J. DIFFERENTIAL GEOMETRY
38 (1993) 235-261

NONUNIFORM HYPERBOLIC LATTICES AND EXOTIC SMOOTH STRUCTURES

F. T. FARRELL & L. E. JONES

0. Introduction

Let Θ_m denote the group of homotopy m-spheres where $m > 4$. Elements in Θ_m are equivalence classes of oriented manifolds homeomorphic to S^m. Two such manifolds Σ_1^m and Σ_2^m are equivalent provided there exists an orientation-preserving diffeomorphism between them. In this paper, $\mathbb{D}^{m+1}$ and S^m respectively denote the unit ball and unit sphere in $\mathbb{R}^{m+1}$; i.e.,

$$
\begin{aligned}
\mathbb{D}^{m+1} &= \{x \in \mathbb{R}^{m+1} \mid |x| \leq 1\}, \\
S^m &= \partial \mathbb{D}^{m+1} = \{x \in \mathbb{R}^{m+1} \mid |x| \leq 1\}.
\end{aligned}
\tag{0.01}
$$

Kervaire and Milnor proved in [13] that Θ_m is a finite abelian group.

Let M^m be a smooth m-dimensional manifold. A possible way to change its smooth structure, without changing its homeomorphism type, is to take its connected sum $M^m \# \Sigma^m$ with a homotopy sphere Σ^m. We showed in [9] that it is sometimes possible to change the smooth structure on a closed (real) hyperbolic manifold M^m in this way and still to have a negatively curved Riemannian metric on $M^m \# \Sigma^m$. But when M^m is noncompact (and connected), this method *never* changes the smooth structure on M^m. (See the proof of Corollary 1.5 for an argument verifying this statement.)

We use a different method in this paper, which can sometimes change the smooth structure on a noncompact manifold M^m. The method is to remove an embedded tube $S^1 \times \mathbb{D}^{m-1}$ from M^m and then reinsert it with a "twist". To be more precise, pick a smooth embedding $f: S^1 \times \mathbb{D}^{m-1} \to M^m$ and an orientation-preserving diffeomorphism $\varphi: S^{m-2} \to S^{m-2}$. Then a new smooth manifold $M_{f,\varphi}$ is obtained as a quotient space of the disjoint union

$$S^1 \times \mathbb{D}^{m-1} \amalg M^m - f(S^1 \times \operatorname{Int} \mathbb{D}^{m-1}), \tag{0.02}$$

Received September 9, 1991. Both authors were supported in part by the National Science Foundation.

where we identify points (x, v) and $f(x, \varphi(v))$ if $(x, v) \in S^1 \times S^{m-2}$. (Here, $\operatorname{Int}\mathbb{D}^{m-1}$ denotes the interior of D^{m-1}; i.e., it is $\mathbb{D}^{m-1} - S^{m-2}$.) The smooth manifold $M_{f,\varphi}$ is canonically homeomorphic to M^m but is not always diffeomorphic to M^m. We obtain the following result in this way. In this paper, $\mathbb{H}^m$ denotes real hyperbolic m-space, and $\operatorname{Iso}(\mathbb{H}^m)$ its group of isometries.

Theorem 0.1. *Let m be any integer such that $\Theta_{m-1} \neq 1$, and ε be any positive real number. Then there exists an m-dimensional complete Riemannian manifold M^m with finite volume and all its sectional curvatures contained in the interval $[-1-\varepsilon, -1+\varepsilon]$, and satisfying the following. M^m is not diffeomorphic to any complete Riemannian locally symmetric space; but it is homeomorphic to $\mathbb{H}^m/\Gamma$ where Γ is a torsion-free nonuniform lattice in $\operatorname{Iso}(\mathbb{H}^m)$.*

Let us recall some qualitative facts about Θ_n. Kervaire and Milnor [13] and Browder [4] showed that Θ_{2n-1} is nontrivial for every integer $n > 2$ which does *not* have the form $n = 2^i - 1$. On the other hand Θ_{12} is trivial. Therefore if M^{12} is a closed hyperbolic manifold, $M^{12} \# \Sigma^{12}$ must be diffeomorphic to M^{12} where Σ^{12} is any homotopy 12-sphere. The results in [9] consequently fail to yield an exotic smooth structure on a negatively curved 12-manifold which is homeomorphic to a (real) hyperbolic manifold. But our present technique does. In particular, we have the following result.

Theorem 0.2. *Let m be any integer such that either Θ_m or Θ_{m-1} is not trivial; e.g., m can be any integer greater than 6 provided it has neither of the following two forms; $m = 2^j - 2$ nor $m = 2^j - 3$, where $j \in \mathbb{Z}$. Then there exists a closed Riemannian manifold M^m whose sectional curvatures are all pinched within ε of -1 and such that*

1. $\dim M^m = m$;
2. *M^m is homeomorphic to a real hyperbolic manifold;*
3. *M^m is not diffeomorphic to any Riemannian locally symmetric space.*

We end this introduction with the following comments. The exotic Riemannian manifolds M^m constructed in this paper via Theorem 4.2 and Addendum 4.3 are all different from those exotic Riemannian manifolds $\mathcal{N}^m$ previously constructed via [9, Proposition 1.2]. That is, the only time M^m is diffeomorphic to $\mathcal{N}^m$ is when they are both diffeomorphic to a locally symmetric space and hence neither is exotic.

Finally, the results of this article should be useful in extending those of [10].

We wish to thank Ronnie Lee, whose question motivated this paper.

1. Exotic smooth structures

Let (M^m, f, φ) be a triple where M^m is a smooth manifold, $f\colon S^1 \times \mathbb{D}^{m-1} \to M^m$ is a smooth embedding (i.e., a *tube*) and $\varphi\colon S^{m-2} \to S^{m-2}$ is an orientation-preserving diffeomorphism. (We assume throughout this paper that $\dim M^m = m > 6$.) This data determines the new smooth manifold $M_{f,\varphi}$ constructed in §0. Recall that $M_{f,\varphi}$ is canonically homeomorphic to M^m. The purpose of this section is to give useful sufficient conditions which guarantee that $M_{f,\varphi}$ is not diffeomorphic to M^m.

Recall that two smooth structures N_0 and N_1 on a topological manifold N are *concordant* if there exists a smooth structure $\overline{N}$ on $N \times [0, 1]$ such that N_i, for $i = 0, 1$, is the induced smooth structure on $N \times i$. Two concordant structures are diffeomorphic. (See [14, pp. 24, 113–116].) The tube f determines a framed simple closed curve $\alpha\colon S^1 \to M^m$ where $\alpha(y) = f(y, 0)$ for each $y \in S^1$. The framing of α consists of the vector fields $X_1, X_2, \cdots, X_{m-1}$ where $X_i(y)$ is the vector tangent to the curve $t \mapsto f(y, te_i)$ at $t = 0$. Here e_i denotes the point in $\mathbb{R}^{m-1}$ whose i-th coordinate is 1 and all other coordinates are 0. We use α to denote the curve equipped with this framing. It is called the *core* of f. The concordance class of $M_{f,\varphi}$ depends only on M^m, the core α of f, and the (pseudo-)isotopy class of φ denoted by x. We consequently denote the concordance class of $M_{f,\varphi}$ by $M(\alpha, x)$. Recall that the isotopy classes of orientation-preserving diffeomorphisms of S^{m-2} are in one-one correspondence with the elements in the abelian group Θ_{m-1} which is also identified with $\pi_{m-1}(\mathrm{Top}/0)$; therefore, $x \in \Theta_{m-1}$.

Assume now that M^m is a complete (connected) Riemannian manifold with finite volume and whose sectional curvatures are all -1. The universal cover of M^m is real hyperbolic m-space $\mathbb{H}^m$ and $\pi_1 M^m$ is identified via the group of all deck transformations with a torsion-free lattice $\Gamma \subseteq \mathrm{Iso}\,\mathbb{H}^m$. We say that an element $\gamma \in \pi_1 M$ is *cuspidal* if there are arbitrarily short closed curves in M^m which are freely homotopic to a curve representing γ. This condition is equivalent to either $\gamma = 1$ or γ corresponds to a nonsemisimple matrix under the identifications

$$\pi_1 M = \Gamma \subset \mathrm{Iso}\,\mathbb{H}^m = O^+(m, 1, \mathbb{R}) \subset \mathrm{GL}_{m+1}(\mathbb{C}). \tag{1.0}$$

Note that the identity $1 \in \pi_1 M$ is the only cuspidal element when M is compact.

We now state the main result of this section. Recall that a π-manifold is a smooth manifold whose tangent bundle is stably trivial. Throughout this paper $\mathbb{Z}$ denotes the ring of all (rational) integers and $\mathbb{Z}_+$ denotes its additive group.

Theorem 1.1. *Let M^m be a complete (connected) Riemannian manifold with finite volume and all sectional curvatures -1. Let α be a framed simple closed curve in M^m and $[\alpha] \subset \pi_1 M$ be the fundamental group element it determines. Assume that M^m is a π-manifold and there exists a homomorphism $\eta\colon \pi_1 M \to \mathbb{Z}_+$ such that*

1. *$\eta([\alpha]) = 1$ and*
2. *$\eta(\gamma)$ is divisible by the order of Θ_{m-1} for every cuspidal element γ in $\pi_1 M$.*

Then M is diffeomorphic to a manifold in the concordance class $M(\alpha, x)$ only when $x = 0$.

The proof of this result requires some preliminaries. Represent $M(\alpha, x)$ by the manifold $M_{f,\varphi}$ and let $g\colon M_{f,\varphi} \to M$ be the canonical homeomorphism. Mostow's rigidity theorem [19], as extended by G. Prasad [21] to the finite volume situation, can be used together with its topological analogue [8, Corollary 10.3] and the topological analogue of Bieberbach's work on flat Riemannian manifolds [6], [7] to reduce Theorem 1.1 to the assertion that g is topologically concordant to a diffeomorphism only when $x = 0$. See the proof of [9, Lemma 2.1] for how to accomplish this reduction.

By [14, Theorem 10.1] there is an identification of the concordance classes of smooth structures on (the topological manifold) M and the homotopy classes of maps from M to Top/O, denoted by $[M, \mathrm{Top}/O]$, with the hyperbolic structure on M corresponding to the class of the constant map. The identification of Θ_m and $\pi_m(\mathrm{Top}/O)$, with S^m corresponding to 0 also follows from [14, Theorem 10.1]. Let $\hat{\alpha}\colon M^m \to S^{m-1}$ be the result of applying the Pontryagin-Thom construction to the framed 1-manifold α. It is explicitly described by

$$\hat{\alpha}(f(y, v)) = q(v), \tag{1.1.1}$$

where $(y, v) \in S^1 \times \mathbb{D}^{m-1}$, and $q\colon \mathbb{D}^{m-1} \to \mathbb{D}^{m-1}/\partial\mathbb{D}^{m-1} = S^{m-1}$ is the canonical quotient map; if $y \notin$ image f, then

$$\hat{\alpha}(y) = q(\partial\mathbb{D}^{m-1}).$$

The naturality of [14, Theorem 10.1] yields the following result.

