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THE MOMENT MAP AND LINE BUNDLES OVER PRESYMPLECTIC TORIC MANIFOLDS

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Abstract

We apply symplectic methods in studying smooth toric varieties with a closed, invariant 2-form ω that may have degeneracies. Consider the push-forward of Liouville measure by the moment map. We show that it is a "twisted polytope" in $\mathfrak{t}^*$ which is determined by the winding numbers of a map $S^{n-1} \to \mathfrak{t}^*$ around points in $\mathfrak{t}^*$. The index of an equivariant, holomorphic line-bundle with curvature ω is a virtual T-representation which can easily be read from this "twisted polytope".

1. Introduction

A *symplectic manifold* is a smooth manifold M with a closed 2-form ω which is everywhere nondegenerate. Let T be a compact torus which acts effectively, preserving ω. A *moment map* for (M, T, ω) is a map $\Phi: M \to \mathfrak{t}^*$ such that $\langle d\Phi, \xi\rangle = -i(\xi_M)\omega$ for every $\xi \in \mathfrak{t}$, where ξ_M denotes the corresponding vector field on M. By the Atiyah-Guillemin-Sternberg convexity theorem [1], [12], the image of the moment map is a convex polytope Δ. For an excellent introduction to this subject, see [3].

If (M, T, ω) admits a moment map, then the dimension of T cannot exceed half of the dimension of M. If $\dim T = \frac{1}{2}\dim M$, then the action is *completely integrable.* Delzant [5] classifies these spaces; the polytope Δ determines (M, T, ω) up to equivariant symplectomorphism. Moreover, he shows that (M, T) is equivariantly diffeomorphic to a *toric manifold*, i.e., a smooth toric variety.

In particular, M admits a complex structure such that T acts holomorphically. Let L be an equivariant holomorphic line bundle over M with curvature ω, where ω is the imaginary part of a Kähler form on M. Denote the sheaf of holomorphic sections of L by $\mathscr{O}_L$. Then $H^i(M, \mathscr{O}_L)$ is a representation of T. Danilov [4] shows that the weights which occur in $H^0(M, \mathscr{O}_L)$ are exactly the lattice points in Δ (with multiplicity one), whereas $H^i(M, \mathscr{O}_L) = 0$ for $i > 0$.

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We extend these results to *presymplectic* forms. A presymplectic form on (M, T) is a closed, invariant 2-form ω which may be degenerate. Although Φ is still defined, $\operatorname{Im}\Phi$ behaves badly. Instead, we consider the push-forward of Liouville measure, $\Phi_*\omega^n$, which was introduced by Duistermaat and Heckman in [6]. It is a measure on $\mathfrak{t}^*$ which is supported on Δ. As was proved in [6], for symplectic ω, $\Phi_*\omega^n$ is equal to Lebesgue measure times a piecewise polynomial function. In particular, in the completely integrable case $\Phi_*\omega^n$ is equal to Lebesgue measure on Δ—up to a universal constant which we shall ignore for the remainder of this introduction. Even for presymplectic ω, one can prove that the density function is piecewise polynomial; $\Phi_*\omega^n$ can be expressed as a sum of polynomial measures on cones [2], [10], [11]. In this case, $\Phi_*\omega^n$ is a signed measure on $\mathfrak{t}^*$.

In this paper, we give an explicit description of $\Phi_*\omega^n$. M/T is homeomorphic to a ball. The moment map descends to the quotient, and, restricting to $\partial(M/T) \simeq S^{n-1}$, we get a map

$$\overline{\Phi}: S^{n-1} \to \mathfrak{t}^*. \tag{1.1}$$

For $\alpha \in \mathfrak{t}^*$, let $d(\alpha)$ be the winding number of (1.1) around α. d has the shape of a "twisted polytope", as is illustrated in Figure 4 (p. 474). It is bounded by hyperplanes; however, some faces may go right through other faces, thus creating a region with a negative density; also, faces may "wrap" several times around a region which then "counts with multiplicity". Theorem 1 in §5 states that $\Phi_*\omega^n$ is equal to Lebesgue measure times d. If ω is *symplectic*, then $d(\alpha)$ is simply one or zero, depending on whether α lies or does not lie in $\operatorname{Im}\Phi$, in agreement with the standard theorem.

Let L be a holomorphic line bundle with curvature form ω. Although Danilov [4] has a recipe for determining $H^i(M, \mathcal{O}_L)$, there is no obvious relationship to the moment map. However, consider the index $\sum(-1)^i H^i(M, \mathcal{O}_L)$ as a virtual representation of T; Theorem 2 in §7 states that the weight $\alpha \in \mathfrak{t}^*$ occurs with a multiplicity $d(\alpha)$ wherever the latter is defined. Again, this agrees with the standard theorem. Theorem 3 in §10 tells us the multiplicity of α when $d(\alpha)$ is not defined.

Here is a prototypical example; although it is not compact, it illustrates these theorems. Let $M = \mathbb{C}$ and $T = S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Identify $\mathfrak{t}^*$ with $\mathbb{R}$ by sending $(\partial/\partial\theta)^*$ to 1, where (r, θ) are polar coordinates. The moment map $\Phi: \mathbb{C} \to \mathbb{R}$ is determined by $d\Phi = -i(\partial/\partial\theta)\omega$.

(i) Take the symplectic form, $\omega = -r\,dr \wedge d\theta$. Then $\Phi(re^{i\theta}) = -\frac{1}{2}r^2$ and $\operatorname{Im}\Phi$ is $\mathbb{R}^- = \{\alpha \in \mathbb{R} \mid \alpha \leq 0\}$. To compute the push-forward measure,

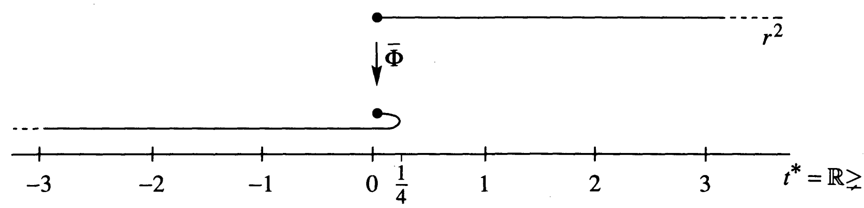
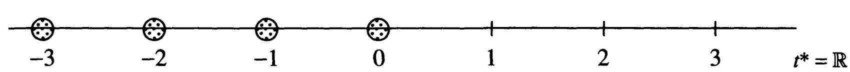


FIGURE 1

write $\omega = d(-\frac{1}{2}r^2) \wedge d\theta = d\alpha \wedge d\theta$. Integrating over the θ coordinate, we have $\Phi_*\omega = (-2\pi)d\alpha$ on $\mathbb{R}^-$.

(ii) Take the presymplectic form $\omega = (1 - r^2)rdr \wedge d\theta$, which is positive inside the unit disc and negative outside. Then $\Phi(re^{i\theta}) = \overline{\Phi}(r^2) = \frac{r^2}{4}(2 - r^2)$. The map $\overline{\Phi}$: $\mathbb{R}^+ \to \mathbb{R}$ "folds" at $r^2 = 1$ as shown in Figure 1. The image of the moment map is $(-\infty, \frac{1}{4}]$, but in $\Phi_*\omega$ the contributions of the overlapping pieces cancel; again, $\Phi_*\omega = (-2\pi)d\alpha$ on $\mathbb{R}^-$.

Consider the space of holomorphic functions on $\mathbb{C}$ as a representation of S^1 under the action $(\lambda f)(z) = f(\lambda^{-1}z)$. In particular, for $f(z) = z^n$ we have $(\lambda f)(z) = \lambda^{-n}f(z)$, so z^n spans a one-dimensional weight space corresponding to the weight $-n$. The multiplicity diagram of this representation can be drawn as



Notice its similarity to the measure $\Phi_*\omega$.

The paper is organized as follows. In §2, we introduce toric manifolds (M, T). In §3, we describe the quotient M/T. In §4, given a presymplectic form ω on M, we define a function d on $\mathfrak{t}^*$. In §5, we prove that the push-forward of Liouville measure by the moment map is given by the function d (Theorem 1). In §6, we give an alternative description of d, as a "twisted polytope", and show that it only depends on the cohomology class of ω. In §7, we state Theorem 2, that the index of a line bundle over M is given by the function d. In §8, we establish the relationship between the index over M and an index over a subset $U_\Sigma \subseteq \mathbb{C}^N$. In §9, we compute the index over U_Σ. In §10, we complete the proof of Theorem 2 and Theorem 3.

