C}(u,v,w) + \frac{2}{n+2}\mathcal{Q}(u,v,w)\\ &\quad - \frac{1}{n+2}(\mathcal{P}(u,\mathcal{C}(v,w)) + (\mathcal{P}(v,\mathcal{C}(u,w)) + (\mathcal{P}(w,\mathcal{C}(u,v)))\end{aligned}$$

This equation involves only the cubic ground form. We can now deduce the evolution equation for the modulus of the cubic ground form:

Theorem 3.6. *The modulus of the cubic ground form evolves as follows:*

$$\begin{aligned}\frac{\partial}{\partial t}|\mathcal{C}|^2 =& \frac{1}{n+2}\tilde{\Delta}|\mathcal{C}|^2 - \frac{2}{n+2}|\tilde{\nabla}\mathcal{C}|^2\\ &+ \frac{4}{n+2}\mathcal{Z} - \frac{6}{n+2}|\mathcal{P}|^2,\end{aligned}$$

where $\mathcal{Z}$ is given by contracting $\mathcal{Q}$ with $\mathcal{C}$, and all norms are with respect to the affine metric $\tilde{g}$.

This gives the following result by the parabolic maximum principle:

Theorem 3.7. *The following holds for any solution to* (1.1):

$$\sup_{S^n\times\{t\}}|\mathcal{C}|^2 \le \left(\left(\sup_{S^n\times\{0\}}|\mathcal{C}|^2\right)^{-1} + \frac{2t}{n(n+2)}\right)^{-1}.$$

Proof. The term involving $\mathcal{Z}$ can be estimated using the following identity:

$$|\mathcal{P}|^2 - \mathcal{Z} = \frac{1}{2}|\mathcal{Y}|^2 \ge 0$$

where $\mathcal{Y}(u,v,w,z) = \tilde{g}(\mathcal{C}(u,v),\mathcal{C}(w,z)) - \tilde{g}(\mathcal{C}(u,w),\mathcal{C}(v,z))$.

We can also estimate $|\mathcal{P}|^2$ in terms of $|\mathcal{C}|^2 = \operatorname{tr}\mathcal{P}$:

$$|\mathcal{P}|^2 \ge \frac{1}{n}|\mathcal{C}|^2$$

Combining these estimates, and applying the parabolic maximum principle, we obtain the inequality:

$$\frac{d}{dt}\sup|\mathcal{C}|^2 \le -\frac{2}{n(n+2)}\sup_{S^n\times t}|\mathcal{C}|^4. \tag{3.8}$$

The result follows.

Since we wish to control the shape of the rescaled hypersurfaces with constant enclosed volume, it is important to note that the quantity

$|\mathcal{C}|^2 V(M)^{\frac{2}{n+2}}$ is scaling-invariant, and tends to zero even faster than the unrescaled quantity $|\mathcal{C}|^2$.

4. The estimate on the affine curvature

In this section we establish an estimate on the affine curvature $\mathcal{A}$. This is related to an estimate obtained by Chow, which is the key to the proof of the Harnack inequality in [13]. Indeed, in affine-geometric language Chow's estimate is a lower bound on the affine mean curvature; this may be obtained by taking a trace of the estimate obtained here for the full affine curvature tensor.

First, we compute an evolution equation for the affine curvature:

Lemma 4.1. *Under Equation* (1.1), *the affine curvature* $\mathcal{A}$ *satisfies:*

$$\frac{\partial}{\partial t}\mathcal{A}(u,v) = \frac{1}{n+2}\Big(\operatorname{Hess}_{\tilde{\nabla}}\mathcal{H}(u,v) + \tilde{\nabla}_{\mathcal{C}(u,v)}\mathcal{H} + \mathcal{H}\mathcal{A}(u,v)\Big)$$

for any tangent vectors u and v.

This follows from the expression (2.5) for the affine curvature and the evolution Equations (2.7) and (2.8). As in the previous section, we need to write this in a parabolic form. The following affine Simons' identity allows us to do this:

Lemma 4.2. *The following identity holds on any smooth, strictly convex hypersurface:*

$$\begin{aligned}\tilde{\Delta}\mathcal{A}(u,v) =& \operatorname{Hess}_{\tilde{\nabla}}\mathcal{H}(u,v) + \tilde{\nabla}_{\mathcal{C}(u,v)}\mathcal{H} + 2\operatorname{tr}\Big(\tilde{\nabla}\mathcal{C}(u,v)\circ\mathcal{A}\Big)\\ &+ \mathcal{P}(u,\mathcal{A}(v)) + \mathcal{P}(v,\mathcal{A}(u)) + 2\operatorname{tr}(\mathcal{C}(\mathcal{C}(u,v))\circ\mathcal{A})\\ &- 2\operatorname{tr}(\mathcal{C}(u)\circ\mathcal{C}(v)\circ\mathcal{A}) + n\mathcal{A}^2(u,v) - |\mathcal{A}|^2\tilde{g}(u,v)\end{aligned}$$

for any tangent vectors u and v.