Lemma 1.2. *The map $x \mapsto M(\alpha, x)$ is the homomorphism $\hat{\alpha}^*$: $\pi_{m-1}(\mathrm{Top}/O) \to [M, \mathrm{Top}/O]$ under the identifications of the previous paragraph.*

Hence Theorem 1.1 is equivalent to the assertion that $\hat{\alpha}^*$ is injective. We now embark on verifying this assertion.

Given an embedding $h\colon N \to \mathcal{N}$ where N and $\mathcal{N}$ are manifolds of the same dimension with N compact, there is a dual map $h'\colon \mathcal{N}/\partial\mathcal{N} \to N/\partial N$ defined by

$$(1.2.1)\qquad \begin{aligned} h'(h(y)) &= y, &&\text{if } y \in \operatorname{Int} N,\\ h'(y) &= \infty, &&\text{otherwise.}\end{aligned}$$

Here ∞ denotes the point corresponding to ∂N in the decomposition space $N/\partial N$. We also, when convenient, abbreviate $N/\partial N$ to N/∂.

Proposition 1.3. *Let N^m be a closed π-manifold, and $h\colon S^1 \times \mathbb{D}^{m-1} \to N^m$ be a tube with core $\alpha\colon S^1 \to N$. Assume there exists a map $\psi\colon N^m \to S^1$ such that the composite $\psi \circ \alpha\colon S^1 \to S^1$ has degree ± 1. Then*

$$(h')^*\colon [S^1 \times \mathbb{D}^{m-1}/\partial, \operatorname{Top}/O] \to [N, \operatorname{Top}/O]$$

is monic.

Proof. Recall that Top/O is an ∞-loop space (see [2, p. 215]). Let Y denote the $(m+1)$-fold delooping of Top/O; i.e., $\Omega^{m+1}Y = \operatorname{Top}/O$. Recall there is a natural bijection between $[\Sigma^{m+1}X, Y]$ and $[X, \Omega^{m+1}Y] = [X, \operatorname{Top}/O]$. Here $[\ ,\ ]$ denotes the homotopy classes of (base-point-preserving) maps and X is a space (with base point). Consequently, to prove Proposition 1.3, it suffices to show that

$$(1.3.1)\qquad (\Sigma^{m+1}h')^*\colon [\Sigma^{m+1}(S^1 \times \mathbb{D}^{m-1}/\partial), Y] \to [\Sigma^{m+1}N, Y]$$

is monic. Consider the codimension-0 embedding

$$(1.3.2)\qquad h \times \mathrm{id}\colon (S^1 \times \mathbb{D}^{m-1}) \times \mathbb{D}^{m+1} \to N^m \times \mathbb{D}^{m+1}$$

and observe that $(h \times \mathrm{id})'$ factors through $\Sigma^{m+1}(h')$. Hence it suffices to show that

$$(1.3.3)\qquad (h \times \mathrm{id})'^*\colon [S^1 \times \mathbb{D}^{m-1} \times \mathbb{D}^{m+1}/\partial, Y] \to [N \times \mathbb{D}^{m+1}/\partial, Y]$$

is monic. Let $F\colon N^m \times \mathbb{D}^{m+1} \to S^1 \times \mathbb{D}^{2m}$ be a codimension-0 embedding such that the composites $p \circ F$ and $\psi \circ q$ are homotopic, where p and q respectively denote the projections onto the first factors of $S^1 \times \mathbb{D}^{2m}$ and $N^n \times \mathbb{D}^{m+1}$. Note that F exists because N is a π-manifold. Using the fact that $(F \circ (h \times \mathrm{id}))' = (h \times \mathrm{id})' \circ F'$, we easily see that

$$(1.3.4)\qquad (h \times \mathrm{id})' \circ F'\colon (S^1 \times \mathbb{D}^{2m})/\partial \to (S^1 \times \mathbb{D}^{m-1} \times \mathbb{D}^{m+1})/\partial$$

is a homotopy equivalence. This completes the proof of Proposition 1.3. q.e.d.

An elaboration of this argument yields the following extension.

Addendum 1.4. *Let N^m be a compact connected π-manifold with (possibly) nonempty boundary. Let $h: S^1 \times \mathbb{D}^{m-1} \to \mathrm{Int}(N^m)$ be a tube with core $\alpha: S^1 \to N^m$. Suppose there exists a map $\psi: N^m \to S^1$ satisfying the following two properties.*

1. *The composite $\psi \circ \alpha : S^1 \to S^1$ has degree ± 1.*
2. *For any map $\beta: S^1 \to \partial N$, the degree of $\psi \circ \beta : S^1 \to S^1$ is divisible by the order of the group Θ_{m-1}.*

Then the composite

$$h'^* \circ p^* : \Theta_{m-1} = \pi_{m-1}(\mathrm{Top}/O) \to [\mathrm{Int}\, N^m , \mathrm{Top}/O]$$

is monic, where $p: S^1 \times \mathbb{D}^{m-1}/\partial \to \mathbb{D}^{m-1}/\partial$ is determined by projection onto the second factor of $S^1 \times \mathbb{D}^{m-1}$.

Proof. We can assume that $\psi \circ \alpha$ has degree one by composing ψ with a degree-one self map of S^1 if necessary. Let $F: N^m \times \mathbb{D}^{m+1} \to S^1 \times \mathbb{D}^{2m}$ be an embedding as in the proof of Proposition 1.3. We can easily arrange that F satisfies the following two additional properties:

$$\text{(1.4.1)} \qquad \mathrm{Image}\, F \cap (S^1 \times S^{2m-1}) = F(\partial N^m \times \mathbb{D}^{m+1}),$$

$$\text{(1.4.2)} \qquad F(h(u, v), w) = (u, (v/2, w/2))$$

for all $u \in S^1$, $v \in \mathbb{D}^{m-1}$ and $w \in \mathbb{D}^{m+1}$. Let $k: S^1 \times \mathbb{D}^{m-1} \to N^m$ denote the composition of h with the inclusion map $\mathrm{Int}(N^m) \subset N^m$. The argument given before shows that

$$(k \times \mathrm{id})'^* : [S^1 \times \mathbb{D}^{m-1} \times \mathbb{D}^{m+1}/\partial , Y] \to [N^m \times \mathbb{D}^{m+1}/\partial , Y]$$

is monic. (Note that $\partial(N^m \times \mathbb{D}^{m+1}) = \partial N^m \times \mathbb{D}^{m+1} \cup N^m \times S^m$.)

Let s denote the order of Θ_{m-1}, and $\Phi: S^1 \times \mathbb{D}^{2m}/\partial \to S^1 \times \mathbb{D}^{2m}/\partial$ be the map induced by the function $(z, u) \mapsto (z^s, u)$ where $z \in S^1$ and $u \in \mathbb{D}^{2m}$. Likewise let

$$P: S^1 \times \mathbb{D}^{m-1} \times \mathbb{D}^{m+1}/\partial \to \mathbb{D}^{m-1} \times \mathbb{D}^{m+1}/\partial$$

be determined by projection onto the last two factors of $S^1 \times \mathbb{D}^{m-1} \times \mathbb{D}^{m+1}$. Property (1.4.2) yields the following identity

$$\text{(1.4.3)} \qquad P \circ (F \circ (k \times \mathrm{id}))' \circ \Phi = P \circ (F \circ (k \times \mathrm{id}))'.$$

Let $\sigma: \partial N^m \times [0, 1] \to N^m$ be a collaring of N^m. To complete the proof, it suffices to show that the equation

$$\text{(1.4.4)} \qquad (\sigma \times \mathrm{id})'^*(a) = (k \times \mathrm{id})'^*(P^*(b))$$

only has solutions a and b when the element $b \in \Theta_{m-1}$ is zero.

To show that $b = 0$, apply $(F' \circ \Phi)^*$ to (1.4.4) and use identity (1.4.3) yielding

$$(1.4.5) \qquad ((F \circ (\sigma \times \mathrm{id}))' \circ \Phi)^*(a) = (F \circ (k \times \mathrm{id}))'^*(P^*(b)).$$

It can be shown, using hypothesis 2 of Addendum 1.4, that

$$((F \circ (\sigma \times \mathrm{id}))' \circ \Phi)^*(a)$$

is divisible by s. (Hint. The map $F \circ (\sigma \times \mathrm{id})\colon \partial N^m \times [0, 1] \times \mathbb{D}^{m+1} \to S^1 \times \mathbb{D}^{2m}$ lifts to the connected s-sheeted cover of $S^1 \times \mathbb{D}^{2m}$.) Since $(F \circ (k \times \mathrm{id}))'^*$ is an isomorphism, $P^*(b)$ is also divisible by s. But P^* is a monomorphism onto a direct summand; therefore, $b \in \Theta_{m-1}$ is divisible by s and hence $b = 0$. This completes the proof of Addendum 1.4. q.e.d.

Recall that M^m is the interior of a compact smooth manifold $\overline{M}^m$ and observe that the cuspidal elements in $\pi_1 M = \pi_1 \overline{M}$ are precisely those representable by curves in $\overline{M}$ which are freely homotopic to curves in $\partial \overline{M}$. If we set $N^m = \overline{M}^m$ and $h = f$, then $\hat{\alpha} = p \circ h'$ and hence Addendum 1.4 verifies the assertion, made in the sentence following Lemma 1.2, that $\hat{\alpha}^*$ is injective. This completes the proof of Theorem 1.1.

The first comment made at the end of the Introduction is explained by the following remark.

Corollary 1.5. *Let M^m be a manifold satisfying the hypotheses of Theorem* 1.1. *Let the homotopy sphere Σ^m represent an element in Θ_m and $x \in \Theta_{m-1}$. If $M^m \# \Sigma^m$ is diffeomorphic to a manifold in the concordance class $M(\alpha, x)$, then $M^m \# \Sigma^m$ is diffeomorphic to M^m.*

Proof. Let $M_{f,\varphi}$ be a manifold in $M(\alpha, x)$ where α is the core of f and the isotopy class of φ is x. Further, suppose $M^m \# \Sigma^m$ is diffeomorphic to $M_{f,\varphi}$. We first consider the case where M^m is not compact; then every map $M^m \to S^m$ is homotopic to a constant map. But the concordance class of $M^m \# \Sigma^m$ is in the image of

$$\hat{\gamma}^*\colon [S^m, \mathrm{Top}/O] \to [M^m, \mathrm{Top}/O],$$

where $\hat{\gamma}\colon M^m \to S^m$ is the result of the Pontryagin-Thom construction applied to a framed point $\gamma\colon * \to M^m$. Therefore $M^m \# \Sigma^m$ is concordant and hence diffeomorphic to M^m. (This argument, showing that $M^m \# \Sigma^m$ is diffeomorphic to M^m, is valid for any noncompact connected manifold M^m.)