2. Toric manifolds

A toric manifold is a smooth toric variety. Although this an algebraic object, we shall only consider its complex analytic structure. For instance, let M be any real $2n$-dimensional manifold with (1) an n-dimensional compact torus T which acts effectively, and (2) an invariant symplectic form ω which is Hamiltonian. By a theorem of Delzant [5], (M, T) is equivariantly diffeomorphic to a toric manifold. In contrast, some toric manifolds do not admit any invariant symplectic form.

Toric manifolds can explicitly be constructed as subquotients of $\mathbb{C}^N$. Let us review this construction, following Michèle Audin [3]:

Let $\mathfrak{t}$ be an n-dimensional real vector space with a lattice ℓ. Consider a set $\{x_1, \cdots, x_N\}$ of primitive elements in ℓ which span $\mathfrak{t}$. Let $\mathbb{R}^+$ denote the nonnegative real numbers, and denote $\{1, \cdots, N\}$ by $\mathbf{N}$.

Definition 2.1. For $I \subseteq \mathbf{N}$, the *cone over* $\{x_i\}_{i\in I}$ is $\Delta x_I = \sum_{i\in I} \mathbb{R}^+ x_i$; Δx_I is a *smooth cone* if $\{x_i\}_{i\in I}$ can be extended to a $\mathbb{Z}$-basis of ℓ.

Definition 2.2. A (*smooth*) *fan* Σ *over* $\{x_1, \cdots, x_N\}$ is a collection of smooth cones of the form Δx_I such that:

(i) Any face of a cone in Σ is itself a cone in Σ, i.e., $\Delta x_I \in \Sigma$, $J \subseteq I \Rightarrow \Delta x_J \in \Sigma$;
(ii) The intersection of two cones in Σ is a common face, i.e., $\Delta x_I, \Delta x_J \in \Sigma \Rightarrow \Delta x_I \cap \Delta x_J = \Delta x_{I\cap J}$;
(iii) $\Delta x_{\{i\}} \in \Sigma \ \forall i$.

Definition 2.3. The fan Σ is *complete* if $\bigcup_{\Delta x_I \in \Sigma} \Delta x_I = \mathfrak{t}$.

A toric manifold is constructed from a fan Σ as follows. Define a linear projection $\pi\colon \mathbb{R}^N \to \mathfrak{t}$ by $\pi(e_i) = x_i$; let $\mathfrak{k} = \ker \pi$. Then we have dual exact sequences:

$$(2.4)\qquad \begin{gathered} 0 \to \mathfrak{k} \to \mathbb{R}^N \xrightarrow{\pi} \mathfrak{t} \to 0, \\ 0 \to \mathfrak{t}^* \xrightarrow{\pi^*} (\mathbb{R}^N)^* \xrightarrow{p} \mathfrak{k}^* \to 0. \end{gathered}$$

Identify $\mathbb{R}^N/\mathbb{Z}^N$ with $(S^1)^N$ and $\mathbb{C}^N/\mathbb{Z}^N$ with $(\mathbb{C}^\times)^N$ by the map $\widehat{\exp}\colon (\zeta_1, \cdots, \zeta_N) \mapsto (e^{2\pi i\zeta_1}, \cdots, e^{2\pi i\zeta_N})$; then π induces a map $(S^1)^N \to \mathfrak{t}/\ell$ and, similarly, $(\mathbb{C}^\times)^N \to \mathfrak{t}_{\mathbb{C}}/\ell$, where $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}$. Denote the kernel by K and G respectively. Then,

$$(2.5)\qquad \begin{gathered} K = \{\widehat{\exp}(\zeta) | \zeta \in \mathbb{R}^N, \ \pi(\zeta) \in \ell\}; \\ G = \{\widehat{\exp}(\zeta) | \zeta \in \mathbb{C}^N, \ \pi(\zeta) \in \ell\}. \end{gathered}$$

Now define

$$U_I = \{z \in \mathbb{C}^N | z_i \neq 0 \ \forall i \notin I\} = \mathbb{C}^I \times (\mathbb{C}^\times)^{N\setminus I},$$

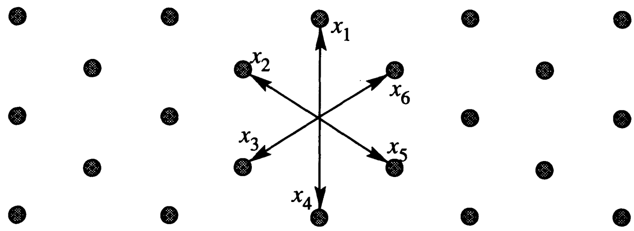


FIGURE 2

and

$$U_\Sigma = \bigcup \{U_I | \Delta x_I \in \Sigma\} .$$

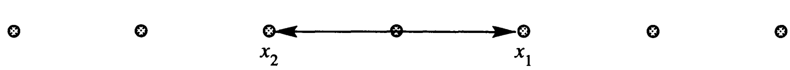
Let $T = (S^1)^N/K \cong \mathfrak{t}/\ell$; let $T_{\mathbb{C}} = (\mathbb{C}^\times)^N/G \cong \mathfrak{t}_{\mathbb{C}}/\ell$. The toric manifold associated to Σ is (M, T), where $M = U_\Sigma/G$. One can prove (see [3]) that M is an n-dimensional complex manifold; T acts effectively and analytically on M ; and M is compact if and only if Σ is a complete fan. Additionally,

(i) $H^1(M) = \{0\}$;

(ii) $\mathrm{Stab}(p) \subseteq T$ is connected for every $p \in M$.

Remark 2.6. One can construct a fan Σ from any rational polytope $\Delta \subset \mathfrak{t}^*$. This fan encodes the directions of the faces of Δ but not their location in $\mathfrak{t}^*$; it also specifies which faces intersect; see [3]. Faces of Δ correspond to cones in Σ of the complementary dimension. Although some fans do not arise in this way, this intuition is useful. If (M, T) is the toric manifold associated to Σ, ω is an invariant Kähler form, Φ is a moment map, and $\Delta = \mathrm{Im}(\Phi)$, then Σ is the fan which corresponds to Δ.

Example 2.7. The following fan produces the manifold $\mathbb{CP}^1 \simeq S^2$ with $T = S^1$ acting by rotations; in homogeneous coordinates, $\lambda \cdot [z_0, z_1] = [\lambda z_0, z_1]$.



Example 2.8. $T = (S^1)^2$ acts in a standard way on $\mathbb{CP}^2$; $(\lambda_1, \lambda_2) \cdot [z_1, z_2, z_3] = [\lambda_1 z_1, \lambda_2 z_2, z_3]$. In Figure 2, take the fan which contains every two-dimensional cone generated by two consecutive vectors. This

fan produces a manifold M which is the blowup of $\mathbb{CP}^2$ at the three fixed points; the action of T extends to M.

Example 2.9. An interesting class of toric manifolds is the Bott-Samelson manifolds; these arise in the study of Lie groups and their representations; see [8], [9].

3. The structure of M/T

Local structure. Let (M, T) be a toric manifold. The smooth structure of M/T is defined by declaring a function smooth if its pullback to M is smooth. A diffeomorphism is, by definition, a homeomorphism which induces a bijection on the sets of smooth functions. For example, any S^1 invariant smooth function on $\mathbb{C}$ is of the form $f(|z|^2)$ where f is smooth on $\mathbb{R}$. Therefore, $z \mapsto |z|^2$ is a diffeomorphism $\mathbb{C}/S^1 \to \mathbb{R}^+$, where the smooth functions on $\mathbb{R}^+$ are the restrictions of smooth functions on $\mathbb{R}$.