Proof. First apply the Codazzi identity for the affine curvature:

$$\tilde{\Delta}\mathcal{A}(u,v) = \operatorname{tr}\Big(\tilde{\nabla}\tilde{\nabla}_u\mathcal{A}(v) + \tilde{\nabla}\mathcal{A}(\mathcal{C}(u,v)) - \tilde{\nabla}\mathcal{C}(v,\mathcal{A}(u))\Big)$$

Now commute the derivatives, introducing terms involving the curvature of the metric $\tilde{g}$:

$$\begin{aligned}\tilde{\Delta}\mathcal{A}(u,v) =& \tilde{\nabla}_u\operatorname{tr}\tilde{\nabla}\mathcal{A}(v) + \operatorname{tr}\Big[\tilde{\nabla},\tilde{\nabla}_u\Big]\mathcal{A}(v)\\ &+ \operatorname{tr}\tilde{\nabla}(\mathcal{A}(\mathcal{C}(u,v)) - \mathcal{C}(v,\mathcal{A}(u)))\end{aligned}$$

Now the Codazzi identity (2.3) can be applied again to move the trace off the derivatives, giving a leading term $\mathrm{Hess}_{\tilde{\nabla}}\,\mathcal{H}(u,v)$. The curvature terms are given by the identity (2.4), and the results are simplified slightly by applications of the Codazzi identities (2.2) and (2.3).

Theorem 4.3. *The following holds for solutions to* (1.1):

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}(u,v) =& \frac{1}{n+2}\tilde{\Delta}\mathcal{A}(u,v) - \frac{2}{n+2}\operatorname{tr}\left(\tilde{\nabla}\mathcal{C}(u,v)\circ\mathcal{A}\right)\\ &- \frac{1}{n+2}\mathcal{P}(u,\mathcal{A}(v)) - \frac{1}{n+2}\mathcal{P}(v,\mathcal{A}(u))\\ &- \frac{2}{n+2}\operatorname{tr}(\mathcal{C}(\mathcal{C}(u,v))\circ\mathcal{A}) + \frac{2}{n+2}\operatorname{tr}(\mathcal{C}(u)\circ(\mathcal{C}(v)\circ\mathcal{A})\\ &+ \frac{\mathcal{H}}{n+2}\mathcal{A}(u,v) + \frac{|\mathcal{A}|^2}{n+2}\tilde{g}(u,v) - \frac{n}{n+2}\mathcal{A}^2(u,v)\end{aligned}$$

for any tangent vectors u and v.

Note that if we consider the change in $\mathcal{A}$ with respect to vectors u and v which evolve to preserve the metric (as in Equation (3.4)) then the last three terms become

$$\frac{2}{n+2}\mathcal{A}^2(u,v) + \frac{1}{n+2}|\mathcal{A}|^2\tilde{g}(u,v) + \frac{2\mathcal{H}}{n+2}\mathcal{A}(u,v).$$

The Codazzi identity (2.2) gives an expression for the traceless part of the affine curvature in terms of the gradient of the cubic ground form, so the above terms can be bounded below by

$$\left(\frac{3n+2}{n+2} - \varepsilon\right)\mathcal{A}^2(u,v) - k_0|\tilde{\nabla}\mathcal{C}|^2\tilde{g}(u,v),$$

where k_0 is a constant depending only on n and ϵ, for any $\epsilon > 0$. The remaining terms, involving $\mathcal{C}$, $\mathcal{P}$, and $\tilde{\nabla}\mathcal{C}$, can also be bounded from below using the Cauchy-Schwartz inequality, yielding the following estimate for the evolution:

$$\frac{\partial}{\partial t}\mathcal{A}(u,v) \geq \frac{1}{n+2}\tilde{\Delta}\mathcal{A}(u,v) - k_1\left(|\tilde{\nabla}\mathcal{C}|^2 + |\mathcal{C}|^4\right)\tilde{g}(u,v)$$

where k_1 is a constant depending only on n. Now note that from section 3 we have the estimate:

$$\frac{\partial}{\partial t}|\mathcal{C}|^2 \leq \frac{1}{n+2}\tilde{\Delta}|\mathcal{C}|^2 - \frac{2}{n+2}|\tilde{\nabla}\mathcal{C}|^2 - \frac{2}{n(n+2)}|\mathcal{C}|^4.$$

Now we can combine these two estimates to give the following:

$$\frac{\partial}{\partial t}\left(\mathcal{A} - k_2|\mathcal{C}|^2\,\mathrm{Id}\right) \geq \frac{1}{n+2}\tilde{\Delta}\left(\mathcal{A} - k_2|\mathcal{C}|^2\,\mathrm{Id}\right),$$

where Id is the identity map and k_2 is a constant depending on n. We deduce the following very useful result:

Theorem 4.4. *For any solution to* (1.1), *the affine curvature* $\mathcal{A}$ *satisfies the following estimate:*

$$\inf_t \mathcal{A} \geq \inf_0 \Big(\mathcal{A} - k_2 |\mathcal{C}|^2 \,\mathrm{Id} \Big).$$

for some constant k_2 *depending only on* n.

Remarks. It is possible, in view of the bound already obtained on the size of the cubic ground form $\mathcal{C}$, to produce an even better evolution equation for $\mathcal{A}$ which has a quadratic growth term. This gives a lower bound on $\mathcal{A}$ independent of any initial bound. This stronger estimate is not necessary for the results obtained here, since Theorem (4.4) already gives a very strong lower bound on the affine curvature after rescaling the solution to constant enclosed volume.

Note also that Equation (4.1) leads directly to an evolution equation for the affine mean curvature $\mathcal{H}$:

$$\frac{\partial}{\partial t}\mathcal{H} = \frac{1}{n+2}\tilde{\Delta}\mathcal{H} + \frac{2\mathcal{H}^2}{n+2} + |\mathcal{A}|^2. \tag{4.5}$$

This equation is the key to the Harnack inequality proved by Chow. The estimate on $\mathcal{A}$ obtained above may be seen as a "matrix Harnack inequality" for the flow (1.1). Other such estimates have been obtained for the heat equation [27], the Kähler Ricci flow [14], and the Ricci flow for Riemannian metrics [26].