We now assume that M^m is compact. Let $\gamma\colon \mathbb{D}^m \to S^1 \times \mathbb{D}^{m-1}$ be an orientation-preserving embedding. Also, let the maps $q\colon S^1 \times \mathbb{D}^{m-1}/\partial \to \mathbb{D}^{m-1}/\partial$ and $\omega\colon \mathbb{D}^{m-1}/\partial \to S^1 \times \mathbb{D}^{m-1}/\partial$ be respectively determined by projection onto the second factor of $S^1 \times \mathbb{D}^{m-1}$ and the inclusion

$$\mathbb{D}^{m-1} = 1 \times \mathbb{D}^{m-1} \subset S^1 \times \mathbb{D}^{m-1}.$$

Lemma 1.2 yields that $M(\alpha, x) = f'^*(q^*(x))$. By looking at [9, Proof of Proposition 1.2], we also see that $M \# \Sigma^m$ and $M \# (-\Sigma^m)$ are in the concordance classes of $f'^*(\gamma'^*(y))$ and $f'^*(\gamma'^*(-y))$, respectively, where $y \in \Theta_m$ is the concordance class of Σ^m. The argument proving [9, Addendum 2.3] yields that $M_{f,\varphi}$ is concordant to either $M \# \Sigma^m$ or $M \# (-\Sigma^m)$; therefore, either $f'^*(q^*(x))$ is equal to $f'^*(\gamma'^*(y))$ or to $f'^*(\gamma'^*(-y))$. Now f'^* is monic by Proposition 1.3 in which we set $N^m = M^m$ and $h = f$. Consequently, $q^*(x) = \gamma'^*(z)$ for some $z \in \Theta_m$. Since the composite $q \circ \omega = \mathrm{id}$, we have

$$x = \omega^*(q^*(x)) = (\gamma' \circ \omega)^*(z). \tag{1.5.1}$$

The map $\gamma' \circ \omega\colon S^{m-1} \to S^m$ is homotopic to a constant. Therefore, (1.5.1) implies that $x = 0$, which completes the proof of Corollary 1.5. q.e.d.

We end this section with a corollary of the Mostow-Prasad strong rigidity theorem.

Proposition 1.6. *Let M^m, with $m > 2$, be a complete (connected) Riemannian manifold with finite volume and all sectional curvatures -1. Let N^m be a complete Riemannian locally symmetric space. If M and N are homeomorphic, then they are isometrically equivalent (after rescaling the metric on N by a positive constant).*

Proof. This is an immediate consequence of the Mostow-Prasad strong rigidity theorem when N has finite volume [19, §24] and [21]; cf. [16, p. 334, Theorem 7.24]. Note that N must have finite volume when M is compact. Hence we now assume that M is not compact.

To show that N has finite volume, in this case, we argue as follows. Let $\widetilde{N}$ be the universal cover of N; then

$$\widetilde{N} = \mathbb{E}^k \times H_1, \tag{1.6.1}$$

where $\mathbb{E}^k$ is k-dimensional flat Euclidean space and H_1 is a symmetric space of noncompact type. (The DeRham decomposition of $\widetilde{N}$ has no compact factor since N is aspherical.) We proceed to show that $k = 0$.

Let

$$\Gamma \subseteq \mathrm{Iso}(\mathbb{E}^k \times H_1) = \mathrm{Iso}(\mathbb{E}^k) \times \mathrm{Iso}(H_1) \tag{1.6.2}$$

be the group of all deck transformations of $\widetilde{N} \to N$. (We denote by $\mathrm{Iso}(X)$ the group of all isometries of a Riemannian manifold X.) Recall that $\mathrm{Iso}(\mathbb{E}^k)$ is a semidirect product $\mathbb{R}^k \rtimes O(k)$. Hence the abelian Lie group $\mathbb{R}^k$ is a closed normal connected subgroup of $\mathrm{Iso}(\mathbb{E}^k \times H_1)$. Let

$$\pi\colon \mathrm{Iso}(\mathbb{E}^k \times H_1) \to \mathrm{Iso}(\mathbb{E}^k \times H_1)/\mathbb{R}^k \tag{1.6.3}$$

be the natural map and $U = \overline{\pi(\Gamma)}$ denote the closure of $\pi(\Gamma)$. The identity component U^0 of U is solvable by [22, Theorem 8.24]. If U^0 is not trivial, then $\pi(\Gamma) \cap U^0$ is a nontrivial normal solvable subgroup of $\pi(\Gamma)$, and Γ would consequently contain a nontrivial normal abelian subgroup. But this is impossible since Γ is isomorphic to a torsion-free lattice in $O^+(m, 1, \mathbb{R})$. Hence, $\pi\colon \Gamma \to \pi(\Gamma)$ is monic and $\pi(\Gamma)$ is a discrete subgroup of $\mathrm{Iso}(H_1)$. By looking at the cohomological dimension of Γ, it is now easily seen that $k = 0$.

Note that $\Gamma \cap \mathrm{Iso}(H_1)^0$ contains a free abelian subgroup A of rank $m-1$ since M^m has at least one cusp. Suppose A contains an element γ with nontrivial semisimple Jordan component s. Since A centralizes s, H_1 contains a *proper* A-invariant totally geodesic subspace H_2 which must be flat by [15] and have codimension one since H_2/A is compact. This forces $\widetilde{N} = H_1 = \mathbb{H}^2$. But this is impossible since $\dim \widetilde{N} > 2$. We therefore conclude that A contains only unipotent elements. Hence [16, Proposition 1.5] shows that A is contained in the unipotent radial $R_u(P)$ of a parabolic subgroup P of $\mathrm{Iso}(H_1)$. For cohomological dimension reasons, A is cocompact in $R_u(P)$. This forces the $\mathbb{R}$-split rank of $\mathrm{Iso}(H_1)$ to be 1 and $R_u(P)$ to be abelian by [22, p. 34, Corollary 2]. Hence $H_1 = \mathbb{H}^m$ after rescaling the metric on H_1 by a positive constant. It is now a routine exercise to see that H_1/Γ has finite volume. This completes the proof of Proposition 1.6.

2. Negative curvature and $M(\alpha, x)$

The symbol M^m will denote, for the rest of this paper, a complete (connected) Riemannian manifold of finite volume (possibly compact) and with all sectional curvatures -1. Also α will denote a simple orthonormally framed geodesic in M^m. It determines an immersion $\bar{\alpha}\colon S^1 \times \mathbb{R}^{m-1} \to M^m$ defined by

$$\bar{\alpha}(y\,;t_1, t_2, \cdots, t_{m-1}) = \exp_{\alpha(y)}\left(\sum_{i=1}^{m-1} t_i e_i(\alpha(y))\right), \tag{2.0}$$

where $y \in S^1$, $t_i \in \mathbb{R}$ and $e_1, e_2, \cdots, e_{m-1}$ is the *orthonormal framing* of α. For each nonnegative real number r and each subset $T \subseteq \mathbb{R}^s$, let rT denote the subset of $\mathbb{R}^s$ defined by

$$rT = \{ry \mid y \in T\}. \tag{2.0.1}$$

We say that α is the core of a *geometric tube of radius* r if the restriction of $\bar{\alpha}$ to $S^1 \times r\mathbb{D}^{m-1}$ is a smooth embedding. Denote the arc length of α by $|\alpha|$ and let the orthogonal matrix $A_\alpha \in O(m-1, \mathbb{R})$ be the holonomy around α. It is explicitly defined as follows where we regard $\alpha\colon \mathbb{R} \to M$ as a periodic function of period 2π and speed $|\alpha|/2\pi$. Let $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{m-1}$ be the parallel vector fields along α satisfying $\bar{e}_i(0) = e_i(0)$ for $i = 1, 2, \cdots, m-1$. Define a matrix $A_\alpha(t) \in O(m-1, \mathbb{R})$, for each $t \in \mathbb{R}$, by

$$e_i(t) = \sum_j (A_\alpha(t))_{ij} \bar{e}_j(t), \tag{2.0.2}$$

and then set $A_\alpha = A_\alpha(2\pi)$. We call the correspondence $t \mapsto A_\alpha(t)$ the *holonomy function associated to* α. The purpose of this section is to prove the following result.

Theorem 2.1. *Given real numbers $\varepsilon, l > 0$, an integer $m > 6$ and a map $A(t)\colon \mathbb{R} \to O(m-1, \mathbb{R})$, there exists a real number $r > 0$ such that the following statement is true for any pair (M^m, α) as above but subject to the following extra constraints*:

1. $|\alpha| = l$;
2. $A_\alpha(t) = A(t)$ *for all* $t \in \mathbb{R}$;
3. α *is the core of a geometric tube of radius* $2r$.

For any $x \in \Theta_{m-1}$, there exists a complete negatively curved Riemannian manifold N^m of finite volume in the concordance class $M(\alpha, x)$ and such that all the sectional curvatures of N^m lie in the interval $[-1-\varepsilon, -1+\varepsilon]$.

The following is the strategy used to prove this result. Pick a representative diffeomorphism $\varphi_x\colon S^{m-2} \to S^{m-2}$ for each element x in the *finite* group Θ_{m-1}. Let $f\colon S^1 \times \mathbb{D}^{m-1} \to M^m$ be the tube defined by

$$f(y, v) = \bar{\alpha}(y, 2rv). \tag{2.1.1}$$

The core of f is the geodesic α equipped with the framing $2re_1, 2re_2, \cdots, 2re_{m-1}$. But it is easily seen that M_{f,φ_x} is a smooth manifold in the concordance class $M(\alpha, x)$. It is the underlying smooth manifold of the posited Riemannian manifold N^m. To put the Riemannian metric on M_{f,φ_x}, we express it as the union of three manifolds

$$(2.1.2)\quad M^m - f(S^1 \times \operatorname{Int} \mathbb{D}^{m-1}), \quad S^1 \times \tfrac{1}{2}\mathbb{D}^{m-1}, \quad S^1 \times (\mathbb{D}^{m-1} - \tfrac{1}{2} \operatorname{Int} \mathbb{D}^{m-1}).$$

The Riemannian metric on M^m induces the Riemannian metrics on the first two submanifolds via the inclusion map and the embedding f, respectively. The third submanifold, which can be identified with the cylinder $(S^1 \times S^{m-2}) \times [\frac{1}{2}, 1]$, thus inherits a Riemannian metric on its top and bottom boundary components. We taper these Riemannian metrics together over the interior. The larger r is, the close to -1 is pinched the sectional curvatures in this tapered Riemannian metric. This tapering is accomplished by using Lemma 2.2 which we proceed to formulate.