Lemma 3.1. *Topologically, M/T is a manifold with boundary M_{sing}/T, where M_{sing} is the set of points with nontrivial stabilizers. Differentiably, it is a manifold with corners, i.e., it is locally diffeomorphic to $\mathbb{R}^{n-l} \times (\mathbb{R}^+)^l$.*

Proof. Choose any $p \in M$ and let $H = \text{Stab}(p)$. The normal bundle of the orbit $\mathscr{O} = T \cdot p$ in M is $T \times_H V$, where $V = T_pM/T_p\mathscr{O}$ and H acts on V by the isotropy action. By the "slice theorem" [3], a neighborhood of $\mathscr{O}$ in M is equivariantly diffeomorphic to a neighborhood of the zero section in $T \times_H V$, where T acts on the latter from the left. Therefore, a neighborhood of $[p]$ in M/T is diffeomorphic to V/H. Because H is a torus which acts effectively on V, we can identify V with $\mathbb{R}^{n-l} \oplus \mathbb{C}^l$ and H with T^l, where T^l acts on $\mathbb{C}^l$ in the standard way and fixes $\mathbb{R}^{n-l}$. Then, $V/H = \mathbb{R}^{n-l} \times (\mathbb{R}^+)^l$.

Global structure. If (M, T) admits an invariant symplectic form with a moment map $\Phi\colon M \to \mathfrak{t}^*$, then Φ descends to a homeomorphism $\overline{\Phi}\colon M/T \to \Delta$, where $\Delta = \operatorname{Im}\Phi$ is a convex polytope in $\mathfrak{t}^*$. More generally, we have

Lemma 3.2. *Let (M, T) be a compact toric manifold. Then M/T is homeomorphic to a closed ball with boundary M_{sing}/T.*

Proof. Let Σ be a complete fan and let $M = U_\Sigma/G$ be the corresponding toric manifold. Consider the map $\widetilde{\varphi}\colon \widehat{\exp}(\zeta + i\mu) \mapsto \pi(\mu)$ from $(\mathbb{C}^\times)^N$ onto $\mathfrak{t}$. The preimage of every point is, by (2.5), an orbit of the group generated by $(S^1)^N$ and G. Therefore, $\widetilde{\varphi}$ descends to a homeomorphism

$$(M \setminus M_{\mathrm{sing}})/T = ((\mathbb{C}^{\times})^N/G)/T \to \mathfrak{t}. \tag{3.3}$$

Now define a map from $\mathfrak{t}$ into $\mathfrak{t}$ as follows; on $\measuredangle x_I \in \Sigma$,

$$\sum_{i\in I} \mu_i x_i \mapsto \sum_{i\in I}(1 - e^{-\mu i})x_i. \tag{3.4}$$

This defines a homeomorphism of $\mathfrak{t}$ with D: a bounded star-shaped domain around 0, which is homeomorphic to an open ball. Let φ : $(M \setminus M_{\mathrm{sing}})/T \to D$ be the composition of (3.3) with (3.4). We will extend φ to a homeomorphism of M/T with the closure of D. We first need

Definition 3.5. Let Σ be a fan in $\mathfrak{t}$ and fix $\measuredangle x_J \in \Sigma$. Let $\hat{\mathfrak{t}} = \mathfrak{t}/(\operatorname{span} \measuredangle x_J)$. Let $\hat{x}_i$ be the image of x_i in $\hat{\mathfrak{t}}$. Let $L = \{l \in \mathbf{N} | \measuredangle x_{J\cup\{l\}} \in \Sigma\}$, and let $\widehat{L} = L \setminus J$. Define $\widehat{\Sigma}$ as follows: $\measuredangle \hat{x}_I \in \widehat{\Sigma}$ if and only if $J \cap I = \varnothing$ and $\measuredangle x_{I\cup J} \in \Sigma$. This is a fan over $\{\hat{x}_l\}_{l\in\widehat{L}}$, and it is called the *fan relative to* $\measuredangle x_J$.

Remark 3.6. Think of the relative fan as what you see if you stand on $\measuredangle x_J$ and look around in $\mathfrak{t}$. Alternatively, if Σ is the fan associated to a polytope Δ, then $\widehat{\Sigma}$ is the fan associated to the Jth face of Δ.

To complete the proof, take any $w \in U_\Sigma$. Let $J = \{j | w_j = 0\}$; then $\measuredangle x_J \in \Sigma$. Write $w_k = e^{2\pi i(\zeta_k' + i\mu_k')}$ for $k \notin J$ and consider $\sum_{k\in\mathbf{N}\setminus J} \mu_k' \hat{x}_k$ in $\hat{\mathfrak{t}}$. It lies in some cone $\measuredangle \hat{x}_I \in \widehat{\Sigma}$ and is equal to $\sum_{i\in I} \mu_i \hat{x}_i$ for some $\mu_i > 0$. If $[w]$ is the image of w in M/T, then define $\varphi([w]) = \sum_{j\in J} x_j + \sum_{i\in I}(1 - e^{-\mu_i})x_i$. One can check that φ is a homeomorphism, though not in general a diffeomorphism.

4. Degree of the moment map

Let (M, T) be a toric manifold; let ω be any closed, invariant 2-form on M. As in the symplectic case, a *moment map* is a map Φ: $M \to \mathfrak{t}^*$ such that

$$\langle d\Phi, \eta\rangle = -i(\eta_M)\omega \quad \text{for all } \eta \in \mathfrak{t},$$

where η_M is the vector field on M corresponding to η. This condition determines Φ up to a translation in $\mathfrak{t}^*$. For a toric manifold $H^1(M) = \{0\}$; therefore, such a Φ exists.

As in the symplectic case, Φ is a T-invariant. Therefore it splits as

$$M \to M/T \xrightarrow{\overline{\Phi}} \mathfrak{t}^*.$$

Definition 4.1. Take $\alpha \in \mathfrak{t}^*$, $\alpha \notin \Phi(\partial(M/T))$. Denote $\{\alpha\}$ by $\boldsymbol{\alpha}$. Define $d(\alpha)$ be the degree of the map $\overline{\phi}\colon \partial(M/T) \to \mathfrak{t}^* \setminus \boldsymbol{\alpha}$.

Explicitly, $\overline{\Phi}$ induces a map $[\overline{\Phi}]$ from the reduced homology group $\widetilde{H}_{n-1}(\partial(M/T))$ to $\widetilde{H}_{n-1}(\mathfrak{t}^* \setminus \boldsymbol{\alpha})$. Both of these groups are isomorphic to $\mathbb{Z}$; $d(\alpha)$ is the image of 1 under the map $[\overline{\Phi}]$.

Of course, $d(\alpha)$ depends on the orientations chosen; we use the following conventions. As a complex manifold, M is oriented. Any orientation for T induces an orientation on $\mathfrak{t}$, and hence on $\mathfrak{t}^*$. For later convenience, let the orientation of M/T, followed by that of T, be equal to that of M times $(-1)^{n(n-1)/2}$. An outward normal to M/T followed by the orientation of $\partial(M/T)$ gives the orientation of M/T; a similar relation picks a generator of $\widetilde{H}_{n-1}(\mathfrak{t}^* \setminus \boldsymbol{\alpha})$. Then $d(\alpha)$ does not depend on the orientation of T.

Additionally, $\overline{\Phi}$ induces a map from $H_n(M/T, \partial(M/T))$ to $H_n(\mathfrak{t}^*, \mathfrak{t}^* \setminus \boldsymbol{\alpha})$. These groups are also isomorphic to $\mathbb{Z}$ and, by a standard homological argument, $d(\alpha)$ is the image of 1 under this map.

Let α be a regular value of Φ. A fortiori, α is not in the image of $\partial(M/T)$. Near $\overline{\Phi}^{-1}(\alpha)$, M/T is an n-dimensional manifold, and $\overline{\Phi}$ is smooth in the usual sense. Regularity implies that for any $[p] \in \overline{\Phi}^{-1}(\alpha)$, $d\overline{\Phi}_{[p]}\colon T_{[p]}(M/T) \to T_\alpha(\mathfrak{t}^*)$ is an isomorphism. Therefore, there exists some neighborhood U of α such that $\overline{\Phi}^{-1}(U)$ is a disjoint union of open sets which are mapped diffeomorphically to U by $\overline{\Phi}$. Therefore, we have

Lemma 4.2. *If $\alpha \in \mathfrak{t}^*$ is a regular value for Φ, then*

$$\mathrm{d}(\alpha) = \sum_{[p] \in \overline{\Phi}^{-1}(\alpha)} \operatorname{sign}(\det \mathrm{d}\overline{\Phi}|_{[p]}).$$

5. Push-forward of Liouville measure

We define a signed measure on M, called *Liouville measure*, by assigning the number $\int_U \omega^n$ to the set $U \subset M$. Its push-forward $\Phi_* \omega^n$ assigns the number $\int_{\Phi^{-1}A} \omega^n$ to the set $A \subseteq \mathfrak{t}^*$.