5. Uniform estimates

In this section we obtain uniform estimates for the solutions to the rescaled solutions, after appropriate affine transformations. The methods here rely on the estimates on the cubic ground form and the affine curvature obtained in sections 3 and 4.

We first control the Gauss curvature, using an estimate of Tso [33].

Lemma 5.1. *Let* $\varphi : S^n \times [0,T) \to \mathbf{R}^{n+1}$ *be a solution to Equation* (1.1). *Then for any* $t > 0$ *there exists an affine transformation* L *of* $\mathbf{R}^{n+1}$ *such that the image solution* $\varphi_L = L \circ \varphi_t$ *has enclosed volume* $V = V(S^n)$, *and such that the following estimate holds*:

$$\sup_{S^n} K_L(x,t) \leq C\Big(1 + t^{-\frac{n}{2(n+1)}}\Big), \tag{5.2}$$

where K_L *is the curvature of* φ_L, *and* $C = C(n)$.

Proof. Fix a time $\tau \geq 0$. We choose an affine transformation L for which the hypersurface $\varphi_L(S^n)$ is contained between the spheres about the origin with radii $\frac{1}{n+1}$ and $n+1$. To see that this is possible, consider the $(n+2)$-hedron of maximal volume enclosed by $\varphi(S^n)$, and take an affine transformation which sends this to a regular $(n+2)$-hedron centred at the origin, with vertices at distance 1 from the origin.

Since the evolving smaller sphere is a barrier, the transformed solution φ_L lies outside the sphere of radius $\frac{1}{2(n+2)}$ about the origin on the time interval $[\tau, \tau+\sigma]$, where

$$\sigma = \frac{n+2}{2(n+1)}(n+1)^{-\frac{2(n+1)}{n+2}}\left(1 - 2^{-\frac{2(n+1)}{n+2}}\right).$$

Now from the Equations (2.6) and (2.8). we deduce the evolution equation for the quantity $Q = \frac{K^{\frac{1}{n+2}}}{s - \frac{1}{4(n+1)}}$:

(5.3)
$$\begin{aligned}\frac{\partial}{\partial t}Q =& \frac{1}{n+2}K^{\frac{1}{n+2}}(A[s]^{-1})^{ij}\bar{\nabla}_i\bar{\nabla}_j Q + \frac{2Q}{n+2}(A[s]^{-1})^{ij}\bar{\nabla}_i s\bar{\nabla}_j Q\\ &- \left(\frac{H}{4(n+1)} - \frac{2(n+1)}{n+2}\right)Q^2,\end{aligned}$$

where H is the Euclidean mean curvature. The last bracket here can be estimated as follows:

$$\begin{aligned}\frac{H}{4(n+1)} - \frac{2(n+1)}{n+2} &\geq \frac{nK^{\frac{1}{n}}}{4(n+1)} - \frac{2(n+1)}{n+2}\\ &\geq \frac{n}{4(n+1)}Q^{1+\frac{2}{n}}\left(s - \frac{1}{4(n+1)}\right)^{1+\frac{2}{n}} - \frac{2(n+1)}{n+2}\\ &\geq n(4n+4)^{-2(1+\frac{1}{n})}Q^{1+\frac{2}{n}} - \frac{2(n+1)}{n+2}\\ &\geq \frac{1}{2}n(4n+4)^{-2(1+\frac{1}{n})}Q^{1+\frac{2}{n}}\end{aligned}$$

provided $Q \geq (4(n+1))^{\frac{3n+2}{n+2}}(n(n+2))^{-\frac{n}{n+2}}$. Here we have used the fact that $s - \frac{1}{4(n+1)} \geq \frac{1}{4(n+1)}$ on the time interval $[\tau, \tau+\sigma]$. The parabolic maximum principle then gives the following estimate:

$$\sup_{S^n\times\{\tau+\epsilon\}} Q \leq \max\left\{\frac{(4(n+1))^{\frac{3n+2}{n+2}}}{(n(n+2))^{\frac{n}{n+2}}}, 4(n+1)^{\frac{n+2}{2(n+2)}}\epsilon^{-\frac{n}{2(n+1)}}\right\}.$$

Since $s - \frac{1}{4(n+1)}$ is bounded above and below on the time interval we consider, this estimate also controls the Gauss curvature.

Now to prove the theorem, choose any time $t > 0$, and take τ to be the larger of $t - \sigma$ and 0. Then we can apply the estimate above with $\epsilon = \min\{\sigma, t\}$, obtaining control on the Gauss curvature of the solution φ_L at the time t, where L is chosen as above using the solution at the time τ. But now we can perform a dilation to ensure that the enclosed volume is equal to that of the sphere; since we have bounds above and below on the support function, we also have bounds above and below on the enclosed volume, and the Gauss curvature remains bounded after this dilation.

Next we combine this result with the bound below on the affine curvature, to obtain a bound below on the Gauss curvature for times sufficiently close to the final time.

