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L_2-COHOMOLOGY OF KÄHLER VARIETIES WITH ISOLATED SINGULARITIES

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0. Introduction

Let V be a complex projective variety. If V is smooth, we may apply the de Rham-Hodge theory to represent the cohomology by harmonic forms. As a consequence of this and the Kähler identities, we obtain a Hodge decomposition (or *Hodge structure*) on the cohomology,

$$H^k(V;\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}, \qquad H^{p,q} = \overline{H^{q,p}}.$$

When V is singular, such a decomposition no longer exists in general. However, by imposing restrictions on the intersections of chains with singular strata, Goresky and MacPherson [21], [22] defined an alternate cohomology theory for singular spaces. This *intersection cohomology*, $IH^{\cdot}(V;\mathbb{C})$, has many of the properties of ordinary cohomology on manifolds. For example, there is a nondegenerate Poincaré duality pairing, and by using the theory of $\mathscr{D}$-modules, Saito [38]–[40] (see also [41]) has shown that $IH^{\cdot}(V;\mathbb{C})$ admits a natural Hodge decomposition. (We are always referring to the *middle perversity* intersection cohomology. For an excellent historical introduction to intersection homology theory and its many ramifications, see Kleiman's article [28]; for an overview of recent advances in Hodge theory, see [10].)

It is natural to try to represent intersection cohomology analytically by a de Rham-type theory, and obtain another, more classical, proof of the existence of a Hodge decomposition on $IH^{\cdot}(V;\mathbb{C})$. In a number of contexts [11], [52] it has been conjectured that the appropriate de Rham theory to consider is the L_2-cohomology of some metric on $V\setminus \operatorname{Sing}(V)$, the set of regular points of V. Given a Riemannian metric on $V\setminus\operatorname{Sing}(V)$, the *$L_2$-cohomology* $H^{\cdot}_{(2)}(V\setminus\operatorname{Sing}(V))$ is the cohomology of the complex of L_2 differential forms whose exterior derivatives are also L_2. L_2-cohomology

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is an invariant of the quasi-isometry (§0.8) class of the metric. (For an introductory survey of L_2-cohomology and its topological interpretations (both conjectural and realized) see [46].)

A natural metric to consider on $V \setminus \mathrm{Sing}(V)$ is one induced from the Fubini-Study metric by a projective embedding; the quasi-isometry class of this metric is independent of the embedding. In this case, Cheeger, Goresky, and MacPherson [13] have conjectured that

$$H^{\bullet}_{(2)}(V \setminus \mathrm{Sing}\, V) \cong IH^{\bullet}(V; \mathbb{C}), \tag{0.1}$$

and that a decomposition of harmonic forms yields a Hodge structure. Recently, Ohsawa [35] has used our main result (Theorem 0.1) to give a proof of (0.1) for V having only isolated singularities.

The conjectured existence of a Hodge decomposition is more subtle, however, due to the incompleteness of the metric. For a *complete* Kähler metric, on the other hand, a Hodge decomposition would immediately follow from the isomorphism (0.1) (see §9). Unfortunately, there does not seem to be a unique natural complete Kähler metric associated to a general variety V. In this paper, when V has only isolated singularities, we introduce a family of complete metrics on $V \setminus \mathrm{Sing}\, V$ which we call *distinguished metrics on* V. The quasi-isometry classes of distinguished metrics on V are in one-to-one correspondence with the resolutions $\pi : \widetilde{V} \to V$ whose exceptional set is a divisor with smooth irreducible components intersecting in normal crossings; as we will show below, the definition of distinguished metrics is motivated by the metrics of locally symmetric varieties. Our first main result is (Theorem 8.4 and Corollary 8.8):

Theorem 0.1. *Let V be a projective variety with only isolated singular points. Then:*

(i) *The quasi-isometry class of distinguished metrics on V associated to a resolution $\pi : \widetilde{V} \to V$ for which π is a projective morphism contains a Kähler metric.*

(ii) *For any distinguished metric on V, we have the natural isomorphism*

$$H^{\bullet}_{(2)}(V \setminus \mathrm{Sing}\, V) \cong IH^{\bullet}(V; \mathbb{C}). \tag{0.2}$$

Remark. By a result of Hironaka [26], any modification $V' \to V$ is dominated by a resolution satisfying the condition in (i).

This theorem is a generalization of our previous work [42] which covered the cases where $\dim_{\mathbb{C}} V = 2$ or where the singularities could be resolved with a smooth exceptional divisor. In this latter case, Ohsawa [34] has given another proof of (0.2), with a different (nonquasi-isometric) metric.

As a corollary, we obtain for each Kähler distinguished metric an L_2-Hodge structure on $IH^\cdot(V;\mathbb{C})$. In fact, the (p,q) part is given by the corresponding L_2-$\bar\partial$-cohomology $H^{p,q}_{(2),\bar\partial}(V\backslash \operatorname{Sing} V)$. A priori these Hodge structures may depend on the specific metrics. However, Zucker has shown that this is not the case:

Theorem 0.2 (Zucker [55]). *For V a projective variety with isolated singularities, the L_2-Hodge structure on $IH^\cdot(V;\mathbb{C})$ arising from any Kähler distinguished metric agrees with the canonical Hodge structure given by Saito.*

When V has isolated singularities, the existence of a natural Hodge decomposition on $IH^\cdot(V;\mathbb{C})$ was originally proven by Steenbrink [48].

Using Zucker's result, we are able to prove (see Theorem 10.2):

Theorem 0.3. *Let V be a projective variety with isolated singularities. For any Kähler distinguished metric on V and for any resolution $\widetilde V$ of V,*

$$H^{0,q}_{(2),\bar\partial}(V\backslash \operatorname{Sing} V) \cong H^q(\widetilde V, \mathscr{O}_{\widetilde V}).$$

In particular, the L_2-$\bar\partial$-index

$$\chi_{(2)}(V\backslash \operatorname{Sing} V) \equiv \sum_q (-1)^q \dim H^{0,q}_{(2),\bar\partial}(V\backslash \operatorname{Sing} V),$$

equals the arithmetic genus $\chi(\widetilde V)$ of any resolution $\widetilde V$ of V.

Remark. The requirement that V be projective in Theorems 0.1–0.3 can be weakened to admit at least the class of compact Kähler varieties; see (8.5) and Remark 8.5.

In other contexts, the equality $\chi_{(2)}(V\backslash \operatorname{Sing} V) = \chi(\widetilde V)$ was conjectured by MacPherson [30]. In particular, the case of projective curves and surfaces where $V\backslash \operatorname{Sing} V$ is given the (incomplete) metric induced by the Fubini-Study metric has been studied by Haskell [24], [25] and Pardon [36]. Recently, Pardon and Stern [37] have given a proof for arbitrary projective varieties if one uses L_2-$\bar\partial$-cohomology for the Fubini-Study metric with Dirichlet boundary conditions.

In the remainder of this introduction, we present an overview of the paper, with the aim of motivating many of the constructions.

0.1. We first look at a class of examples where natural metrics do exist, the locally symmetric varieties; this will motivate the definition of distinguished metrics.

Let X be a bounded symmetric domain and let Γ be an arithmetic group of automorphisms of X acting freely (for a simple example, take the upper half-plane $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ for X, and a finite index, torsion-free subgroup of $\mathrm{SL}(2,\mathbb{Z})$ for Γ). The complex manifold $\Gamma\backslash X$ has a natural complete Kähler metric induced from the Bergman metric on X. (Equivalently, a metric on X invariant under the automorphism group descends to $\Gamma\backslash X$.)

Such manifolds arise in many areas of mathematics, often as moduli spaces. One difficulty in studying them is that $\Gamma\backslash X$ is usually noncompact. However, $\Gamma\backslash X$ is a quasi-projective variety, called a *locally symmetric variety*; in fact there is a natural projective compactification, the Baily-Borel-Satake compactification $\Gamma\backslash X^*$ [3]. In general $\Gamma\backslash X^*$ is very singular; however, Zucker conjectured [52] that

$$H^{\cdot}_{(2)}(\Gamma\backslash X) \cong IH^{\cdot}(\Gamma\backslash X^*; \mathbb{C}).$$

(More generally one also considers cohomology with coefficients in a local system arising from a finite-dimensional complex representation of the automorphism group of X.) The conjecture was proven independently by Saper and Stern [44], [45] and by Looijenga [29].

What do the metrics on locally symmetric varieties look like? For simplicity, assume that $\Gamma\backslash X^*$ has only a single isolated singular point. Let U^c be a closed neighborhood of the singular point with the singularity itself deleted (a *punctured closed neighborhood*; see §0.8). We may choose U^c such that it decomposes as $[c, \infty) \times Z$, where the coordinate r on $[c, \infty)$ tends to ∞ near the singular point and Z is a compact manifold called the *link* having a doubly fibered structure. Specifically, there is an integer p, $1 \le p \le n = \dim_{\mathbb{C}} X$, such that

(1) Z is a flat bundle over a compact locally symmetric space M of dimension $p-1$, with a nilmanifold fiber S,

$$\begin{tikzcd} S \arrow[r] & Z \arrow[d, "\Phi"] \\ & M, \end{tikzcd} \tag{0.3}$$

and

(2) S is a principle $(S^1)^p$-bundle (equipped with a connection) over an abelian variety N_1 of complex dimension $n-p$,

$$\begin{tikzcd} (S^1)^p \arrow[r] & S \arrow[d, "\pi_1"] \\ & N_1. \end{tikzcd} \tag{0.4}$$

The metric on U^c may be written quasi-isometrically as

$$dr^2 + ds_M^2 + e^{-r}\pi_1^* ds_{N_1}^2 + e^{-2r}\sum_{i=1}^{p}\tau_i^2, \tag{0.5}$$

where the $(\tau_i)_{i=1}^p$ are connection forms for π_1. (This is a special case of Borel's formula [5], [6].) Simply put, a vector field tangent to the level sets of r, which is invariant under translation in r, may have its norm-squared decay at one of three possible rates toward the singularity: $1 = e^{0r}$, e^{-r}, e^{-2r}; these different "weights" play a crucial role in L_2-cohomology.

Before turning back to a general variety V, it will be helpful to view the above structure and metric in the setting of a smooth *toroidal compactification* $\overline{\Gamma\backslash X}$ of $\Gamma\backslash X$ ([2], [31]); for this we assume Γ is *neat* (every Γ contains neat subgroups of finite index, so this is not a severe restriction). The compactification $\overline{\Gamma\backslash X}$ is not unique, but depends on the choice of a certain type of triangulation of M. By decomposing M into its top-dimensional simplices $\{M_\alpha\}$ (over each of which Φ becomes trivial), U^c may be decomposed into regions $U_\alpha^c \cong [c, \infty) \times M_\alpha \times S$ with disjoint interiors. Projecting onto the third factor and applying π_1, we obtain a real analytic fibration of U_α^c over the abelian variety N_1, whose fibers turn out to be holomorphic submanifolds of $\Gamma\backslash X$. (It is possible to decompose U^c somewhat differently so as to obtain an actual holomorphic bundle, but we shall not need this.)

The point now is that this fibration sits inside a principle $(\mathbb{C}^*)^p$-bundle over N_1. Explicitly, the embedding of fibers

$$[c, \infty) \times M_\alpha \times (S^1)^p \to (\mathbb{C}^*)^p \tag{0.6}$$

is given by

$$(r, (t_i)_{i=1}^p, (\theta_i)_{i=1}^p) \mapsto (e^{-t_1 e^r/2} e^{i\theta_1}, \cdots, e^{-t_p e^r/2} e^{i\theta_p}), \tag{0.7}$$

where the $(t_i)_{i=1}^p$ are barycentric coordinates on M_α. (Actually, for this formula to hold exactly, the exhaustion function r must be chosen carefully, and may in fact be only piecewise smooth on U^c, though smooth on each U_α^c.)

The image of (0.6) lies in the product of the punctured closed disks $(\bar\Delta^*)^p$. A partial compactification $\overline{U_\alpha^c}$ is obtained by embedding each fiber further into $\bar\Delta^p$ and taking the closure; gluing together the different $\overline{U_\alpha^c}$ along their boundaries yields the toroidal compactification $\overline{\Gamma\backslash X}$. For appropriate triangulations of M, one can show that $\overline{\Gamma\backslash X}$ is a compact complex manifold and that there is a natural map $\overline{\Gamma\backslash X} \to \Gamma\backslash X^*$ which is a resolution of singularities, with exceptional set $D = \overline{\Gamma\backslash X} - \Gamma\backslash X$ a divisor with normal crossings.

Using (0.7) we may reexpress the locally symmetric metric (0.5) quasi-isometrically as

$$\left|\log \prod_{k=1}^{p} |z_k|^2\right|^{-2} \sum_{i=1}^{p} \frac{1}{|z_i|^2} dz_i d\bar{z}_i + \left|\log \prod_{k=1}^{p} |z_k|^2\right|^{-1} \sum_{j=1}^{n-p} dw_j d\bar{w}_j, \tag{0.8}$$

where $(z_i)_{i=1}^{p}$ and $(w_j)_{j=1}^{n-p}$ are coordinates on $\bar{\Delta}^p$ and N_1, respectively. Alternatively, the same expression holds when $((z_i)_{i=1}^{p}, (w_j)_{j=1}^{n-p})$ are any holomorphic local coordinates on $\overline{\Gamma\backslash X}$ for which $D = \bigcup_{i=1}^{p} z_i^{-1}(0)$.

0.2. We now return to a general projective variety V of dimension n which we assume to have only isolated singularities. We would like to use the locally symmetric metrics as models for a metric on $V\backslash \operatorname{Sing} V$, so we need to find near the singularities of V some of the structure present in the cusps of a locally symmetric variety.

As before, we express a punctured closed neighborhood U^c of a singular point as $[c, \infty) \times Z$; the link Z now no longer has the global structure of (0.3) and (0.4) for a fixed p. It is possible however to decompose U^c into regions (with disjoint interiors) $U^c_{[p]} = [c, \infty) \times Z_{[p]}$ for $p = 1, \cdots, n$, such that for each p, $U^c_{[p]}$ has structure mimicing that on U^c_α for some locally symmetric variety with the given value of p. In fact, $U^c_{[p]} \cong [c, \infty) \times M_{[p]} \times S^*_{[p]}$, where $M_{[p]}$ is a ($p-1$)-dimensional simplex and $\pi_{[p]} : S^*_{[p]} \to D^*_{[p]}$ is an $(S^1)^p$-bundle (equipped with a connection) over a Kähler manifold (with corners) $D^*_{[p]}$ of dimension $n-p$.

To do this, the theory of toroidal compactifications suggests that we consider a resolution $\pi : \widetilde{V} \to V$ whose exceptional set D is a divisor with normal crossings; identify $\widetilde{V}\backslash D = V\backslash \operatorname{Sing} V$. For simplicity we assume the components of D are smooth. Then $D^*_{[p]}$ and $U^c_{[p]}$ may be defined inductively: Assume $U^c_{[p+1]}, \cdots, U^c_{[n]}$ have already been defined. Then $D^*_{[p]}$ will consist of points of D where exactly p local components of D intersect, minus those in the interior of $\overline{U^c_{[p+1]}} \cup \cdots \cup \overline{U^c_{[n]}}$ (here the closures are taken in $\widetilde{V}$); $\overline{U^c_{[p]}}$ will be a certain closed tubular neighborhood of $D^*_{[p]}$. The fibration structure and connection on $U^c_{[p]} = \overline{U^c_{[p]}} \cap (\widetilde{V}\backslash D)$ are constructed using a theorem of Clemens [14] (Theorem 5.1).

For each $p = 1, \cdots, n$, we would now like to use (0.8) as a model metric in $U^c_{[p]}$. However such a model near a p-fold intersection will not restrict to the corresponding model at nearby points where fewer components intersect; to cure this we use a perturbation of the model:

Definition 0.4. A Riemannian metric on $\widetilde{V}\backslash D$ is called *distinguished* if near a p-fold intersection of local components of D it has the quasi-isometric form

$$\begin{aligned}(0.9)\qquad &\left|\log\prod_{k=1}^{p}|z_k|^2\right|^{-2}\sum_{i=1}^{p}\left(\frac{1}{|z_i|^2}+\left|\log\prod_{k=1}^{p}|z_k|^2\right|\right)dz_i d\bar{z}_i\\ &+\left|\log\prod_{k=1}^{p}|z_k|^2\right|^{-1}\sum_{j=1}^{n-p}dw_j d\bar{w}_j,\end{aligned}$$

where $((z_i)_{i=1}^{p}, (w_j)_{j=1}^{n-p})$ are holomorphic local coordinates for which $D = \bigcup_{i=1}^{p} z_i^{-1}(0)$. A metric ds^2 on $V\backslash\operatorname{Sing}V$ is called a *distinguished metric on* V if there exists a resolution $\pi : \widetilde{V} \to V$ for which $\pi^* ds^2$ is distinguished.

Remark. To be precise we should call a metric on $\widetilde{V}\backslash D$ with the form (0.9) *distinguished relative to* π; this is because if V were to have nonisolated singularities, the correct description of a distinguished metric would depend not only on the configuration of $D \subset \widetilde{V}$, but also on how π blows down D. Since we are only considering isolated singularities at present, we will usually suppress the reference to π.

Also note that if $\widetilde{V} \supset D$ is *any* compact complex manifold containing a divisor with normal crossings, (0.9) defines a quasi-isometry class of metrics on $\widetilde{V}\backslash D$. We will also call these metrics distinguished (relative to $\pi : \widetilde{V} \to V$, where now V is merely a topological pseudomanifold formed by collapsing D to a finite number of points). As we shall see however (Example 7.10), (0.2) does not always hold in this more general situation.

In terms of the decomposition $U_{[p]}^{c} \cong [c, \infty) \times M_{[p]} \times S_{[p]}^{*}$, a distinguished metric is quasi-isometric to (Proposition 6.5):

$$(0.10)\qquad dr^2 + ds^2_{M_{[p]}}(r) + e^{-r}\pi^*_{[p]}ds^2_{D^*_{[p]}} + e^{-2r}\sum_{i=1}^{p}(1+e^{-t_i e^r}e^r)\tau_i^2,$$

where the $(t_i)_{i=1}^{p}$ are barycentric coordinates on $M_{[p]}$ and

$$ds^2_{M_{[p]}}(r) = \sum_{i=1}^{p}(1+e^{-t_i e^r}e^r)dt_i^2.$$

Compare this with the locally symmetric model (0.5); note that in a compact subset of the interior of $M_{[p]}$, $(1+e^{-t_i e^r}e^r) \sim 1$ and the two metrics are quasi-isometric.

It is easy to see from (0.10) that a distinguished metric is complete and has finite volume; unlike a locally symmetric metric, however, whose Ricci curvature is a constant negative multiple of the metric, a distinguished metric in general may have unbounded Ricci curvature.

0.3. The theory of locally symmetric varieties not only motivates the definition of distinguished metrics, but also suggests how to construct *Kähler* ones (Theorem 0.1(i)).

For example, let $H = \{ u \in \mathbb{C} \mid \operatorname{Im} u > 0 \}$ be the upper half-plane and Γ the modular group. The cusp of $\Gamma\backslash H$ is biholomorphic to a punctured neighborhood of $0 \in \Delta$ via $z = e^{2\pi i u}$, and a compactification is obtained by gluing in $z = 0$. Since a Kähler potential for the Bergman metric on H is $-\log \operatorname{Im} u$, the resulting locally symmetric metric on Δ^* near the origin has Kähler potential (modulo constants) $-\log(\log|z|^2)^2$.

Less trivially, let $X = B^{q_1+1} \times \cdots \times B^{q_p+1}$ be the product of complex balls of dimensions $q_1 + 1, \cdots, q_p + 1$, and let Γ be an arithmetically defined group of automorphisms of X. There is a biholomorphism $X \cong \{ (u, v) \in \mathbb{C}^p \times \mathbb{C}^{n-p} \mid \operatorname{Im} u_\alpha - |v_\alpha|^2 > 0,\ 1 \le \alpha \le p \}$, where $n = p + \sum q_\alpha$ is the dimension of X, and we write $u = (u_\alpha)_{\alpha=1}^p$, $v = (v_\alpha)_{\alpha=1}^p$, $u_\alpha \in \mathbb{C}$, $v_\alpha \in \mathbb{C}^{q_\alpha}$; this is a realization of X as a Siegel domain of the second kind [47]. Over a 0-dimensional stratum of $\Gamma\backslash X^*$, a smooth toroidal compactification of $\Gamma\backslash X$ has the following local structure ([2], [31]): Let $\sigma \subseteq (\mathbb{R}^+)^p$ be a closed simplicial cone with nonempty interior, that is, $\sigma = \sum_{i=1}^p \mathbb{R}^+\xi_i$, where $\xi_1, \cdots, \xi_p$ is a basis of $\mathbb{R}^p$ consisting of vectors with positive coordinates, and for $c \in \mathbb{R}$ let σ^c denote the truncated cone $\{\sum x_i\xi_i \in \sigma \mid \sum x_i \ge e^c\}$. For appropriate σ and c large, Γ acts on $\{ (u, v) \mid \operatorname{Im} u \in \sigma^c,\ v \in \Delta^{n-p} \} \subseteq X$ merely through translation on u via the lattice $\sum \mathbb{Z}\xi_i$. Thus this set descends in $\Gamma\backslash X$ to a segment of the cusp U_σ^c which is diffeomorphic to $i\sigma^c \times (\mathbb{R}^p/\sum \mathbb{Z}\xi_i) \times \Delta^{n-p}$. (Warning: this is not quite the same as our previous region and decomposition.) We now have an open holomorphic imbedding $U_\sigma^c \hookrightarrow (\Delta^*)^p \times \Delta^{n-p}$ given by $(u, v) \mapsto (e^{2\pi i l_1(u)}, \cdots, e^{2\pi i l_p(u)}, v)$, where $l_1, \cdots, l_p$ is the dual basis to $\xi_1, \cdots, \xi_p$. A partial toroidal compactification is obtained by taking the closure of the image in $\Delta^p \times \Delta^{n-p}$.

The potential for the Bergman metric on X is $\sum_\alpha -\log(\operatorname{Im} u_\alpha - |v_\alpha|^2)$ (note that the $v = 0$ section of X is merely H^p). Consequently the potential on $(\Delta^*)^p \times \Delta^{n-p}$ may be written (modulo constants) as

$$\sum_{\alpha=1}^{p} -\log\left(\log \prod_i |z_i|^{2a_{\alpha i}} h_\alpha\right)^2, \tag{0.11}$$

where $z_i = e^{2\pi i l_i(u)}$ are coordinates on $(\Delta^*)^p$, $u_\alpha = \sum_i a_{\alpha i} l_i(u)$, and $h_\alpha = e^{-|v_\alpha|^2}$.