Remark 5.1. We say that $\omega^n > 0$ if and only if it is compatible with the orientation of M. A typical situation is that $\omega^n = 0$ along a hypersurface and has opposite signs on each side. Liouville measure takes negative values in the region where $\omega^n < 0$.

Theorem 1. *Let (M, T) be a toric manifold. Let ω be an invariant, closed 2-form; let Φ be its moment map. Then*

$$\Phi_* \omega^n = (-2\pi)^n n! \cdot d(\alpha) \cdot (\textit{Lebesgue measure on } \mathfrak{t}^*), \tag{5.2}$$

where $d(\alpha)$ *is the degree as in Definition* 4.1.

Remark 5.3. Lebesgue measure on $\mathfrak{t}^*$ is normalized so that the quotient of $\mathfrak{t}^*$ by ℓ^* has volume 1. The right-hand side of (5.2) is well defined because the singular values of Φ have measure zero.

Proof. By Lemma 4.2 it suffices to show that if p is a regular point of Φ, then

(i) in a neighborhood of p, T acts freely and ω is nondegenerate,
(ii) there exists an invariant neighborhood U of $T \cdot p$ such that

$$\Phi_*(\omega^n|_U) = (-2\pi)^n n! \cdot \operatorname{sign}(\det d\overline{\Phi}|_{[p]}) \cdot (\text{Lebesgue measure on } \Phi(U)). \tag{5.4}$$

Proof of (i). Let $p \in M$ be a regular point of Φ. Because $d\Phi|_p$ is onto, for any nonzero $\eta \in \mathfrak{t}$, $i(\eta_M)\omega|_p = \langle d\Phi|_p, \eta \rangle \neq 0$, so $\eta_M|_p \notin \operatorname{Null}(\omega|_p)$. In particular, $\eta_M|_p \neq 0$, so the orbit of p is n dimensional. Since $\operatorname{Stab}(p)$ is connected, T acts freely at p. In addition, the tangent to the orbit at p descends to an n-dimensional subspace of $T_pM/\operatorname{Null}(\omega|_p)$. This subspace is isotropic because the restriction of ω to an orbit is zero, just as in the symplectic case. Since an isotropic subspace of a symplectic space is at most half the dimension of the vector space, $\operatorname{Null}(\omega|_p) = 0$.

Proof of (ii). By (i) and invariance, ω is symplectic in a neighborhood of the orbit of p. Because the signs of both sides of (5.4) depend in the same way on the orientation of U, we can assume that this orientation is compatible with the symplectic structure. The rest is standard; by the Darboux-Weinstein "local normal form" [14], U is equivariantly symplectomorphic to a neighborhood of $T \times \{0\}$ in the cotangent bundle $T \times \mathfrak{t}^*$, where T acts by left translation on the first factor, and ω is the standard symplectic form on the cotangent bundle. The moment map is projection to the second factor. The Liouville measure ω^n is the product of the volume form on T with total measure $(-2\pi)^n n!$, and Lebesgue measure on $\mathfrak{t}^*$. q.e.d.

We now describe the function d for various examples.

Example 5.5 (*Archimedes*). Let $T = S^1$ act on $M = S^2$ by rotations around the z-axis, as in Example 2.7, and take $-\omega$ to be the standard area form. Then the moment map is the height function on S^2. For a general ω, d is supported on an interval whose length is $\frac{1}{2\pi}|\int_M \omega|$, and the value of d on this interval is $\operatorname{sign}(-\int_M \omega)$ (see Figure 3, next page).

Example 5.6. Let M be the blow-up of $\mathbb{CP}^2$ at three points, as in Example 2.8. Figure 4 shows several possibilities for d for various ω's.

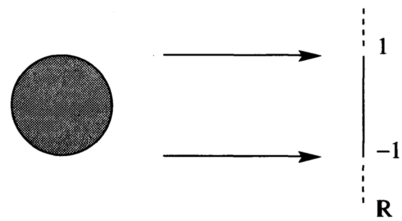


FIGURE 3

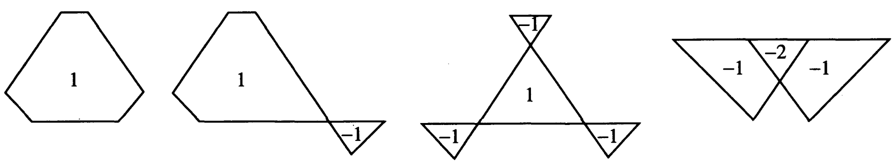


FIGURE 4

Notice that the locations of the "faces" change but their slopes do not change.

6. Twisted polytopes

In this section we show that $\Phi_* \omega^n$ only depends on the cohomology class of ω. (This was proved in [13] under more general assumptions). In the process, we obtain an algorithm for constructing $\Phi_* \omega^n$ from $[\omega]$. We describe $\Phi_* \omega^n$ as a "twisted polytope", which we compute by induction on the dimension of M. First, we stratify M_{sing} by lower dimensional toric manifolds.

In the symplectic case, $\Delta = \operatorname{Im} \Phi$ is a polytope. Each face is itself a lower dimensional polytope which spans an affine plane, F, in $\mathfrak{t}^*$. Let M_F denote $\Phi^{-1}(F)$. Let H be the subtorus of T perpendicular to F. Then T/H acts effectively on M_F, and $\omega|_{M_F}$ is an invariant symplectic form. By Delzant [5], M_F is a toric manifold. M is the union of the M_F's.

A similar stratification holds in general. Let $M = U_\Sigma/G$ as in §2. For J such that $\Delta x_J \in \Sigma$, consider the coordinate plane $E_J = \{0\}^J \times \mathbb{C}^{N\setminus J}$. Let M_J be $(U_\Sigma \cap E_J)/G$. Let H be the image of span Δx_J in T. Then, T/H acts effectively on M_J. M is stratified by the sets $(M_J \setminus (M_J)_{\text{sing}})$. We must show the following lemma:

Lemma 6.1. *$(M_J, T/H)$ is a toric manifold.*

Proof. Let $\widehat{\Sigma}$ be the fan relative to Δx_J. As in Definition 3.5, we will always use "$\widehat{\ }$" to denote objects associated to this fan. Following §2, we construct the associated toric manifold $(U_{\widehat{\Sigma}}/\widehat{G}, \widehat{T})$. We claim that $(M_J, T/H)$ is equivariantly diffeomorphic to $(U_{\widehat{\Sigma}}/\widehat{G}, \widehat{T})$.

First, the natural embedding of $(S^1)^{\widehat{L}}$ in $(S^1)^{\mathrm{N}}$ descends to an isomorphism of $\widehat{T}$ with T/H. Similarly, the embedding of $\mathbb{C}^{\widehat{L}}$ in $\mathbb{C}^N$ by $\mathbb{C}^{\widehat{L}} \to \{0\}^J \times \mathbb{C}^{\widehat{L}} \times \{1\}^{\mathrm{N}\setminus L}$ also embeds $U_{\widehat{\Sigma}}$ into $U_\Sigma \cap E_J$. Define an injection from $\widehat{G}$ to G as follows: Given $g \in \widehat{G}$ and $l \in \widehat{L}$ there exists ξ_l such that $e^{2\pi i \xi_l} = g_l$. Then there exist $(\xi_j)_{j\in J}$ such that $\sum_{l\in\widehat{L}} \xi_l x_l + \sum_{j\in J} \xi_j x_j$ is in l. Let $\xi_i = 0$ $\forall i \notin L$; then simply send g to $\widehat{\exp}(\xi)$. Together, these two maps define an isomorphism of $U_{\widehat{\Sigma}}/\widehat{G}$ with $M_J \subset U_\Sigma/G$. q.e.d.