Lemma 5.4. *There exist a time $\tau < T$ and a constant $k > 0$ such that for each time $t \in [\tau, T)$ there exists an affine transformation L of $\mathbf{R}^{n+1}$ with $V(L \circ \varphi_t(S^n)) = V(S^n)$ and $\inf_{S^n \times \{t\}} K_L \geq k$.*

Proof. We choose the same affine transformation L as in the previous Lemma. Theorem 4.4 ensures that the following estimate holds for some constant B:

$$\mathcal{A} \geq -BV(\varphi_t(S^n))^{\frac{2}{(n+1)(n+2)}} \operatorname{Id}. \tag{5.5}$$

Since we know the hypersurface contracts to a point at the time T, the right hand side converges to zero as the final time is approached. We can rewrite this using the expressions for the affine curvature and metric as follows:

$$A\left[K^{\frac{1}{n+2}} + BsV(\varphi_t(S^n))^{\frac{2}{(n+1)(n+2)}}\right] \geq 0. \tag{5.6}$$

Thus the function $\mathcal{X} = K^{\frac{1}{n+2}} + BsV(\varphi_t(S^n))^{\frac{2}{(n+1)(n+2)}}$ is the support function of a convex region. Furthermore, Lemma 5.1 gives an upper bound on $\mathcal{X}$, and s and $\mathcal{X}$ are positive for the choice of L as above.

Now we have the following identity:

$$\int_{S^n} K^{-1} d\mu = |L \circ \varphi(S^n)|,$$

from which we can control the measure of the set on S^n where K is small:

$$|\{K \leq \varepsilon\}| \leq \varepsilon |L \circ \varphi(S^n)| \leq \varepsilon r_+^n |S^n|,$$

since $L \circ \varphi(S^n)$ is enclosed by a sphere $r_+ S^n$, and the area is a monotonic functional on convex regions (see for example [9]). Since s is positive, this also gives a bound on the size of the region where $\mathcal{X}$ is small:

$$|\{\mathcal{X} \leq \varepsilon\}| \leq \varepsilon^{n+2} r_+^n |S^n|. \tag{5.7}$$

Now suppose there is some point $z \in S^n$ with $\mathcal{X}(z) \leq \sigma$, where σ will be chosen later. Then the convexity of $\mathcal{X}$ gives the following estimate for $\mathcal{X}$ at points z' with $\langle z, z'\rangle \geq 0$:

$$\mathcal{X}(z') \leq \sigma\langle z, z'\rangle + d\sqrt{1-\langle z, z'\rangle^2}, \tag{5.8}$$

where d is a bound for $\mathcal{X}$, given by Lemma 5.1. This estimate follows because $\mathcal{X}$ is the support function of a region which lies inside a cylinder of radius d with axis z, truncated by the plane $\{y : \langle y, z\rangle = \sigma\}$; thus $\mathcal{X}$ is bounded by the support function of this region.

Now consider the size of the region where $\mathcal{X}$ is no greater than 2σ: From (5.8) it follows that this region contains the set $\{\sqrt{1-\langle z, z'\rangle^2} \leq \sigma\}$. But now an estimate of the area of this set (which is a spherical cap of radius σ) gives the result:

$$|\{\mathcal{X} \leq 2\sigma\}| \geq \frac{1}{n}\left(\frac{2}{\pi}\right)^{n-1} |S^{n-1}|\sigma^n. \tag{5.9}$$

This is a contradiction to the estimate (5.7) provided we choose σ such that

$$\sigma^2 < \sigma_0^2 = \frac{|S^{n-1}|}{8n|S^n|r_+^n\pi^{n-1}}.$$

Hence we have $\mathcal{X}$ bounded below by $2\sigma_0$. Since s is also uniformly bounded, this also gives a bound below for K for times close to the final time, in view of the definition of $\mathcal{X}$.

Next we obtain control over all of the principal curvatures, by combining this result with the estimate on the cubic ground form:

Lemma 5.10. *There exist positive constants $\underline{C}$, $\bar{C}$ and $\tau < T$ such that the Weingarten curvature $\mathcal{W}$ of the embeddings $L \circ \varphi_t$, with L as in Lemma 5.1, satisfy for $t \in [\tau, T)$*

$$0 < \underline{C}\,\mathrm{Id} \leq \mathcal{W} \leq \bar{C}\,\mathrm{Id} < \infty.$$

Proof. The following identity holds for any vector u on S^n with $\bar{\nabla}_u u = 0$:

$$\bar{\nabla}_u \ln\left(K^{\frac{3}{n+2}} A[s](u,u)\right) = \frac{2\mathcal{C}(u,u,u)}{\tilde{g}(u,u)}. \tag{5.11}$$

But Theorem 3.7 gives the estimate

$$|\mathcal{C}| \leq kV(\varphi_t(S^n))^{\frac{1}{n+2}},$$

and we have $\mathcal{C}(u,u,u) \leq |\mathcal{C}|\tilde{g}(u,u)^{3/2}$. This gives the estimate:

$$\left|\bar{\nabla}_u \ln\left(K^{\frac{3}{n+2}} A[s](u,u)\right)\right| \leq kV(\varphi_t(S^n))^{\frac{1}{n+2}}\tilde{g}(u,u)^{1/2}$$

for some constant k. Now integrate along an arbitrary great circle γ in S^n, taking u to be the unit tangent vector of γ:

$$\text{(5.12)} \qquad \frac{\sup_\gamma\left(K^{\frac{3}{n+2}}A[s](u,u)\right)}{\inf_\gamma\left(K^{\frac{3}{n+2}}A[s](u,u)\right)} \leq \exp\left\{kL_{\tilde{g}}[\gamma]V(\varphi_t(S^n))^{\frac{1}{n+2}}\right\}.$$