Let $S^{m-2} \times \mathbb{R}$ denote the Riemannian symmetric space which is the metric product of S^{m-2} and the flat line $\mathbb{R}$. Let ψ be the isometry of S^{m-2} induced by $A(2\pi)$; i.e.,

$$(2.1.3)\qquad \psi(y) = yA(2\pi), \qquad y \in S^{m-2},$$

and let $T\colon \mathbb{R} \to \mathbb{R}$ be the translation $T(y) = y + l$, $y \in \mathbb{R}$. Then $\psi \times T$ is an isometry of $S^{m-2} \times \mathbb{R}$. Let $\mathscr{N}^{m-1}$ denote the orbit space of $S^{m-2} \times \mathbb{R}$ under the action of the infinite cyclic group of isometries generated by $\psi \times T$. It is a compact Riemannian locally symmetric space. Let $\pi\colon S^{m-2} \times \mathbb{R} \to \mathscr{N}^{m-1}$ denote the covering projection, and $\omega\colon \mathscr{N}^{m-1} \to (\frac{l}{2\pi})S^1$ be the map induced by projection onto the second factor of $S^{m-2} \times \mathbb{R}$. Here the circle $(\frac{l}{2\pi})S^1$ is identified with the orbit space of $\mathbb{R}$ under the action of the group generated by T. Consider the cylinder $\mathscr{N}^{m-1} \times [1, 2]$. Let ξ and γ be the distributions respectively tangent to the foliations $\{\mathscr{N}^{m-1} \times t \mid t \in [1, 2]\}$ and $\{y \times [1, 2] \mid y \in \mathscr{N}^{m-1}\}$. Let ξ_1 and ξ_2 be the subdistributions of ξ respectively tangent to the foliations

$$(2.1.4)\qquad \begin{gathered} \{\pi(S^{m-2} \times y) \times t \mid y \in \mathbb{R},\ t \in [1, 2]\} \quad \text{and} \\ \{\pi(y \times \mathbb{R}) \times t \mid y \in S^{m-2},\ t \in [1, 2]\}. \end{gathered}$$

Let B be any Riemannian metric on $\mathscr{N}^{m-1} \times [1, 2]$ satisfying

$$
\text{(2.1.5)} \qquad \begin{aligned} &(1)\quad \xi \perp \gamma \text{ and}\\ &(2)\quad B\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) \equiv 1, \end{aligned}
$$

where t is the second coordinate in the product structure $\mathcal{N}^{m-1} \times [1, 2]$. Given a positive real number r, construct a new Riemannian metric B_r on $\mathcal{N}^{m-1} \times [1, 2]$ by requiring the following properties:

$$
\text{(2.1.6)} \qquad \begin{aligned} &(1)\quad \xi \perp \gamma;\\ &(2)\quad B_r\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = r^2;\\ &(3)\quad B_r(u, v) = \sinh^2(rt)B(u, v), \text{ if } u, v \in \xi_1;\\ &(4)\quad B_r(u, v) = \sinh^2(rt)B(u, v), \text{ if } u \in \xi_1 \text{ and } v \in \xi_2;\\ &(5)\quad B_r(u, v) = \sinh^2(rt)B(u, v) + d\omega(u) \cdot d\omega(v), \text{ if } u, v \in \xi_2. \end{aligned}
$$

Lemma 2.2. *Let P denote an arbitrary 2-plane tangent to $\mathcal{N}^{m-1} \times [1, 2]$. Then*

$$
\lim_{r \to \infty} K_{B_r}(P) = -1
$$

uniformly in P, where $K_{B_r}(P)$ denotes the sectional curvature of P relative to B_r.

Before proving Lemma 2.2, we use it to prove Theorem 2.1 by implementing in detail the strategy outlined after the statement of this theorem. Define a diffeomorphism $\widehat{\Psi}: \mathbb{R} \times S^{m-2} \to S^{m-2}$ by the formula

$$
\widehat{\Psi}(t, y) = \left(yA(t), \frac{l}{2\pi}t\right). \tag{2.2.0.1}
$$

(Recall that $A_\alpha(t) = A(t)$.) It induces a diffeomorphism $\Psi: S^1 \times S^{m-2} \to \mathcal{N}^{m-1}$ since $\psi(y) = yA(2\pi)$; cf. (2.0.2) and (2.1.3).

Fix a Riemannian metric $B(\ ,\ )$ on $\mathcal{N}^{m-1} \times [1, 2]$ satisfying (2.1.5) together with the following additional properties where $B^t(\ ,\ )$ denotes the induced Riemannian metric on the hypersurface $\mathcal{N}^{m-1} \times t$, $t \in [1, 2]$:

$$
\text{(2.2.0.2)} \qquad \begin{aligned} &(1)\quad B^1 \text{ is the given locally symmetric metric on } \mathcal{N}^{m-1}.\\ &(2)\quad B^2 \text{ is the pullback of } B^1 \text{ under the composite}\\ &\qquad \text{diffeomorphism } \Psi \circ (\mathrm{id} \times \varphi_x) \circ \Psi^{-1}.\\ &(3)\quad B^t \text{ is constant in } t \text{ near } t = 1, 2. \end{aligned}
$$

If $s \in \mathbb{R}$ and $u = (t, y) \in S^1 \times \mathbb{R}^{m-1}$, define $s \cdot u = (t, sy)$. Let $h, h_x : \mathcal{N}^{m-1} \times (0, 2] \to M^m$ be the two embeddings defined by

$$\text{(2.2.0.3)} \qquad \begin{aligned} h(y, t) &= f(\tfrac{1}{2}t \cdot \Psi^{-1}(y)) \quad \text{and} \\ h_x(y, t) &= f(\tfrac{1}{2}t \cdot \mathrm{id} \times \varphi_x(\Psi^{-1}(y))), \end{aligned}$$

where $y \in \mathcal{N}^{m-1}$ and $t \in (0, 2]$. (Recall f was defined in (2.1.1).) Note that M_{f, φ_x} can be constructed by gluing $\mathcal{N}^{m-1} \times [1, 2]$ to $M^m - h(\mathcal{N}^{m-1} \times (1, 2))$ along the maps

$$\text{(2.2.0.4)} \qquad \begin{aligned} h &: \mathcal{N}^{m-1} \times 1 \to M^m - h(\mathcal{N}^{m-1} \times (1, 2)), \\ h_x &: \mathcal{N}^{m-1} \times 2 \to M^m - h(\mathcal{N}^{m-1} \times (1, 2)). \end{aligned}$$

Put a Riemannian metric $\langle \ , \ \rangle$ on M_{f, φ_x} as follows. It is the given hyperbolic metric on $M^m - h(\mathcal{N}^{m-1} \times (1, 2))$; while, restricted to $\mathcal{N}^{m-1} \times [1, 2]$, $\langle \ , \ \rangle$ is the Riemannian metric $B_r(\ , \ )$ of (2.1.6) constructed using the Riemannian metric $B(\ , \ )$ fixed in (2.2.0.2). We leave the reader an exercise to show that these two Riemannian metrics fit together. Use the following hints. The Jacobi fields in $\mathbb{H}^m$ can be explicitly calculated. This calculation shows that $\mathbb{H}^m$ is isometric to the warped product $\mathbb{H}^m \times_g \mathbb{R}$ where $g(p) = \cosh(\rho(p))$, $p \in \mathbb{H}^{m-1}$, and $\rho(p)$ denotes the distance between p and a fixed point $p_0 \in \mathbb{H}^{m-1}$. The set of points $p_0 \times \mathbb{R}$ is a geodesic line, and $\mathbb{H}^{m-1} \times 0$ is a totally geodesic subspace (isometric to $\mathbb{H}^{m-1}$) meeting this line perpendicularly at $p_0 \times 0$. Furthermore, $\mathbb{H}^{m-1} - p_0$ is isometric to the warped product $(0, +\infty) \times_h S^{m-2}$ where $h(t) = \sinh(t)$, $t \in (0, +\infty)$. Now use (2.2.0.2) and (2.1.6) together with the trigonometric identity $\cosh^2(t) = \sinh^2(t) + 1$ to show the metrics agree at $\mathcal{N}^{m-1} \times 1$. To show they agree at $\mathcal{N}^{m-1} \times 2$ use, in addition, the following matrix identity. Let P denote the $(m-1) \times (m-1)$ matrix whose only nonzero entry is $P_{m-1, m-1} = 1$, and let A be any $(m-1) \times (m-1)$ matrix whose bottom row is $(0, 0, \cdots, 0, 1)$. Then $A^t P A = P$ where A^t denotes the transpose of A. The conclusion of Theorem 2.1 now follows from Lemma 2.2.

Proof of Lemma 2.2. Smooth coordinate functions $x_1, \cdots, x_{m-2}$, x_{m-1}, t defined in an open neighborhood of a point $(p_0, t_0) \in \mathcal{N}^{m-1} \times [1, 2]$ are said to form a *regular coordinate system* about (p_0, t_0) if there exist coordinate functions $y_1, y_2, \cdots, y_{m-1}$, s defined in an open neighborhood of a point $(q_0, t_0) \in (S^{m-2} \times \mathbb{R}) \times [1, 2]$ such that

(2.2.1)

(1) $\pi(q_0) = p_0$;

(2) the composite $x_i \circ (\pi \times \mathrm{id}) = y_i$, for $i = 1, 2, \cdots, m-1$;

(3) s and t are the $[1, 2]$ coordinates in the two product structures;

(4) y_{m-1} is the $\mathbb{R}$ coordinate in the product structure $S^{m-2} \times \mathbb{R} \times [1, 2]$; and

(5) y_i is constant on each leaf of the foliation $\{z \times \mathbb{R} \times [1, 2] \mid z \in S^{m-2}\}$ provided $1 \leq i \leq m-2$.

It is easy to find a regular coordinate system about a given point in $\mathscr{N}^{m-1} \times [1, 2]$. By composing the coordinate functions $y_1, y_2, \cdots, y_m$ for this system with members from a precompact set of isometries of the symmetric space $S^{m-2} \times \mathbb{R} \times \mathbb{R}$, we construct a family of regular coordinate systems (one for each point $(p_0, t_0) \in \mathscr{N}^{m-1} \times [1, 2]$) satisfying the following properties:

(2.2.2)

(1) $B(\ ,\ ) = g_{ij}\, dx_i\, dx_j + dt^2$. We denote these functions by $g_{ij}^{(p_0, t_0)}(\ )$ when we need to make explicit the dependence of the functions $g_{ij}(\ )$ on the (base) point (p_0, t_0).