Because H acts trivially on M_J, $\Phi(M_J)$ lies in a plane $F_J \subseteq \mathfrak{t}^*$, which is perpendicular to Δx_J. As an affine space, F_J is isomorphic to $\hat{\mathfrak{t}}^*$. The restriction $\Phi|_{M_J}: M_J \to F_J$ is a moment map for the triple $(M_J, T/H, \omega|_{M_J})$.

ω determines the location of the F_J's in $\mathfrak{t}^*$ up to a global translation. Their exact location can be encoded by an element $c \in (\mathbb{R}^N)^*$; choose c so that

$$F_i = F_{\{i\}} = \{\alpha \in \mathfrak{t}^* | \langle \alpha, x_i \rangle = c_i\}. \tag{6.2}$$

In fact, the map $\omega \mapsto c$ gives an isomorphism of $H^2(M, \mathbb{R})$ with $\mathfrak{k}^* = (\mathbb{R}^N)^*/\mathfrak{t}^*$ (see [5, §3]).

Remark 6.3. Fix $\alpha \in \mathfrak{t}^*$ which is not in any F_i. Let ω' be another closed 2-form—one which has a moment map Φ' which corresponds to a $c' \in (\mathbb{R}^N)^*$ such that (1) $\langle \alpha, x_i \rangle \neq c_i'$, and (2) $\langle \alpha, x_i \rangle < c_i'$ exactly if $\langle \alpha, x_i \rangle < c_i$. Let d' be the associated degree function. Then Φ and Φ' induce homotopic maps from $\partial(M/T)$ to $\mathfrak{t}^* \setminus \alpha$. Therefore, $d'(\alpha) = d(\alpha)$.

Fix $\alpha \in \mathfrak{t}^*$ which is not in any F_i. Choose $\beta \in S^{n-1} \subset \mathfrak{t}^*$ so that the ray $r = \alpha + \mathbb{R}^+\beta$ avoids all F_J's for $|J| \geq 2$. This is possible because these are planes of codimension ≥ 2 in $\mathfrak{t}^*$.

Lemma 6.4.

$$d(\alpha) = \sum_{i\,:\, r\cap F_i \neq \emptyset} \operatorname{sign}\langle \beta, x_i\rangle d_i(r\cap F_i),$$

where d_i is defined with respect to the map $\Phi|_{M_i}: M_i \to F_i$, as in Definition 4.1.

Proof. Define $\tilde{\Phi}: M_{\text{sing}}/T \to S^{n-1}$ by $\tilde{\Phi}(p) = (\overline{\Phi}(p) - \alpha)/\|\overline{\Phi}(p) - \alpha\|$. Assume β is a regular value of $\tilde{\Phi}$. Then, the degree of $\overline{\phi}$ is given by

$$d(\alpha) = \sum_{[p]\in\tilde{\Phi}^{-1}(\beta)} \operatorname{sign}(\det d\tilde{\Phi}|_{[p]}).$$

Finally, $\operatorname{sign}(\det d\tilde{\Phi}|_{[p]}) = \operatorname{sign}\langle \beta, x_i\rangle \operatorname{sign}(\det d\overline{\Phi}|_{[p]})$ and $\tilde{\Phi}^{-1}(\beta) = \bigcup_i \overline{\Phi}^{-1}(r\cap F_i)$.

Remark 6.5. (1) By Lemma 6.4, d is locally constant on $\mathfrak{t}^* \setminus \bigcup F_i$. (2) Additionally, suppose that for $\alpha_1, \alpha_2 \in \mathfrak{t}^*$ the interval $\overline{\alpha_1\alpha_2}$ intersects the wall F_i transversely at γ, and does not intersect any other F_j. Then

$$d(\alpha_2) - d(\alpha_1) = \operatorname{sign}\langle \alpha_1 - \alpha_2, x_i\rangle d_i(\gamma).$$

Two walls may coincide; in this case, the right-hand side should be replaced by the sum of their individual contributions. Any function which satisfies (1) and (2) differs from d by a global constant.

The following definition is a formal recipe for constructing the pictures in Figure 4.

Definition 6.6. A *twisted polytope* Δ consists of the following data:

(i) a fan Σ in $\mathfrak{t}^*$ over $\{x_1, \cdots, x_N\}$;
(ii) a vector $c \in (\mathbb{R}^N)^*$.

This data determines a "degree" function which is constructed inductively in the following way. For $i \in \mathbf{N}$, let $F_i = \{\alpha \in \mathfrak{t}^* | \langle \alpha, x_i\rangle = c_i\}$. Let $\hat{\Sigma}$ be the fan relative to $\measuredangle x_i$ as in Definition 3.5. Choose any $\beta \in F_i$; this induces an isomorphism of F_i with $\hat{\mathfrak{t}}^*$, which sends β to zero. Let $\hat{c}_l = c_l - \langle \beta, x_l\rangle$ for all $l \in \hat{L}$. Then by hypothesis we can construct the twisted polytope Δ_i in F_i from $\hat{\Sigma}$, $\hat{c}$. Note that Δ_i are independent of the choice of β. The Δ_i are the "faces" of Δ. If we denote by d and d_i the degree functions on Δ and Δ_i respectively, then d is defined on $\mathfrak{t}^* \setminus \bigcup F_i$ by the formula in Lemma 6.4.

Example 6.7. Take the fan in Figure 2 and let $c = (2, -1, 2, -1, 2, -1)$. To get the twisted polytope, first draw the hyperplanes F_i in

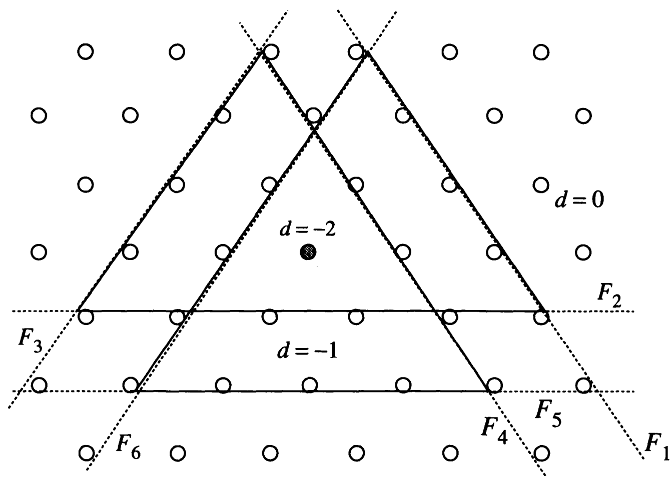


FIGURE 5

$\mathfrak{t}^* = \mathbb{R}^2$, see Figure 5. We drew them inaccurately; not through the lattice dots but on their edges; so that F_i is pushed a bit in the direction of x_i. The reason for this will be revealed in Theorem 3. The vertices are $v_i = F_{i-1} \cap F_i$ cyclically, and the edges are $\overline{v_i v_{i+1}}$.

Lemma 6.8. *Let (M, T) be a toric manifold. Let ω_1, ω_2 be invariant, closed 2-forms on M and let Φ_1, Φ_2 be the corresponding moment maps. Assume that $\Phi_1(p) = \Phi_2(p)$ for every $p \in M$ which is fixed by T. Then $\Phi_{1*}\omega_1^n = \Phi_{2*}\omega_2^n$.*

Proof. Assume this for toric manifolds of lower dimensions. Noting that both maps send M_i to the same hyperplane F_i, we apply the induction hypothesis; $\Phi_{1*}(\omega_1^{n-1}|_{M_i}) = \Phi_{2*}(\omega_2^{n-1}|_{M_i})$ as measures on F_i. Therefore, the result follows from Lemma 6.4 and Theorem 1.