Here $L_{\tilde{g}}[\gamma]$ is the length of γ, with respect to the affine metric $\tilde{g}$. We have the following estimate:

$$\begin{aligned} L_{\tilde{g}}[\gamma] &= \int_\gamma \left(\frac{A[s](u,u)}{K^{\frac{1}{n+2}}}\right)^{\frac{1}{2}} \\ &\leq B^{-1/(n+2)}\int_\gamma A[s](u,u)^{\frac{1}{2}} \\ &\leq B'\left(\int_\gamma A[s](u,u)\right)^{\frac{1}{2}} \end{aligned}$$

for some constant B', by the Hölder inequality. The integral in the last line is just the length of the projection of $L\circ\varphi(S^n)$ onto the plane containing γ, which is finite since the diameter of $L\circ\varphi(S^n)$ is bounded. Hence we have a bound on the affine diameter of γ, and hence the ratio in Equation (5.12) is uniformly bounded. But now integration along γ shows that $A[s](u,u)$ is comparable to $\int_\gamma A[s](u,u)$; but again this is just the length of the projection onto the plane containing γ, which is bounded above and below.

This result ensures that the evolution Equation (1.1) is uniformly parabolic (after a suitable affine transformation). We can therefore apply the regularity estimates of Krylov [30], to deduce the following theorem:

Theorem 5.13. *There exist constants C_ℓ, $\ell = 0, 1, \ldots$, such that for each $t < T$ there exists an affine transformation L of $\mathbf{R}^{n+1}$ with $V(L\circ\varphi_t(S^n)) = V(S^n)$, and $\|L\circ\varphi_t\|_{C^\ell(S^n)} \leq C_\ell$.*

Remark. In the case of curves ($n = 1$) the estimate on the cubic ground form is vacuous, and the estimate on the affine curvature is just the usual Harnack estimate.

6. Convergence to ellipsoids

In this section we show that the rescaled solutions to Equation (1.1) converge in C^∞, without affine correction, to ellipsoids. We begin with a weaker result which follows easily from the estimates of the previous section:

Lemma 6.1. *There exists a subsequence of times $\{T_i\}$ approaching T, and a sequence of affine transformations L_i of $\mathbf{R}^{n+1}$ such that the transformed solutions $L_i \circ \varphi_{t_i}$ converge in C^∞ to a limiting embedding with image an ellipsoid.*

Proof. The existence of a subsequence which converges in C^∞ follows directly from the estimates in the previous section. In the case $n \geq 2$, we deduce that the limit is an ellipsoid since the cubic ground form $\mathcal{C}$ vanishes identically.

Next consider the case of curves: The entropy estimate (2.9) implies that the limit is homothetic, since the entropy integral strictly decreases otherwise. It remains to show that the only smooth homothetic solutions to (1.1) are ellipses. This is well known, but I include a proof here for completeness.

The condition for homothety of a solution is the equation $K^{\frac{1}{3}} = s$ (possibly after translating and rescaling). Since $K^{-1} = A[s]$, we may write this as follows:

$$s^3(s_{\theta\theta} + s) = 1 \tag{6.2}$$

where $s_{\theta\theta}$ denotes the second derivative of s with respect to the angle parameter θ on S^1. A first integral of this equation is:

$$s^2 + s_\theta^2 + s^{-2} = E \tag{6.3}$$

Now we compute:

$$\begin{aligned}(s^2)_{\theta\theta} + 4s^2 &= 2(s_\theta^2 + s^2) + 2s(s_{\theta\theta} + s) \\ &= 2(E - s^{-2}) + 2s^{-2} \\ &= 2E,\end{aligned} \tag{6.4}$$

so that the solutions are precisely $s^2 = \frac{1}{2}E + A\sin 2(\theta - \theta_0)$, which are just the support functions of ellipses centred at the origin.

Now we proceed to the problem of showing stronger convergence:

Lemma 6.5. *Let φ be a smooth solution to (1.1), with $n \geq 2$ and final point $p \in \mathbf{R}^{n+1}$. Then the rescaled solutions*

$$\varphi_t' = \left(\frac{V(S^n)}{V(\varphi_t(S^n))}\right)^{\frac{1}{n+1}} (\varphi_t - p)$$

converge in C^∞ to a limit φ_∞ such that $\varphi_\infty(S^n)$ is an ellipsoid centred at the origin.

Proof. Section 3 gives the estimate:

$$\sup_{S^n\times\{t\}} |\mathcal{C}| \leq kV(\varphi_t(S^n))^{\frac{n+1}{n+2}}$$

We also have the following evolution equation for the enclosed volume:

$$\frac{\partial}{\partial t}V(\varphi_t(S^n)) = -|\varphi_t(S^n)|_{\tilde{g}}.$$

The affine isoperimetric inequality may be written as follows:

$$|\varphi_t(S^n)|_{\tilde{g}} \leq \left(\frac{V(\varphi_t(S^n))}{V(S^n)}\right)^{\frac{n}{n+2}} |S^n|. \tag{6.6}$$