(2) The closure of the set

$$\{g_{ij}^{(p_0, t_0)}(p_0, t_0) \mid (p_0, t_0) \in \mathscr{N}^{m-1} \times [1, 2]\}$$

is a compact space K of positive definite symmetric matrices.

(3) There is a positive real number C, which is independent of (p_0, t_0), such that for all integers $k, s \geq 0$, with $k + s \leq 2$, the following inequalities hold:

$$\left| \frac{\partial^{k+s} g_{ij}^{(p_0, t_0)}}{\partial x_1 \cdots \partial x_{i_k} \partial t^s}(p_0, t_0) \right| < C.$$

For each positive real number r, define a new coordinate system $\bar{x}_i$, $\bar{t}$ about (p_0, t_0) by setting

$$\bar{t} = rt \quad \text{and} \quad \bar{x}_i = \sinh(rt_0) x_i. \tag{2.2.3}$$

We then have the following equalities:

$$\text{(2.2.4)}\qquad \begin{aligned} d\bar{t} &= r\,dt, & d\bar{x}_i &= \sinh(rt_0)\,dx_i, \\ \frac{\partial}{\partial \bar{t}} &= \frac{1}{r}\frac{\partial}{\partial t}, & \frac{\partial}{\partial \bar{x}_i} &= \frac{1}{\sinh(rt_0)}\frac{\partial}{\partial x_i}. \end{aligned}$$

From (2.2.4) and the definition (2.1.6) of B_r in terms of B it follows that

$$\text{(2.2.5)}\qquad \begin{aligned} &B_r(\ ,\ ) = \bar{g}_{ij}\,d\bar{x}_i\,d\bar{x}_j + d\bar{t}^2, \\ &\qquad\qquad \text{where } \bar{g}_{ij} = \frac{\sinh^2(rt)}{\sinh^2(rt_0)} g_{ij}, \text{ if } 1 \le i, j \le m-2; \\ &\bar{g}_{i,m-1} = \bar{g}_{m-1,i} = \frac{\sinh^2(rt)}{\sinh^2(rt_0)} g_{i,m-1}, \quad \text{if } 1 \le i \le m-2; \\ &\bar{g}_{m-1,m-1} = \frac{\sinh^2(rt)}{\sinh^2(rt_0)} g_{m-1,m-1} + \frac{1}{\sinh^2(rt_0)}. \end{aligned}$$

Let $X_{i,j;i_1,\cdots,i_k;s}$ denote the partial derivatives (0th through 2nd order)

$$\text{(2.2.6)}\qquad \frac{\partial^{k+s}\bar{g}_{ij}}{\partial \bar{x}_{i_1}\cdots\partial \bar{x}_{i_k}\,\partial \bar{t}^s}(p_0, t_0).$$

In particular, $X_{ij} = \bar{g}_{ij}(p_0, t_0)$. It follows from properties (2.2.2), (2.2.4) and (2.2.5) that

$$\text{(2.2.7)}\qquad \lim_{r\to\infty} X_{i,j;i_1,\cdots,i_k;s} = 0^k 2^s g_{ij}^{(p_0,t_0)}(p_0, t_0)$$

uniformly in (p_0, t_0). Here $0^0 = 1$ and $0^k = 0$ if $k \ge 1$.

Choose an orthonormal basis $\{v_1, v_2\}$ for the 2-plane P and write

$$\text{(2.2.8)}\qquad v_i = a_{ik}\partial/\partial\bar{x}_k + a_{im}\partial/\partial\bar{t}$$

where we sum over the index k. It is a consequence of the classical relation between the coefficients of the curvature tensor and of the first fundamental form (cf. [12, §§5.3 and 6.2]) that $K_{B_r}(P)$ is a polynomial $f(\ )$ in the set of variables $\{X_{i,j;i_1,\cdots,i_k;s}, a_{ij}, \det(X_{ij})^{-1}\}$. The set $\mathscr{L}$ of limiting values (as $r \to +\infty$) of these variables

$$\text{(2.2.9)}\qquad \mathscr{L} = \{0^k 2^s g_{ij}^{(p_0,t_0)}(p_0, t_0), a_{ij}, \det(g_{ij}^{(p_0,t_0)}(p_0, t_0))^{-1}\}$$

is a precompact (i.e., bounded) subset of the domain of f because of (2.2.2) and (2.2.7). It consequently suffices to show that $f|\mathscr{L} \equiv -1$ in order to complete the proof of Lemma 2.2.

Fix a point $(p_0, t_0) \in \mathscr{N}^{m-1} \times [1, 2]$. Let $x_1, x_2, \cdots, x_{m-1}$ be the standard coordinates on $\mathbb{R}^{m-1}$ and let $A(\ ,\ )$ be the flat Riemannian metric on $\mathbb{R}^{m-1}$ described by

$$A(\ ,\ ) = g_{ij}(p_0, t_0)\, dx_i\, dx_j, \tag{2.2.10}$$

where $g_{ij}(\ ,\ )$ is an abbreviated notation for $g_{ij}^{(p_0, t_0)}(\ ,\ )$. Now form the warped product $\mathbb{R} \times_\eta \mathbb{R}^{m-1}$, where $\eta: \mathbb{R} \to \mathbb{R}$ is the function $\eta(t) = e^t$ and $\mathbb{R}$ has its standard (flat) Riemannian metric. (See [20, pp. 204–211] for the definition and properties of the warped product.) Let t be the first coordinate in the product structure $\mathbb{R} \times \mathbb{R}^{m-1}$. Then the Riemannian metric $\overline{A}(\ ,\ )$ on the warped product $\mathbb{R} \times_\eta \mathbb{R}^{m-1}$ is explicitly described by the formula

$$\overline{A}(\ ,\ ) = \hat{g}_{ij}\, dx_i\, dx_j + dt^2, \tag{2.2.11}$$

where $\hat{g}_{ij}(x_1, \cdots, x_{m-1}, t) = e^{2t} g_{ij}(p_0, t_0)$. The sectional curvatures of $\mathbb{R} \times_\eta \mathbb{R}^{m-1}$ are easily calculated using [20, Proposition 4.2, p. 210]. They are all -1. Consider the point $0 = (0; 0, 0, \cdots, 0) \in \mathbb{R} \times \mathbb{R}^{m-1}$. The value of the partial derivatives (0th through second order) of $\hat{g}_{ij}$ at 0 are equal to the limiting values of $X_{i,j;i_1,\cdots,i_k;s}$ as $r \to +\infty$; i.e., are $0^k 2^s g_{ij}(p_0, t_0)$. We consequently have that the value of f restricted to $\mathscr{L}$ is identically -1. This completes the proof of Lemma 2.2.

3. Relevant group theory

We intend to use the results of §§1 and 2 to construct the Riemannian manifolds posited in Theorems 0.1 and 0.2. We will use examples due to Millson [17] and a theorem of Sullivan [23]. Some specific group-theoretic facts are needed to enable us to assemble these results. The purpose of this section is to state and prove these results in group theory.

Let $\overline{\mathbb{Q}} \subseteq \mathbb{C}$ denote the algebraic closure (inside the complex numbers $\mathbb{C}$) of the field of rational numbers $\mathbb{Q}$.

Lemma 3.1. *Let $\Gamma \subseteq \mathrm{GL}_n(\overline{\mathbb{Q}})$ be a finitely generated subgroup, and $A, B \in \Gamma$ be a pair of noncommuting elements. Also assume that A is a semisimple matrix in $\mathrm{GL}_n(\mathbb{C})$. Then there exists a homomorphism $\varphi: \Gamma \to G$ where G is a finite group such that $\varphi(B)$ is not an integral power of $\varphi(A)$; i.e., the equation $\varphi(B) = \varphi(A)^n$ has no integral solution n.*

Proof. Since Γ is finitely generated, there exists an algebraic number field $k \subseteq \overline{\mathbb{Q}}$ such that $\Gamma \subseteq \mathrm{GL}_n(k)$. (Recall that an algebraic number field is a finite field extension of $\mathbb{Q}$.) We can even pick k so that A is diagonalizable over k since the eigenvalues of A are also in $\overline{\mathbb{Q}}$. Hence, we may assume, after applying an inner automorphism, that A is represented by a diagonal matrix in $\mathrm{GL}_n(k)$ under the embedding $\Gamma \subseteq \mathrm{GL}_n(k)$. But B is *not* a diagonal matrix since the set of all diagonal matrices form an abelian subgroup of $\mathrm{GL}_n(k)$; in particular, $B_{ij} \neq 0$ for some pair of unequal indices i and j. Let $\mathscr{O}$ be the ring of all algebraic integers inside of k. Recall the following properties of $\mathscr{O}$:

(3.1.1)

1. $\mathscr{O}$ is a Dedekind domain.
2. $\mathscr{O}$ is a finitely generated free $\mathbb{Z}$-module.
3. $\mathscr{O} \otimes_{\mathbb{Z}} \mathbb{Q} = k$.

Using property 3 and the fact that Γ is finitely generated, we conclude that $\Gamma \subseteq \mathrm{GL}_n(\mathscr{O}[\frac{1}{m}])$ where $m \in \mathbb{Z}$ and $m \neq 0$. In particular, $B_{ij} = bm^s$ where $b \in \mathscr{O}$, $b \neq 0$ and $s \in \mathbb{Z}$. Use property 1 to pick a maximal ideal $\mathfrak{A}$ in $\mathscr{O}$ such that both $b \notin \mathfrak{A}$ and $m \notin \mathfrak{A}$. Let $\psi\colon \mathscr{O}[\frac{1}{m}] \to \mathscr{O}/\mathfrak{A}$ be the canonical factor homomorphism. Note that $\mathscr{O}/\mathfrak{A}$ is a finite field by property 2 and that $\psi(bm^s) \neq 0$. Let $\hat{\psi}\colon \mathrm{GL}_n(\mathscr{O}[\frac{1}{m}]) \to \mathrm{GL}_n(\mathscr{O}/\mathfrak{A})$ be the induced group homomorphism. Then the posited φ is the composite of the inclusion of Γ into $\mathrm{GL}_n(\mathscr{O}[\frac{1}{m}])$ with $\hat{\psi}$. q.e.d.

A closed geodesic $\gamma\colon S^1 \to M^m$ is said to be *t-simple* if $\dot{\gamma}\colon S^1 \to TM$ is simple, i.e., a one-to-one function. Recall M^m has the same meaning here as it has in §2 and $m > 6$. Also TM denotes the tangent bundle of M. Let $1 \in S^1 \subseteq \mathbb{R}^2 = \mathbb{C}$ be the complex number one.