Returning to our general variety V with isolated singularities, we consider $\pi : \widetilde{V} \to V$, a resolution of singularities with exceptional divisor D; assume that π is a projective morphism and that $D = D_1 \cup \cdots \cup D_m$ has normal crossings. To construct a Kähler potential on $\widetilde{V}\backslash D$ analogous to (0.11) near a p-fold intersection of components of D, we search for independent metrized line bundles $L_1, \cdots, L_n \in -\sum_{i=1}^m \mathbb{Z}^+[D_i]$ which have positive curvature in a neighborhood of D. The desired potential is then

$$(0.12) \qquad \sum_{\alpha=1}^{n} -\log(\log|s_\alpha^*|^2)^2,$$

where $s_\alpha^* \in \Gamma(\mathscr{O}_{\widetilde{V}}(L_\alpha^*))$ vanishes only on D. Such line bundles are obtained in §8.1 as $\pi^{-1}\mathscr{I}_\alpha$, where $\mathscr{I}_1, \cdots, \mathscr{I}_n$ are ideals in $\mathscr{O}_{V, \operatorname{Sing} V}$, all of whose blow-ups yield the morphism $\pi : \widetilde{V} \to V$. The discrepancy between (0.11) and (0.12), due to the different curvature hypotheses, corresponds precisely to the difference between (0.8) and (0.9) (likewise (0.5) and (0.10)).

0.4. We now give some indication of how to prove Theorem 0.1(ii),

$$H_{(2)}^{\cdot}(V\backslash \operatorname{Sing} V) \cong IH^{\cdot}(V; \mathbb{C}),$$

for a distinguished metric. To do this, it suffices by the local characterization of intersection cohomology to prove the local vanishing condition

$$(0.13) \qquad H_{(2)}^k(U^c) = 0, \qquad k \geq n,$$

where U^c, for c large, ranges over a cofinal system of punctured neighborhoods of $\operatorname{Sing} V$.

We first consider the abstract situation of a domain X in a Riemannian manifold Y together with a free S^1-action T on X (an S^1-*domain*). Our main tool for understanding the L_2-cohomology of Y is the calculation (Theorem 2.4), under certain technical conditions, of a spectral sequence converging to $H_{(2)}^{\cdot}(Y)$ which is a combined analogue of the Leray spectral sequence and the exact sequence of the pair. Briefly,

$$E_1^{-p,q} = \begin{cases} H_{(2)}^{q-2}(X/T; |\tau|) & \text{for } p = 1, \\ H_{(2)}^{q}(\widehat{Y\backslash X}) & \text{for } p = 0, \\ H_{(2)}^{q+1}(X/T, (\mathrm{bd}_Y X)/T; |\tau|^{-1}) & \text{for } p = -1, \end{cases}$$

where τ is the connection form for T and $\widehat{Y\backslash X}$ is $Y\backslash X$ with the T-orbits in the boundary $\mathrm{bd}_Y X$ filled in by disks; for $p = \pm 1$ we are using here L_2-cohomology with the *weight function* $|\tau|^{\pm 1}$ (see §1.1).

Now let $Y = U^c$, equipped with a distinguished metric associated to a resolution $\pi : \widetilde{V} \to V$; let $D_1, \cdots, D_m$ be the smooth components of the normal crossing exceptional divisor. For each $i \in \{1, \cdots, m\}$, we can construct a free S^1-action (and thus an S^1-domain X^i) in a punctured neighborhood of D_i out of the $(S^1)^p$-actions on each $U^c_{[p]}$ (§§7.1, 7.2). If we apply the above argument repeatedly with $X = X^1, \cdots, X^m$, we obtain a spectral sequence converging to $H^{\cdot}_{(2)}(U^c)$ (Theorem 7.7); this spectral sequence is very similar to the weight spectral sequence (Theorem 7.5) for the mixed Hodge structure on the ordinary cohomology $H^{\cdot}(U^c)$, with one important difference: aside from a potentially infinite-dimensional row at $q = n+1$, the E_1-term is truncated so that $E_1^{-p,q} = 0$ for $q \geq n$.

To see why this happens, consider (0.10) and define a 1-form in $T^*M_{[p]}$ (resp. $\pi^*_{[p]}T^*D^*_{[p]}$, $\sum_{i=1}^p \mathbb{C}\tau_i$) to have *weight* 0 (resp. 1, 2); a pure wedge product is given the weight equal to the sum of the weights of its factors. It turns out that $E_1^{-p,q}$ may be represented by forms $\phi = \phi_0 + dr \wedge \phi_1$, with ϕ_0 of pure weight q and ϕ_1 of pure weight $q-1$ (see the proof of Theorem 7.7, particularly (7.17)). On the other hand, the volume form contributes a factor of e^{-nr} to the integrand of the L_2-norm $\|\phi\|$. The weight truncation of E_1 then follows from the calculation of the weighted L_2-cohomology of $\mathbb{R}^+$ (Lemma 1.8(i))—basically the point is that $e^{(q-n)r}$ is not integrable on $\mathbb{R}^+$ for $q \geq n$.

However a truncation by weight is not enough; for (0.13) we need a truncation by degree: $E_2^{-p,q} = 0$ for $q - p \geq n$. Fortunately the mixed Hodge structure on $H^k(U^c)$ satisfies a semipurity condition (Theorem 8.6; see [19], [33], or [48]): in degrees $k = q - p \geq n$, there is no cohomology with weight $q \leq k$. In other words, for $k = q - p \geq n$, $E_2^{-p,q} = 0$ for $q \leq k$. The previous weight truncation together with semipurity prove the desired degree truncation.

0.5. It is natural to wonder whether the class of distinguished metrics could be enlarged so that Theorem 0.1 remained valid. This is certainly the case; in fact, if V is a locally symmetric variety, we have not in general admitted the locally symmetric metric as distinguished. It may be possible to generalize the construction of a Kähler potential in §0.3 (for example, to treat projective morphisms that are not resolutions), and modifications of some of the techniques used here may apply to studying the L_2-cohomology of these more general distinguished metrics. We hope to consider this, as well as the case of nonisolated singularities, in a future paper.

It should be stressed, however, that the complete Kähler metric considered more commonly on $\widetilde{V}\backslash D$, the *Poincaré metric*, should not be admitted as distinguished for V, since our aim is to distinguish metrics relative to which analysis (in particular, L_2-cohomology) will reflect invariants of V. The Poincaré metric has the local quasi-isometric form

$$\sum_{i=1}^{p} \frac{dz_i d\bar{z}_i}{|z_i|^2(\log|z_i|^2)^2} + \sum_{j=1}^{n-p} dw_j d\bar{w}_j;$$

its L_2-cohomology is easily seen to be isomorphic to the ordinary cohomology of $\widetilde{V}$ [51]. In analogy to the previous notation, we would call the Poincaré metric a distinguished metric relative to the identity map $\widetilde{V} \to \widetilde{V}$, or simply, a distinguished metric on $\widetilde{V}$.

0.6. The sections of the paper fall naturally into two parts; the first (§§1–4) deals with basic tools for calculating the L_2-cohomology and the spectral sequence of an S^1-domain, the second (§§5–10) deals with L_2-cohomology of distinguished metrics. The reader may wish to begin with the second part and refer back to the first part when the results are needed.

After presenting some basic tools for calculating L_2-cohomology in §1, we present in §2 the calculation (Theorem 2.4) of the spectral sequence associated to an S^1-domain. The somewhat technical proof of this result is relegated to §3 and a generalization to multiple overlapping S^1-domains (Theorem 4.7) is presented in §4.

In §§5–7, $\widetilde{V}$ is a complex manifold of dimension n and $D = \bigcup_{i=1}^m D_i \subset \widetilde{V}$ is a divisor with smooth components intersecting with normal crossings. Of course our main interest is when D blows down to the isolated singular points of a variety V, but this assumption will not be needed until §8. In §5 we decompose a punctured neighborhood U^c of D into regions with cusp-like structure, and thus are able to define S^1-domains $X^1, \cdots, X^m$. We next define distinguished metrics in §6 and give several quasi-isometric expressions for one. Lastly we calculate in §7 the spectral sequence (from §4) associated to the S^1-domains $\{X^i\}_{1\leq i\leq m}$ for a distinguished metric on U^c and observe the weight truncation.

Beginning in §8, V is a Kähler variety with isolated singularities and $\pi : \widetilde{V} \to V$ is a resolution of singularities. We assume that the morphism π is projective and that the exceptional divisor D has smooth components intersecting with normal crossings. The proof of Theorem 0.1 is finished by showing that Kähler distinguished metrics exist (Theorem 8.4) and that the weight truncation may be replaced by a degree truncation (Theorem 8.7).

In §9 we discuss L_2-Hodge structures and in §10 we give the proof of Theorem 0.3.

0.7. Acknowledgments. Theorem 0.1 was proven in the fall of 1985; it was announced and an outline of the proof was presented in [43]. Theorem 0.3 was proven in the fall of 1986 after seeing Zucker's proof of Theorem 0.2 and after discussions with Bill Pardon concerning MacPherson's conjecture. We would like to thank W. Pardon and S. Zucker for their interest in this work. We also wish to thank N. Habegger and D. Morrison for helpful conversations. This manuscript was revised, on and off, while visiting the University of California at San Diego, the University of Illinois at Chicago, and Harvard University; we would like to thank these institutions for their hospitality. Finally, we would like to thank Shing-Tung Yau for his continual encouragement.

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0.8. Notation. Two positive functions (resp. Riemannian metrics, positive $(1, 1)$ forms, etc.) g and g' are said to be *quasi-isometric*, denoted $g \sim g'$, if there exists a constant $C > 0$ such that $C^{-1}g' \leq g \leq Cg'$. We write $g \lesssim g'$ if $g \leq Cg'$.

If C is a complex, let $C[p]$ denote the shifted complex so that $C[p]^i = C^{i+p}$. For W an increasing filtration on C, we denote the *associated graded complex* $W_{\bullet}C/W_{\bullet-1}C$ by $\mathrm{Gr}^W_{\bullet} C$.

Denote the boundary of a piecewise-smooth (ps) manifold with boundary Y by ∂Y; if $X \subset Y$ is a closed ps domain, the set of its topological boundary points relative to Y, $\overline{\partial X \backslash \partial Y}$, is denoted $\mathrm{bd}_Y X$.

If $A \subseteq Y$ is a subset of a topological space, a *punctured* (*closed*) *neighborhood* of A is a (closed) neighborhood of A with A itself removed.

If v is a vector field, we denote the corresponding operations of interior multiplication and Lie derivation by ι_v and L_v, respectively.

1. Basic calculations of L_2-cohomology

In order to establish notation, we briefly recall the definition of L_2-cohomology in §1.1; references are [11], [13], [52]. In the remainder of this section, we present some tools for calculating L_2-cohomology, all based on the use of homotopy formulas (§1.2); many of the arguments are inspired by techniques of Cheeger and Zucker. In §1.3 we restate Zucker's theorem

[52] on the L_2-cohomology of warped products in a form convenient for our needs; in §1.4 this is extended to some simple cases of warped bundles. Finally, §1.5 deals with the problem of extending homotopy formulas outside a domain.

1.1. L_2-cohomology. Let Y be a piecewise-smooth (ps) Riemannian manifold with boundary. In other words, Y is a ps manifold with boundary equipped with compatible Riemannian metrics on each closed top-dimensional simplex of some ps triangulation; compatible here means that the restriction of the metric to vectors tangent to the codimension one simplices is well defined. Let $F \subseteq \partial Y$ be a closed domain and $g > 0$ a ps function on Y (called a *weight* function). We shall refer to this type of data as a *triple* (Y, F, g). Let $A^i(Y, F)$ (resp. $A^i_c(Y, F)$) denote the space of ps (resp. ps compactly supported) i-forms on Y with zero Dirichlet boundary data on F. That is, if $j : \partial Y \hookrightarrow Y$ denotes the inclusion, then $\phi \in A^i(Y, F)$ satisfies $j^*\phi = 0$ on F.

The closure of $A^i_c(Y, F)$ under the weighted L_2-norm

$$\|\phi\|^2 = \int_Y |\phi|^2 g\, dV$$

forms a Hilbert space, denoted $L^i_2(Y; g)$ (it is clearly independent of F). Let $d_{Y,F;g}$ denote the exterior differential operator on the domain

$$\mathrm{Dom}(d_{Y,F;g}) = \{\phi \in A^{\cdot}(Y, F) \cap L^{\cdot}_2(Y; g) \mid d\phi \in L^{\cdot+1}_2(Y; g)\}.$$

The L_2-*cohomology* $H^{\cdot}_{(2)}(Y, F; g)$ is the cohomology of the complex $(\mathrm{Dom}(d_{Y,F;g}), d_{Y,F;g})$; it is an invariant of the quasi-isometry class of g and the metric on Y.

It is often more convenient to work with closed operators on Hilbert space. Thus let $\bar{d}_{Y,F;g}$ be the closure of the operator $d_{Y,F;g}$ in the sense of unbounded operators on Hilbert space; that is, the graph of $\bar{d}_{Y,F;g}$ is the closure of the graph of $d_{Y,F;g}$ in the graph norm. Then by [11] the L_2-cohomology may also be defined as the cohomology of $(\mathrm{Dom}(\bar{d}_{Y,F;g}), \bar{d}_{Y,F;g})$.

We will omit F (resp. g) from the above notation when $F = \varnothing$ (resp. $g \equiv 1$). Furthermore we will omit all the subscripts from $\bar{d}_{Y,F;g}$ when this will not cause confusion.

Decompose the L_2-cohomology orthogonally as

$$\begin{aligned} H^{\cdot}_{(2)}(Y, F; g) &= \mathrm{Ker}(\bar{d})/\mathrm{Range}(\bar{d}) \\ &\cong \mathrm{Ker}(\bar{d})/\overline{\mathrm{Range}(\bar{d})} \oplus \overline{\mathrm{Range}(\bar{d})}/\mathrm{Range}(\bar{d}). \end{aligned} \tag{1.1}$$

The second term in the right-hand side of (1.1) can be seen by the open mapping theorem to be either 0 or infinite dimensional. The first term is called the *reduced L_2-cohomology*, $\bar{H}^{\cdot}_{(2)}(Y, F; g)$; it admits a representation by harmonic forms. Namely, let $\overline{d}^*$ denote the Hilbert space adjoint of $\overline{d}$ and let $\Delta = \overline{d}\,\overline{d}^* + \overline{d}^*\overline{d}$ be the corresponding Laplacian with the induced domain. Then, since $\operatorname{Ker}(\overline{d}^*) = \operatorname{Range}(\overline{d})^\perp$,

$$\bar{H}^{\cdot}_{(2)}(Y, F; g) = \operatorname{Ker}(\overline{d})/\overline{\operatorname{Range}(\overline{d})} \cong \operatorname{Ker}(\overline{d}) \cap \operatorname{Ker}(\overline{d}^*) = \operatorname{Ker}(\Delta).$$

Combining this with (1.1), we have

$$H^{\cdot}_{(2)}(Y, F; g) = \operatorname{Ker}(\Delta) \oplus \overline{\operatorname{Range}(\overline{d})}/\operatorname{Range}(\overline{d}). \tag{1.2}$$

Finally, we remark that if Y is complete, a theorem of Gaffney [20] states that $A^{\cdot}_c(Y, F)$ is dense in $\operatorname{Dom}(\overline{d})$ for the graph norm. In other words, if Y is complete, one has the equality $\overline{d} = \overline{d}_c$, where d_c is the exterior differential operator with domain $A^{\cdot}_c(Y, F)$. This sometimes reduces computations on $\operatorname{Dom}(\overline{d})$ to ones on $A^{\cdot}_c(Y, F)$.

1.2. Homotopy formulas.

Definition 1.1. Let C be a complex with differential d and let P be a map of C. P is *homotopic* to the identity map if there exists a *homotopy operator* $H : C \to C$ satisfying the *homotopy formula*

$$dH + Hd = I - P.$$

We call H a *good* homotopy if $HP = PH = 0$.

If P is homotopic to the identity, it induces the identity map on the cohomology $H(C)$; if, in addition, C is a normed complex and P is bounded (but not necessarily H), then P induces the identity map on the *reduced cohomology* $\overline{H}(C) = \operatorname{Ker}(d)/\overline{\operatorname{Range}(d)} \cap \operatorname{Ker}(d)$. By a *normed complex*, we simply mean that as a vector space C is endowed with a norm; C may not be complete and the differential d may not be bounded.

Example 1.2. Let (Y, F, g) be a triple as in §1.1 and let $(C, d) = (\operatorname{Dom}(\overline{d}), \overline{d})$. If $\operatorname{Range}(\overline{d})$ is closed, then orthogonal projection P of $\operatorname{Dom}(\overline{d})$ onto $\operatorname{Ker}(\Delta)$ (harmonic projection) is homotopic to the identity via a good bounded homotopy operator H. Here H is the pseudoinverse $\overline{d}^{-1}$, that is, $H\phi$ is 0 if $\phi \in \operatorname{Dom}(\overline{d}) \ominus \operatorname{Range}(\overline{d})$ and is the unique solution of $\overline{d}\psi = \phi$, $\psi \perp \operatorname{Ker}(\overline{d})$, if $\phi \in \operatorname{Range}(\overline{d})$. The harmonic projection P induces the isomorphism $H^{\cdot}_{(2)}(Y, F; g) \cong \operatorname{Ker}(\Delta)$ of (1.2).

Remark. Good homotopies are useful when lifting a homotopy on an associated graded complex $\operatorname{Gr}^W C$ to a homotopy on C (see Proposition 3.7 and Lemma 4.11). Note that any operator P which is homotopic to the identity via a good homotopy is necessarily a projection operator, i.e., $P = P^2$.

Definition 1.3. A *homotopy equivalence* between two complexes C and D consists of a pair of maps $C \underset{S}{\overset{R}{\rightleftarrows}} D$ such that SR and RS are homotopic to the identity via homotopies H_C and H_D respectively. The homotopy equivalence is *good* if both H_C and H_D are; the homotopy equivalence is *filtered* (for some filtration W on C and D) if R, S, H_C, and H_D are.

Homotopy equivalences may be composed:

Lemma 1.4. *Let* $C \underset{S^1}{\overset{R^1}{\rightleftarrows}} D$ *and* $D \underset{S^2}{\overset{R^2}{\rightleftarrows}} E$ *be homotopy equivalences, with homotopies* H_C^1, H_D^1, H_D^2, *and* H_E^2. *Then*

(i) $C \underset{S^1S^2}{\overset{R^2R^1}{\rightleftarrows}} E$ *is a homotopy equivalence.*

(ii) *If the given homotopy equivalences are good, then so is the homotopy equivalence in* (i), *provided* $[H_D^1, S^2R^2] = [H_D^2, R^1S^1] = 0$ (*we may also need to replace* R^1 *by* $R^1S^1R^1$, *etc.*).

(iii) *If the given homotopy equivalences are filtered for some filtration, then so is the homotopy equivalence in* (i).

Proof. For $S^1S^2R^2R^1$, use the homotopy $H_C^1 + S^1H_D^2R^1$.

Remark. We frequently use the special case where S^1 and S^2 are inclusion maps (in which case the parenthetical comment of (ii) is unnecessary).

If C has a norm, a usual way to establish a homotopy formula is to prove it on a subcomplex which is dense in the graph norm:

Lemma 1.5. *Let* C *be a normed complex, which is complete in the graph norm of* d. *Let* $C_0 \subset C$ *be a subcomplex, dense in the graph norm. Say* H *and* P *are bounded operators from* C_0 *into* C *such that*

$$dH + Hd = I - P \tag{1.3}$$

on C_0. *Then* H *and* P *extend to operators from* C *to* C *such that* (1.3) *still holds. If* H *is a good homotopy on* C_0, *so is its extension to* C.

Proof. Extend H and P to bounded operators on $\widetilde{C}$, the Banach space completion of C. Since C_0 is graph norm dense in C, we may approximate $\phi \in C$ by a sequence $\{\phi_i\}$ in C_0 such that $\phi_i \to \phi$ and $d\phi_i \to d\phi$. Since H and P are bounded, we see that $H\phi_i \to H\phi$ and $d(H\phi_i) = \phi_i - P\phi_i - Hd\phi_i \to \phi - P\phi - Hd\phi$ in $\widetilde{C}$. Since C is closed in graph norm we see that $H\phi \in C$ and (1.3) holds. It follows that $P\phi \in C$. The final statement is clear.

Remark. In practice, C will be a subcomplex of $\mathrm{Dom}(\overline{d}_{Y,F;g})$, for some triple $(Y, F; g)$. For example, C may be defined by invariance under a group action. C_0 will typically be $A^{\cdot}(Y, F) \cap C$, or $A_c^{\cdot}(Y, F) \cap C$ if Y is complete.

1.3. L_2-cohomology of warped products. Let (M, F_M, g_M) and (N, F_N, g_N) be triples, each consisting of a ps Riemannian manifold with boundary, a domain in the boundary, and a positive ps weight function. Given $w : M \to \mathbb{R}^+$, a positive ps function (called a *warping function*), define the *warped product triple* to be $(M \times_w N, F_M \times N \cup M \times F_N, g_M g_N)$, where $M \times_w N$ is the manifold $M \times N$ equipped with the metric

$$ds_M^2 + w(m)^2 ds_N^2.$$

By examining compactly supported decomposable forms one can see that [52, (2.10)]

$$(1.4) \qquad L_2^{\cdot}(M \times_w N; g_M g_N) \cong \bigoplus_k \left(L_2^{\cdot}(M; g_k g_M) \widehat{\otimes} L_2^k(N; g_N)[-k]\right),$$

where $\widehat{\otimes}$ denotes the completed tensor product and $g_k = w^{\nu - 2k}$ ($\nu = \dim_{\mathbb{R}} N$).