Integrating and using the fact that the enclosed volume converges to zero at the final time, we have $V(\varphi_t(S^n)) \leq |S^n|\left(\frac{2(T-t)}{n+2}\right)^{\frac{n+2}{2}}$. This gives the following estimate:

$$\sup_{S^n\times\{t\}} |\mathcal{C}| \leq k(T-t)^{\frac{n+1}{2}}. \tag{6.7}$$

Now since we have uniform estimates on the higher derivatives, we can use standard interpolation inequalites to deduce estimates of the following form:

$$|\tilde{\nabla}^{(\ell)}\mathcal{C}| \leq C_\ell(T-t)^{\alpha_\ell}, \tag{6.8}$$

for some positive constants C_ℓ and α_ℓ, $\ell = 1, 2, \ldots$. In particular, we have $|\tilde{\nabla}^{(2)}\mathcal{C}| \leq C_2(T-t)^{\alpha}$. The Codazzi identity for the affine curvature gives

$$\begin{aligned}
\tilde{\nabla}\mathcal{A}(w,u) - \tilde{\nabla}\mathcal{A}(u,w) &= \mathcal{C}\left(\mathcal{A}(w),u\right) - \mathcal{C}\left(\mathcal{A}(u),w\right)\\
&= \mathcal{C}\left(\mathcal{A}(w) - \frac{\mathcal{H}}{n}w, u\right) - \mathcal{C}\left(\mathcal{A}(u) - \frac{\mathcal{H}}{n}u, w\right)\\
&= \mathcal{C}(\operatorname{tr}\tilde{\nabla}\mathcal{C}(w),u) - \mathcal{C}(\operatorname{tr}\tilde{\nabla}\mathcal{C}(u),w).
\end{aligned}$$

Thus the antisymmetric part of $\tilde{\nabla}\mathcal{A}$ can be written in terms of $\mathcal{C}$ and $\tilde{\nabla}\mathcal{C}$, which are bounded by (6.8). This leads to an estimate on the norm of the traceless part of $\tilde{\nabla}\mathcal{A}$ as in [28, Lemma 2.2]:

$$|\tilde{\nabla}\mathcal{A}|^2 - \frac{1}{n}|\tilde{\nabla}\mathcal{H}|^2 \geq \left(\frac{2(n-1)}{3n^2} - \varepsilon\right)|\tilde{\nabla}\mathcal{H}|^2 - k(T-t)^{\alpha},$$

where k depends on ε, for any $\varepsilon > 0$. This gives a bound for the norm of the gradient of the affine mean curvature $\mathcal{H}$ in terms of the norm of the gradient of the traceless part of the affine mean curvature, which is bounded in terms of the second derivatives of $\mathcal{C}$ since $\mathcal{A}-\frac{\mathcal{H}}{n}\tilde{g}=\frac{2}{n}\operatorname{tr}\tilde{\nabla}\mathcal{C}$. This gives the estimate:

$$\sup\mathcal{H}-\inf\mathcal{H}\leq k(T-t)^{\alpha}$$

since the affine diameter of the solution is uniformly bounded. This shows that the affine mean curvature is nearly constant. Integrating over the manifold gives:

$$\begin{aligned}\mathcal{H}&\leq\frac{\int\mathcal{H}d\tilde{\mu}}{\int d\tilde{\mu}}+k(T-t)^{\alpha}\\&\leq\frac{n\int d\tilde{\mu}}{(n+1)V}+k(T-t)^{\alpha}\end{aligned}$$

where we have used the Aleksandrov-Fenchel inequality (see [4] or [9]).

The rescaled Gauss curvature evolves as follows:

$$\frac{\partial}{\partial t}(KV^{\frac{n}{n+1}})=KV^{\frac{n}{n+1}}\left(\mathcal{H}-\frac{n\int d\tilde{\mu}}{(n+1)\int d\tilde{\mu}}\right).$$

Consequently we have

$$\frac{\partial}{\partial t}\ln K'\leq k(T-t)^{\alpha}. \tag{6.9}$$

Since the time of existence is finite, this gives a bound above on the Gauss curvature of the rescaled solution, and furthermore shows that the Gauss curvature is almost monotonic, and hence convergent. It follows that this solution converges in C^{∞}, without modification by affine transformations, to an ellipsoid.

It remains to prove the result for $n=1$. In this case we do not have such a strongly decreasing quantity as the cubic ground form in higher dimensions. Instead we consider the evolution of some integral quantities. We first show that the affine-geometric quantities converge as the final time is approached:

Lemma 6.10. *The affine isoperimetric ratio $V^{-\frac{1}{3}}|\varphi_t(S^1)|_{\tilde{g}}$ converges to $2\pi^{\frac{2}{3}}$ as t approaches T. The affine curvature $\mathcal{A}$ approaches a constant in C^{∞} with respect to the affine metric and connection.*

Proof. The convergence of the affine isoperimetric ratio is clear, because it is increasing (by the entropy estimate (2.9)) and we have convergence for a subsequence of times. In order to control the affine curvature and its derivatives, we consider the evolution of the following scaling invariant quantity:

$$\frac{d}{dt}\left\{\int_{S^1}\mathcal{A}d\tilde{\mu}\int_{S^1}d\tilde{\mu}\right\}=\frac{2}{3}\left\{\left(\int_{S^1}\mathcal{A}d\tilde{\mu}\right)^2-\int_{S^1}d\tilde{\mu}\int_{S^1}\mathcal{A}^2d\tilde{\mu}\right\}. \tag{6.11}$$

Note that this quantity is bounded above: By the Aleksandrov-Fenchel inequality we have:

$$\int_{S^1} \mathcal{A} d\tilde{\mu} V - \frac{1}{2}\left(\int_{S^1} d\tilde{\mu}\right)^2 \leq 0.$$

Multiplying by $V^{-1} \int_{S^1} d\tilde{\mu}$, and applying the affine isoperimetric inequality, we obtain:

$$\int_{S^1} \mathcal{A} d\tilde{\mu} \int_{S^1} d\tilde{\mu} \leq \frac{1}{2} V^{-1}\left(\int_{S^1} d\tilde{\mu}\right)^3 \leq 4\pi^2. \tag{6.12}$$