Corollary 3.2. *Assume M^m is orientable and $\gamma\colon S^1 \to M^m$ is a t-simple closed geodesic. Let $x_0 = \gamma(1)$ and $A \in \pi_1(M, x_0)$ be the homotopy class of γ. Let B be any other element in $\pi_1(M, x_0)$ which is not an integral power of A. Then there exists a homomorphism $\varphi\colon \pi_1(M, x_0) \to G$ where G is a finite group and such that $\varphi(B)$ is not an integral power of $\varphi(A)$.*

Proof. The group Γ of all deck transformations of the universal covering space $\mathbb{H}^m$ of M^m can be identified with $\pi_1(M, x_0)$. Using the fact that A leaves invariant a geodesic line in $\mathbb{H}^m$, one sees that A and B do *not* commute. Since Γ acts via isometries on $\mathbb{H}^m$, we can identify Γ as a lattice in the Lie group $\mathrm{SO}^+(m, 1, \mathbb{R}) \subseteq \mathrm{GL}_{m+1}(\mathbb{R})$ such that A is represented by a diagonalizable matrix in $\mathrm{GL}_{m+1}(\mathbb{C})$. We can apply the weak arithmeticity result of Garland and Raghunathan [11] which generalizes to nonuniform lattices earlier results of Selberg; cf. [22, Proposition 6.6], and Calabi [5]. There consequently exist an algebraic number field

$k \subseteq \mathbb{R}$ and an element $g \in \mathrm{SO}^+(m, 1, \mathbf{R})$ such that

$$g\Gamma g^{-1} \subseteq \mathrm{SO}^+(m, 1, k) \subseteq \mathrm{GL}_{m+1}(\overline{\mathbb{Q}}).$$

The composite of the inner automorphism determined by g with the embedding $\mathrm{SO}^+(m, 1, k) \subseteq \mathrm{GL}_{m+1}(\overline{\mathbb{Q}})$ hence gives an embedding of Γ into $\mathrm{GL}_{m+1}(\overline{\mathbb{Q}})$ which satisfies the hypotheses of Lemma 3.1. An application of Lemma 3.1 thus completes the proof of Corollary 3.2.

Corollary 3.3. *Let $\gamma: S^1 \to M^m$ be an orthonormally framed t-simple closed geodesic, and r be a positive real number. Then there exist a (connected) finite sheeted cover $p: \widetilde{M} \to M$ and an orthonormally framed simple closed geodesic $\alpha: S^1 \to \widetilde{M}$ such that*

1. *$p \circ \alpha = \gamma$ and*
2. *α is the core of a geometric tube of radius r.*

Proof. Note first that we can assume M^m is orientable since γ lifts to the oriented cover of M^m when M^m is nonorientable.

Let $x_0 = \gamma(1)$, and $\tilde{x}_0$ be a lift of x_0 to the universal cover $q: \mathbb{H}^m \to M^m$; i.e., $q(\tilde{x}_0) = x_0$. Identify $\pi_1(M, x_0)$ with the group Γ of all deck transformations of $q: \mathbb{H}^m \to M^m$ via these choices, and let $A \in \Gamma$ correspond to the homotopy class of α in $\pi_1(M, x_0)$. Let L be a geodesic segment of finite length such that $\tilde{x}_0 \in L$ and $q(L) = \gamma(S^1)$. A compactness argument shows that these are only a finite number of elements $B_1, B_2, \cdots, B_n$ in Γ such that, for each index $1 \le i \le n$, the following hold:

1. B_i is not an integral power of A, and
2. some point on $B_i(L)$ is within distance $2r + |L|$ of L where $|L|$ denotes the length of L.

Now apply Corollary 3.2 to obtain a group homomorphism $\varphi: \pi_1(M, x_0) \to G$ where G is a finite group and such that none of the elements $\varphi(B_i)$ is an integral power of $\varphi(A)$, where $i = 1, 2, \cdots, n$. Let S be the cyclic subgroup of G consisting of all the integral powers of $\varphi(A)$, and let $p: \widetilde{M} \to M$ be the covering space corresponding to the subgroup $\varphi^{-1}(S) \subseteq \pi_1(M, x_0)$. It is now routine to verify that $p: \widetilde{M} \to M$ satisfies the properties posited in Corollary 3.3.

Lemma 3.4. *Assume M^m is orientable and $\varphi: \pi_1(M^m) \to \mathbb{Z}_+$ is an epimorphism. Then there is a t-simple closed geodesic $\gamma: S^1 \to M^m$ such that $\varphi([\gamma]) \neq 0$ where $[\gamma]$ denotes the free homotopy class of γ.*

Proof. Recall that every conjugacy class of a nonidentity element in $\pi_1 M^m$ is represented by a closed geodesic when M^m is compact. So

Lemma 3.4 is obviously true when M^m is compact. We argue as follows in the general situation.

Identify $\pi_1 M^m$ with the group of all deck transformations of the universal covering space $p\colon \mathbb{H}^m \to M^m$. Since Γ acts via isometries on $\mathbb{H}^m$, we can further identify it to a lattice in $\mathrm{SO}^+(m, 1, \mathbb{R}) \subseteq \mathrm{GL}_{m+1}(\mathbb{R})$. The individual elements $\beta \in \Gamma - \{1\}$ are partitioned into two disjoint classes *semisimple* and *cuspidal* depending on whether the matrix representing β in $\mathrm{GL}_{m+1}(\mathbb{C})$ is semisimple (i.e., diagonalizable) or not. We now recall a few facts about the Jordan decomposition of β; cf. [19, p. 10]. It decomposes uniquely as a product $\beta = pku$ where p, k, u are pairwise commuting matrices in $\mathrm{SO}^+(m, 1, \mathbb{R})$ with u unipotent and both p and k semisimple with positive real and length 1 eigenvalues, respectively. When $p \neq 1$, it has exactly two eigenvalues λ and λ^{-1} different from 1, and each eigenvalue has a one-dimensional eigenspace. It is a consequence of [19, Lemma 5.2(i)] that if $p \neq 1$, then $u = 1$; hence, β is cuspidal if and only if $p = 1$. Also [19, Lemma 5.2(i)] yields the following useful criterion.

(3.4.1) If trace $\beta > m + 1$, then β is semisimple.

On the geometric side, the elements in $\Gamma - \{1\}$ whose conjugacy class is represented by a closed geodesic are precisely the semisimple elements, while the cuspidal elements are those representable by curves of arbitrarily short arc length.

We proceed to find a semisimple element $\beta \in \Gamma$ with $\varphi(\beta) \neq 0$. Pick an element $c \in \Gamma$ such that $\varphi(c) \neq 0$. If c is semisimple, then we are done. Hence assume that c is cuspidal. Using the fact that the set of all closed geodesics is dense in the set of all geodesics [19, Lemma 8.3′], we can find a semisimple element $\beta \in \Gamma - \{1\}$ such that trace $(\beta^n c) \to +\infty$ as $n \to +\infty$. (Hint: β should "point towards" the cusp.) Hence $\beta^n c$ is also semisimple by (3.4.1), when n is sufficiently large. Note that $\varphi(\beta^n c) = n\varphi(\beta) + \varphi(c)$. Therefore either $\varphi(\beta) \neq 0$ or $\varphi(\beta^n c) \neq 0$. Thus we have accomplished the goal stated in sentence one of this paragraph.

Note that for every closed geodesic β there exist an integer n and a t-simple closed geodesic γ such that β is the composite $\gamma \circ P_n$ where $P_n\colon S^1 \to S^1$ is the function $z \to z^n$. Hence we can find a t-simple closed geodesic γ with $\varphi([\gamma]) \neq 0$. q.e.d.

We will need the following technical fact about the group $\mathrm{SO}^+(m, 1, \mathbb{Z})$ in order to prove Theorem 0.1.

Proposition 3.5. *Given a semisimple element $A \in \mathrm{SO}^+(m, 1, \mathbb{Z})$ of infinite order and a positive integer n, there exist a finite group G and a* *homomorphism* ψ: $\mathrm{SO}^+(m, 1, \mathbb{Z}) \to G$ *with the following properties*:

1. *The order of* $\psi(A)$ *is divisible by* n.

2. *Let* β *be any unipotent element in* $\mathrm{SO}^+(m, 1, \mathbb{Z})$ *such that* $\psi(B) = \psi(A)^s$ *where* $s \in \mathbb{Z}$; *then* n *divides* s.

Fix an algebraic number field k such that k contains all the eigenvalues of A as well as all the nth roots of unity. Note that A has real eigenvalues λ and λ^{-1} with $\lambda > 1$, and its other eigenvalues are complex numbers of length 1. Furthermore, the eigenspaces corresponding to λ and λ^{-1} are both one-dimensional. Let $\mathscr{O}$ denote the ring of all algebraic integers in k, and let units $\mathscr{O}$ denote the group of units of this ring. Notice that units $\mathscr{O}$ contains all the eigenvalues of A and all the nth roots of unity. Fix a positive (rational) prime q which divides n, and let Ω denote a specific choice of a primitive qth root of unity. The proof of Proposition 3.5 requires the following preliminary result.

Lemma 3.6. *Given a positive* (*rational*) *integer* s, *there exists a prime ideal* $\mathfrak{A}$ *in* $\mathscr{O}$ *such that the coset* $\lambda + \mathfrak{A}$ *is a unit in the finite field* $\mathscr{O}/\mathfrak{A}$ *and its order is divisible by* q^s.

We now complete the proof of Proposition 3.5 using Lemma 3.6, and after that we prove Lemma 3.6. Let $n = q_1^{s_1} q_2^{s_2} \cdots q_r^{s_r}$ be the prime factorization of n where the numbers q_i are distinct positive primes and each $s_i > 0$. We will use Lemma 3.6 to construct finite groups G_i and homomorphisms ψ_i: $\mathrm{SO}^+(m, 1, \mathbb{Z}) \to G_i$, for $i = 1, 2, \cdots, r$, with the following two properties:

(3.6.1)
1. The order of $\psi_i(A)$ is divisible by $q_i^{s_i}$.
2. Let B be any unipotent element in $\mathrm{SO}^+(m, 1, \mathbb{Z})$ such that $\psi_i(B) = \psi_i(A)^s$, then $q_i^{s_i}$ divides s.

The proof of Proposition 3.5 is completed by setting $G = G_1 \times G_2 \times \cdots \times G_r$ and $\psi = \psi_1 \times \psi_2 \times \cdots \times \psi_r$.

It remains to construct G_i and ψ_i satisfying (3.6.1). Let $\mathfrak{A}_i$ be the prime ideal $\mathfrak{A}$ posited in Lemma 3.6 relative to setting $q = q_i$ and $s = s_i$. Let $G_i = SL_{m+1}(\mathscr{O}/\mathfrak{A}_i)$, and ψ_i be the composite of the inclusion $\mathrm{SO}^+(m, 1, \mathbb{Z}) \subseteq SL_{m+1}(\mathscr{O})$ with the group homomorphism

$$\eta_i: SL_{m+1}(\mathscr{O}) \to SL_{m+1}(\mathscr{O}/\mathfrak{A}_i)$$

induced by the coset homomorphism $x \mapsto x + \mathfrak{A}_i$, $x \in \mathscr{O}$.