Theorem 1.6 (Zucker [52,(2.29)]). *Assume that*

(i) $M \times_w N$ *is complete* (*or, more generally,* $\overline{d} = \overline{d}_c$),
(ii) $\overline{d}_{N, F_N; g_N}$ *has closed range, and*
(iii) *the warping function* w *is bounded.*

Then the operator P *induced on* (1.4) *by harmonic projection in* $L_2^{\cdot}(N; g_N)$ (*with zero boundary conditions on* F_N) *preserves* $\mathrm{Dom}(\overline{d})$. *The restriction of* P *to this complex is homotopic to the identity via a agood bounded homotopy*

$$\begin{aligned} H : L_2^{\cdot}(M \times_w N; g_M g_N) &\to L_2^{\cdot - 1}(M \times_w N; w^{-2} g_M g_N) \\ &\subseteq L_2^{\cdot - 1}(M \times_w N; g_M g_N). \end{aligned}$$

Remark. The fact that H is bounded into a space with an additional weight of w^{-2} will be crucial for our applications, particularly in order to satisfy the hypotheses of Lemma 1.11 below. Neither this fact, nor the fact that H is good, is explicitly stated in [52], however they are clear from the proof.

Corollary 1.7 [52,(2.34)]. *Under the assumptions of Theorem* 1.6, *if* $H^k_{(2)}(N, F_N; g_N)$ *is finite dimensional for each* k, *then*

$$\begin{aligned} &H^{\cdot}_{(2)}(M \times_w N, F_M \times N \cup M \times F_N; g_M g_N) \\ &\quad \cong \bigoplus_k (H^{\cdot}_{(2)}(M, F_M; g_k g_M) \otimes H^k_{(2)}(N, F_N; g_N)[-k]). \end{aligned}$$

With the metrics we will be considering, Corollary 1.7 often reduces the computation of L_2-cohomology to knowing the L_2-cohomology of a half-line with exponential weights. For convenience, we recall this here (see, for example, [42, Propositions 3.2, 3.3]):

Lemma 1.8. *Let* $\mathbb{R}^+ = [0, \infty)$ *have the usual metric* dr^2, *where* r *is a coordinate on* $\mathbb{R}^+$, *and let* $\lambda \in \mathbb{R}$. *Then*

(i)
$$H^0_{(2)}(\mathbb{R}^+; e^{\lambda r}) = \begin{cases} 0 & \lambda \geq 0, \\ \mathbb{C} & \lambda < 0, \end{cases}$$
$$H^1_{(2)}(\mathbb{R}^+; e^{\lambda r}) = \begin{cases} 0 & \lambda \neq 0, \\ \text{infinite dimensional} & \lambda = 0. \end{cases}$$

(ii)
$$H^0_{(2)}(\mathbb{R}^+, \{0\}; e^{\lambda r}) = 0,$$
$$H^1_{(2)}(\mathbb{R}^+, \{0\}; e^{\lambda r}) = \begin{cases} \mathbb{C} & \lambda > 0, \\ \text{infinite dimensional} & \lambda = 0, \\ 0 & \lambda < 0. \end{cases}$$

1.4. Warped S^1-bundles. Let (M, F_M, g_M) be a triple and let $\pi : Y \to M$ be a principal S^1-bundle with S^1-action T. Let $\frac{\partial}{\partial\theta}$ denote the vector field on Y defined by T. Given a warping function $w : M \to \mathbb{R}^+$ and a connection on Y with connection form τ, the associated *warped S^1-bundle* is the manifold Y equipped with the metric

$$\pi^* ds_M^2 + (\pi^* w)^2 \tau^2.$$

Here, the connection form τ may be any T-invariant ps 1-form on Y such that $\tau(\frac{\partial}{\partial\theta}) \equiv 1$; this corresponds to a T-invariant ps choice of a horizontal subspace of TY, namely $\operatorname{Ker}(\tau)$. The 1-form $d\tau$ is then the lift of the curvature form on M. Note that any T-invariant metric on Y gives Y the structure of a warped S^1-bundle.

Similarly, if $\hat{\pi} : \widehat{Y} \to M$ is the associated disk bundle to π, with radial coordinate x, the *warped disk bundle* metric is defined to be

$$\pi^* ds_M^2 + (\pi^* w)^2 (dx^2 + x^2 \tau^2).$$

Proposition 1.9. *Assume the warping function* w *is bounded. Then there exists a bounded projection operator* P *of* $L_2^{\cdot}(Y; \pi^* g_M)$ *onto the subspace*

$$\pi^* L_2^{\cdot}(M; w g_M) \oplus \tau \wedge \pi^* L_2^{\cdot-1}(M; w^{-1} g_M) \tag{1.5}$$

which preserves the complex $\mathrm{Dom}(\overline{d}_{Y, \pi^{-1}F_M; \pi^* g_M})$. *The restriction of* P *to this complex is homotopic to the identity via a good bounded homotopy operator*

$$H : L_2^{\cdot}(Y; \pi^* g_M) \to L_2^{\cdot-1}(Y; w^{-2}\pi^* g_M) \subseteq L_2^{\cdot-1}(Y; \pi^* g_M).$$

If we also assume that

$$|d\tau|^2 < w^{-4}, \tag{1.6}$$

then the analogous result holds for $\widehat{Y}$ *with the subspace* (1.5) *replaced by* $\hat{\pi}^* L_2^{\cdot}(M; w^2 g_M)$.

Remark. Note that we do *not* need M, Y, or $\widehat{Y}$ to satisfy $\overline{d} = \overline{d}_c$.

Proof. By Lemma 1.5, it suffices to consider just ps forms; for simplicity, we will also assume $F_M = \emptyset$, $g_M \equiv 1$.

Let T_θ denote the action of $\theta \in S^1$ on Y and define

$$P(\phi) = \frac{1}{2\pi} \int_0^{2\pi} T_\theta^*(\phi)\, d\theta,$$
$$H(\phi) = \frac{1}{2\pi} \int_0^{2\pi} (\theta - \pi)(\iota_{\partial/\partial\theta} T_\theta^*(\phi))\, d\theta,$$

for $\phi \in A^{\cdot}(Y)$. P and H preserve piecewise smoothness and, since $|\tau|^2 = w^{-2}$, clearly have the required boundedness properties. The homotopy formula follows for ps forms from the standard formula $d \circ \iota_{\partial/\partial\theta} + \iota_{\partial/\partial\theta} \circ d = L_{\partial/\partial\theta}$ and integration by parts.

For the case of $\widehat{Y}$, denote the zero section of $\widehat{Y}$ also by M and consider the warped S^1-bundle $\widehat{Y} - M \to (0, 1] \times M$ (the projection is $p \mapsto (x(p), \hat{\pi}(p))$ and the warping function is xw). By applying the first part of the theorem we obtain operators P_θ and H_θ and a homotopy formula on $\mathrm{Dom}(\overline{d}_{\widehat{Y}-M})$. However, since $M \subseteq \widehat{Y}$ has codimension 2, a result of Cheeger [11, Lemma 1.1] implies that $\overline{d}_{\widehat{Y}-M} = \overline{d}_{\widehat{Y}}$ if we identify $L_2^{\cdot}(\widehat{Y} - M) = L_2^{\cdot}(\widehat{Y})$. Now let $\phi \in \mathrm{Dom}(d_{\widehat{Y}})$ and decompose in $\widehat{Y} - M$

$$P_\theta(\phi) = \alpha_0 + dx \wedge \alpha_1 + \tau \wedge (\alpha_2 + dx \wedge \alpha_3), \tag{1.7}$$

where $\iota_{\partial/\partial x}\alpha_i = \iota_{\partial/\partial\theta}\alpha_i = 0$. Since the α_i are T-invariant we have

$$\begin{aligned} dP_\theta(\phi) &= \tilde{d}\alpha_0 + d\tau\wedge\alpha_2 + dx\wedge\left(\frac{\partial\alpha_0}{\partial x} - \tilde{d}\alpha_1 + d\tau\wedge\alpha_3\right) \\ &\quad + \tau\wedge\left(-\tilde{d}\alpha_2 + dx\wedge\left(-\frac{\partial\alpha_2}{\partial x} + \tilde{d}\alpha_3\right)\right); \end{aligned} \tag{1.8}$$

here we have decomposed

$$d = dx\wedge\frac{\partial}{\partial x} + \tau\wedge\frac{\partial}{\partial\theta} + \tilde{d}.$$

Define

$$P\phi = \int_0^1 (2x\alpha_0 - (x^2-1)\,d\tau\wedge\alpha_3),$$

$$H\phi = H_\theta\phi - \int_x^1 \alpha_1 + \int_0^1 x^2\alpha_1 - \tau\wedge\int_0^x \alpha_3,$$

where the integrals are with respect to x. Using the fact that ϕ is ps on $\widehat{Y}$ (and thus in particular that α_2 and α_3 vanish when $x=0$) one easily computes the desired homotopy formula. The boundedness of H and P follows from the Cauchy-Schwarz inequality and Fubini's theorem by the type of arguments used for [52, (2.39)]; the hypothesis (1.6) is needed for the second term of P.

The fact that H is a good homotopy in both cases is a simple calculation from the definitions.

1.5. Extending homotopy operators. In many situations, the previous results may only apply in a subdomain of a larger space. We will show, under certain conditions, that operators homotopic to the identity on a domain $X\subseteq Y$ can be extended to operators homotopic to the identity on Y.

Definition 1.10. Let X be a closed ps domain in Y, a ps manifold with boundary. Let $M = \mathrm{bd}_Y X$ and assume M is a ps manifold with boundary $\partial M = M\cap\partial Y$. A *tubular boundary neighborhood* (N, p, t) of X consists of a tubular neighborhood N of M with normal projection $p:(N, N\cap\partial Y)\to(M,\partial M)$ and normal variable $t:(N, N\cap X, M)\to([-b,b],[0,b],\{0\})$, where $b>0$. We identify N with $[-b,b]\times M$ via $t\times p$.

Terminology. In later sections, we will say an object on N (e.g., a map or a group action) is *independent of* t if it is induced from some object on M via the identification $N\cong[-b,b]\times M$.

Lemma 1.11. *Let (Y, F, g) be a triple. Let $X\subseteq Y$ be a closed ps domain and assume there exists a tubular boundary neighborhood (N, p, t)*

of X satisfying:

(i) $\rho(F \cap N\backslash(X \cap N)) \subseteq F \cap N \cap X$, *where ρ denotes the reflection of N, $(t, m) \mapsto (-t, m)$.*

(ii) $g|_N \sim p^*(g|_M)$.

(iii) $ds_Y^2|_N \sim (p^*w)^2 dt^2 + p^* ds_M^2$, *where $w(m) > 0$ is a bounded function on M and ds_M^2 is a ps metric on M.*

Let P_X be a bounded operator on $\mathrm{Dom}(\overline{d}_{X, F\cap X; g|_X})$, *homotopic to the identity via a bounded operator*

$$H_X : L_2^{\cdot}(X; g) \to L_2^{\cdot-1}(X; \tilde{w}^{-2} g) \subseteq L_2^{\cdot-1}(X; g),$$

where $\tilde{w} > 0$ is a bounded function on Y with $\tilde{w}|_N \sim p^* w$.

Then there exists a bounded operator P_Y on $\mathrm{Dom}(\overline{d}_{Y, F; g})$ *satisfying*

$$P_Y \phi = \begin{cases} P_X(\phi|_X) & \text{on } X, \\ \phi|_{Y\backslash(X\cup N)} & \text{on } Y \backslash (X \cup N), \end{cases}$$

which is homotopic to the identity via a bounded homotopy operator

$$H_Y : L_2^{\cdot}(Y; g) \to L_2^{\cdot-1}(Y; \tilde{w}^{-2} g) \subseteq L_2^{\cdot-1}(Y; g),$$

satisfying

$$H_Y \phi = \begin{cases} H_X(\phi|_X) & \text{on } X, \\ 0 & \text{on } Y\backslash(X \cup N). \end{cases}$$

If H_X is a good homotopy, then so is H_Y.

Proof. Let $\eta(t)$ be a ps cutoff function on N such that

$$\eta(t) \equiv 0 \quad \text{for } t \leq -2b/3, \qquad \eta(t) \equiv 1 \quad \text{for } t \geq -b/3.$$

For $\phi \in L_2^{\cdot}(Y; g)$ define

$$H_Y \phi = \begin{cases} H_X(\phi|_X) & \text{in } X, \\ \eta \rho^* H_X(\phi|_X) & \text{in } N\backslash(X \cap N), \\ 0 & \text{in } Y\backslash(X \cup N), \end{cases}$$

$$P_Y \phi = \begin{cases} P_X(\phi|_X) & \text{in } X, \\ \phi - \eta \rho^*(\phi|_X) + \eta \rho^* P_X(\phi|_X) & \\ \quad - d\eta \wedge \rho^* H_X(\phi|_X) & \text{in } N\backslash(X \cap N), \\ \phi & \text{in } Y\backslash(X \cup N). \end{cases}$$

By our hypotheses, the operation $\psi \mapsto \eta \rho^* \psi$ defines a bounded map

$$A^{\cdot}(X, F \cap X) \cap L_2^{\cdot}(X; g) \\ \to A^{\cdot}(N\backslash(X \cap N), F \cap N\backslash(X \cap N)) \cap L_2^{\cdot}(N\backslash(X \cap N); g)$$

(resp. for the weight g replaced by $\tilde{w}^{-2}g$). Thus to see that H_Y and P_Y are bounded into the desired spaces, it remains to examine $d\eta \wedge \rho^* H_X(\phi|_X)$. But this is bounded since $|d\eta|^2 \lesssim w^{-2}$ and H_X is bounded into a space with an extra weight of $\tilde{w}^{-2}$.

Since extending a form ψ on X by $\eta\rho^*\psi$ preserves piecewise smoothness, it is easy to check that the desired homotopy formula holds on piecewise smooth forms and that if H_X is a good homotopy, so is H_Y. By Lemma 1.5, H_Y and P_Y preserve $\mathrm{Dom}(\overline{d}_{Y,F;g})$ and the homotopy formula extends.

2. A spectral sequence for S^1-domains

Let (Y, F, g) be a triple as in §1.1, consisting of a ps Riemannian manifold with boundary Y, $F \subseteq \partial Y$ a domain, and g a weight function on Y. Assume Y contains a closed domain X possessing a free S^1-action T (an *S^1-domain*; see below for the precise definition). Associated to the S^1-domain is a filtration W on $C = \mathrm{Dom}(\overline{d}_{Y,F;g})$. Formally this filtration is composed of two well-known filtrations. The first stage, $W_1 C \supset W_0 C$, is associated to the Leray spectral sequence of the quotient map

$$Y \to (Y \backslash X) \cup (X/T),$$

obtained by collapsing the fibers of the S^1-action on X. The second stage, $W_0 C \supset W_{-1} C$, corresponds to the exact sequence of the pair

$$((Y\backslash X) \cup (X/T), (Y\backslash X) \cup (M/T)),$$

where $M = \mathrm{bd}_Y X$. The main result of this section (Theorem 2.4) is the computation of the W-spectral sequence converging to $H^{\cdot}_{(2)}(Y, F; g)$ under the technical hypothesis that X be *admissible* (Definition 2.3). Briefly, the result is what one would expect from the formal description above, except for changes in the weight functions due to the possible decay of $|\frac{\partial}{\partial\theta}|$.

2.1. Definitions.

Definition 2.1. An *S^1-domain* in Y is a closed ps domain $X \subseteq Y$, together with a free S^1-action T on X (preserving both ∂X and $M = \mathrm{bd}_Y X$), such that X, ∂X, and M are all principal ps S^1-bundles over ps manifolds with boundary.

Let X be an S^1-domain in Y. We denote the projection $X \to X/T$ by π and the vector field on X induced by T by $\frac{\partial}{\partial\theta}$. If $Z \subseteq X$ is invariant under T, we write $T|_Z$ for the induced S^1-action.

Denote the complex $\mathrm{Dom}(\overline{d}_{Y,F;g})$ by C.

Definition 2.2. Let X be an S^1-domain in Y. The filtration W on C *associated* to X is defined by

$$W_1 C = C,$$
$$W_0 C = \left\{ \phi \in C \mid \int_0^{2\pi} (\iota_{\partial/\partial\theta} T_\theta^* \phi|_X)\, d\theta \equiv 0 \right\},$$
$$W_{-1} C = \{ \phi \in W_0 C \mid \phi|_{Y \setminus X} \equiv 0 \},$$
$$W_{-2} C = 0,$$

where T_θ denotes the action of $\theta \in S^1$ on X.

Definition 2.3. The pair $(X, (N, p, t))$ consisting of an S^1-domain X and a tubular boundary neighborhood (N, p, t) (Definition 1.10) is said to be *admissible relative to* (Y, F, g) if the following conditions hold:

(i) $N \cap X$ is invariant under T and the S^1-action $T|_{N\cap X}$ is induced from $T|_M$ independently of t.

(ii) $F \cap X$ is T-invariant and $F \cap N \cong [-b, b] \times (F \cap M)$.

(iii) Extend T and $\frac{\partial}{\partial\theta}$ to all of $X \cup N$, independently of t in N (this is possible by (i)). Let τ denote the ps 1-form on $X \cup N$ given by

$$\tau\left(\frac{\partial}{\partial\theta}\right) = 1, \qquad \tau|_{(\partial/\partial\theta)^\perp} = 0.$$

Then ds_Y^2 and g satisfy:

(2.1) $ds_Y^2|_{X\cup N}$ and $g|_{X\cup N}$ are quasi-isometrically T-invariant, that is,
$$ds_Y^2|_{X\cup N} \sim T_\theta^* ds_Y^2|_{X\cup N}$$
uniformly in θ, and similarly for g,

$$g|_N \sim p^*(g|_M), \tag{2.2}$$

$$ds_Y^2|_N \sim \begin{cases} f(t, m)(dt^2 + w(m)^4\tau^2) \\ \quad + p^*(\pi|_M)^* ds_{M/T}^2 & \text{in } X \cap N, \\ w(m)^2(dt^2 + \tau^2) + p^*(\pi|_M)^* ds_{M/T}^2 & \text{in } N\setminus X, \end{cases} \tag{2.3}$$

where $w(m) > 0$ is a T-invariant function on M, $f(t, m) > 0$ is a T-invariant function on $[0, b] \times M$, and $ds_{M/T}^2$ is a ps metric on M/T,

$$\int_0^b f(s, \cdot)\, ds \sim 1 \quad \text{in } M, \qquad 1 \lesssim f \lesssim (p^* w)^{-2} \quad \text{in } [0, b] \times M, \tag{2.4}$$

$$f \sim 1 \quad \text{in } [\varepsilon, b] \times M \text{ for } \varepsilon > 0, \qquad f|_{\{0\}\times M} \sim w^{-2}, \tag{2.5}$$

$$(2.6) \qquad |\tau|^{-2} \lesssim 1 \quad \text{in } X \cup N$$

(in N this already follows from (2.3) and (2.4)), and

$$(2.7) \qquad |d\tau|^2 \lesssim |\tau|^2 \quad \text{in } X \backslash N, \qquad |d\tau|^2 \lesssim w^{-4} \quad \text{in } N.$$

(iv) (N, p, t) may be extended to a larger tubular boundary neighborhood $(\widetilde{N}, \tilde{p}, t)$ in which (i)–(iii) continue to hold (say with b replaced by $2b$).

When we speak of an *admissible S^1-domain* X, we are assuming that we have fixed a tubular boundary neighborhood (N, p, t) such that $(X, (N, p, t))$ is admissible relative to (Y, F, g) (in particular, we are assuming such a tubular boundary neighborhood exists).

2.2. Main results. In order to state the main theorems of this section we need the following construction. Let X be an admissible S^1-domain. Define

$$\hat{\pi}_M : \widehat{M} \to M/T$$

to be the closed disk bundle associated to $\pi|_M : M \to M/T$. We give $\widehat{M}$ the warped disk bundle metric

$$w(m)^2(dt^2 + (1-t)^2\tau^2) + \hat{\pi}_M^* ds^2_{M/T}$$

(here we identify $\widehat{M} \backslash \{\text{zero section}\} \cong [0, 1) \times M$ and let t denote the first coordinate). Define

$$(2.8) \qquad \widehat{Y \backslash X} = \overline{Y \backslash X} \cup_M \widehat{M},$$

where we identify the copies of M lying in the boundaries. Similarly define $(F \backslash (F \cap X))^\frown$. We give $\widehat{Y \backslash X}$ the ps metric which restricts to those on $Y \backslash X$ and $\widehat{M}$. Let $\hat{g}$ denote the extension of $g|_{Y \backslash X}$ to $\widehat{Y \backslash X}$ which is independent of t in $\widehat{M}$.

Define the triples

$$(2.9) \quad (Y_j, F_j, g_j) = \begin{cases} (X/T, (F \cap X)/T \cup (M/T), |\tau|^{-1} g) & \text{if } j = -1, \\ (\widehat{Y \backslash X}, (F \backslash (F \cap X))^\frown, \hat{g}) & \text{if } j = 0, \\ (X/T, (F \cap X)/T, |\tau| g) & \text{if } j = 1. \end{cases}$$

The tubular neighborhood N (resp. $\widetilde{N}$) of M induces tubular neighborhoods

$$(2.10) \qquad N_j = \begin{cases} (N \cap X)/T & \text{if } j = -1, \\ (N \backslash (N \cap X))^\frown & \text{if } j = 0, \\ (N \cap X)/T & \text{if } j = 1 \end{cases}$$

(resp. $\widetilde{N}_j = \cdots$) of M/T viewed as a subset of Y_j; let

$$p_j : N_j \to M/T \qquad (j = -1, 0, 1) \tag{2.11}$$

(resp. $\tilde{p}_j$) be the induced projections. Let $\eta^{\pm}(t)$ be ps cutoff functions on $\widetilde{N}_j$ satisfying

$$\begin{aligned} \eta^+(t) &\equiv 1 \quad \text{for } t \le b, & \eta^+(t) &\equiv 0 \quad \text{for } t \ge 3b/2, \\ \eta^-(t) &\equiv 0 \quad \text{for } t \le -3b/2, & \eta^-(t) &\equiv 1 \quad \text{for } t \ge -b. \end{aligned} \tag{2.12}$$

Theorem 2.4. *Let X be an admissible S^1-domain relative to (Y, F, g), and let W be the associated filtration on $C = \mathrm{Dom}(\bar{d}_{Y,F;g})$. Then the W-spectral sequence for $H^{\cdot}_{(2)}(Y, F; g)$ has*

$$E_1^{-p,q} = H^{q-p}(\mathrm{Gr}_p^W C) = H_{(2)}^{q-p-\delta_{p1}}(Y_p, F_p; g_p),$$

and the differential

$$d_1 : E_1^{-j,q} \to E_1^{-j+1,q}$$

is induced by the map

$$q_1 : \gamma \mapsto \begin{cases} (-1)^{\deg\gamma}(\eta^- d\tau + d\eta^- \wedge \tau) \wedge \tilde{p}_0^*(\gamma|_{M/T}) & \text{if } j = 1, \\ d\eta^+ \wedge \tilde{p}_{-1}^*(\gamma|_{M/T}) & \text{if } j = 0. \end{cases}$$

In these formulas it is assumed that $\gamma|_{N_j}$ is the pullback via p_j^* of a form on M/T, so that $\gamma|_{M/T}$ is well defined. (The fact that a class in $E_1^{-p,q}$ can be represented by such a γ will follow from the proof.)