Equality holds on ellipses. Since we have convergence on a subsequence of times, it follows that $\int_{S^1} \mathcal{A} d\tilde{\mu} \int_{S^1} d\tilde{\mu}$ converges uniformly to $4\pi^2$ as t approaches T. This allows us to obtain a bound on the Aleksandrov-Fenchel difference:

$$V \int_{S^1} \mathcal{A} d\tilde{\mu} - \frac{1}{2}\left(\int_{S^1} d\tilde{\mu}\right)^2 = \frac{V}{\int_{S^1} d\tilde{\mu}}\left\{\int_{S^1} \mathcal{A} d\tilde{\mu} \int_{S^1} d\tilde{\mu} - \frac{(\int_{S^1} d\tilde{\mu})^3}{2V}\right\}.$$

Now both terms in the bracket on the right hand side converge uniformly to $4\pi^2$, so the bracket converges uniformly to zero:

$$\frac{1}{2}\left(\int_{S^1} d\tilde{\mu}\right)^2 - V \int_{S^1} \mathcal{A} d\tilde{\mu} = o(1) \tag{6.13}$$

as t approaches T. To make use of this we apply a stability estimate for the Aleksandrov-Fenchel inequality (see for example [32, Theorem 6.6.10]):

Lemma 6.14. *Let s be the support function of a bounded, smooth, strictly convex region D in $\mathbf{R}^2$, and let F be the support function of a convex bounded region. Let $\Delta[F]$ be the Aleksandrov- Fenchel difference $V[F,s]^2 - V[F,F]V[s,s]$. Then the following inequality holds:*

$$\Delta[F] \geq BV[F,s]^2 \int_{S^1} (\bar{s} - \bar{F})^2 d\theta \tag{6.15}$$

where B depends only on the diameter and inradius of D, V is the area of D, and $\bar{s}$ and $\bar{F}$ are obtained from s and F by translating to move the centre of mass of each region to the origin, and rescaling to give integrals equal to 1:

$$\int_{S^1} \bar{s} z d\theta = \int_{S^1} \bar{F} z d\theta = 0$$
$$\int_{S^1} \bar{s} d\theta = \int_{S^1} \bar{F} d\theta = 1$$

We apply this result to the function $F = A^{-\frac{1}{3}}$, after applying an affine transformation of $\mathbf{R}^2$ so that the estimates of the previous chapter apply. Now Δ is bounded by the estimate (6.13). In order to apply the lemma, we require F to be convex; this is guaranteed for sufficiently large times because $\mathcal{A}$ is eventually positive, since it converges to a positive constant on a subsequence of times, and is increasing. Thus $A[F] = A[s]\mathcal{A}$ is also positive as required. This gives the following estimate if we translate s appropriately:

$$V^{-\frac{4}{3}} \int_{S^1} |sA^{\frac{1}{3}} - C_1|^2 d\theta = o(1) \tag{6.16}$$

for some constant C_1 (which depends on time). But now (since we have bounds on all higher derivatives, and control on the convergence of the integrals V and $|\varphi(S^1)|_{\tilde{g}}$) we obtain by interpolation the estimate $|V^{-\frac{2}{3}} sA[s]^{\frac{1}{3}} - (2\pi)^{-\frac{2}{3}}| = o(1)$ as t approaches T. This shows that solution becomes close to satisfying the homothety condition.

Now consider the following identity:

$$\tilde{\Delta}\sigma + \mathcal{A}\sigma = 1 \tag{6.17}$$

where $\sigma = sA[s]^{\frac{1}{3}}$. He have bounds on the oscillation of σ, and hence (by interpolation) bounds on higher derivatives of σ : $\operatorname{osc}(\tilde{\Delta}\sigma) = o(1)$ as t approaches T. Thus (6.17) yields $\operatorname{osc}\mathcal{A} = o(1)$. The convergence of higher derivatives of $\mathcal{A}$ follows by interpolation (see for example [23], section (6.1)).

Note that we modified the solutions by affine transformations in this analysis; thus for the unmodified solutions we have control only on the affine-invariant quantities.

We can now obtain an estimate on the rate of convergence:

Lemma 6.18. *There exist positive constants α and k such that the following estimate holds:*

$$V^{\frac{2}{3}} \operatorname{osc}\mathcal{A} \le k(T-t)^{\alpha}$$

Proof. A straightforward calculation gives the following evolution equation:

$$\begin{aligned}\frac{d}{dt}\int_{S^1} |\tilde{\nabla}\mathcal{A}|^{2p} d\tilde{\mu} = &-\left(\frac{4}{3} + \frac{2}{3p}\right)\int_{S^1} |\tilde{\nabla}|\tilde{\nabla}\mathcal{A}|^p|^2 d\tilde{\mu} \\ &+ \frac{2(10p-1)}{3}\int_{S^1} \mathcal{A}|\tilde{\nabla}\mathcal{A}|^{2p} d\tilde{\mu}\end{aligned}$$

Assume that t is close enough to T so that $|V^{\frac{2}{3}}\mathcal{A} - (2\pi)^{\frac{2}{3}}| \le \varepsilon$, where ε will be chosen later. Then we have the following inequality for the evolution of the integral normalised by the volume V:

$$\frac{d}{dt}\left(V^{2p-\frac{1}{3}}\int_{S^1}|\tilde{\nabla}\mathcal{A}|^{2p}d\tilde{\mu}\right) \le -\left(\frac{4}{3}-\frac{2}{3p}\right)V^{2p-\frac{1}{3}}\int_{S^1}|\tilde{\nabla}|\tilde{\nabla}\mathcal{A}|^p|^2 d\tilde{\mu} + \left(\frac{8p}{3}+C\varepsilon(1+p)\right)V^{2p-1}\int_{S^1}|\tilde{\nabla}\mathcal{A}|^{2p}d\tilde{\mu}$$