We must now show that (3.6.1) is satisfied. Note first that $\lambda + \mathfrak{A}_i$ is an eigenvalue of the matrix $\psi_i(A)$. Consequently, $\psi_i(A)$ is conjugate to a blocked upper triangular matrix $\mathscr{A}$ of the form

$$\left(\begin{array}{c|c} \lambda+\mathfrak{A}_i & \star \\ \hline 0 & \star \end{array}\right), \tag{3.6.2}$$

where the top diagonal block is a 1×1 matrix whose entry is $\lambda + \mathfrak{A}_i$. Hence the order of $\mathscr{A}$ is divisible by the order of the unit $\lambda + \mathfrak{A}_i$ in the field $\mathscr{O}/\mathfrak{A}_i$. Lemma 3.6 now shows that $q_i^{s_i}$ divides the order of $\psi_i(A)$ since $\mathscr{A}$ and $\psi_i(A)$ have the same order. This verifies property 1 of (3.6.1).

To verify property 2, note that $\psi_i(A)^s$ is conjugate to $\mathscr{A}^s$. Hence $\mathscr{A}^s$ is a unipotent matrix; i.e., all its eigenvalues are 1. But $\lambda^s+\mathfrak{A}_i$ is clearly an eigenvalue of $\mathscr{A}^s$. The order of $\lambda+\mathfrak{A}_i$ in units$(\mathscr{O}/\mathfrak{A}_i)$ therefore divides s. A second application of Lemma 3.6 now shows that $q_i^{s_i}$ divides s. This completes the proof of Proposition 3.5.

Proof of Lemma 3.6. Recall that $\mathscr{O}$ satisfies the three properties of (3.1.1). For each element $x \in \mathscr{O}$, let (x) denote the principal ideal it generates; i.e., $(x) = x\mathscr{O}$. Each ideal in $\mathscr{O}$ is the product of prime ideals since $\mathscr{O}$ is a Dedekind domain. In particular, we have that

$$(\Omega - 1) = \mathfrak{A}_1^{m_1}\mathfrak{A}_2^{m_2}\cdots\mathfrak{A}_r^{m_r}, \tag{3.6.3}$$

where each $\mathfrak{A}_i$ is a prime ideal and each m_i is a nonnegative (rational) integer. Let

$$\nu = \max\{m_1, m_2, \cdots, m_r\} + 1. \tag{3.6.4}$$

Recall that prime ideals are all maximal in Dedekind domains; i.e., if $\mathfrak{A}$ is a prime ideal, then $\mathscr{O}/\mathfrak{A}$ is a field. Furthermore, $\mathscr{O}/\mathfrak{A}$ has finite cardinality since the additive group of $\mathscr{O}$ is finitely generated. We see that $\mathscr{O}/\mathfrak{A}^n$ is a finite ring for each nonnegative (rational) integer n by arguing in this way. We can clearly make the following assumption in proving Lemma 3.6; namely,

$$q^s > \text{cardinality of } \mathscr{O}/\mathfrak{A}_i^{\nu} \tag{3.6.5}$$

for each $i = 1, 2, \cdots, r$.

There is now the following fact which we will verify after first using it to complete the proof of Lemma 3.6.

Claim 3.7. *There exist a prime ideal $\mathfrak{A}$ in $\mathscr{O}$ and a positive (rational) integer j such that $b^j - \Omega \in \mathfrak{A}$ but $\Omega - 1 \notin \mathfrak{A}$, where $b = \lambda^{q^{s-1}}$.*

As observed above, $\mathscr{O}/\mathfrak{A}$ is a finite field. Claim 3.7 can be rephrased as the following statement about elements in this field

$$b^j + \mathfrak{A} = \Omega + \mathfrak{A} \neq 1 + \mathfrak{A}. \tag{3.7.1}$$

Consequently, $b^j + \mathfrak{A}$ has order q in units $(\mathscr{O}/\mathfrak{A})$. But since $b^j = \lambda^{jq^{s-1}}$, the order of $\lambda + \mathfrak{A}$ must be divisible by q^s. This proves Lemma 3.6.

We now proceed to formulate and verify an auxiliary result needed to prove Claim 3.7. Fix a $\mathbb{Z}$-basis for $\mathscr{O}$. Then multiplication determines a faithful representation

$$\eta: \mathscr{O} \to M_n(\mathbb{Z}), \tag{3.7.2}$$

where $M_n(\mathbb{Z})$ is the ring of all $n \times n$ matrices with entries in $\mathbb{Z}$ and $n = [k : \mathbb{Q}]$. Composing η with the determinant function $\det: M_n(\mathbb{Z}) \to \mathbb{Z}$ defines a norm on $\mathscr{O}$. Denote by $N(x)$ the norm of an element $x \in \mathscr{O}$.

Assertion 3.8.

$$\limsup_{j \to +\infty} |N(b^j - \Omega)| = +\infty.$$

To verify this assertion, we start by analyzing the eigenvalues of the two matrices $\eta(\lambda)$ and $\eta(\Omega)$. Since the field $\mathbb{Q}(\lambda)$ is contained in k, we see that λ is an eigenvalue of $\eta(\lambda)$ and also that $\eta(\lambda)$ is diagonalizable in $M_n(\mathbb{C})$. Since λ is a root of the characteristic polynomial of A, this polynomial also annihilates $\eta(\lambda)$. The eigenvalues of $\eta(\lambda)$ are consequently a subset of the eigenvalues of A (not counting their multiplicities). By the same reasoning, $\eta(\Omega)$ is also diagonalizable in $M_n(\mathbb{C})$ and its eigenvalues are all primitive qth roots of unity.

Note that $\eta(\Omega)$ and $\eta(\lambda)$ are simultaneously diagonalizable in $M_n(\mathbb{C})$ since $\Omega\lambda = \lambda\Omega$. We consequently have the following formula for the norm of $b^j - \Omega$:

$$N(b^j - \Omega) = \prod_{i=1}^{n} (\lambda_i^j - \Omega_i), \tag{3.8.1}$$

where $\lambda_1 = b$, each $\lambda_i \in \{b, b^{-1}\} \cup S^1$, and each $\Omega_i \in S^1 - \{1\}$ with $\Omega_i^q = 1$. Observe the following two facts:

$$\begin{aligned} &1.\ \lim_{j \to +\infty} |b^j - z| = +\infty, \text{ and} \\ &2.\ \lim_{j \to +\infty} |b^{-j} - z| = 1 \end{aligned} \tag{3.8.2}$$

for each complex number $z \in S^1$. Furthermore, there exists an infinite set $\mathscr{S}$ of positive integers such that

$$|\lambda_i^j - 1| \le \sin(\pi/q) \tag{3.8.3}$$

for each $j \in \mathscr{S}$, and such that $\lambda_i \in S^1$ for each index i. One now easily shows that

$$\lim_{\substack{j\in\mathcal{S} \\ j\to+\infty}} |N(b^j - \Omega)| = +\infty \tag{3.8.4}$$

by using facts (3.8.2) and (3.8.3) in conjunction with formula (3.8.1). Thus Assertion 3.8 is verified.

We now establish Claim 3.7 via proof by contradiction; hence we assume Claim 3.7 is false. The prime factorization of each ideal $(b^j - \Omega)$, $j > 0$, consequently has the following form:

$$(b^j - \Omega) = \mathfrak{A}_1^{m_{1,j}} \mathfrak{A}_2^{m_{2,j}} \cdots \mathfrak{A}_r^{m_{r,j}} \tag{3.8.5}$$

where each $m_{i,j}$ is a nonnegative (rational) integer. (Recall the prime ideals $\mathfrak{A}_i$ come from the factorization (3.6.3).) Suppose all the numbers $m_{i,j} < \nu$; then there would only be a finite number of distinct ideals in the list $(b^j - \Omega)$, $j > 0$. Consequently only a finite number of integers in the set $\{N(b^j - \Omega) \mid j > 0\}$, contradicting Assertion 3.8. (Note that if $(x) = (y)$, then $N(x) = \pm N(y)$.) Hence there exists a pair of positive numbers i, j such that $m_{i,j} \geq \nu$. Consider the finite ring $R = \mathscr{O}/\mathfrak{A}_i^\nu$. Then the following is true about certain elements in R:

$$b^j + \mathfrak{A}_i^\nu = \Omega + \mathfrak{A}_i^\nu \neq 1 + \mathfrak{A}_i^\nu. \tag{3.8.6}$$

Therefore $b^j + \mathfrak{A}_i^\nu$ has order q in $\operatorname{units} R$. Recall again that $b^j = \lambda^{jq^{s-1}}$. Hence the order of the element $\lambda + \mathfrak{A}_i^\nu$ in $\operatorname{units} R$ must be divisible by q^s. In particular, the cardinality of $\mathscr{O}/\mathfrak{A}_i^\nu$ must be greater than q^s which contradicts assumption (3.6.5). This proves Claim 3.7.

4. Proof of Theorem 0.1 and 0.2

Recall that in this section, as in the previous two, M^m still denotes a complete (connected) Riemannian manifold with finite volume (possibly compact), all sectional curvatures -1 and $\dim M^m = m > 6$. Our object is to combine §§1, 2 and 3 with earlier work of Millson [17] and Sullivan [23] to prove Theorems 0.1 and 0.2 formulated in the introduction. We start by recalling Sullivan's result.

Theorem 4.1 (Sullivan [23]). *Each lattice Γ in $O^+(m, 1, \mathbb{R})$ contains a torsion-free subgroup of finite index $\widehat{\Gamma}$ such that $\mathbb{H}^m/\widehat{\Gamma}$ is a π-manifold.*

The following is a consequence of Theorem 4.1 and results from the previous sections.

Theorem 4.2. *Assume that M^m is closed and has positive first Betti number. Given $\varepsilon > 0$ and an infinite order element $y \in H_1(M^m, \mathbb{Z})$, there exist a (connected) finite sheeted covering space $p\colon \mathscr{M}^m \to M^m$ and* *a simple framed geodesic* α *in* $\mathcal{M}^m$ *with the following properties:*

1. *Some multiple of the homology class represented by* α *maps to a nonzero multiple of* y *via* $p_*: H_1(\mathcal{M}^m, \mathbb{Z}) \to H_1(M^m, \mathbb{Z})$.