Similarly, the differential

$$d_2 : E_2^{-1,q} \to E_2^{1,q-1}$$

is induced by the map

$$\gamma \mapsto q_1\psi + q_2\gamma,$$

where

$$q_2 : \gamma \mapsto (-1)^{\deg\gamma} d\tau \wedge (\gamma - \eta^+ \tilde{p}_{-1}^*(\gamma|_{M/T})),$$

and ψ satisfies $d\psi = -q_1\gamma$.

In order to compute L_2-cohomology of X/T (and thus, by the theorem, $E_1^{\pm 1,q}$), it is useful to replace the metric and weight function by simpler ones in which the function $f(t, m)$ (see (2.3)) has been replaced by 1. By (2.4), this is equivalent to starting with a new metric on X which has been averaged with respect to t in $X \cap N$. Sometimes this yields an isomorphism on L_2-cohomology:

Theorem 2.5. *Let X be an admissible S^1-domain relative to (Y, F, g). Let $\widetilde{X/T}$ denote the manifold X/T equipped with the metric*

$$ds^2_{\widetilde{X/T}} \sim \begin{cases} ds^2_{X/T} & \text{on } (X\backslash N)/T, \\ dt^2 + p^* ds^2_{M/T} & \text{on } (X \cap N)/T, \end{cases} \tag{2.13}$$

and let $\widetilde{|\tau|}$ denote:

$$\widetilde{|\tau|} \sim \begin{cases} |\tau| & \text{on } (X\backslash N)/T, \\ (p^* w)^{-2} & \text{on } (X \cap N)/T. \end{cases} \tag{2.14}$$

Then there are isomorphisms of cohomology

$$\begin{aligned} &H^{\cdot}_{(2)}(X/T, (F \cap X)/T \cup M/T; |\tau|^{-1} g) \\ &\qquad \cong H^{\cdot}_{(2)}(\widetilde{X/T}, (F \cap X)/T \cup M/T; \widetilde{|\tau|}^{-1} g), \end{aligned}$$

$$H^{\cdot}_{(2)}(X/T, (F \cap X)/T; |\tau| g) \cong H^{\cdot}_{(2)}(\widetilde{X/T}, (F \cap X)/T; \widetilde{|\tau|} g),$$

which are induced by bounded inclusions of L_2-forms.

2.3. Preliminaries for the proofs. In order to prove the theorems, we will show that (C, W) may be replaced by a filtered complex (C^X, W) in which W is naturally split. As vector spaces,

$$C^X = \bigoplus_j C_j^X$$

and

$$W_p C^X = \bigoplus_{j \le p} C_j^X, \tag{2.15}$$

where

$$C_j^X = \operatorname{Dom}(\overline{d}_{Y_j, F_j; g_j})[-\delta_{j1}]. \tag{2.16}$$

Intuitively, C_{-1}^X and C_1^X represent the terms without τ and the coefficient of τ respectively of a form on Y restricted to X; this accounts for the different weight functions and the shift in degree. C_0^X represents the restriction of a form to $\overline{Y\backslash X}$, extended to $\widehat{Y\backslash X}$; this extension allows us to avoid the special boundary conditions on $\overline{Y\backslash X}$ that were used in [42].

The differential d_{C^X} on (C^X, W) is given by the following proposition.

Proposition 2.6. *There exists a differential d_{C^X} on (C^X, W) such that*

(i) *$d_{C^X} = \bigoplus d_{C_j^X} + q$, where $\{d_{C_j^X}\}$ are the respective $\overline{d}$-operators on $\{C_j^X\}$ and $q = q_1 + q_2$ strictly lowers W-weight.*

(ii) *There exists a good filtered homotopy equivalence*

$$(C, W) \underset{S}{\overset{R}{\rightleftarrows}} (C^X, W).$$

The proof of this proposition will occupy the next section; the proof of Theorem 2.5 will appear along the way. We conclude this section by giving the

Proof of Theorem 2.4. By Proposition 2.6(ii), $\mathrm{Gr}^W C$ is homotopy equivalent to $\mathrm{Gr}^W C^X$, thus it suffices to prove the result for the spectral sequence of (C^X, W). In this case, however, it is obvious given Proposition 2.6(i) and (2.16).

3. Proof of Proposition 2.6

We retain the notation established in §2. Since the weight function g and the boundary conditions on F play no role in the following arguments due to their T-invariance and independence of t, we will omit mentioning them.

3.1. In this subsection we head towards the proof of Proposition 2.6 by showing that C is filtered homotopy equivalent to the subcomplex

$$(3.1)\quad \begin{aligned} C_{\mathrm{inv}} = \{\phi \in C \mid \phi|_X &\text{ is } T\text{-invariant and } \phi|_N \text{ is the} \\ &\text{pullback under } p : N \to M \text{ of a } T\text{-invariant form on } M\}. \end{aligned}$$

Lemma 3.1. *There exist maps P_θ and H_θ of C (of degree 0 and -1, respectively) such that*

(i) $dH_\theta + H_\theta d = I - P_\theta$,
(ii) $P_\theta\phi$ *is T-invariant in* $X \cup N$ *for* $\phi \in C$,
(iii) H_θ *is a good homotopy,*
(iv) P_θ *and* H_θ *preserve the filtration* $\{W_pC\}$.

Proof. $X \cup N$ is (quasi-isometric to) a warped S^1-bundle by (2.1) and the warping function $|\tau|^{-2}$ is bounded according to (2.6). Assertions (i)–(iii) now follow by applying Proposition 1.9 (to construct P_θ and H_θ on $X \cup N$) and Lemma 1.11 (to extend P_θ and H_θ from $X \cup N$ to Y). (The hypotheses of Lemma 1.11 are fulfilled due to (2.3), (2.6), and Definition 2.3(iv).) Assertion (iv) follows by inspection.

Lemma 3.2. *There exist maps P_t and H_t of $P_\theta C$ such that*

(i) $dH_t + H_t d = I - P_t$,
(ii) $P_t(P_\theta C) \subseteq C_{\mathrm{inv}}$,

(iii) H_t *is a good homotopy,*

(iv) P_t *and* H_t *preserve the filtration* $\{W_p C \cap P_\theta C\}$.

Proof. We begin by operating on forms in N. Set

$$C_N = \{\phi \in \mathrm{Dom}(\overline{d}_N) \mid \phi \text{ is } T\text{-invariant}\}.$$

For $\phi \in C_N$ we decompose

$$\phi = \alpha_0 + dt \wedge \alpha_1 + \tau \wedge (\alpha_2 + dt \wedge \alpha_3) \tag{3.2}$$

analogously to (1.7).

In order to motivate the construction of a homotopy operator to forms pulled back from M, we recall that Cheeger [11] used the operator

$$\phi \mapsto H_a \phi = \int_a^t (\iota_{\partial/\partial t} \phi)$$

on metrical collars and cones to obtain the homotopy formula

$$dH_a\phi + H_a d\phi = \phi - p^*(\phi|_{\{a\}\times M}).$$

But $\phi \mapsto \phi|_{\{a\}\times M}$ does not define a bounded operator. To get around this, we average the formula (and H_a) in a. Furthermore, in order to preserve the W filtration, we must treat the terms of a form with and without τ separately: terms with τ will be homotoped in the negative t direction, and terms without τ will be homotoped in the positive t direction.

Thus for $\phi \in A^{\cdot}(N) \cap C_N$, define

$$H^+\phi = \frac{1}{b}\int_{-b}^{0}\left(\int_a^t \alpha_1\right) da$$

and

$$H^-\phi = \tau \wedge \frac{1}{b}\int_0^b \left(\int_t^a \alpha_3\right) da,$$

where unmarked integrals are over the normal variable in N. From Definition 2.3(i), (ii) we see that $H^{\pm}\phi \in A^{\cdot}(N) \cap C_N$. One easily computes (using the analogue of (1.8)) that

$$\begin{aligned} dH^+\phi + H^+ d\phi = \phi - \Bigg[\frac{1}{b}\int_{-b}^{0} \alpha_0 + \frac{1}{b}\int_{-b}^{0}\left(\int_a^t d\tau \wedge \alpha_3\right) da \\ + \tau \wedge (\alpha_2 + dt \wedge \alpha_3)\Bigg] \\ = \phi - P^+\phi \end{aligned} \tag{3.3}$$

and

$$
\begin{aligned}
dH^{-}\phi + H^{-}d\phi = \phi - \Big[\alpha_0 + \frac{1}{b}\int_0^b \Big(\int_t^a d\tau \wedge \alpha_3\Big)\, da \\
\qquad + dt \wedge \alpha_1 + \tau \wedge \frac{1}{b}\int_0^b \alpha_2\Big] \qquad (3.4) \\
= \phi - P^{-}\phi,
\end{aligned}
$$

where we are defining $P^{\pm}$ by these equations. It is also easy to check that $H^{\pm}$ are good homotopies.

We now claim that $H^{\pm}$ and $P^{\pm}$ are bounded; consequently, by Lemma 1.5, they define maps of C_N (of degree -1 and 0, respectively) for which the good homotopy formulas (3.3) and (3.4) remain valid.

To prove the claim, first note that due to (2.3) we have

$$
|dt|^{-2} \sim \begin{cases} f & \text{in } X \cap N, \\ w^2 & \text{in } N \backslash X, \end{cases}
$$

and

$$
|\tau|^{-1}|dt|^{-1} \sim \begin{cases} f w^2 & \text{in } X \cap N, \\ w^2 & \text{in } N \backslash X. \end{cases}
$$

Thus since $\int_0^b f(s, \cdot)ds \sim 1$, $1 \lesssim f$, and $w^2 \lesssim 1$ (by (2.4)), we have

$$
\int_{-b}^{b} |dt|^{-2} \lesssim 1, \tag{3.5}
$$

and

$$
\int_{-b}^{b} |\tau|^{-1}|dt|^{-1} \sim w^2 \lesssim |\tau|^{-1}\,|dt|^{-1} \quad \text{in } N. \tag{3.6}
$$

We also have

$$
dV_N = |\tau|^{-1}|dt|^{-1} dt \wedge \tau \wedge dV_{M/T},
$$

although since ϕ is T-invariant we will suppress the integration in the

T-orbits from our formulas. Now compute

$$\int_{-b}^{b}\int_{M/T}\left(\int_{-b}^{b}|\alpha_1|\right)^2|\tau|^{-1}|dt|^{-1}\,dV_{M/T}\,dt$$

(3.7)

$$\begin{aligned}
&\lesssim \int_{-b}^{b}\int_{M/T}\left(\int_{-b}^{b}|dt\wedge\alpha_1|^2\right)\left(\int_{-b}^{b}|dt|^{-2}\right)|\tau|^{-1}|dt|^{-1}\,dV_{M/T}\,dt \\
&\hspace{20em}\text{(by Cauchy-Schwartz)} \\
&\lesssim \int_{M/T}\left(\int_{-b}^{b}|dt\wedge\alpha_1|^2\right)\left(\int_{-b}^{b}|\tau|^{-1}|dt|^{-1}\,dt\right)dV_{M/T} \qquad \text{(by (3.5))} \\
&\lesssim \int_{M/T}\left(\int_{-b}^{b}|dt\wedge\alpha_1|^2|\tau|^{-1}|dt|^{-1}\,dt\right)dV_{M/T} \qquad \text{(by (3.6))} \\
&\lesssim \|\phi\|^2
\end{aligned}$$

and

$$\int_{-b}^{b}\int_{M/T} w^{-4}\left(\int_{-b}^{b}|\alpha_3|\right)^2|\tau|^{-1}|dt|^{-1}\,dV_{M/T}\,dt$$

(3.8)

$$\begin{aligned}
&\lesssim \int_{-b}^{b}\int_{M/T} w^{-4}\left(\int_{-b}^{b}|\tau\wedge dt|\cdot|\alpha_3|^2\right) \\
&\qquad\cdot\left(\int_{-b}^{b}|\tau|^{-1}|dt|^{-1}\right)|\tau|^{-1}|dt|^{-1}\,dV_{M/T}\,dt \\
&\hspace{20em}\text{(by Cauchy-Schwartz)} \\
&\lesssim \int_{M/T} w^{-4}\left(\int_{-b}^{b}|\tau\wedge dt|\cdot|\alpha_3|^2\right)\left(\int_{-b}^{b}|\tau|^{-1}|dt|^{-1}\right)^2 dV_{M/T} \\
&\lesssim \int_{M/T}\left(\int_{-b}^{b}|\tau\wedge dt\wedge\alpha_3|^2|\tau|^{-1}|dt|^{-1}\,dt\right)dV_{M/T} \qquad \text{(by (3.6))} \\
&\lesssim \|\phi\|^2.
\end{aligned}$$

If we note that

$$|\tau|^2,\ |d\tau|^2 \lesssim w^{-4} \qquad \text{(by (2.3), (2.4), and (2.7))},$$

then it is easy to see that all terms in $H^{\pm}$ and $P^{\pm}$ involving integrals of α_1 and α_3 have their L_2-norm squared dominated by (3.7) and (3.8) respectively. As for the other terms, if we replace α_1 by α_0 in (3.7) and change the application of the Cauchy-Schwarz inequality to merely

$(\int_{-b}^{b}|\alpha_0|)^2 \lesssim (\int_{-b}^{b}|\alpha_0|^2)$, we obtain

$$\int_{-b}^{b}\int_{M/T}\left(\int_{-b}^{b}|\alpha_0|\right)^2 |\tau|^{-1}|dt|^{-1}\,dV_{M/T}\,dt \lesssim \|\phi\|^2. \tag{3.9}$$

On the other hand, if we replace in (3.8) α_3 by α_2 and the inner $\int_{-b}^{b}$ by $\int_0^b$, then we have

$$\int_{-b}^{b}\int_{M/T} w^{-4}\left(\int_{0}^{b}|\alpha_2|\right)^2 |\tau|^{-1}|dt|^{-1}\,dV_{M/T}\,dt \lesssim \|\phi\|^2, \tag{3.10}$$

since $|dt|^2 \lesssim 1$ for $t > 0$ (by (2.3) and (2.4)). Thus the remaining two nontrivial terms in $P^{\pm}$, involving integrals of α_0 and α_2, are bounded by (3.9) and (3.10), respectively. This completes the proof of our claim.

Since one may easily check that $[P^{\pm}, H^{\mp}] = 0$, Lemma 1.4 shows that $\widetilde{P}_t = P^+P^-$ is homotopic in C_N to the identity via the good homotopy $\widetilde{H}_t = H^- + H^+P^-$.

To extend these results to $P_\theta C$ we use Lemma 1.11; this can be done provided we show that $H^{\pm}$ (and thus $\widetilde{H}_t$) is bounded into $L_2^{\bullet-1}(N; |dt|^2)$. For H^- this is clear from (3.8) since $|dt|^2|\tau|^2 \lesssim w^{-4}$ (see (3.6)). For H^+ the estimate is obvious when $t > 0$, where $|dt|^2 \sim f^{-1} \lesssim 1$ (2.4). If $t < 0$, we note that $H^+\phi$ only depends on $\phi|_{[-b,0]\times M}$, so in (3.7) all integrals in t can be restricted to $[-b, 0]$. Then an extra weight of $|dt|^2$ can be cancelled by applying $\int_{-b}^{0}|dt|^{-2} \sim w^2 \sim |dt|^{-2}$ (instead of (3.5)) to the second line of (3.7).

Thus we have proven (i) and (iii). We leave it to the reader to check that (ii) and (iv) are satisfied (note that the choice of $\int_0^b$ or $\int_{-b}^{0}$ in $H^{\pm}$ is essential for (iv)). q.e.d.

Combining Lemmas 3.1 and 3.2 with Lemma 1.4 we find:

Proposition 3.3. *There exist maps P_C and H_C of C such that*

(i) $dH_C + H_Cd = I - P_C$,
(ii) $P_C(C) \subseteq C_{\mathrm{inv}}$,
(iii) H_C *is a good homotopy,*
(iv) P_C *and* H_C *preserve the filtration* $\{W_pC\}$.

Corollary 3.4. *We have a good filtered homotopy equivalence*

$$(C_{\mathrm{inv}}, W) \underset{p_C}{\overset{i_C}{\rightleftarrows}} (C, W),$$

where i_C is the inclusion map and P_C is a projection.

3.2. We now show that a subspace C^X_{inv} of C^X, defined similarly to C_{inv}, has a differential for which it is isomorphic to C_{inv}.

Recall we have tubular neighborhoods (N_j, p_j) of $M/T \subset Y_j$, defined in (2.10) and (2.11). For $j = 0$, (N_j, p_j) is a warped disk bundle with warping factor w^2 (this is bounded by (2.4)), while for $j = -1$ or 1 it is a trivial $[0, b]$-bundle. Although in the latter cases, the metric is not the product metric (due to the dependence of $f(t, m)$ on t), estimate (2.4) shows this is irrelevant when computing L_2-norms of forms lifted from M/T with weights $|\tau|^{\pm 1} = \sqrt{f}^{\mp 1} w^{\mp 2}$. Thus we have quasi-isometric embeddings defined by lifting forms:

$$
\begin{aligned}
&p_{-1}^* : L_2^\cdot(M/T; w^2) \to L_2^\cdot(N_{-1}; |\tau|^{-1}), \\
(3.11) \qquad &p_0^* : L_2^\cdot(M/T; w^2) \to L_2^\cdot(N_0), \\
&p_1^* : L_2^\cdot(M/T; w^{-2}) \to L_2^\cdot(N_1, |\tau|)
\end{aligned}
$$

(the change in the weight function for p_0^* is due to the effect of $|\tau|$ on the volume form). The analogous embeddings defined by lifting to $\tilde{N}_j$ will be denoted $\tilde{p}_j^*$.

Define $C^X_{\mathrm{inv}} \subseteq C^X$ (merely considered as graded vector spaces, for now) by

$$C^X_{\mathrm{inv}} = \bigoplus_j C^X_{\mathrm{inv}, j},$$

where

$$(3.12) \qquad C^X_{\mathrm{inv}, j} = \{ \phi \in C^X_j \mid \phi|_{N_j} \in \mathrm{Range}(p_j^*) \}.$$

Clearly $C^X_{\mathrm{inv}, j}$ is a subcomplex of C^X_j. The advantage of $C^X_{\mathrm{inv}, j}$ is that a form $\phi \in C^X_{\mathrm{inv}, j}$ has a well-defined restriction $\phi|_{M/T}$ to the submanifold M/T. By (2.16), (3.11), and (3.12), the restriction $\phi \mapsto \phi|_{M/T}$ defines bounded cochain maps:

$$
\begin{aligned}
&C^X_{\mathrm{inv}, -1} \to 0, \\
(3.13) \qquad &C^X_{\mathrm{inv}, 0} \to \mathrm{Dom}(\bar{d}_{M/T; w^2}), \\
&C^X_{\mathrm{inv}, 1} \to \mathrm{Dom}(\bar{d}_{M/T; w^{-2}})[-1]
\end{aligned}
$$

($C^X_{\mathrm{inv}, -1}|_{M/T} = 0$ due to the boundary conditions in C^X_{-1}).

Proposition 3.5. *There exists a differential* $d_{C^X_{\mathrm{inv}}}$ *on* (C^X_{inv}, W) *such that:*

(i) $d_{C^X_{\mathrm{inv}}} = \bigoplus d_{C^X_j} + q_{\mathrm{inv}}$, *where* $q_{\mathrm{inv}} : C^X_{\mathrm{inv}} \to C^X_{\mathrm{inv}}$ *strictly lowers* W-*weight.*

(ii) *There exist filtered isomorphisms*

$$(C_{\mathrm{inv}}, W) \underset{S_{\mathrm{inv}}}{\overset{R_{\mathrm{inv}}}{\rightleftarrows}} (C^X_{\mathrm{inv}}, W)$$

such that R_{inv} *and* S_{inv} *are inverses of each other.*

Proof. For part (i), define

$$d_{C^X_{\mathrm{inv}}} = \bigoplus d_{C^X_j} + q_{\mathrm{inv}}, \tag{3.14}$$

where

$$\begin{aligned} q_{\mathrm{inv}}(\gamma_{-1}, \gamma_0, \gamma_1) = (&d\eta^+ \wedge \tilde{p}^*_{-1}(\gamma_0|_{M/T}) \\ &+ ((-1)^{\deg \gamma_1} d\tau \wedge (\gamma_1 - \eta^+ \tilde{p}^*_{-1}(\gamma_1|_{M/T}))), \\ &(-1)^{\deg \gamma_1}(\eta^- d\tau + d\eta^- \wedge \tau) \wedge \tilde{p}^*_0(\gamma_1|_{M/T}), 0). \end{aligned} \tag{3.15}$$

Here $\tilde{p}^*_i$ is defined following (3.11) and the $\eta^\pm$ are the ps cutoff functions defined in (2.12).