Corollary 6.9. *Let (M, T) be a toric manifold. Let ω_1, ω_2 be invariant, closed 2-forms on M which represent the same class in $H^2(M)$. Let Φ_1, Φ_2 be the corresponding moment maps. Then $\Phi_{1*}\omega_1^n$ and $\Phi_{2*}\omega_2^n$ differ by a translation in $\mathfrak{t}^*$.*

Proof. Write $\omega_2 = \omega_1 + d\theta$, for some T-invariant 1-form θ. Define $\tilde{\Phi}_2 \colon M \to \mathfrak{t}^*$ by

$$\langle \tilde{\Phi}_2, \xi \rangle = \langle \Phi_1, \xi \rangle + i(\xi_M)\theta \quad \text{for all } \xi \in \mathfrak{t}.$$

$\tilde{\Phi}_2$ is a moment map for (M, T, ω_2), so it differs from Φ_2 by a translation. By the preceding lemma, it is enough to show that $\Phi_1(p) = \tilde{\Phi}_2(p)$ for every fixed point p. But this is easy; if p is a fixed point, then $\xi_M|_p = 0$, so $\langle \Phi_1 - \tilde{\Phi}_2, \xi \rangle_p = -i(\xi_M)\theta|_p = 0$.

7. The index

Let (M, T) be a toric manifold. Let L be a T-equivariant holomorphic line bundle over M and let $\mathscr{O}_L$ be the sheaf of holomorphic sections.

Definition 7.1. The *index* of L is $\sum_{i=0}^{N}(-1)^i H^i(M, \mathscr{O}_L)$. The function $\nu: \ell^* \to \mathbb{Z}$ assigns to each weight α its multiplicity in the index, considered as a virtual representation of T.

Let θ be any invariant connection 1-form on L with curvature ω. The lifting of the T-action from M to L determines a moment map $\Phi: M \to \mathfrak{t}^*$ for (M, T, ω) by $\langle \Phi, \eta \rangle = i_{\eta_L} \theta$ for all $\eta \in \mathfrak{t}$.

Theorem 2. *If* $\alpha \in \ell^* \setminus \operatorname{Im}(S^{n-1})$, *then* $\nu(\alpha) = d(\alpha)$.

As stated, this theorem only applies to $\alpha \notin \operatorname{Im}(S^{n-1})$. In fact, using a small technical trick, we can determine $\nu(\alpha)$ for all $\alpha \in \ell^*$; see Theorem 3 in §10.

Remark 7.2. Let L be a T-equivariant holomorphic line bundle over M. Then the action can be uniquely extended to a holomorphic action of $T_\mathbb{C}$ generated by the vector fields ξ_M and $(i\xi)_M = J\xi_M$, where $\xi \in \mathfrak{t}$, and $J: TM \to TM$ is the complex structure. Therefore we may restrict our attention to $T_\mathbb{C}$-equivariant holomorphic bundles.

8. Upstairs/downstairs

In this section we show that we can carry out computations in U_Σ instead of in M. It is easier to work with the space U_Σ.

Remember that $(\mathbb{C}^\times)^N$ acts naturally on $U_\Sigma \subset \mathbb{C}^N$, $G \subseteq (\mathbb{C}^\times)^N$ acts freely on U_Σ, $M = U_\Sigma/G$, and $T_\mathbb{C} = (\mathbb{C}^\times)^N/G$. Therefore, we can pull back any holomorphic $T_\mathbb{C}$-equivariant line bundle over M to a holomorphic $(\mathbb{C}^\times)^N$-equivariant line bundle over U_Σ. Conversely, if L is any $(\mathbb{C}^\times)^N$-equivariant line bundle over U_Σ, then L/G is a $T_\mathbb{C}$-equivariant line bundle over M. These constructions give an isomorphism between the equivariant Picard groups of M and U_Σ.

Let c be any weight of $(\mathbb{C}^\times)^N$, i.e., $c \in (\mathbb{Z}^N)^*$. Let ρ be the character with weight c, i.e., $\rho(\lambda) = \lambda^c = \lambda_1^{c_1} \cdots \lambda_N^{c_N}$ for any $\lambda \in (\mathbb{C}^\times)^N$. Then we construct an equivariant line bundle L_c over U_Σ: As a holomorphic line bundle, $L_c = U_\Sigma \times \mathbb{C}$; $(\mathbb{C}^\times)^N$ acts by $\lambda(z, x) = (\lambda z, \rho(\lambda)x)$ for any $\lambda \in (\mathbb{C}^\times)^N$.

Remark 8.1. Fix $i \in \mathbf{N}$, and embed $\mathbb{C}^\times = \mathbb{C}_i^\times \subseteq (\mathbb{C}^\times)^N$ as the ith factor. If $z \in U_\Sigma$ and $z_i = 0$, then $\lambda \in \mathbb{C}_i^\times$ acts on the fiber above

z as multiplication by λ^{c_i}. Moreover, let p be the image of z in M, i.e., $p \in M_i$. The image of $\mathbb{C}_i^\times$ in $T_{\mathbb{C}}$ is $\widehat{\exp}(\mathbb{C}x_i)$, and acts on the fiber $(L_c/G)|_P$ with weight c_i.

Lemma 8.2. *Let L be an equivariant holomorphic line bundle over U_Σ. Then L is isomorphic to L_c for some weight $c \in (\mathbb{Z}^N)^*$.*

Proof. Let $z \in U_\Sigma$. If $z_i = 0$, then $\mathbb{C}_i^\times$ acts on the fiber above z by $\lambda: x \mapsto \lambda^{c_i}x$ for some $c_i \in \mathbb{Z}$. c_i is independent of the choice of z_i. In this way we determine $c = (c_i) \in (\mathbb{Z}^N)^*$. It suffices to show that $L \otimes L_c^{-1}$ is trivial, i.e., that is has a global, invariant, nonvanishing holomorphic section. It is easy to find such a section over the subset $(\mathbb{C}^\times)^N$; take any nonzero $(\mathbb{C}^\times)^N$ orbit. Moreover, this section extends continuously to a section over all of U_Σ with the desired properties.

Remark 8.3. Recall, from §6, that $\Phi_*\omega^n$ is determined by a vector $c \in (\mathbb{R}^N)^*$. As we shall see in Lemma 10.1, this is the same as the $c \in (\mathbb{Z}^N)^*$ associated to a line bundle L over M, when ω is the curvature of L.

Let $\mathscr{O}$ be the sheaf of holomorphic functions on U_Σ (with $(\mathbb{C}^\times)^N$ acting trivially on the fiber). For any representation R and weight α, denote the corresponding weight space by R_α. Recall that $\pi^*: \mathfrak{t}^* \to (\mathbb{R}^N)^*$ sends ℓ^* into $(\mathbb{Z}^N)^*$.

Lemma 8.4. *For $\alpha \in \ell^*$, $H^0(M, \mathscr{O}_L)_\alpha = H^0(U_\Sigma, \mathscr{O})_{\pi^*(\alpha)-c}$.*

Proof. The sections of $L = L_c/G$ are exactly the G-invariant sections of L_c. A section of L_c is given by a holomorphic function f on U_Σ. $(\mathbb{C}^\times)^N$ acts on sections by $(\lambda f)(z) = \rho(\lambda)f(\lambda^{-1}z)$. f is given by its Laurent series, and it is G-invariant if and only if each monomial in the series is invariant.

Consider $f(z) = z^{-\xi}$ where $\xi \in (\mathbb{Z}^N)^*$. Then $(\lambda f)(z) = \lambda^{\xi+c}f(z)$; this monomial is an eigenvector with weight $\xi + c$. Therefore f is G-invariant if and only if $\lambda^{\xi+c} = 1$ for all $\lambda \in G$. Equivalently, by (2.5), $(\widehat{\exp}(\zeta))^{\xi+c} = e^{2\pi i\langle\zeta, \xi+c\rangle} = 1$ for all $\zeta \in \mathbb{C}^N$ such that $\pi(\zeta) \in \ell$. So f is G invariant if and only if Mike's dog really ate his frog [8] if and only if $\pi(\zeta) \in \ell$ implies $\langle\zeta, \xi + c\rangle \in \mathbb{Z}$, i.e., $\xi + c = \pi^*(\alpha)$ for some $\alpha \in \ell^*$. The weight for the action of T on f as a section of L is α. In contrast, $\xi = \pi^*(\alpha) - c$ is the weight of $(\mathbb{C}^\times)^N$ on f as a section on the trivial bundle over U_Σ.