2. *There is no manifold diffeomorphic to* $N^m \# \Sigma^m$ *in the concordance class* $\mathcal{M}(\alpha, x)$ *provided* N^m *is a Riemannian locally symmetric space,* Σ^m *represents an element in* Θ_m *and* x *is a nonzero element in* Θ_{m-1}.

3. *Each concordance class* $\mathcal{M}(\alpha, x)$ *contains a complete and finite volume Riemannian manifold whose sectional curvatures are all in the interval* $[-1-\varepsilon, -1+\varepsilon]$.

Proof. Theorem 4.1 yields a (connected) finite sheeted covering space $p_1: \mathcal{M}_1 \to M$ such that $\mathcal{M}$ is a π-manifold for every covering space $\mathcal{M}$ of $\mathcal{M}_1$. There is clearly an orthonormally framed t-simple closed geodesic α_1 in $\mathcal{M}_1$ whose homology class $[\alpha_1]$ satisfies the following two conditions:

(4.2.1)
1. Some integral multiple of $p_{1*}[\alpha_1]$ is a nonzero multiple of y.
2. There is a homomorphism $\eta_1: \pi_1\mathcal{M}_1 \to \mathbb{Z}_+$ with $\eta_1([\alpha_1]) = 1$.

Let l denote the length of α_1, and $A_{\alpha_1}: S^1 \to O(m-1)$ be the holonomy function associated to α_1 via formula (2.0.2). Let r be the positive real number posited in Theorem 2.1 relative to the given data ε, l, m and $A(t) = A_{\alpha_1}(t)$. Next apply Corollary 3.3, with $M = \mathcal{M}_1$ and $\gamma = \alpha_1$, to get a covering space $p_2: \mathcal{M} \to \mathcal{M}_1$ and an orthonormally framed simple closed geodesic α in $\mathcal{M}$ such that $p_2 \circ \alpha = \alpha$ and α_1 is the core of a geometric tube of radius $2r$. We now set $p = p_2 \circ p_1: \mathcal{M}^m \to M^m$. Condition 1 of Theorem 4.2 is obviously satisfied. If we let $\eta: \pi_1\mathcal{M} \to \mathbb{Z}_+$ in Theorem 1.1 be the composite of η_1 with $p_{2\#}: \pi_1\mathcal{M} \to \pi_1\mathcal{M}_1$, then condition 2 is an immediate consequence of Theorem 1.1 together with Corollary 1.5 and Proposition 1.6. Theorem 2.1 shows that condition 3 is satisfied since $A_\alpha = A_{\alpha_1}$ and the length of α is the same as the length of α_1. This proves Theorem 4.2.

The pattern of the above proof yields the following weaker version when M^m is not compact.

Addendum 4.3. *Assume that* M^m *is a* π*-manifold (not necessarily compact),* γ *is a* t*-simple framed closed geodesic in* M, *and* $\lambda: \pi_1 M^m \to \mathbb{Z}_+$ *is a homomorphism such that*

1. $\lambda([\gamma]) = 1$ *where* $[\gamma]$ *denotes the free homotopy class of* γ *and*
2. $\lambda(\beta)$ *is divisible by the order of* Θ_{m-1} *for each cuspidal element* β *in* $\pi_1 M^m$.

Given a positive real number ε, *there exist a (connected) finite sheeted covering space* $p: \mathcal{M}^m \to M^m$ *and a simple framed geodesic* α *in* $\mathcal{M}^m$

such that the composite $p \circ \alpha = \gamma$, *and conditions* 2 *and* 3 *of Theorem* 4.2 *are satisfied.*

We next recall some examples due to Millson which we then use together with Theorem 4.2 and Addendum 4.3 to prove Theorem 0.2 and 0.1.

Theorem 4.4 (Millson [17]). *For each integer* $n > 1$, *there exist two complete (connected) finite volume Riemannian manifolds* K^n *and* N^n *of dimension* n *which satisfy the following properties*:

1. *All the sectional curvatures of both* K^n *and* N^n *are* -1.
2. *Both* K^n *and* N^n *have positive first Betti number.*
3. K^n *is compact.*
4. N^n *is not compact.*
5. $\pi_1 N^n$ *is isomorphic to a finite index subgroup of* $\mathrm{SO}^+(n, 1, \mathbb{Z})$.

Proof of Theorem 0.2. When Θ_m is nontrivial, this result follows from [9, Theorem1.1] and Proposition 1.6. When Θ_{m-1} is nontrivial, it follows from Theorem 4.2 by setting M^m (in Theorem 4.2) equal to the manifold K^m of Theorem 4.3.

Proof of Theorem 0.1. Define two sequences of positive integers a_n and b_n as follows. Let a_n be the order of the finite group Θ_n and let b_n be the least common multiple of the orders of the holonomy groups of lattices in the Lie group of all rigid motions of Euclidean n-dimensional space. Bieberbach [1] showed that b_n exists and divides the order of the finite group $\mathrm{GL}_n(\mathbb{Z}_3)$ because of Minkowski's theorem [18]. Let N^m be the Millson manifold in Theorem 4.4. Because of Theorem 4.1, there is a finite sheeted (connected) covering space $p: \mathcal{N}^m \to N^m$ such that every covering space of $\mathcal{N}^m$ is a π-manifold. There is an epimorphism $\varphi: \pi_1\mathcal{N}^m \to \mathbb{Z}_+$ since N^m has positive first Betti number. Because of Lemma 3.4, there is a t-simple framed closed geodesic ω in $\mathcal{N}^m$ such that $\varphi([\omega]) \neq 0$ where $[\omega]$ denotes the fundamental group element corresponding to ω. (Note that $[\omega]$ is well defined up to conjugacy.) Let $A \in \mathrm{SO}^+(m, 1, \mathbb{Z})$ denote the semisimple matrix corresponding to $[\omega]$ under an identification of $\pi_1\mathcal{N}^m$ with a subgroup of $\mathrm{SO}^+(m, 1, \mathbb{Z})$. Let $\psi: \mathrm{SO}^+(m, 1, \mathbb{Z}) \to G$ be a homomorphism satisfying the conclusions of Proposition 3.5 relative to A and $n = a_{m-1}b_{m-1}$. Note that $g^{b_{m-1}}$ is unipotent for every cuspidal element $g \in \pi_1\mathcal{N}^m$. Hence conclusion 2 of Proposition 3.5 yields the following fact.

(4.4.1) If $\psi(B) = \psi(A)^s$ where B is a cuspidal element of $\pi_1\mathcal{N}^m$, then a_{m-1} divides s.

Consider the homomorphism $\varphi \times \psi: \pi_1\mathcal{N}^m \to \mathbb{Z}_+ \times G$ and let $\mathcal{C}$

denote the infinite cyclic subgroup of $\mathbb{Z}_+ \times G$ generated by $(\varphi \times \psi)(A) = (\varphi(A), \psi(A))$. Let $q: M^m \to \mathcal{N}^m$ be the covering space corresponding to $(\varphi \times \psi)_\#^{-1}(\mathscr{C})$, and $\lambda: \pi_1 M^m \to \mathbb{Z}_+$ be the composite of $q_\#$, $\varphi \times \psi$ and the identification of $\mathscr{C}$ with $\mathbb{Z}_+$ determined by making $(\varphi(A), \psi(A))$ correspond to $1 \in \mathbb{Z}_+$. Let γ be a lift of ω to M^m. Since the conditions of Addendum 4.3 are clearly satisfied, the examples posited in Theorem 0.1 can now be drawn from the conclusions of Addendum 4.3.

References

[1] L. Bieberbach, *Über die Bewegungsgruppen des n-dimensional Euklidischen Räumes mit einen endlichen Fundamentalbereich*, Göttingen Nachr. (1910) 75–84.

[2] J. M. Boardman & R. M. Vogt, *Homotopy invariant structures on topological spaces*, Lecture Notes in Math., Vol. 347, Springer, Berlin, 1973.

[3] A. Borel & Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2) **75** (1962) 485–535.

[4] W. Browder, *The Kervaire invariant of framed manifolds and its generalization*, Ann. of Math. (2) **90** (1969) 157–186.

[5] E. Calabi, *On compact Riemannian manifolds with constant curvature.* I, *Differential geometry*, Proc. Sympos. Pure Math., Vol. 3, Amer. Math. Soc., Providence, RI, 1961, 155–180.

[6] F. T. Farrell & W. C. Hsiang, *The topological-Euclidean space form problem*, Invent. Math. **45** (1978) 181–192.

[7] ——, *Topological characterization of flat and almost flat Riemannian manifolds* M^n $(n \neq 3, 4)$, Amer. J. Math. **105** (1983) 641–672.

[8] F. T. Farrell & L. E. Jones, *A topological analogue of Mostow's rigidity theorem*, J. Amer. Math. Soc. **2** (1989) 257–370.

[9] ——, *Negatively curved manifolds with exotic smooth structures*, J. Amer. Math. Soc. **2** (1989) 899–908.

[10] ——, *Smooth non-representability of* $\operatorname{Out}\pi_1 M$, Bull. London Math. Soc. **22** (1990) 485–488.

[11] H. Garland & M. S. Raghunathan, *Fundamental domains for lattices in* ($\mathbb{R}$-)*rank* 1, Semisimple Lie groups, Ann. of Math. (2) **92** (1970) 279–326.

[12] N. Hicks, *Notes on differential geometry*, Van Nostrand, Princeton, NJ, 1965.

[13] M. Kervaire & J. W. Milnor, *Groups of homotopy spheres.* I, Ann. of Math. (2) **77** (1963) 504–537.

[14] R. C. Kirby & L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Annals of Math. Studies, No. 88, Princeton University Press, Princeton, NJ, 1977.

[15] B. Lawson & S. T. Yau, *Compact manifolds of nonpositive curvature*, J. Differential Geometry **7** (1972) 211–228.

[16] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer, Berlin, 1991.

[17] J. Millson, *On the first Betti number of a constant negatively curved manifold*, Ann. of Math. (2) **104** (1976) 235–247.

[18] H. Minkowski, *Zur Theorie der positiven quadratischen Formen*, J. Reine Angew. Math. **101** (1887), 196–202.

[19] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Annals of Math. Studies, Vol. 78, Princeton University Press, Princeton, NJ, 1973.

[20] B. O'Neill, *Semi-Riemannian geometry*, Academic Press, New York, 1983.

[21] G. Prasad, *Strong rigidity of* $\mathbb{Q}$*-rank* 1 *lattices*, Invent. Math. **21** (1973) 255–286.
[22] M. Raghunathan, *Discrete subgroups of Lie groups*, Springer, Berlin, 1972.
[23] D. Sullivan, *Hyperbolic geometry and homeomorphisms*, Geometric Topology (J. Cantrell, ed.), Academic Press, New York, 1979, 543–555.

STATE UNIVERSITY OF NEW YORK, BINGHAMTON

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