It is straightforward to see that the condition for $d_{C^X_{\mathrm{inv}}}$ to be a differential,

$$\left(\bigoplus d_{C^X_i}\right) q_{\mathrm{inv}} + q_{\mathrm{inv}} \left(\bigoplus d_{C^X_i}\right) + q_{\mathrm{inv}} q_{\mathrm{inv}} = 0, \tag{3.16}$$

holds on $A^{\cdot} \cap C^X_{\mathrm{inv}}$. If we show that q_{inv} is bounded, it will follow that q_{inv} preserves C^X_{inv} and that (3.16) holds on C^X_{inv} (compare Lemma 1.5); note that the -1 component of q_{inv} has zero boundary values on M/T as desired. Thus the following estimates (which follow from the form of the metric in (2.3) and the noted references) will finish the proof of part (i):

$$\begin{aligned} &\|d\eta^+ \wedge \tilde{p}^*_{-1}(\gamma_0|_{M/T})\|_{X/T;|\tau|^{-1}} \\ &\quad\lesssim \|\gamma_0|_{M/T}\|_{M/T;w^2} \qquad \text{(by (2.5) and (3.11))} \\ &\quad\lesssim \|\gamma_0\|_{\widehat{Y\setminus X}} \qquad \text{(by (3.13))}, \end{aligned} \tag{a}$$

$$
\begin{aligned}
&\|d\tau \wedge (\gamma_1 - \eta^+ \tilde{p}_{-1}^*(\gamma_1|_{M/T}))\|_{X/T;|\tau|^{-1}} \\
&\quad \lesssim \|d\tau \wedge \gamma_1\|^{-1}_{(X\setminus N)/T;|\tau|} + \|d\tau \wedge \tilde{p}_{-1}^*(\gamma_1|_{M/T})\|_{N_{-1};|\tau|^{-1}} \\
\text{(b)}\qquad &\quad \lesssim \|\gamma_1\|_{(X\setminus N)/T;|\tau|} + \|\gamma_1|_{M/T}\|_{M/T;w^{-2}} \quad \text{(by (2.7) and (3.11))} \\
&\quad \lesssim \|\gamma_1\|_{X/T;|\tau|} \quad \text{(by (3.13))},
\end{aligned}
$$

$$
\begin{aligned}
\text{(c)}\qquad \|\eta^- d\tau \wedge \tilde{p}_0^*(\gamma_1|_{M/T})\|_{\widehat{Y\setminus X}} &\lesssim \|d\tau \wedge (\gamma_1|_{M/T})\|_{M/T;w^2} \quad \text{(by (3.11))} \\
&\lesssim \|\gamma_1|_{M/T}\|_{M/T;w^{-2}} \quad \text{(by (2.7))} \\
&\lesssim \|\gamma_1\|_{X/T;|\tau|} \quad \text{(by (3.13))},
\end{aligned}
$$

$$
\begin{aligned}
&\|d\eta^- \wedge \tau \wedge \tilde{p}_0^*(\gamma_1|_{M/T})\|_{\widehat{Y\setminus X}} \\
\text{(d)}\qquad &\quad \lesssim \|\gamma_1|_{M/T}\|_{M/T;w^{-2}} \quad \text{(by (2.3) and (3.11))} \\
&\quad \lesssim \|\gamma_1\|_{X/T;|\tau|} \quad \text{(by (3.13))}.
\end{aligned}
$$

For part (ii), decompose $\phi \in C_{\text{inv}}$ as

$$\phi|_X = \phi_{-1} + \tau \wedge \phi_1 ,$$

where $\iota_{\partial/\partial\theta}\phi_j = 0$ $(j = -1, 1)$. From the definition of C_{inv} (3.1), ϕ_j is T-invariant, so we may write

$$\phi_j = \pi^* \psi_j ,$$

where $\pi : X \to X/T$ denotes the quotient map and ψ_j is a form on X/T. Furthermore, since ψ_j is independent of t in N, $\psi_j|_{M/T}$ is well defined.

Define $R_{\text{inv}} = (R_{\text{inv},-1}, R_{\text{inv},0}, R_{\text{inv},1}) : C_{\text{inv}} \to C^X_{\text{inv}}$ formally by

$$
\begin{aligned}
&R_{\text{inv},-1}(\phi) = \psi_{-1} - \eta^+ \tilde{p}_{-1}^*(\psi_{-1}|_{M/T}), \\
\text{(3.17)}\qquad &R_{\text{inv},0}(\phi) = \begin{cases} \phi|_{Y\setminus X} - (-1)^{\deg \phi_1} \eta^- \tau \\ \quad \wedge \tilde{p}_0^*(\psi_1|_{M/T})|_{Y\setminus X} & \text{in } Y \setminus X, \\ \tilde{p}_0^*(\psi_{-1}|_{M/T}) & \text{in } \widehat{M}, \end{cases} \\
&R_{\text{inv},1}(\phi) = (-1)^{\deg \phi_1} \psi_1 ,
\end{aligned}
$$

where $\tilde{p}_j^*$ is defined following (3.11) and $\eta^\pm$ is defined in (2.12).

Define $S_{\mathrm{inv}} : C^X_{\mathrm{inv}} \to C_{\mathrm{inv}}$ formally by

$$(3.18)\qquad S_{\mathrm{inv}}((\gamma_j)_{j\in\{-1,0,1\}}) = \begin{cases} \pi^*(\gamma_{-1} + \eta^+ \tilde{p}^*_{-1}(\gamma_0|_{M/T})) & \\ \quad + (-1)^{\deg \gamma_1} \tau \wedge \pi^* \gamma_1 & \text{in } X, \\ \gamma_0|_{Y\setminus X} + (-1)^{\deg \gamma_1} \eta^- \tau & \\ \quad \wedge \tilde{p}^*_0(\gamma_1|_{M/T})|_{Y\setminus X} & \text{in } Y\setminus X. \end{cases}$$

We leave it to the reader to verify that R_{inv} and S_{inv} formally commute with the differentials, are inverses of each other, and preserve W. It remains to see that they are bounded.

By (2.1), we have quasi-isometric embeddings

$$(3.19)\qquad \begin{aligned} \pi^* &: L_2(X/T; |\tau|^{-1}) \to L_2(X), \\ \pi^* &: L_2(X/T; |\tau|) \to L_2(X; |\tau|^2). \end{aligned}$$

From (3.19), (3.11), (3.13), and (for the last term in S_{inv}) the estimate $1 \lesssim w^{-2}$ (2.4), one easily checks that R_{inv} and S_{inv} are bounded.

3.3. We now show that $\mathrm{Gr}^W C^X$ is homotopy equivalent to $\mathrm{Gr}^W C^X_{\mathrm{inv}}$, which enables us to prove Theorem 2.5. Next we extend the differential on C^X_{inv} (Proposition 3.5) to a differential on all of C^X in such a way that C^X is filtered homotopy equivalent to C^X_{inv}. We then finally give the proof of Proposition 2.6.

Lemma 3.6. *There exist maps P_j and H_j of C^X_j $(j = -1, 0, 1)$ such that*

(i) $d_{C^X_j} H_j + H_j d_{C^X_j} = I - P_j$,

(ii) $P_j(C^X_j) \subseteq C^X_{\mathrm{inv},j}$,

(iii) *H_j is a good homotopy.*

Proof. We first construct the required maps over N_j (defined at the begining of §3.2). For $j = 0$, they exist by Proposition 1.9. For $j = -1$, 1, we adapt the maps constructed in the proof of Lemma 3.2. That is, when $j = -1$, write $\phi \in C^X_{\mathrm{inv},-1}$ as $\phi = \alpha_0 + dt \wedge \alpha_1$ and use

$$H^+\phi = \int_0^t \alpha_1 \quad \text{and} \quad P^+\phi = 0$$

for H_j and P_j; when $j = 1$, write $\phi \in C^X_{\mathrm{inv},1}$ as $\phi = \alpha_2 + dt \wedge \alpha_3$ and use

$$H^-\phi = \frac{1}{b}\int_0^b \left(\int_t^a \alpha_3\right) da \quad \text{and} \quad P^-\phi = \frac{1}{b}\int_0^b \alpha_2.$$

The proof that these maps are bounded and satisfy (i)–(iii) is similar to the proof of Lemma 3.2. We now conclude by applying Lemma 1.11. q.e.d.

We pause briefly to present the

Proof of Theorem 2.5. The function f of (2.3) appears in the norm integral for forms in $L_2^\bullet(X/T; |\tau|^{-1})$ either to the 1st power or not at all (for terms of a form without and with dt respectively), while for forms in $L_2^\bullet(X/T; |\tau|)$, f appears to the -1st power or not at all. Since $f \gtrsim 1$ (2.4), there are thus bounded inclusions

$$L_2^\bullet(X/T; |\tau|^{-1}) \to L_2^\bullet(\widetilde{X/T}; \widetilde{|\tau|}^{-1}),$$
$$L_2^\bullet(X/T; |\tau|) \leftarrow L_2^\bullet(\widetilde{X/T}; \widetilde{|\tau|})$$

(note that the metric and weight functions on the right are obtained by replacing f by 1).

To see the isomorphism on cohomology, define $\widetilde{C_j^X}$ and $\widetilde{C_{\mathrm{inv},j}^X}$ ($j = -1, 1$) analogously to C_j^X and $C_{\mathrm{inv},j}^X$, but with $\widetilde{X/T}$ and $\widetilde{|\tau|}$. Then the analogue of Lemma 3.6 still holds. However, $C_{\mathrm{inv},j}^X = \widetilde{C_{\mathrm{inv},j}^X}$, by (2.4) (see the remark before (3.11)).

Proposition 3.7. *There exists a differential d_{C^X} on (C^X, W) such that:*

(i) $d_{C^X} = \bigoplus d_{C_j^X} + q$, *where q strictly lowers W-weight and* $q|_{C_{\mathrm{inv}}^X} = q_{\mathrm{inv}}$.

(ii) *We have a good filtered homotopy equivalence*

$$(C_{\mathrm{inv}}^X, W) \underset{P_{C^X}}{\overset{i_{C^X}}{\rightleftarrows}} (C^X, W),$$

where i_{C^X} is the inclusion map and P_{C^X} is a projection.

Proof. Define the differential d_{C^X} of C^X as $\bigoplus_j d_{C_j^X} + q$, where

$$q = q_{\mathrm{inv}} \circ \left(\bigoplus_j P_j \right). \tag{3.20}$$

The differential condition

$$\left(\bigoplus d_{C_j^X}\right) q + q \left(\bigoplus d_{C_j^X}\right) + qq = 0 \tag{3.21}$$

follows from (3.16) and Lemma 3.6 (note that $(\bigoplus P_j) q_{\mathrm{inv}} = q_{\mathrm{inv}}$). Clearly part (i) holds.

For part (ii), let $P_{C^X} = \bigoplus P_j$ and $H_{C^X} = \bigoplus H_j$. The homotopy formula

$$d_{C^X} H_{C^X} + H_{C^X} d_{C^X} = I - P_{C^X} \tag{3.22}$$

holds after applying Gr^W by Lemma 3.6. To see that (3.22) is actually valid on C^X, we must show that the terms which do not respect the grading, qH_{C^X} and $H_{C^X}q$, vanish. This follows from (3.20), since the H_j are good homotopies. q.e.d.

Finally we may give the

Proof of Proposition 2.6. Apply Corollary 3.4, Proposition 3.5, and Propostion 3.7. The proposition follows upon setting $R = i_{C^X} R_{\mathrm{inv}} P_C$ and $S = i_C S_{\mathrm{inv}} P_{C^X}$. By (3.15) and (3.20), q has the desired form (on C^X_{inv}).

4. A generalization of Theorem 2.4 to a family of S^1-domains

In this section we extend Theorem 2.4 to handle a family

$$\mathscr{X} = \{(X^i, (N^i, p^i, t_i))\}_{1 \leq i \leq m}$$

of (possibly overlapping) admissible S^1-domains of (Y, F, g). We require the family to be *associative* (Definition 4.4). The main result is Theorem 4.7, which computes the spectral sequence of a convolution $W^{\mathscr{X}}$ (§4.1) of the filtrations $\{W^i\}_{1 \leq i \leq m}$ associated to $\mathscr{X}$.

For brevity, we will sometimes write $\mathscr{X}$ simply as $\{X^i\}_{1 \leq i \leq m}$.

4.1. Convolution of filtrations.

Definition 4.1 [49, (1.4)]. If W and W' are filtrations on a complex C, their *convolution* $W * W'$ is the filtration defined by

$$(W * W')_j C = \sum_{p+q=j} W_p C \cap W'_q C.$$

We note the

Proposition 4.2 [53, (A.1)]. *If W and W' are filtrations of C, bounded from below, there is a natural isomorphism*

$$\mathrm{Gr}_j^{W*W'} C \cong \bigoplus_{p+q=j} \mathrm{Gr}_p^W \mathrm{Gr}_q^{W'} C.$$

The induced filtrations W and W' on the left-hand side are given by truncating the sum in the obvious manner.

Definition 4.3. Let $\mathscr{X} = \{X^i\}_{1 \leq i \leq m}$ be a family of admissible S^1-domains (relative to (Y, F, g)). The filtration $W^{\mathscr{X}}$ on the complex $C = \mathrm{Dom}(\bar{d}_{Y,F;g})$ *associated* to $\mathscr{X}$ is the convolution

$$W^1 * (\cdots * (W^{m-1} * W^m) \cdots),$$

where the $\{W^i\}$ are the filtrations associated to $\{X^i\}$.

Since convolution is commutative but not associative, the definition of $W^{\mathscr{X}}$ depends on an ordering of $\mathscr{X}$. When Theorem 4.7 below applies, however, it will follow that the filtered homotopy equivalence class of $(C, W^{\mathscr{X}})$ is naturally independent of the choice of ordering.

4.2. Associative families of S^1-domains. In order to apply §2 successively for all X^i, we need to impose some conditions on our family of S^1-domains. Let $\{T^i\}$ denote the S^1-actions corresponding to $\{X^i\}$.

Definition 4.4. $\mathscr{X} = \{(X^i, (N^i, p^i, t_i))\}_{1\leq i\leq m}$ is an *associative* family of admissible S^1-domains (relative to (Y, F, g)) if the following hold for all u and v, $1 \leq u \neq v \leq m$:

(i) $M^u \cap N^v = (p^v)^{-1}(M^u \cap M^v)$, $N^u \cap N^v = (p^v)^{-1}(N^u \cap M^v)$, and $p^v|_{N^u\cap N^v}$ is induced from $p^v|_{M^u\cap N^v}$ independently of t_u.

(ii) $M^u \cap X^v$ and $N^u \cap X^v$ are invariant under T^v, and $T^v|_{N^u\cap X^v}$ agrees with the action induced from $T^v|_{M^u\cap X^v}$ independently of t_u.

(iii) $\partial/\partial\theta^u$ is quasi-isometrically perpendicular to $\partial/\partial\theta^v$ on $X^u \cap X^v$, that is, the angle between $\partial/\partial\theta^u$ and $\partial/\partial\theta^v$ is uniformly bounded away from 0.

(iv) For $I \subseteq \{1, \cdots, m\}$, the S^1-actions T^i $(i \in I)$ on $X^I = \bigcap_{i\in I} X^i$ (which is invariant under T^i $(i \in I)$ by (ii)) commute, and define a free action T^I of $(S^1)^I$. With this action, X^I is a ps principal $(S^1)^I$-bundle over a ps manifold with boundary.

Remark 4.5. The last part of condition (i) is equivalent to $t_u|_{N^u\cap N^v} = (p^v)^*(t_u|_{N^u\cap M^v})$ and $p^u p^v = p^v p^u$ on $N^u \cap N^v$.

Let $\mathscr{X} = \{X^i\}_{1\leq i\leq m}$ be an associative family of admissible S^1-domains (relative to (Y, F, g)). For $u \in \{1, \cdots, m\}$, define

$$(Y_j^u, F_j^u, g_j^u) \quad (j = -1, 0, 1) \tag{4.1}$$

as in (2.9), with respect to X^u.

Now for $i \in \{1, \cdots, m\}$, $i \neq u$, the S^1-domain X^i of Y induces S^1-domains $(X^i)_j^u$ of Y_j^u $(j = -1, 0, 1)$. Namely, define

$$(X^i)_j^u = \begin{cases} (X^u \cap X^i)/T^u & (j = -1, 1), \\ (X^i \setminus (X^u \cap X^i))^{\wedge} & (j = 0), \end{cases}$$

with the induced T^i action (for $j = 0$, extend T^i in $\widehat{M^u \cap X^i}$ independently of t_u). Furthermore, the tubular boundary neighborhood (N^i, p^i, t_i) of X^i in Y induces one of $(X^i)_j^u$ in Y_j^u (with $N = (N^i)_j^u$, etc.). The following proposition is now easy to check (to verify that $g_j^u = |\tau_u|^j g$, $j = -1, 1$, satisfies (2.2), use Definition 4.4(iii)).

Proposition 4.6. *Let $\{X^i\}_{1\le i\le m}$ be an associative family of admissible S^1-domains (relative to (Y, F, g)) and fix $u \in \{1, \cdots, m\}$, $j \in \{-1, 0, 1\}$. Then $\{(X^i)^u_j\}_{1\le i\le m, i\ne u}$ is an associative family of admissible S^1-domains (relative to (Y^u_j, F^u_j, g^u_j)).*

Thus we may iterate the above discussion. For $I \subseteq \{1, \cdots, m\}$, and $J = (j_i)_{i\in I} \in \{-1, 0, 1\}^I$, choose any $u \in I$ and define $I' = I\setminus\{u\}$ and $J' = (j_i)_{i\in I'}$. Let the triple

$$(Y^I_J, F^I_J, g^I_J) \tag{4.2}$$

be the result of applying the j_uth line of (2.9) to the triple $(Y^{I'}_{J'}, F^{I'}_{J'}, g^{I'}_{J'})$ with admissible S^1-domain $(X^u)^{I'}_{J'}$. It is straightforward to verify that this definition is independent of the choice of $u \in I$ (equivalently, the order in which the constructions of (2.9) are applied for $u \in I$ is irrelevant).

For $J \in \{-1, 0, 1\}^m$, denote

$$(Y^{\mathscr{X}}_J, F^{\mathscr{X}}_J, g^{\mathscr{X}}_J) = (Y^{\{1,\cdots,m\}}_J, F^{\{1,\cdots,m\}}_J, g^{\{1,\cdots,m\}}_J).$$

4.3. Main results.

Theorem 4.7. *Let $\mathscr{X} = \{X^i\}_{1\le i\le m}$ be an associative family of admissible S^1-domains relative to (Y, F, g) and let $W^{\mathscr{X}}$ be the associated filtration on $C = \mathrm{Dom}(\bar{d}_{Y,F;g})$. Then the $W^{\mathscr{X}}$-spectral sequence for $H^{\cdot}_{(2)}(Y, F; g)$ has*

$$E_1^{-p,q} = H^{q-p}(\mathrm{Gr}^{W^{\mathscr{X}}}_p C) \cong \bigoplus_{\substack{J\in\{-1,0,1\}^m\\ |J|=p}} H_{(2)}^{q-p-|\{s|j_s=1\}|}(Y^{\mathscr{X}}_J, F^{\mathscr{X}}_J; g^{\mathscr{X}}_J),$$

and the differential $d_1 : E_1^{-p,q} \to E_1^{-p+1,q}$ is induced by the map

$$(\gamma_J)_{|J|=p} \mapsto \Bigg(\sum_{\{s|j_s<1\}} q_1^s \gamma_{(j_1,\cdots,j_s+1,\cdots,j_m)}\Bigg)_{|J|=p-1},$$

where q_1^s is as in Theorem 2.4, relative to X^s.

Similarly, the differential $d_2 : E_2^{-p,q} \to E_2^{-p+2,q-1}$ is induced by the map

$$(\gamma_J)_{|J|=p} \mapsto \Bigg(\sum_{\{s|j_s<1\}} q_1^s \psi_{(j_1,\cdots,j_s+1,\cdots,j_m)} + \sum_{\{s|j_s=-1\}} q_2^s \gamma_{(j_1,\cdots,j_s+2,\cdots,j_m)}\Bigg)_{|J|=p-2},$$

where $(\psi_J)_{|J|=p-1}$ *satisfies*

$$d\psi_J = -\sum_{\{s|j_s<1\}} q_1^s \gamma_{(j_1, \cdots, j_s+1, \cdots, j_m)}.$$

Theorem 4.7 follows from Proposition 4.8, below. To begin with, let $(C_J^{\mathscr{X}}, d_{C_J^{\mathscr{X}}})$ denote the $\overline{d}$-complex associated to the triple $(Y_J^{\mathscr{X}}, F_J^{\mathscr{X}}, g_J^{\mathscr{X}})$ shifted by $|\{s \mid j_s = 1\}|$, and set (as a vector space)

$$C^{\mathscr{X}} = \bigoplus_{J\in\{-1,0,1\}^m} C_J^{\mathscr{X}},$$

with filtrations

$$W_j^i C^{\mathscr{X}} = \bigoplus_{\substack{J\in\{-1,0,1\}^m \\ j_i \leq j}} C_J^{\mathscr{X}}.$$

This is one case where convolution is associative, thus the filtration $W^{\mathscr{X}} = W^1 * \cdots * W^m$ on $C^{\mathscr{X}}$ is well defined. We assume $\mathrm{Gr}^{W^{\mathscr{X}}} C^{\mathscr{X}}$ is given the differential $\bigoplus_J d_{C_J^{\mathscr{X}}}$; the next proposition will define a compatible differential on $C^{\mathscr{X}}$.

Proposition 4.8. *There exists a differential* $d_{C^{\mathscr{X}}}$ *on* $(C^{\mathscr{X}}, W^{\mathscr{X}})$ *such that*:

(i) $d_{C^{\mathscr{X}}} = \bigoplus_J d_{C_J^{\mathscr{X}}} + \sum_{i=1}^m q^i$, *where* $q^i = q_1^i + q_2^i$ *strictly decreases* W^i*-weight and is graded with respect to* $W^{i'}$, $i' \neq i$.

(ii) *There exists a good filtered homotopy equivalence*

$$(C, W^{\mathscr{X}}) \underset{S}{\overset{R}{\rightleftarrows}} (C^{\mathscr{X}}, W^{\mathscr{X}}).$$

Proof. We first make some preliminary constructions. For $i \in \{1, \cdots, m\}$, let $\mathscr{X}' = \mathscr{X} \setminus \{X^i\}$ and apply Proposition 2.6 (with S^1-domain $(X^i)_{J'}^{\mathscr{X}'}$) simultaneously to all $(C_{J'}^{\mathscr{X}'}, d_{C_{J'}^{\mathscr{X}'}})$. The result, after taking the direct sum over J', is a differential $\bigoplus_J d_{C_J^{\mathscr{X}}} + q^i$ on $\mathrm{Gr}^{W^{\mathscr{X}'}} C^{\mathscr{X}}$, such that

$$q^i = q_1^i + q_2^i : \mathrm{Gr}^{W^{\mathscr{X}'}} C^{\mathscr{X}} \to \mathrm{Gr}^{W^{\mathscr{X}'}} C^{\mathscr{X}},$$

and such that there is a good W^i-filtered homotopy equivalence

$$\mathrm{Gr}^{W^{\mathscr{X}'}} C^{\mathscr{X}'} \underset{S^i}{\overset{R^i}{\rightleftarrows}} \mathrm{Gr}^{W^{\mathscr{X}'}} C^{\mathscr{X}},$$

via homotopies

$$H^i_{\mathscr{X}'} : \mathrm{Gr}^{W^{\mathscr{X}'}} C^{\mathscr{X}'} \to \mathrm{Gr}^{W^{\mathscr{X}'}} C^{\mathscr{X}'} ,$$
$$H^i_{\mathscr{X}} : \mathrm{Gr}^{W^{\mathscr{X}'}} C^{\mathscr{X}} \to \mathrm{Gr}^{W^{\mathscr{X}'}} C^{\mathscr{X}} .$$

Precisely, we have the following lemma.