Lemma 8.5. *For $\alpha \in \ell^*$, $H^i(M, \mathscr{O}_L)_\alpha = H^i(U_\Sigma, \mathscr{O})_{\pi^*(\alpha)-c}$.*

Proof. Define an open cover for U_Σ.

$$\mathfrak{A} = \{U_I | \Delta_I \in \Sigma\}, \quad \text{where } U_I = \mathbb{C}^I \times (\mathbb{C}^\times)^{\mathrm{N}\setminus I}.$$

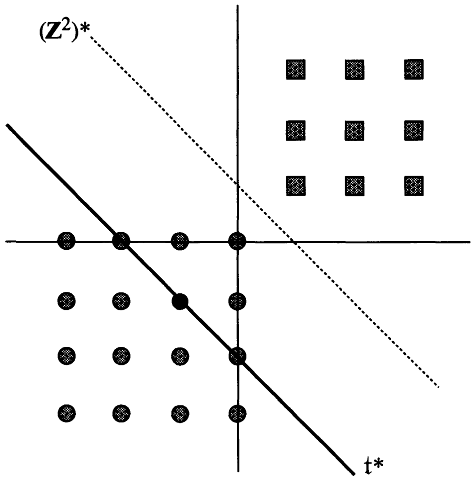


FIGURE 6

The Čech cochains corresponding to this cover are $\check{C}^i(\mathfrak{A}, \mathcal{O}) = \bigoplus H^0(U_{I_0} \cap \cdots \cap U_{I_i}, \mathcal{O})$. Arguing as in Lemma 8.4, $\check{C}^i(\mathfrak{A}/G, \mathcal{O}_L)_\alpha = \check{C}^i(\mathfrak{A}, \mathcal{O})_{\pi^*(\alpha)-c}$. These isomorphism commute with the boundary maps, so

$$\check{H}^i(\mathfrak{A}/G, \mathcal{O}_L)_\alpha = \check{H}^i(\mathfrak{A}, \mathcal{O})_{\pi^*(\alpha)-c}.$$

Moreover, $U_{I_0} \cap \cdots \cap U_{I_i}$ and $(U_{I_0} \cap \cdots \cap U_{I_i})/G$ are products of $\mathbb{C}$'s and $\mathbb{C}^\times$'s (see [3, §5.2]); thus $\mathfrak{A}$ and $\mathfrak{A}/G$ are good covers. Therefore, by Leray's theorem [7, §0.3],

$$\check{H}^i(\mathfrak{A}/G, \mathcal{O}_L) = H^i(M, \mathcal{O}_L) \quad \text{and} \quad \check{H}^i(\mathfrak{A}, \mathcal{O}) = H^i(U_\Sigma, \mathcal{O}).$$

Definition 8.6. The function $\mu\colon (\mathbb{Z}^N)^* \to \mathbb{Z}$ associates to each weight ξ its multiplicity in the index over U_Σ; $\mu(\xi) = \Sigma(-1)^i \dim(H^i(U_\Sigma, \mathcal{O})_\xi)$.

By Lemma 8.2, the equivariant line bundle L over M gives rise to an embedding $j\colon \mathfrak{t}^* \to \mathbb{R}^{N^*}$ which sends α to $\pi^*(\alpha) - c$. Then, for $\alpha \in \ell^*$,

$$\nu(\alpha) = \mu(j(\alpha)) \tag{8.7}$$

by Lemma 8.5. Therefore, it will be sufficient to compute the function μ.

Example 8.8. Consider the action of S^1 on $\mathbb{CP}^1$ as in Examples 2.7 and 5.5. The map $\pi^*\colon \mathbb{R} \to \mathbb{R}^2$ sends α to $(\alpha, -\alpha)$. Let L be the tangent bundle of $\mathbb{CP}^1$. The S^1 action naturally lifts to L. Let $c = (1, 1) \in (\mathbb{Z}^2)^*$, i.e., let $(\mathbb{C}^\times)^2$ act on $U_\Sigma \times \mathbb{C}$ by $(\lambda_0, \lambda_1)(z_0, z_1, x) = (\lambda_0 z_0, \lambda_1 z_1, \lambda_0\lambda_1 x)$. Then $L = L_c/G$. Therefore, $j(\alpha) = (\alpha - 1, -\alpha - 1)$

embeds $\mathbb{R}$ in $\mathbb{R}^2$ as the solid diagonal line in Figure 6 where the black dot is the origin of $\mathbb{R}$.

9. The index over U_Σ

In this section we compute the function μ defined in Definition 8.6. Because each $\check{C}^i(\mathfrak{A}, \mathcal{O})_\xi$ is finite dimensional, $\mu(\xi) = \Sigma(-1)^i \check{C}^i(\mathfrak{A}, \mathcal{O})_\xi$; this is easier to compute.

Example 9.1. In Example 8.8, $U_\Sigma = \mathbb{C}^2 \setminus \{0\}$. Consider the covering $\mathfrak{A} = \{U_1, U_2\}$, where $U_1 = \mathbb{C} \times \mathbb{C}^\times$ and $U_2 = \mathbb{C}^\times \times \mathbb{C}$. The essential idea is very simple: z is a holomorphic function on $\mathbb{C}^\times$ and on $\mathbb{C}$. In contrast, z^{-1} is holomorphic on $\mathbb{C}^\times$ but is not holomorphic on $\mathbb{C}$. The monomial $z_0^{-i} z_1^{-j}$ is holomorphic on U_1 if and only if $i \le 0$ and it is holomorphic on U_2 if and only if $j \le 0$. Every monomial is holomorphic on $U_1 \cap U_2$. Therefore, $\dim \check{C}^1(\mathfrak{A}, \mathcal{O})_{i,j} = 1$ for all i, j and

$$\dim \check{C}^0(\mathfrak{A}, \mathcal{O})_{i,j} = \begin{cases} 2 & \text{if } i \le 0 \text{ and } j \le 0, \\ 1 & \text{if } i > 0 \text{ and } j \le 0, \text{ or vice versa}, \\ 0 & \text{if } i > 0 \text{ and } j > 0. \end{cases}$$

Taking the alternating sum:

$$\mu(i, j) = \begin{cases} 1 & \text{if } i \le 0 \text{ and } j \le 0, \\ -1 & \text{if } i > 0 \text{ and } j > 0, \\ 0 & \text{otherwise}. \end{cases}$$

This is illustrated in Figure 6, where circles represent multiplicity 1, and squares represent multiplicity -1. Notice that the index of the tangent bundle is three-dimensional. As an additional example, the dotted line represents the tautological bundle over $\mathbb{CP}^1$, for which $H^i = 0$ for all i.

In the general case, let H_i be the half-space $\{\zeta \in (\mathbb{Z}^N)^* | \zeta_i \le 0\}$. Let $H_I = \bigcap_{i \in I} H_i$. The monomial $z^{-\xi}$ is holomorphic on U_I exactly if ξ is in H_I. Any holomorphic function on U_I is given by its Laurent series:

$$\sum_{\xi \in H_I} \lambda_\xi z^{-\xi}.$$

Therefore, the multiplicity of ξ in the representation $\Gamma(U_I, \mathcal{O})$ is 1 if $\xi \in H_I$, and is 0 otherwise. Since $U_{I_0} \cap \cdots \cap U_{I_i} = U_{I_0 \cap \cdots \cap I_i}$, we have

Lemma 9.2. *$H^i(U_\Sigma, \mathscr{O})_\xi$, and hence $\mu(\xi)$, depends only on whether $\xi_i \leq 0$ or $\xi_i > 0$ for $i \in \mathbf{N}$.*

We now determine how $\mu(\xi)$ changes as ξ passes through the coordinate hyperplanes. Let $\xi, \xi' \in (\mathbb{Z}^N)^*$. Without loss of generality, $\xi'_i = \xi_i$ for all $i \neq 1$, but $\xi'_1 \leq 0$ whereas $\xi_1 > 0$. Let $\widehat{\Sigma}$ be the fan relative to x_1, as in Definition 3.5; let $U_I = \mathbb{C}^I \times (\mathbb{C}^\times)^{\widehat{L}\setminus I}$, and let $\widehat{\mathfrak{A}} = \{\widehat{U}_I | \mathcal{A}_I \in \widehat{\Sigma}\}$. Define $\hat{\xi} \in (\mathbb{Z}^{\widehat{L}})^*$ by $\hat{\xi}_l = \xi_l$ for all $l \in \widehat{L}$, and $\hat{\mu}\colon (\mathbb{Z}^{\widehat{L}})^* \to \mathbb{Z}$ by $\hat{\mu}(\hat{\xi}) = \Sigma(-1)^i \check{C}^i(\widehat{\mathfrak{A}}, \mathscr{O})_{\hat{\xi}}$. This is the multiplicity of $\hat{\xi}$ in the index of $U_{\widehat{\Sigma}}$.