Now for any function f the Hölder inequality gives

$$\int_{S^1}|\tilde{\nabla}f|^2 d\tilde{\mu} \ge \frac{\left(\int_{S^1}|\tilde{\nabla}f|d\tilde{\mu}\right)^2}{\int_{S^1}d\tilde{\mu}} \ge \frac{(\operatorname{osc} f)^2}{\int_{S^1}d\tilde{\mu}}$$

Taking $f=|\tilde{\nabla}\mathcal{A}|^p$, we have $\operatorname{osc} f=\sup f$ since the gradient of $\mathcal{A}$ is zero somewhere. The affine isoperimetric inequality gives:

$$\int_{S^1}|\tilde{\nabla}f|^2 d\tilde{\mu} \ge \frac{1}{4\pi^{\frac{4}{3}}}V^{-\frac{2}{3}}\int_{S^1}f^2 d\tilde{\mu}.$$

Substituting this in the above evolution equation, we find:

$$\frac{d}{dt}\left(V^{2p-\frac{1}{3}}\int_{S^1}|\tilde{\nabla}\mathcal{A}|^{2p}d\tilde{\mu}\right) \le -\left[\frac{\pi^{\frac{4}{3}}}{4}\left(\frac{4}{3}+\frac{2}{3p}\right)-\frac{8p}{3}-\varepsilon(1+p)\right] V^{2p-1}\int_{S^1}|\tilde{\nabla}\mathcal{A}|^{2p}d\tilde{\mu}$$

Choosing p and ε sufficiently small, we have:

$$\frac{d}{dt}B \le -\alpha V^{-\frac{2}{3}}B \tag{6.19}$$

where B is the quantity in the evolution equation above, and $\alpha>0$. Using the estimate for V in terms of $T-t$ and integrating, we obtain the desired estimate:

$$B \le k(T-t)^{\alpha} \tag{6.20}$$

The Lemma now follows by interpolation between L^p spaces (see for example [23], Equation (7.10)) to obtain a decay estimate on the L^1 norm of f, which then controls the oscillation, and hence the L^∞ norm of f. Since f is just the norm of the derivatives of $\mathcal{A}$, this controls the oscillation of $\mathcal{A}$.

The convergence of the rescaled solutions without modification by affine transformations now follows exactly as in the higher-dimensional case above.

This completes the proof of the main theorem.

7. Affine isoperimetric inequalities

In this last section we show how the main result can be used to prove several affine-geometric isoperimetric inequalities. The first of these, which has already been mentioned in the last section, is the following:

Theorem 7.1. *Let M be any smooth, strictly convex, bounded hypersurface in $\mathbf{R}^{n+1}$. Then the following inequality holds:*

$$V(M)^{-\frac{n}{n+2}}|M|_{\tilde{g}} \leq (n+1)^{\frac{n}{n+2}}|S^n|^{\frac{2}{n+2}}$$

with equality if and only if M is an ellipsoid.

This follows directly from the convergence result of Theorem 1.1 (or the weaker convergence result of Theorem 6.1, together with the entropy estimate of Theorem 2.7. We deduce the following consequence, which provides an estimate on the total affine mean curvature:

Corollary 7.2. *For M as in Theorem 7.1, the following holds:*

$$\int_M \mathcal{H}d\tilde{\mu} \leq n|M|_{\tilde{g}}^{\frac{n-2}{n}}|S^n|^{\frac{2}{n}},$$

with equality if and only if M is an ellipsoid.

This follows by combining the isoperimetric inequality above with the Aleksandrov-Fenchel inequality, which states:

$$V(M)\int_M \mathcal{H}d\tilde{\mu} \leq \frac{n}{n+1}|M|_{\tilde{g}}^2.$$

Finally, we have the following result:

Theorem 7.3. *For M as above, the following inequality holds:*

$$ZV(M) \leq \frac{n}{n+1}|S^n|^2,$$

where Z is the minimum over points e in the region enclosed by M of the following integral:

$$\int_{S^n}(s(z)-\langle z,e\rangle)^{-(n+1)}d\mu.$$

Equality holds if and only if M is an ellipsoid.

Proof. This follows from the following evolution equation:

$$\frac{d}{dt}\left(V(M)\int_{S^n} s^{-(n+1)}d\mu\right) = \int_{S^n}\sigma d\tilde{\mu}\int_{S^n}\sigma^{-(n+2)}d\tilde{\mu} - \int_{S^n}\sigma^{-(n+1)}d\tilde{\mu}\int_{S^n}d\tilde{\mu}$$

where $\sigma = sS_n^{\frac{1}{n-2}}$. Hence for any choice of origin which is enclosed by the hypersurface, this quantity is increasing (by the Hölder inequality). In particular, if we choose the origin to be the final point, then this quantity increases, and converges to the value for an ellipsoid. Hence the integral with this choice of origin at the initial time is less than that for the ellipsoid; thus certainly the minimum over all choices of origin is less than the value for the ellipsoid.

Remarks. The affine isoperimetric inequality (Theorem 7.1) was proved initially for dimensions $n = 1$ or 2 by Blaschke in [6] and [8], and for higher dimensions by Santaló [31] and Diecke [16].

The result of Theorem 7.3 is known as the Blaschke-Santaló inequality, and is also a classical result (see [7]-[8] and [31]).

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