Lemma 4.9. *There exist W^i-filtered maps q^i, R^i, S^i, $H^i_{\mathscr{X}'}$, and $H^i_{\mathscr{X}}$ (between the spaces indicated above) satisfying*

(i) $$\Big(\bigoplus_J d_{C_J^{\mathscr{X}}}\Big)q^i + q^i\Big(\bigoplus_J d_{C_J^{\mathscr{X}}}\Big) + q^i q^i = 0,$$

(ii) $$q^i R^i = R^i\Big(\bigoplus_{J'} d_{C_{J'}^{\mathscr{X}'}}\Big) - \Big(\bigoplus_J d_{C_J^{\mathscr{X}}}\Big)R^i,$$

(iii) $$S^i q^i = \Big(\bigoplus_{J'} d_{C_{J'}^{\mathscr{X}'}}\Big)S^i - S^i\Big(\bigoplus_J d_{C_J^{\mathscr{X}}}\Big),$$

(iv) $$\Big(\bigoplus_{J'} d_{C_{J'}^{\mathscr{X}'}}\Big)H^i_{\mathscr{X}'} + H^i_{\mathscr{X}'}\Big(\bigoplus_{J'} d_{C_{J'}^{\mathscr{X}'}}\Big) = I - S^i R^i,$$

(v) $$\Big(\bigoplus_J d_{C_J^{\mathscr{X}}} + q^i\Big)H^i_{\mathscr{X}} + H^i_{\mathscr{X}}\Big(\bigoplus_J d_{C_J^{\mathscr{X}}} + q^i\Big) = I - R^i S^i,$$

(vi) $$S^i R^i H^i_{\mathscr{X}'} = H^i_{\mathscr{X}'} S^i R^i = R^i S^i H^i_{\mathscr{X}} = H^i_{\mathscr{X}} R^i S^i = 0.$$

If we do this for all $i \in \{1, \cdots, m\}$, the next lemma is straightforward to verify from the constructions of §3 and Definition 4.4.

Lemma 4.10. *Let $u, v \in \{1, \cdots, m\}$, $u \neq v$. Then*

(i) $$q^u q^v = -q^v q^u,$$

(ii) $$q^u R^v = R^v q^u,$$

(iii) $$S^v q^u = q^u S^v,$$

(iv) $$q^u H^v_{\mathscr{X}} = -H^v_{\mathscr{X}} q^u,$$

(v) $$S^u R^u H^v_{\mathscr{X}'} = H^v_{\mathscr{X}'} S^u R^u, \qquad R^v S^v H^u_{\mathscr{X}} = H^u_{\mathscr{X}} R^v S^v.$$

We now prove the proposition by induction on $m = |\mathscr{X}|$. In the case where $m = 1$, it is merely Proposition 2.6. In general, fix $i = m$ and $\mathscr{X}' = \mathscr{X} \backslash \{X^m\}$; we may assume the proposition has been proven for $\mathscr{X}'$. The fact that

(4.3) $$d_{C^{\mathscr{X}}} = \bigoplus_J d_{C_J^{\mathscr{X}}} + \Big(\sum_{i=1}^m q^i\Big)$$

is a differential on $(C^{\mathscr{X}}, W^{\mathscr{X}})$ follows from Lemmas 4.9(i) and 4.10(i). With this differential (and the corresponding one for $C^{\mathscr{X}'}$), the maps

$$(C^{\mathscr{X}'}, W^{\mathscr{X}}) \underset{S^m}{\overset{R^m}{\rightleftarrows}} (C^{\mathscr{X}}, W^{\mathscr{X}})$$

are filtered chain maps by Lemmas 4.9(ii), (iii) and 4.10(ii), (iii). Furthermore, by Lemmas 4.9(iv)–(vi) and 4.10(iv), they define a good filtered homotopy equivalence. By the inductive hypothesis, Lemma 1.4, and Lemma 4.10(v), this completes the proof. q.e.d.

Actually, Theorem 4.7 and Proposition 4.8 remain valid under weaker conditions on $\mathscr{X}$ than Definition 4.4, but without the homotopies being good and with a less explicit formula for the differentials. For example, we could omit the last part of Definition 4.4(i). Since Lemma 4.10 then no longer holds, the proof of Proposition 4.8 becomes more complicated. One proceeds by induction on $m = |\mathscr{X}|$ and defines

$$d_{C^{\mathscr{X}}} = \bigoplus_J d_{C_J^{\mathscr{X}}} + q^m + R^m\left({}^{\alpha}d_{C^{\mathscr{X}'}} - \bigoplus d_{C_{J'}^{\mathscr{X}'}}\right)S^m,$$

where $\mathscr{X}' = \mathscr{X}\backslash\{X^m\}$ and ${}^{\alpha}d_{C^{\mathscr{X}'}}$ is a conjugate of $d_{C^{\mathscr{X}'}}$ for which $S^m R^m$ is a chain map on $C^{\mathscr{X}'}$.

Since we will not need this more general result, we omit the details except for the following lemma which may be of independent interest. Basically it says that a good homotopy on an associated graded complex $\mathrm{Gr}^W C$ can be lifted to a homotopy on C after perhaps conjugating the differential of C.

Lemma 4.11. *Let (C, W) be a finite filtered complex and assume we have fixed a splitting by which we identify (as filtered vector spaces)*

$$(C, W) \cong (\mathrm{Gr}^W C, W). \tag{4.4}$$

Let P be a projection map of $(\mathrm{Gr}^W C, W)$ which is homotopic to the identity via a good filtered homotopy H. Then there exists a filtered vector space automorphism α of (C, W) satisfying the following conditions:

(i) *P is a cochain map of $({}^{\alpha}C, W)$ which is homotopic to the identity via H, where ${}^{\alpha}C$ is the complex with the same underlying space as C but with the differential ${}^{\alpha}d_C = \alpha\, d_C\, \alpha^{-1}$.*

(ii) $\mathrm{Gr}^W \alpha = I$ *(and hence $\mathrm{Gr}^W({}^{\alpha}C) = \mathrm{Gr}^W C$).*

(iii) *Any filtration W' of C which is preserved by H and is compatible with the identification (4.4) is preserved by α and thus induces a filtration of ${}^{\alpha}C$.*

(iv) *If* $\mathrm{Gr}^{W'} P$ *is a cochain map of* $\mathrm{Gr}^{W'} C$, *where* W' *is as in* (iii), *then*

$$\mathrm{Gr}^{W'}({}^{\alpha}d_C - \mathrm{Gr}^W d_C) = (\mathrm{Gr}^{W'} P)\,\mathrm{Gr}^{W'}(d_C - \mathrm{Gr}^W d_C).$$

Remark 4.12. Note that α yields an isomorphism of complexes $C \cong {}^{\alpha}C$, which is filtered for any W' as in (iii).

Proof. We first prove (i). We are given that

$$(4.5)\qquad (\mathrm{Gr}^W d_C)H + H(\mathrm{Gr}^W d_C) = I - P.$$

Define q by the equation

$$d_C = \mathrm{Gr}^W d_C + q$$

(recall our splitting (4.4)). From $d_C^2 = 0$ we see that

$$(4.6)\qquad (\mathrm{Gr}^W d_C)q + q(\mathrm{Gr}^W d_C) + qq = 0.$$

(4.5) and (4.6) easily imply

$$(4.7)\qquad (\mathrm{Gr}^W d_C)(I + Hq) = (I + Hq)(\mathrm{Gr}^W d_C) + (I + Hq)q - Pq$$

and

$$(4.8)\qquad (\mathrm{Gr}^W d_C)(I + qH) = (I + qH)(\mathrm{Gr}^W d_C) - q(I + qH) + qP.$$

Since q strictly lowers W-weight, Hq and qH are nilpotent; thus $(I + Hq)$ and $(I + qH)$ are invertible. It follows from (4.7) and (4.8) that

$$(4.9)\qquad (I + Hq)d_C(I + Hq)^{-1} = \mathrm{Gr}^W d_C + Pq(I + Hq)^{-1}$$

and

$$(4.10)\qquad (I + qH)^{-1}d_C(I + qH) = \mathrm{Gr}^W d_C + (I + qH)^{-1}qP$$

respectively. Note that conjugation by either $\alpha = (I + Hq)$ or $\alpha = (I + qH)^{-1}$, as in (4.9) and (4.10) respectively, satisfies (ii)–(iv). (For (iv), we apply $\mathrm{Gr}^{W'}$ to (4.9) and (4.10) and note that the hypothesis implies $\mathrm{Gr}^{W'} P$ and $\mathrm{Gr}^{W'} q$ commute. The result now follows since $PH = HP = 0$ for a good homotopy.) Thus in proving (i), we may replace d_C with the conjugated differential of (4.9); consequently we may assume that $Pq = q$ (since $P^2 = P$). Similarly, we may assume in addition that $qP = q$ by using (4.10).

Thus P becomes a d_C-cochain map. Since H is good, $qH + Hq = qPH + HPq = 0$; thus we may add $qH + Hq$ to the left-hand side of (4.5) to obtain a homotopy for d_C, which establishes (i).

5. Local structure associated to a divisor with normal crossings

Let $\widetilde{V}$ be a complex manifold and $D \subset \widetilde{V}$ a divisor with normal crossings. Write $D = \bigcup_{i=1}^m D_i$, where the D_i are the irreducible components of D which we assume to be smooth, and define

$$D_I = \bigcap_{i \in I} D_i$$

for $I \subseteq \{1, \cdots, m\}$. For $c \in \mathbb{R}$ we will define a punctured closed regular neighborhood $U^c \subset \widetilde{V} \backslash D$ of D; as $c \to \infty$ these will form a cofinal system. The set U^c will be formed as a union $\bigcup_{I \subseteq \{1, \cdots, m\}} U_I^c$, where the sets U_I^c have disjoint interiors and U_I^c is bounded away from $\bigcup_{k \notin I} D_k$. Furthermore, each set U_I^c will be given the cusp-like structure

$$[c, \infty) \times L_I \times S_I^*,$$

where L_I is an $(|I| - 1)$-simplex and S_I^* is a principal $(S^1)^I$-bundle over a subset of D_I. This will allow us later to define a family $\{X^i\}_{i=1}^m$ of S^1-domains, by $X^i = \bigcup_{I \ni i} U_I^c$.

In order to motivate the construction, let us first examine the situation locally. Near a point of $D_I \setminus \bigcup_{k \notin I} D_k$, we can define the following structure. Let $((z_i)_{i \in I}, (w_j)_{j \in \{1, \cdots, n-|I|\}})$ be holomorphic coordinates on Δ^n such that $\Delta^n \cap D = \bigcup_{i \in I} z_i^{-1}(0)$. Provisionally define

$$r = \log \left| \log \prod_{i \in I} |z_i|^2 \right|,$$

$$t_i = (-\log |z_i|^2) e^{-r}, \quad \theta_i = \arg z_i \qquad (i \in I).$$

Then we can define a decomposition

$$\Delta^n \backslash (\Delta^n \cap D) \cong \mathbb{R} \times L_I^\circ \times (S^1)^I \times \Delta^{n-|I|} \tag{5.1}$$

via

$$y \mapsto (r(y), (t_i(y))_{i \in I}, (\theta_i(y))_{i \in I}, (w_j)_{j \in \{1, \cdots, n-|I|\}}),$$

where L_I° is an open $(|I| - 1)$-simplex with barycentric coordinates $(t_i)_{i \in I}$.

For the global version of this structure, we begin with a construction of Clemens [14] of compatible tubular neighborhoods of the D_I. This allows us to repeat the above discussion globally on U_I^c; the factor $(S^1)^I \times \Delta^{n-|I|}$ in (5.1) is replaced by a principal $(S^1)^I$-bundle.

5.1. Compatible tubular neighborhoods and torus actions. The following construction of Clemens [14] applies, even though he was working in a different context.

Theorem 5.1 (Clemens [14, Theorem 5.7]). *For each $I \subseteq \{1, \cdots, m\}$, there exists a tubular neighborhood U_I of D_I and a C^∞ normal projection $\pi_I : U_I \to D_I$ such that:*

(i) *the fibers of π_I are holomorphic submanifolds of U_I,*

(ii) *for all $I, J \subseteq \{1, \cdots, m\}$,*

$$(U_I \cap U_J) = U_{(I \cup J)},$$

(iii) *if $I \supseteq J$, then on U_I,*

$$\pi_I \circ \pi_J = \pi_I.$$

Furthermore, for each $i \in \{1, \cdots, m\}$, there exists a C^∞ one-form ω_i defined on $(U_i \backslash D_i)$ (we set $U_i = U_{\{i\}}$), such that if $x \in D_I$, $i \in I$, and $X = \pi_I^{-1}(x)$ (a complex submanifold by (i)*), then the following hold:*

(iv) *$\omega_i|_X$ is a closed meromorphic one-form on X with simple pole (and residue 1) along $(D_i \cap X)$,*

(v) *if $y_0 \in (X \backslash (D \cap X))$, then the functions*

$$z_{i,x}(y) = \exp\left(2\pi i \int_{y_0}^{y} \omega_i\right), \qquad i \in I,$$

give a system of holomorphic coordinates on X such that

$$z_{i,x} = 0 \quad \textit{defines} \quad D_i \cap X, \tag{5.2}$$

and such that on $U_{\{i,k\}}$ $(i \neq k)$, $z_{i,x}$ is constant on fibers of $\pi_{\{k\}}$,

(vi) *there exist C^∞ functions h_i on U_i with*

$$h_i > 0 \quad on \quad U_i \backslash D_i, \qquad h_i = 0 \quad on\ D_i,$$

and if $x \in D_I$ and $i \in I$, then

$$\frac{1}{2\pi i} d \log h_i|_{\pi_I^{-1}(x)} = (\omega_i - \overline{\omega_i})|_{\pi_I^{-1}(x)}.$$

Proof. Repeat the proof of [14, Theorem 5.7] noting the following. In Clemens's situation there is a global holomorphic function t with D as its zero set. We do not have this but it is not essential for his proof. His construction begins with a collection of holomorphic coordinate patches in which t has a canonical form [14, (5.6)] but it will actually work using any local holomorphic coordinates $((z_i)_{i\in I}, (w_j)_{j\in\{1,\cdots,n-|I|\}})$ for which $z_i = 0$ defines D_i. The only other change necessary is to let $\{m_j\}$ at the bottom of [14, p. 239] be arbitrary positive integers and to ignore the equation at the top of [14, p. 240] involving t. Then (i)–(v) follow (in [14], (v) is the stronger statement that $t|_X$ has a canonical form with respect to $z_{i,x}$). (vi) follows from [14, (6.1), (6.3)]. (There is a misprint in [14, (6.3)]; it should be as in (vi) above.) q.e.d.

We may use this theorem to introduce *special coordinates* in a neighborhood of a point on D_I, which will be useful in several ways. Let $W \subset D_I$ be an open subset and let

$$\sigma : W \to \left(\pi_I^{-1}(W) \setminus \bigcup_{i\in I} D_i\right) \tag{5.3}$$

be a local section of π_I. For each $i \in I$, define

$$z_{i,\sigma}(y) = \exp\left(2\pi i \int_{\sigma(\pi_I(y))}^{y} \omega_i\right) \tag{5.4}$$

analogously to Theorem 5.1(v). These are C^∞ functions on $\pi_I^{-1}(W)$; in fact $z_{i,\sigma} = f_i\tilde{z}_i$, where f_i is a nonvanishing C^∞ function, and $\tilde{z}_i$ is a local holomorphic defining function for D_i (see the end of [14, §5]). If $\vec{w} = (w_j)_{j\in\{1,\cdots,n-|I|\}}$ are C^∞ complex coordinates on W, then

$$((z_{i,\sigma})_{i\in I}, \pi_I^*\vec{w}) \tag{5.5}$$

are our special coordinates on $\pi_I^{-1}(W)$.

Remark 5.2. Note that the special coordinates (5.5) defined relative to π_I are also special coordinates relative to π_K, for $K \subseteq I$. This follows from Theorem 5.1(iii), (v); one uses the local section σ_K of π_K which is defined over $\pi_I^{-1}(W) \cap D_K$ by $z_{i,\sigma} = 1$ $(i \in K)$.

These special coordinates allow us to define a smooth $(S^1)^I$-action T^I on U_I. Namely, let T^I be given in the coordinates (5.5) by

$$(\lambda_i)_{i\in I} \times ((z_{i,\sigma})_{i\in I}, \pi_I^*\vec{w}) \mapsto ((\lambda_i z_{i,\sigma})_{i\in I}, \pi_I^*\vec{w}), \tag{5.6}$$

where $(\lambda_i)_{i\in I} \in (S^1)^I$. This is well defined in U_I, for if σ' is another local section of π_I as in (5.3), then

$$z_{i,\sigma'} = (\pi_I^* f) z_{i,\sigma}, \tag{5.7}$$

where f is a C^∞ function on W.

Note that these torus actions for different $I \subseteq \{1, \cdots, m\}$ are compatible, that is,

$$T^I|_{(S^1)^K} = T^K \quad \text{on } U_I \text{, for } K \subseteq I, \tag{5.8}$$

and that T^I is free on $U_I \setminus \bigcup_{i \in I} D_i$.

Proposition 5.3. *There exist tubular neighborhoods U_I as in Theorem 5.1 such that (in addition to (i)-(vi)), for each $i \in \{1, \cdots, m\}$, there exists a C^∞ one-form τ_i defined on $(U_i \backslash D_i)$ such that for all $I \subseteq \{1, \cdots, m\}$ and $i \in I$:*

(vii) *if σ is a local section of π_I over $W \subset D_I$ as in (5.3) and $z_{i,\sigma}$ is defined as in (5.4), then*

$$\tau_i = d \arg z_{i,\sigma} + \pi_I^* \gamma_{i,\sigma}$$

on $(\pi_I^{-1}(W) \backslash D_i)$, where $\gamma_{i,\sigma}$ is a C^∞ one-form on W.

Proof. Let $\{U_I^{(0)}\}$ and $\{U_I^{(1)}\}$ be two sets of tubular neighborhoods satisfying the conditions of Theorem 5.1 and such that $\overline{U_I^{(0)}} \subset U_I^{(1)}$ for all I. For every $K \subseteq \{1, \cdots, m\}$ and $i \in K$, Theorem 5.1(v), (vi) implies that the function h_i may be expressed locally as

$$h_i = (\pi_K^* c_{i,\sigma}) |z_{i,\sigma}|^2, \tag{5.9}$$

where $z_{i,\sigma}$ is defined as in (5.4) and $c_{i,\sigma}$ is a C^∞ function on $W \subseteq D_K$. Let

$$\tau_{K,i} = d \arg z_{i,\sigma} - \tfrac{1}{2} \pi_K^* J d \log c_{i,\sigma},$$

where J is the almost complex structure. This is independent of the choice of special local coordinates by (5.7) and (5.9) and so defines a C^∞ one-form on $U_K^{(1)} \backslash D_i$. Furthermore, it follows from Theorem 5.1(iii) and Remark 5.2 that $\tau_{K,i}$ satisfies (vii) whenever $I \subseteq K$.

Now define

$$U_I' = U_I^{(1)} \setminus \Big(\bigcup_{k \notin I} U_k^{(0)} \Big)$$

for all I. As in [14, (5.15)], we can find a partition of unity $\{\eta_I\}_{I \subseteq \{1, \cdots, m\}}$ subordinate to the covering $\{U_I'\}_{I \subseteq \{1, \cdots, m\}}$ of $\bigcup_{j=1}^m U_j^{(1)}$ and such that

$$\text{for all } I \subseteq K, \ \eta_K \text{ is constant on the fibers of } \pi_I|_{U_I^{(0)}}. \tag{5.10}$$

Then define

$$\tau_i = \sum_{K \ni i} \eta_K \tau_{K,i}$$

for all $i \in \{1, \cdots, m\}$, and set $U_I = U_I^{(0)}$ for all I. By (5.10) and the construction of the cover $\{U_I'\}$, we may express

$$\tau_i|_{U_I} = \sum_{K \supseteq I} (\pi_I^* \eta_K) \tau_{K,i}|_{U_K};$$

thus (vii) for $\tau_{K,i}$ ($K \supseteq I$) implies (vii) for τ_i.

5.2. A cofinal system of neighborhoods of D. Let $\{U_I\}_{I \subseteq \{1, \cdots, m\}}$ be compatible tubular neighborhoods of $\{D_I\}$ as in Theorem 5.1 and Proposition 5.3. We may assume, after rescaling, that $\{y \in U_i \mid 0 \le h_i(y) \le 1\}$ is a compact subset of U_i. Then define

$$U_I^* = \Big\{ y \in U_I \mid 0 < h_i(y) \le 1 \text{ for all } i \in I,$$

$$\text{and } \prod_{i \in I} h_i(y) < 1 \Big\} \setminus \bigcup_{k \notin I} \{ y \in U_k \mid 0 \le h_k(y) < 1 \},$$

$$D_I^* = \pi_I(U_I^*) = D_I \setminus \bigcup_{k \notin I} \{ y \in U_k \mid 0 \le h_k(y) < 1 \},$$

and

$$U^* = \bigcup_{I \subseteq \{1, \cdots, m\}} U_I^*.$$

The set U^* is an open neighborhood of D with D itself deleted.

Define a piecewise-smooth function r on U^* by

$$r(y) = \log \left| \log \prod_{i \in I} h_i(y) \right| \quad \text{if } y \in U_I^*. \tag{5.11}$$

This is well defined on U^* since

$$h_i(y) = 1 \quad \text{for } y \in U_I^* \cap U_K^*, \text{ if } i \in (I \setminus K) \cup (K \setminus I).$$

For $c \in \mathbb{R}$, let

$$U^c = \bigcup_{I \subseteq \{1, \cdots, m\}} U_I^c,$$

where $U_I^c = \{ y \in U_I^* \mid r(y) \ge c \}$. Since $r \to \infty$ at D, the sets $U^c \cup D$ form a cofinal system of closed neighborhoods of D, with piecewise-smooth boundary.