Lemma 9.3. $\mu(\xi') - \mu(\xi) = \hat{\mu}(\hat{\xi})$.

Proof. Let $I \subseteq \mathbf{N}$, such that $\mathcal{A}_I \in \Sigma$. If $1 \notin I$, then $z^{-\xi}$ is holomorphic on U_I exactly if $z^{-\xi'}$ is holomorphic. If $1 \in I$, then $z^{-\xi}$ is not holomorphic on U_I. In contrast, let $\hat{I} = I \setminus \{1\}$, then, since $\hat{I} \subseteq \widehat{L}$, $z^{-\xi'}$ will be holomorphic on U_I if and only if $z^{-\hat{\xi}}$ is holomorphic on $\widehat{U}_{\hat{I}}$. So $\dim(\Gamma(U_I, \mathscr{O})_{\xi'}) - \dim(\Gamma(U_I, \mathscr{O})_\xi) = \dim(\Gamma(\widehat{U}_{\hat{I}}, \mathscr{O})_{\hat{\xi}})$. Therefore,

$$\sum_{i=1}^{N}(-1)^i \dim(\check{C}^i(\mathfrak{A}, \mathscr{O})_{\xi'}) - \sum_{i=0}^{N}(-1)^i \dim(\check{C}^i(\mathfrak{A}, \mathscr{O})_\xi)$$
$$= \sum_{i=0}^{N}(-1)^i \dim(\check{C}^i(\widehat{\mathfrak{A}}, \mathscr{O})_{\hat{\xi}}).$$

10. Proof of Theorem 2

We can now prove Theorem 2 by induction; assume that $\nu = d$ for $(n-1)$-dimensional toric manifolds.

Let us review some notation. (M, T) is the toric manifold associated to the fan Σ. $L = L_c/G$ is an equivariant holomorphic line bundle over M (§8). Construct ω, θ, and Φ as in §7. For any $i \in \mathbf{N}$, M_i is the corresponding toric submanifold of dimension $n-1$, as in §6. $F_i \subseteq \mathfrak{t}^*$ is the hyperplane perpendicular to x_i which contains $\Phi(M_i)$.

We know how d and ν change as we cross the walls F_i and $j^{-1}(E_i)$ respectively. To show that $\nu = d$, we first need:

Lemma 10.1. *Let E_i be the ith coordinate plane in $(\mathbb{R}^N)^*$; then $j^{-1}(E_i) = F_i$.*

Proof. Choose any $p \in M_i$ and let $\alpha = \Phi(p)$. Let ξ be the vector field on M which generates the action of the circle $(S^1)_i = \widehat{\exp}(\mathbb{R}x_i) \subset T$.

By Remark 8.1, $(S^1)_i$ acts on the fiber over p with weight c_i, so $i_\xi\theta|_p = c_i$. But this is exactly $\langle\Phi(p), x_i\rangle$, by the construction of Φ. Therefore, $\langle\pi^*(\alpha) - c, e_i\rangle = \langle\Phi(p), x_i\rangle - c_i = 0$, i.e., $j(\alpha) \in E_i$. q.e.d.

If $x_i = -x_j$, then it is possible that $F_i = F_j$. For simplicity, we will assume that this does not happen. By Remark 6.5, the following three lemmas imply that $\nu = d$.

Lemma 10.2. *$H^l(M, \mathscr{O}_L)_\alpha$ and $\nu(\alpha)$ only depend on whether $\langle\alpha, x_i\rangle \leq c_i$ or $> c_i$ for all $i \in \mathbf{N}$.*

Proof. This follows immediately from Lemmas 10.1 and 9.2, and (8.7).

Lemma 10.3. *Assume that for $\alpha_1, \alpha_2 \in \mathfrak{t}^*$ the interval $\overline{\alpha_1, \alpha_2}$ intersects the wall F_i transversely at γ, and does not intersect any other F_j. Then*

$$\nu(\alpha_2) - \nu(\alpha_1) = \operatorname{sign}\langle\alpha_1 - \alpha_2, x_i\rangle d_i(\gamma).$$

Proof. Define $\hat{c} \in (\mathbb{Z}^{\widehat{L}})^*$ by $\hat{c}_l = c_l$ $\forall l \in \widehat{L}$. Then $L|_{M_i} = L_{\hat{c}}/\widehat{G}$, in the notation of Lemma 6.1. By the induction hypothesis, for any $\widehat{\alpha} \in \hat{\mathfrak{t}}^*$, $d_i(\widehat{\alpha}) = \hat{\nu}(\widehat{\alpha})$. By (8.7), $\hat{\nu}(\widehat{\alpha}) = \hat{\mu}(\hat{\pi}^*(\widehat{\alpha}) - \hat{c})$. Let $\alpha \in \mathfrak{t}^*$ be the image of $\widehat{\alpha}$ under the natural map from $\hat{\mathfrak{t}}^*$ to $\mathfrak{t}^*$. Let $\xi = \pi^*(\alpha) - c$, and $\hat{\xi} = \hat{\pi}^*(\widehat{\alpha}) - \hat{c}$. Then $\hat{\xi}_l = \xi_l$ for all $l \in \widehat{L}$. The lemma now follows from Lemma 9.3 and (8.7).

Lemma 10.4. *There exists $\alpha \in \mathfrak{t}^*$ such that $\nu(\alpha) = d(\alpha)$.*

Proof. Choose any $\beta \in \mathfrak{t}^*$ such that $\langle\beta, x_i\rangle \neq 0$ for all i. Choose $m \in \mathbb{Z}$ such that $m|\langle\beta, x_i\rangle| > |c_i|$ for all i. By the previous lemma, $n \in \mathbb{Z}$ and $n \geq m$ imply that $H^i(M, \mathscr{O}_L)_{m\beta} = H^i(M, \mathscr{O}_L)_{n\beta}$. Because M is compact, $H^i(M, \mathscr{O}_L)$ is finite dimensional; therefore, $H^i(M, \mathscr{O}_L)_{m\beta} = 0$. On the other hand, $d(m\beta) = 0$ for large m because d is compactly supported. Thus, $\nu(m\beta) = 0 = d(m\beta)$.

We are now almost finished. However, we still wish to determine $\nu(\alpha)$ for $\alpha \in F_i$; we do this by shifting the walls F_j slightly in the "positive" direction. Formally, define c' in $(\mathbb{Z}^N)^*$ by $c'_i = c_i + \frac{1}{2}$ for all $i \in \mathbf{N}$. Remember that a degree function is determined by any N-tuple in $(\mathbb{R}^N)^*$, as in Definition 6.6. Let d' be the degree function associated to c'. Then $d'(\alpha)$ is defined for all $\alpha \in \mathfrak{t}^*$, and $d'(\alpha) = d(\alpha)$ wherever the latter is defined.

Theorem 3. *$\nu(\alpha) = d'(\alpha)$ for all $\alpha \in \mathfrak{t}^*$.*

Proof. Let $\xi = \pi^*(\alpha) - c$. Define $\tilde{c}$ in $(\mathbb{Z}^N)^*$ by $\tilde{c}_i = c_i$ if $\xi_i \neq 0$, $\tilde{c}_i = c_i + 1$ if $\xi_i = 0$. Let $\tilde{d}$ be the degree function associated to $\tilde{c}$. Let $\tilde{\xi} = \pi^*(\alpha) - \tilde{c}$. It is clear from Lemma 9.2 that $\mu(\xi) = \mu(\tilde{\xi})$. Then, $\nu(\alpha) = \mu(\xi) = \mu(\tilde{\xi}) = \tilde{\nu}(\alpha)$ where $\tilde{\nu} = \mu(\pi^*(\alpha) - \tilde{c})$. By Theorem 2, $\tilde{d}(\alpha) = \tilde{\nu}(\alpha)$. Finally, it follows directly from Remark 6.3 that $\tilde{d}(\alpha) = d'(\alpha)$. q.e.d.

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