5.3. Decompositions of the U_I^*. Define

$$S_I = \{ y \in U_I \mid h_i(y) = 1 \text{ for all } i \in I \}$$

and

$$S_I^* = S_I \cap \pi_I^{-1}(D_I^*).$$

Let

$$x_i = -\log h_i \qquad (i \in I) \tag{5.12}$$

and let

$$\rho_I : U_I \to S_I \tag{5.13}$$

be the projection which is given with respect to special local coordinates (5.5) by

$$((z_{i,\sigma})_{i\in I}, \pi_I^* \vec{w}) \mapsto ((z_{i,\sigma} h_i^{-1/2})_{i\in I}, \pi_I^* \vec{w}).$$

These maps allow us to define our first decomposition,

$$U_I^* \cong ((\mathbb{R}^{\geq 0})^I \setminus \{\vec{0}\}) \times S_I^*, \tag{5.14}$$

by the diffeomorphism

$$y \mapsto ((x_i(y))_{i\in I}, \rho_I(y)).$$

Note that the $(S^1)^I$-action T^I operates only on the second factor of (5.14), making $\pi_I : S_I^* \to D_I^*$ a principal $(S^1)^I$-bundle. The forms $(\tau_i)_{i\in I}$ from Proposition 5.3 are pullbacks under ρ_I^* of forms on S_I, which we also denote $(\tau_i)_{i\in I}$; they are T^I-invariant and define a connection on S_I^*.

Our second decomposition of U_I^* breaks apart the first factor of (5.14) further using r. Note that

$$e^r = \sum_{i\in I} x_i. \tag{5.15}$$

So let L_I be the standard simplex in $\mathbb{R}^I$,

$$\left\{ (t_i)_{i\in I} \in (\mathbb{R}^{\geq 0})^I \mid \sum_{i\in I} t_i = 1 \right\},$$

and define

$$t_i = x_i \Big(\sum_{k\in I} x_k\Big)^{-1} = x_i e^{-r} \qquad (i \in I) \tag{5.16}$$

in U_I^*. Now we can further decompose

$$U_I^* \cong \mathbb{R} \times L_I \times S_I^* \tag{5.17}$$

by

$$y \mapsto (r(y), (t_i(y))_{i\in I}, \rho_I(y)),$$

and hence

$$U_I^c \cong [c, \infty) \times L_I \times S_I^*. \tag{5.18}$$

6. Distinguished metrics on the complement of a divisor

Let $\widetilde{V}$ be a complex manifold of dimension n, and $D \subset \widetilde{V}$ a divisor having smooth irreducible components $\{D_i\}_{i=1}^m$ intersecting with normal crossings. Let $\pi : \widetilde{V} \to V$ be a topological quotient map such that $\pi|_{\widetilde{V}\setminus D}$ is a homeomorphism and $\pi(D)$ is a finite set.

In §6.1 we single out a particular class of metrics on $\widetilde{V}\setminus D$ whose L_2-cohomology we wish to study by the method of Theorem 4.7; we will call these metrics *distinguished relative to* π. Our special interest in these metrics is that, as we will see in §8, if π is a projective morphism resolving a Kähler variety V with only isolated singularities and D is the exceptional divisor, then $\widetilde{V}\setminus D$ admits a Kähler distinguished metric relative to π and the L_2-cohomology of the distinguished metric represents the intersection cohomology of V.

In §6.2 we relate the asymptotics of distinguished metrics relative to π to the local structure near D discussed in §5.

6.1. Distinguished metrics.

Definition 6.1. A metric ds^2 on $\widetilde{V}\setminus D$ is *distinguished relative to* π if near all $q \in D$,

$$\begin{aligned} ds^2 \sim &\left|\log\prod_{k\in I}|z_k|^2\right|^{-2} \sum_{i\in I}\left(\frac{1}{|z_i|^2} + \left|\log\prod_{k\in I}|z_k|^2\right|\right) dz_i d\bar{z}_i \\ &+ \left|\log\prod_{k\in I}|z_k|^2\right|^{-1} \sum_{j=1}^{n-|I|} dw_j d\bar{w}_j, \end{aligned} \tag{6.1}$$

where $q \in D_I$ for $I \subseteq \{1, \cdots, m\}$, and $((z_i)_{i\in I}, (w_j)_{j\in\{1,\cdots,n-|I|\}})$ are C^∞ complex coordinates in some neighborhood $\Delta^n \subset \widetilde{V}$ of q such that

$$\Delta^n \cap D_i = \begin{cases} z_i^{-1}(0), & i \in I, \\ \varnothing, & i \notin I. \end{cases} \tag{6.2}$$

Remark. Since it will not cause confusion in the present work, we will simply call such a metric a *distinguished metric* on $\widetilde{V}\setminus D$. In future papers, we intend to consider the case where $\pi(D)$ does not necessarily consist of isolated points, in which case the mention of the map π is necessary.

Lemma 6.2. *The property of a metric being distinguished is independent of the choice of coordinates* (*satisfying* (6.2)) *used to verify* (6.1).

Proof. Let $((z_i)_{i\in I}, (w_j)_{j\in\{1,\cdots,n-|I|\}})$ and $((\tilde{z}_i)_{i\in I}, (\tilde{w}_j)_{j\in\{1,\cdots,n-|I|\}})$ be C^∞ complex coordinates near $q \in D$ satisfying (6.2). Let

$$ds^2 = K\sum_{i\in I}\alpha_{i\bar{i}}dz_i d\bar{z}_i + \sum_{j=1}^{n-|I|}\beta_{j\bar{j}}dw_j d\bar{w}_j$$

and $\widetilde{ds}^2 = \cdots$ be the respective model distinguished metrics as in the right-hand side of (6.1), except that we put in a constant $K > 0$ in front of the $dz_i d\bar{z}_i$ (respectively $d\tilde{z}_i d\bar{\tilde{z}}_i$) terms; the constant K will be determined later. We must show that $ds^2 \sim \widetilde{ds}^2$, perhaps after shrinking the coordinate patch.

By Taylor's theorem and (6.2) we may write

$$\tilde{z}_A = f_{AA}z_A + f_{A\bar{A}}\bar{z}_A, \tag{6.3}$$

where f_{AA} and $f_{A\bar{A}}$ are C^∞ functions. In this formula and in what follows, the indices A and B range over $I\cup\bar{I}$, while C and D range over $\overline{\{1,\cdots,n-|I|,\bar{1},\cdots,n-|I|\}}$. From (6.3) we deduce $|z_i| \sim |\tilde{z}_i|$ and hence

$$\alpha_{i\bar{i}} \sim \tilde{\alpha}_{i\bar{i}} \quad\text{and}\quad \beta_{j\bar{j}} \sim \tilde{\beta}_{j\bar{j}}. \tag{6.4}$$

Furthermore,

$$\begin{aligned} d\tilde{z}_A &= f_{AA}dz_A + f_{A\bar{A}}dz_{\bar{A}} + \sum_B O(|z_A|)dz_B + \sum_D O(|z_A|)dw_D, \\ d\tilde{w}_C &= \sum_D k_{CD}dw_D + \sum_B O(1)dz_B, \end{aligned} \tag{6.5}$$

where

$$\left|\det\begin{pmatrix} f_{AA} & f_{A\bar{A}} \\ f_{\bar{A}A} & f_{\bar{A}\bar{A}}\end{pmatrix}\right| \sim 1 \quad\text{and}\quad |\det(k_{CD})| \sim 1. \tag{6.6}$$

Thus, if we plug (6.5) into the expression for $\widetilde{ds}^2$, we obtain terms quasi-isometric to the corresponding terms of ds^2 (by (6.4) and (6.6)), as well as cross-terms of the form

(6.7) (a) $K\alpha_{i\bar{i}}O(|z_i|)dz_i dy$, (b) $K\alpha_{i\bar{i}}O(|z_i|^2)dy\,dy'$ $(i\in I)$,

(6.8) (a) $\beta_{j\bar{j}}O(1)dw_C dz_B$, (b) $\beta_{j\bar{j}}O(1)dz_A dz_B$ $(j\in\{1,\cdots,n-|I|\})$,

where dy and dy' represent either dz_B or dw_C.

It remains to show that each of these cross-terms is bounded by a small constant times ds^2. First note the following estimates, which are easy to verify:

(6.9) $$|z_i|^2\alpha_{i\bar{i}}/\alpha_{BB} \to 0 \quad \text{at } q,$$

(6.10) $$|z_i|^2\alpha_{i\bar{i}}/\beta_{C\bar{C}} \to 0 \quad \text{at } q,$$

(6.11) $$\beta_{C\bar{C}}/\alpha_{BB} \lesssim 1.$$

Now (6.7)(a) is bounded by

$$\varepsilon\alpha_{i\bar{i}}dz_i d\bar{z}_i + \varepsilon^{-1}K^2 O(|z_i|^2)\alpha_{i\bar{i}}dyd\bar{y},$$

in which the second term can be controlled by (6.9) or (6.10). A similar argument (without ε) applies to (6.7)(b). Also (6.8)(a) is bounded by

$$\varepsilon\beta_{C\bar{C}}dw_C dw_{\bar{C}} + \varepsilon^{-1}O(1)\beta_{C\bar{C}}dz_B dz_{\bar{B}}$$

(note that $\beta_{j\bar{j}}$ is independent of j), in which the second term can be controlled by using (6.11) and choosing K sufficiently large. Again, a similar argument applies to (6.8)(b). q.e.d.

Thus distinguished metrics always exist; one simply patches together local models using a partition of unity.

Proposition 6.3. *Near D, a distinguished metric on $\widetilde{V}\backslash D$ is complete with finite volume.*

We postpone the proof of this proposition to the next subsection.

6.2. Distinguished metrics and decompositions of U_I^*. We use the notation of §5. Recall that in §5.2 we constructed a punctured neighborhood $U^* = \bigcup_{I\subseteq\{1,\cdots,m\}} U_I^*$ of D and a piecewise-smooth exhaustion function r tending to ∞ at D. In this subsection we express a distinguished metric on $U_I^c = \{y \in U_I^* \mid r(y) \geq c\}$ in terms adapted to the product decompositions and bundle structures constructed in §5.3. We begin with (5.14), $U_I^* \cong ((\mathbb{R}^{\geq 0})^I \setminus \{\vec{0}\}) \times S_I^*$.

Lemma 6.4. *For $c \in \mathbb{R}$, a distinguished metric is quasi-isometric in U_I^c to*

$$e^{-2r}\sum_{i\in I}(1 + e^{-x_i}e^r)(dx_i^2 + \tau_i^2) + e^{-r}\pi_I^* ds_{D_I^*}^2.$$

Here $ds_{D_I^*}^2$ is an arbitrary Riemannian metric on D_I^*.

Proof. It suffices to prove the result in the domain of special coordinates $((z_{i,\sigma})_{i\in I}, \pi_I^*\vec{w})$, as in (5.5); we can further assume the section σ maps into S_I^*, and thus that $h_i = |z_{i,\sigma}|^2$. Recall that $x_i = -\log h_i$ and $e^r = \sum_{i\in I} x_i$. This shows that the model distinguished metric of (6.1) is equal to

$$e^{-2r}\sum_{i\in I}(1 + e^{-x_i}e^r)(dx_i^2 + (d\arg z_{i,\sigma})^2) + e^{-r}\pi_I^* ds_{D_I^*}^2.$$

Since $\tau_i = d\arg z_{i,\sigma} + \pi_I^*\gamma_{i,\sigma}$ (Proposition 5.3(vii)), $(d\arg z_{i,\sigma})^2$ may be replaced by τ_i^2, modulo terms that can be controlled precisely like those in (6.7). q.e.d.

Our main concern is how the metric on the level sets of r behaves as $r \to \infty$. So we now consider the decomposition (5.17), $U_I^* \cong \mathbb{R} \times L_I \times S_I^*$. Let $\frac{\partial}{\partial r} = \sum_{i \in I} x_i \frac{\partial}{\partial x_i}$ denote the vector field induced by translation in the $\mathbb{R}$ factor.

Proposition 6.5. *For $c \in \mathbb{R}$, a distinguished metric is quasi-isometric in U_I^c to*

$$dr^2 + ds_{L_I}^2(r) + e^{-2r}\sum_{i\in I}(1 + e^{-t_i e^r} e^r)\tau_i^2 + e^{-r}\pi_I^* ds_{D_I^*}^2,$$

where

$$ds_{L_I}^2(r) = \sum_{i \in I}(1 + e^{-t_i e^r} e^r) dt_i^2$$

is a family of Riemannian metrics on L_I depending on r.

Proof. Set $\alpha_i = e^{-2r}(1 + e^{-x_i} e^r)$. Then in view of (5.16) and Lemma 6.4, it suffices to show that the metric $\sum_{i \in I} \alpha_i dx_i^2$ on $[c, \infty) \times L_I$ is quasi-isometric to $dr^2 + ds_{L_I}^2(r)$. Clearly the restriction of the metric to TL_I is $ds_{L_I}^2(r)$, while $|\frac{\partial}{\partial r}|^2 = e^{-2r}\sum x_i^2 + e^{-r}\sum x_i^2 e^{-x_i} \sim 1$ (since $\sum x_i^2 \sim (\sum x_i)^2$ and $x_i^2 e^{-x_i} \lesssim 1$). Thus it only remains to show that the angle between $\frac{\partial}{\partial r}$ and TL_I is uniformly greater than 0.

Assume this is not the case, that is, for some sequence of points in $[c, \infty) \times L_I$, the angle $\measuredangle(\frac{\partial}{\partial r}, TL_I) \to 0$. Equivalently,

$$\measuredangle\left(\frac{\partial}{\partial r}, \nu\right) \to \frac{\pi}{2}, \tag{6.12}$$

where ν is a normal vector field to L_I. Express $\frac{\partial}{\partial r} = \sum x_i \alpha_i^{1/2} \mathbf{e}_i$ and $\nu = \sum \alpha_i^{-1/2} \mathbf{e}_i$ with respect to the orthonormal frame $\mathbf{e}_i = \alpha_i^{-1/2}\frac{\partial}{\partial x_i}$. Since $|\frac{\partial}{\partial r}| \sim 1$, we can assume, after passing to a subsequence, that $(x_i \alpha_i^{1/2})_{i\in I}$ converges to a nonzero vector. We can also assume that both points in real projective space, $[(x_i\alpha_i^{1/2})_{i\in I}]$ and $[(\alpha_i^{-1/2})_{i \in I}]$, converge. But since the coordinates are nonnegative, (6.12) implies the limit points must lie on oppposing faces of the boundary of the projectivized positive quadrant. In particular, if $i_0 \in I$ is an index such that

$$x_{i_0}\alpha_{i_0}^{1/2} \not\to 0, \tag{6.13}$$

then the corresponding homogeneous coordinate of $[(\alpha_i^{-1/2})_{i\in I}]$ must go to 0, i.e.,

$$\frac{\alpha_{i_0}^{-1/2}}{\alpha_k^{-1/2}} \to 0 \tag{6.14}$$

for some $k \neq i_0$. However (6.13) implies $1 \lesssim x_{i_0}\alpha_{i_0}^{1/2} \lesssim x_{i_0}e^{-r/2}$ (since $\alpha_i \lesssim e^{-r}$), or $x_{i_0} \gtrsim e^{r/2}$. But then

$$\frac{\alpha_{i_0}^{-1/2}}{\alpha_k^{1/2}} = \frac{(1+e^{-x_k}e^r)^{1/2}}{(1+e^{-x_{i_0}}e^r)^{1/2}} \geq \frac{1}{(1+e^{-Ce^{r/2}}e^r)^{1/2}} \gtrsim 1,$$

which contradicts (6.14). q.e.d.

We now give the

Proof of Proposition 6.3. It suffices to prove the result on U_I^c, for each I. Completeness is clear from Proposition 6.5. Now note that since $\sum_{i\in I} t_i = 1$ and $|I| \leq n = \dim_{\mathbb{C}} \widetilde{V}$,

$$\prod_{i\in I}(1+e^{-t_ie^r}e^r) \lesssim (e^{(n-1)r}+e^{-e^r}e^{nr}) \lesssim e^{(n-1)r}.$$

Thus the volume form is dominated by $e^{-r}dr \wedge dV_{L_I}(1) \wedge dV_{S_I^*}$, which is integrable.

7. L_2-cohomology of distinguished metrics

As in the previous two sections, let $\widetilde{V}$ be a complex manifold and $D = \bigcup_{i=1}^m D_i \subset \widetilde{V}$ a divisor with normal crossings and smooth irreducible components. In §5 we constructed a cofinal system U^c ($c \in \mathbb{R}$) of punctured closed regular neighborhoods of D, whereas in §6 we defined the notion of a distinguished metric.

In this section we examine the L_2-cohomology of U^c with respect to a distinguished metric (and certain boundary conditions and weight functions). Our approach is to construct in §7.1 a family $\{X^i\}_{1\leq i\leq m}$ of S^1-domains of U^c; this uses the constructions of §5. We then show in §7.2 that this is an associative family of admissible S^1-domains (Definitions 2.3 and 4.4). Thus the techniques in §§2–4 apply to yield a spectral sequence converging to $H_{(2)}^{\cdot}(U^c, F; g)$; in §7.3 we compute E_1 and d_1 for this spectral sequence using Theorem 4.7. We find that, modulo a row of infinite-dimensional groups, the terms $E_1^{-p,q}$ vanish if $q \geq n$; in the next section, under the hypothesis that D can be blown down to isolated singularities, we will see that this truncation by *weight* (i.e., q) in E_1 becomes a truncation by *degree* (i.e., $p+q$) in E_2.

We freely use the notation of §5.

7.1. S^1-domains in U^c and tubular boundary neighborhoods. Fix $c \in \mathbb{R}$. For all $i \in \{1, \cdots, m\}$ define

$$X^i = \bigcup_{\substack{I \subseteq \{1, \cdots, m\} \\ I \ni i}} U_I^c; \tag{7.1}$$

X^i together with the free ps S^1-action $T^i = T^{\{i\}}$ defined in §5.1 is an S^1-domain of U^c. In the remainder of this subsection we will construct tubular boundary neighborhoods (Definition 1.10) of $X^i \subset U^c$, that is, tubular neighborhoods (N^i, p^i, t_i) of

$$M^i = \mathrm{bd}_{U^c} X^i = \overline{\partial X^i \backslash \partial U^c} = \bigcup_{\substack{I \subseteq \{1, \cdots, m\} \\ I \supsetneq \{i\}}} U_I^c \cap U_{I\backslash\{i\}}^c.$$

It suffices to do this for each $U_I^c \subset U_I^c \cup U_{I\backslash\{i\}}^c$ (where $I \supsetneq \{i\}$) in such a way that the constructions match up piecewise smoothly along the boundaries.

We begin with the following simple lemma, which may be proved similarly to Clemens's Theorem 5.1, although more simply. Recall that L_I denotes the simplex with barycentric coordinates $(t_i)_{i \in I}$. For $K \subseteq I$ we identify L_K with the subsimplex of L_I defined by setting $(t_i)_{i \in I\backslash K}$ to zero.

Lemma 7.1. *There exist $b > 0$ (small) and C^∞ normal projections (for all $I \subseteq \{1, \cdots, m\}$ and $i \in I$)*

$$p_I^i : \{ y \in L_I \mid t_i(y) \in [0, b] \} \to L_{I\backslash\{i\}}, \tag{7.2}$$

which satisfy

$$p_I^i|_{L_K} = p_K^i \qquad (\text{for } i \in K \subseteq I), \tag{7.3}$$

and, for $u, v \in I$, $u \neq v$,

$$p_I^u \circ p_I^v = p_I^v \circ p_I^u, \tag{7.4}$$

$$(p_I^v)^*(t_u) = t_u, \tag{7.5}$$

on $\{ y \in L_I \mid t_u(y), t_v(y) \in [0, b] \}$.

Denote also by p_I^i the projections

$$p_I^i : \{ y \in U_I^c \mid t_i(y) \in [0, b] \} \to U_I^c \cap U_{I\backslash\{i\}}^c \tag{7.6}$$

induced by applying (7.2) to the middle factor of (5.17).

Now for $I \subseteq \{1, \cdots, m\}$ and $i \in I$, extend t_i piecewise smoothly from U_I^c into a neighborhood of $U_I^c \cap U_{I\setminus\{i\}}^c$ in $U_I^c \cup U_{I\setminus\{i\}}^c$ by defining

$$t_i = -\log h_i \tag{7.7}$$

in $\{y \in U_{I\setminus\{i\}}^c \mid h_i(y) \in [1, e)\}$. Also, perhaps after shrinking b, let

$$p_{I\setminus\{i\}}^i : \{y \in U_{I\setminus\{i\}}^c \mid t_i(y) \in [-b, 0]\} \to U_I^c \cap U_{I\setminus\{i\}}^c \tag{7.8}$$

be the projection defined by restricting ρ_I (5.13), and define

$$p^i = \begin{cases} p_I^i & \text{on } U_I^c, \\ p_{I\setminus\{i\}}^i & \text{on } U_{I\setminus\{i\}}^c. \end{cases}$$

Then these definitions fit together for all $I \ni i$ to form a tubular boundary neighborhood (N^i, p^i, t_i) of X^i, where

$$N^i = \bigcup_{\substack{I\subseteq\{1,\cdots,m\}\\ I\supsetneq\{i\}}} \{y \in U_I^c \cup U_{I\setminus\{i\}}^c | t_i(y) \in [-b, b]\}.$$

7.2. Associativity and admissibility.

Lemma 7.2. *Let $i \in \{1, \cdots, m\}$ and consider $I \subseteq \{1, \cdots, m\}$ satisfying $I \supsetneq \{i\}$. Then a distinguished metric on U^c is quasi-isometric in $\{y \in U_I^c \mid t_i(y) \in [0, b]\}$ to*

$$(1 + e^{-t_i e^r} e^r)(dt_i^2 + e^{-2r}\tau_i^2) + (p^i)^* \pi_i^* ds_{M^i/T^i}^2, \tag{7.9}$$

where ds_{M^i/T^i}^2 is a metric on M^i/T^i.

Proof. In view of $(p^i)^*(r) = r$ and Proposition 6.5, it suffices to show

$$\sum_{k\in I}(1 + e^{-t_k e^r} e^r)\tau_k^2 \sim (1 + e^{-t_i e^r} e^r)\tau_i^2 + \sum_{k\in I\setminus\{i\}} (1 + e^{-(p^i)^*(t_k)e^r} e^r)\tau_k^2 \tag{7.10}$$

and

$$\sum_{k\in I}(1 + e^{-t_k e^r} e^r)dt_k^2 \sim (1 + e^{-t_i e^r} e^r)dt_i^2 + (p^i)^* ds_{L_{I\setminus\{i\}}}(r). \tag{7.11}$$

For each $K \subsetneq I\setminus\{i\}$, define $W_K \subset \{y \in U_I^c \mid t_i(y) \in [0, b]\}$ by the conditions

$$\begin{aligned} t_k &< b \qquad (k \in K), \\ t_k &> b/2 \qquad (k \in I\setminus(K \cup \{i\})). \end{aligned} \tag{7.12}$$

Since $\{W_K\}$ is a cover, it suffices to prove (7.10) and (7.11) in each W_K separately. Now it follows from (7.5) and (7.12) that, in W_K,

$$(p^i)^*(t_k) = t_k \qquad (k \in K),$$
$$(p^i)^*(t_k) > b/2 \qquad (k \in I\backslash(K \cup \{i\})),$$

and consequently

$$(1 + e^{-t_k e^r} e^r) \sim \begin{cases} (1 + e^{-(p^i)^*(t_k)e^r} e^r) & (k \in K), \\ 1 & (k \in I\backslash(K \cup \{i\})). \end{cases}$$

(7.10) now follows immediately, while for (7.11) we may write the left-hand side quasi-isometrically as

$$(1 + e^{-t_i e^r} e^r)dt_i^2 + (p^i)^* \left(\sum_{k\in K} e^{-t_k e^r} e^r dt_k^2 \right) + \sum_{k\in I\backslash\{i\}} dt_k^2.$$

Since $\sum_{k\in I} dt_k^2 \sim dt_i^2 + (p^i)^*(\sum_{k\in I\backslash\{i\}} dt_k^2)$ in $t_i^{-1}([0, b])$, (7.11) is proved. q.e.d.

Since

$$ds^2_{D^*_{I\backslash\{i\}}} \sim dt_i^2 + \tau_i^2 + \pi_I^* ds^2_{D_I^*}$$

on $\{x \in D^*_{I\backslash\{i\}} \mid h_i(x) \in [1, e)\}$, the next lemma follows from Lemma 6.4 applied to $I\backslash\{i\}$.

Lemma 7.3. *Let $i \in \{1, \cdots, m\}$ and consider $I \subseteq \{1, \cdots, m\}$ satisfying $I \supsetneqq \{i\}$. A distinguished metric on U^c is quasi-isometric in $\{y \in U^c_{I\backslash\{i\}} \mid t_i(y) \in [-b, 0]\}$ to*

$$e^{-r}(dt_i^2 + \tau_i^2) + (p^i)^* \pi_i^* ds^2_{M^i/T^i},$$

where $ds^2_{M^i/T^i}$ is a metric on M^i/T^i.

Proposition 7.4. *$\{X^i\}_{1\le i\le m}$ is an associative set of admissible S^1-domains relative to (U^c, F, g) (Definitions 2.3 and 4.4), where U^c is given a distinguished metric, g is any function of r, and $F = \varnothing$ or ∂U^c.*

Proof. Let $i \in \{1, \cdots, m\}$. We first check that X^i with the tubular boundary neighborhood (N^i, p^i, t_i) defined in §7.1 is an admissible S^1-domain.

Definition 2.3, parts (i) and (ii), are clear. For part (iii), we must verify the estimates (2.1)–(2.7). Estimate (2.1) follows from Proposition 6.5, while (2.2) is obvious since $(p^i)^*(r) = r$. Note that the form τ of §2 corresponds to our τ_i. Thus estimate (2.3), with $f = 1 + e^{-t_i e^r} e^r$ and

$w^2 = e^{-r}$, follows from Lemmas 7.2 and 7.3. Estimates (2.4) and (2.5) also follow by inspection since $\int_0^b f\,dt_i = (t_i - e^{-t_i e^r})|_0^b = 1 + b - e^{-be^r} \sim 1$ and $e^{-\varepsilon e^r} e^r \lesssim 1$.

From Proposition 6.5, we can estimate $|d\tau_i|^2 \lesssim e^{2r}$ in $X^i \cup N^i$ (since by Proposition 5.3, $d\tau_i = \pi_I^* d\gamma_{i,\sigma}$), and $|\tau_i|^2 \sim e^{2r}$ in $X^i \backslash N^i$. This proves (2.6) and (2.7). Definition 2.3(iv) is satisfied by making b smaller.

The associativity conditions, Definition 4.4(i)–(iv), are all obvious from our construction; the last part of (i) follows from (7.4) and (7.5) by Remark 4.5.

7.3. A spectral sequence converging to $H^{\cdot}_{(2)}(U^c)$. We begin by recalling the weight spectral sequence for the mixed Hodge structure on $H^{\cdot}(U^c;\mathbb{C})$ (see [17] or [18]). Denote

$$D^p = \coprod_{|K|=p} D_K. \tag{7.13}$$

Theorem 7.5 ([17], [18]). *There is a weight spectral sequence converging to $H^{\cdot}(U^c;\mathbb{C})$ and degenerating at E_2. The qth row of the E_1 term is:*

$$\begin{array}{ccccccccc} \to & E_1^{-2,q} & \to & E_1^{-1,q} & \to & E_1^{0,q} & \to & E_1^{1,q} & \to \\ & \| & & \| & & \| & & \| & \\ & H^{q-4}(D^2) & & H^{q-2}(D^1) & & H^q(D^1) & & H^q(D^2) & \end{array} \tag{7.14}$$

The differential $d_1 : E_1^{-p,q} \to E_1^{-p+1,q}$ for $p \geq 2$ is an alternating sum of Gysin maps, the differential for $p = 1$ is an alternating sum of Gysin maps into $H^q(U^c \cup D)$ composed with a map induced by an alternating sum of restriction maps, and the differential for $p \leq 0$ is induced by an alternating sum of restriction maps.

Remark 7.6 ([17], [18]). For $p = 1$, the component of d_1 mapping $H^{q-2}(D_i) \to H^q(D_j)$ $(i \neq j)$ can also be described (modulo sign) as a restriction to $D_i \cap D_j$ composed with the Gysin map to D_j.

We now compute an analogous spectral sequence converging to the L_2-cohomology of U^c. By Proposition 7.4, $\mathscr{X} = \{X^m\}_{1 \leq m \leq N}$ is an associative family of admissible S^1-domains in U^c; let $W^{\mathscr{X}}$ be the associated filtration (Definition 4.3) on $C = \mathrm{Dom}(\overline{d}_{U^c})$. Define the filtration W by

$$W_p = \begin{cases} W_p^{\mathscr{X}} & p > 0, \\ W_{p-1}^{\mathscr{X}} & p \leq 0 \end{cases} \tag{7.15}$$

(we shall see below that $\mathrm{Gr}_0^{W^{\mathscr{X}}} C$ is acyclic).

Theorem 7.7. *The qth row of the E_1 term of the W-spectral sequence for $H^{\cdot}_{(2)}(U^c)$ is equal to* (7.14) *for $0 \le q \le n-1$. For $q = n+1$, it is equal to $H^1_{(2)}([c, \infty))$ (which is infinite dimensional) tensored with* (7.14) *in which we set $q = n$. All other rows are zero.*

Proof. By Theorem 4.7,

$$H^{q-p}(\mathrm{Gr}_p^{W^{\mathscr{X}}} C) \cong \bigoplus_{\substack{J \in \{-1,0,1\}^m \\ |J| = p}} H_{(2)}^{q-p-|\{s | j_s = 1\}|}(Y_J^{\mathscr{X}}, F_J^{\mathscr{X}}; g_J^{\mathscr{X}}). \tag{7.16}$$

In order to evaluate the right-hand side of (7.16), we refer to the construction of $(Y_J^{\mathscr{X}}, F_J^{\mathscr{X}}, g_J^{\mathscr{X}})$ in §4.2 and set $I_j = \{s \mid j_s = j\}$. We find

$$\begin{aligned} Y_J^{\mathscr{X}} &= \left(\left(\bigcap_{u \in I_{-1} \cup I_1} X^u \cap \bigcap_{u \in I_0} (Y \backslash X^u)\right) \Big/ T^{I_{-1} \cup I_1}\right)^{\wedge} \\ &= \left(U^c_{I_{-1} \cup I_1} / T^{I_{-1} \cup I_1}\right)^{\wedge} \\ &= \left([c, \infty) \times L_{I_{-1} \cup I_1} \times D^*_{I_{-1} \cup I_1}\right)^{\wedge} \\ &= [c, \infty) \times L_{I_{-1} \cup I_1} \times D_{I_{-1} \cup I_1}. \end{aligned}$$

Here the "hat" operation denotes the successive applications (for each S^1-action T^u, $u \in I_0$) of the "hat" operation defined in (2.8). In effect, we fill in with solid tori the torus fibers in the boundary of $D^*_{I_{-1} \cup I_1}$ associated to the $(S^1)^{I'}$-action $T^{I'}$, for all $I' \subseteq I_0$. Similarly,

$$F_J^{\mathscr{X}} = [c, \infty) \times \bigcup_{s \in I_{-1}} L_{(I_{-1} \backslash \{s\}) \cup I_1} \times D_{I_{-1} \cup I_1}.$$

By working with the form of the metric on $U^c_{I_{-1} \cup I_1}$ given by Proposition 6.5, we see the metric on $Y_J^{\mathscr{X}}$ is quasi-isometric to

$$dr^2 + ds^2_{L_{I_{-1} \cup I_1}}(r) + e^{-r} ds^2_{D_{I_{-1} \cup I_1}},$$

and the weight function is

$$g_J^{\mathscr{X}} \sim \prod_{i \in I_{-1} \cup I_1} \left((1 + e^{-t_i e^r} e^r)^{-1/2} e^r\right)^{-j_i}.$$

By repeated application of Theorem 2.5, the L_2-cohomology in (7.16) is unchanged if we replace the metric and weight function by

$$dr^2 + ds^2_{L_{I_{-1} \cup I_1}} + e^{-r} ds^2_{D_{I_{-1} \cup I_1}} \quad \text{and} \quad e^{(|I_{-1}| - |I_1|) r}$$

respectively. Thus we can apply Zucker's theorem on L_2-cohomology of warped products, Corollary 1.7. But since

$$H^k\Big(L_{I_{-1}\cup I_1},\ \bigcup_{s\in I_{-1}} L_{(I_{-1}\setminus\{s\})\cup I_1}\Big) \tag{7.17}$$
$$= \begin{cases} \mathbb{C} & \text{for } I_{-1}=\varnothing,\ k=0, \\ \mathbb{C} & \text{for } I_1=\varnothing,\ k=|I_{-1}|-1, \\ 0 & \text{otherwise}, \end{cases}$$

only terms with I_{-1} or I_1 empty (and the other nonempty) contribute to (7.16). Consequently, Corollary 1.7 applied to (7.16) yields

$$H^{q-p}(\mathrm{Gr}_p^{W^{\mathscr{X}}} C) \cong \begin{cases} \bigoplus_{|K|=p} H_{(2)}^{q-2p}([c,\infty)\times D_K;\, e^{pr}) & p>0, \\ 0 & p=0, \\ \bigoplus_{|K|=-p} H_{(2)}^{q+1}([c,\infty)\times D_K;\, e^{pr}) & p<0, \end{cases}$$

$$\cong \begin{cases} \bigoplus_{|K|=p} \bigoplus_i H_{(2)}^i([c,\infty);\, e^{(q-i-n)r}) \otimes H^{q-2p-i}(D_K) & p>0, \\ 0 & p=0, \\ \bigoplus_{|K|=-p} \bigoplus_i H_{(2)}^i([c,\infty);\, e^{(q+1-i-n)r}) \otimes H^{q+1-i}(D_K) & p<0. \end{cases}$$

One now applies (7.13), (7.15), and the calculation of weighted L_2-cohomology of a half-line (Lemma 1.8(i)), to see that

$$E_1^{-p,q} = H^{q-p}(\mathrm{Gr}_p^W C)$$
$$\cong \begin{cases} H^{q-2p}(D^p) & p>0,\ q\le n-1, \\ H^q(D^{-p+1}) & p\le 0,\ q\le n-1, \\ H_{(2)}^1([c,\infty))\otimes H^{n-2p}(D^p) & p>0,\ q=n+1, \\ H_{(2)}^1([c,\infty))\otimes H^n(D^{-p+1}) & p\le 0,\ q=n+1, \\ 0 & \text{otherwise}, \end{cases}$$

as desired.

All that remains is to verify that $d_1 : E_1^{-p,q} \to E_1^{-p+1,q}$ is as claimed. But for $p \neq 1$, d_1 for the W-spectral sequence agrees with d_1 for the $W^{\mathscr{X}}$-spectral sequence (modulo reindexing when $p \le 0$). For $p = 1$, it is induced by d_2 for the $W^{\mathscr{X}}$-spectral sequence. Thus we can apply the calculations of d_1 and d_2 from Theorem 4.7 to conclude the proof. q.e.d.

All the preceding arguments also apply to the complex $\mathrm{Dom}(\overline{d}_{U^c, \partial U^c})$, which computes $H^{\cdot}_{(2)}(U^c, \partial U^c)$, however we need to replace the use of Lemma 1.8(i) by Lemma 1.8(ii). The result is

Theorem 7.8. *For $q \geq n+2$, the qth row of the E_1 term of the W-spectral sequence for $H^{\cdot}_{(2)}(U^c, \partial U^c)$ is equal to* (7.14) *in which q is decreased by one. For $q = n+1$, it is equal to $H^1_{(2)}([c, \infty), \{c\})$ tensored with* (7.14) *in which we set $q = n$. All other rows are zero.*

Remark 7.9. There are analogous spectral sequences for the weighted L_2-cohomology with weight function $g = e^{\lambda r}$ ($\lambda \in \mathbb{R}$). In the case of $H^{\cdot}_{(2)}(U^c; e^{\lambda r})$, the rows of E_1 are equal to (7.14) for $0 \leq q < n - \lambda$, and, if $\lambda \in \mathbb{Z}$, there is an infinite-dimensional row for $q = n + 1 - \lambda$. In the case of $H^{\cdot}_{(2)}(U^c, \partial U^c; e^{\lambda r})$, the rows of E_1 for $q > n + 1 - \lambda$ are equal to (7.14) in which q is decreased by one, and, if $\lambda \in \mathbb{Z}$, there is an infinite-dimensional row for $q = n + 1 - \lambda$.

Example 7.10. We present a simple 3-dimensional example to emphasize the fact that, unlike in the surface case [42], the nonclosedness of $\mathrm{Range}(\overline{d})$ is not the only obstruction to showing $H^i_{(2)}(U^c) = 0$ for $i \geq n$. The example also shows that the L_2-cohomology of a distinguished metric relative to π, where $\pi : \widetilde{V} \to V$ is merely a topological quotient map collapsing D to the isolated singularities of a topological pseudomanifold, may not represent the intersection cohomology of V.

Let $\widetilde{V} = \mathbb{P}^3$ with homogeneous coordinates $[z_0, z_1, z_2, z_3]$ and let $D = \bigcup_{i=1}^3 D_i$, where $D_i = \{z_i = 0\} \cong \mathbb{P}^2$. It is easy to see that the nonzero rows of $E_1^{-p,q}$ reduce to

$$0 \to H^0\Big(\coprod_i D_i\Big) \to H^2\Big(\coprod_i D_i\Big) \xrightarrow{r} H^2\Big(\coprod_{i<j} D_i \cap D_j\Big) \to 0 \qquad (-1 \leq p \leq 1, q = 2)$$

and

$$0 \to H^0\Big(\coprod_i D_i\Big) \to H^0\Big(\coprod_{i<j} D_i \cap D_j\Big) \to H^0(D_1 \cap D_2 \cap D_3) \to 0 \qquad (-2 \leq p \leq 0, q = 0).$$

The potentially infinite-dimensional row vanishes in this case, so we know that $\mathrm{Range}(\overline{d})$ is closed. On the other hand, the matrix of the indicated map $r : \mathbb{C}^3 \to \mathbb{C}^3$ (with respect to the basis induced by restrictions of a

Kähler form on $\mathbb{P}^3$) is just

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix},$$

which is singular. Thus $E_2^{1,2} \neq 0$; since clearly $E_2^{1,2} = E_\infty^{1,2}$, we see that $H_{(2)}^3(U^c)$ is nonzero.

8. L_2-cohomology of isolated singularities

Let V be a complex analytic space of dimension n with only isolated singularities. Let $\pi : \widetilde{V} \to V$ be a resolution of V, with exceptional set D a divisor with normal crossings. Assume the irreducible components of D are smooth and the morphism π is projective. By Hironaka [26], such resolutions exist and any modification $V' \to V$ is dominated by such a resolution. We will study the L_2-cohomology of distinguished metrics on $\widetilde{V} \backslash D \cong V \backslash \mathrm{Sing}(V)$; such a metric is called a *distinguished metric on* V.

There are two important additional features to this case as opposed to the general situation treated in §7. First, under a mild global condition on V (satisfied, say, if V is an open subset of a projective or even Kähler variety (see Remark 8.5)), there exists a distinguished metric which is Kähler; this occupies §8.1. Second, we see in §8.2 that the local L_2-cohomology near $\mathrm{Sing}(V)$ vanishes in the middle and higher degrees. We deduce as a consequence that for compact V, $H_{(2)}^{\bullet}(V \backslash \mathrm{Sing}(V)) \cong IH^{\bullet}(V; \mathbb{C})$, the (middle perversity) intersection cohomology of V.

8.1. Kähler distinguished metrics. Since π is projective, we can find a (not necessarily reduced) complex subspace Z of V, supported on $\mathrm{Sing}(V)$, such that $\pi : \widetilde{V} \to V$ is the monoidal transformation of V with center Z. Write $\pi^{-1}(Z) = \sum_{i=1}^m a_i D_i$, where $a_i \in \mathbb{Z}^+$ and the D_i are the smooth irreducible components of the exceptional divisor D, and let $L = -\sum_{i=1}^m a_i [D_i]$. In other words, if $\mathcal{J} \subseteq \mathcal{O}_{V, \mathrm{Sing}\, V}$ is the ideal defining Z, then $\mathcal{O}_{\widetilde{V}}(L)$ is the locally invertible sheaf $\pi^{-1}\mathcal{J}$ guaranteed by the universal property of monoidal transformations.

Lemma 8.1. *In a small neighborhood* $\widetilde{U}$ *of* D, *there exists a metric on* L *with positive curvature.*

Proof. Embed a small neighborhood U of $\mathrm{Sing}(V)$ as an analytic subvariety of a polydisk Δ^M with coordinates $(x_1, \cdots, x_M)$. By construction of the monoidal transformation, we know that if U is sufficiently small and $\widetilde{U} = \pi^{-1}(U)$, then $\Gamma(\widetilde{U}, \mathcal{O}_{\widetilde{V}}(L))$ defines a map $\iota : \widetilde{U} \to \mathbb{P}^N$ into some projective space such that $\pi \times \iota$ is an embedding $\widetilde{U} \hookrightarrow \Delta^M \times \mathbb{P}^N$. Since $(\pi \times \iota)^* \pi_2^* H = \iota^* H = L$, where H is the hyperplane bundle of $\mathbb{P}^N$, and since there exists a metric on $\pi_2^* H$ with positive curvature in $\Delta^M \times \mathbb{P}^N$, L has a metric with positive curvature.

Explicitly, if $\mathscr{I}_Z \subset \mathscr{O}_{V,\operatorname{Sing} V}$ is the ideal defining the subspace Z and $f_0, \cdots, f_N$ are generators of $\mathscr{I}_Z$, the map $\iota|_{\widetilde{U}\setminus D}$ is given by $[\pi^* f_0, \cdots, \pi^* f_N]$ and the equation $|s|^2 = (\sum_j |\pi^* f_j|^2)^{-1} e^{-|x|^2}$ (where s is a section of L with $(s) = -\sum_i a_i D_i$) defines a metric on L with positive curvature.

Lemma 8.2. *There exist positive integers* $(a_{\alpha i})_{\alpha\in\{1,\cdots,n\}, i\in\{1,\cdots,m\}}$ *such that (after perhaps shrinking* $\widetilde{U}$*):*

(i) *Any* p *columns of* $(a_{\alpha i})$ *are linearly independent for* $p \leq \min(n, m)$.

(ii) *For* $\alpha \in \{1, \cdots, n\}$, *the line bundle* $L_\alpha = -\sum_{i=1}^m a_{\alpha i}[D_i]$ *has a metric with curvature form* Θ_α *positive in* $\widetilde{U}$.

Proof. By Lemma 8.1, we may give the line bundles $[D_i]$ metrics so that the induced metric on L has positive curvature. Let Ω_i be the curvature form of $[D_i]$. We can perturb $(a_1, \cdots, a_m)$ slightly to get rational vectors $(\tilde{a}_{\alpha 1}, \cdots, \tilde{a}_{\alpha m})$ for each $\alpha \in \{1, \cdots, n\}$, such that (i) is satisfied and such that $-\sum_i \tilde{a}_{\alpha i} \Omega_i$ is positive in a slightly smaller $\widetilde{U}'$. Then multiplying $(\tilde{a}_{\alpha i})$ by a common denominator to obtain $(a_{\alpha i})$ finishes the proof.

Remark. Although we will not need it, for N sufficiently large, the locally invertible sheaves $\mathscr{O}_{\widetilde{U}}(L_\alpha^N)$ have the form $\pi^{-1}(\mathscr{I}_\alpha)$ for ideals $\mathscr{I}_1, \ldots, \mathscr{I}_n \subseteq \mathscr{O}_{V,\operatorname{Sing} V}$, each of whose blow-up yields the same morphism $\pi : \widetilde{V} \to V$.

Let $s_\alpha^* \in \Gamma(\widetilde{U}, \mathscr{O}(L_\alpha^*))$ be such that $(s_\alpha^*) = \sum_i a_{\alpha i} D_i$, where $(a_{\alpha i})$ is as in the lemma. Define on $\widetilde{V} \setminus D \cong V \setminus \operatorname{Sing}(V)$ the closed $(1, 1)$-form

$$\omega_1 = -i \sum_{\alpha=1}^{n} \partial\bar{\partial}\, \eta \log(\log |s_\alpha^*|^2)^2, \tag{8.1}$$

where η is a C^∞ cutoff function equal to 1 near $\operatorname{Sing}(V)$, and equal to 0 outside a small neighborhood of $\operatorname{Sing}(V)$.

Proposition 8.3. *The* $(1, 1)$*-form* ω_1 *is positive definite near* D, *and the associated Riemannian metric is distinguished.*

Proof. Let $q \in D$ and let $((z_i)_{i\in I}, (w_i)_{i\in\{1,\cdots,n-|I|\}})$ be holomorphic coordinates in a neighborhood $\Delta^n \subset \widetilde{U}$ satisfying $\Delta^n \cap D_i = z_i^{-1}(0)$ for $i \in I$, and $= \varnothing$ for $i \notin I$. We must show that the hermitian form associated to ω_1 is quasi-isometric to (6.1) near q.

Define

$$u_\alpha = -\log|s_\alpha^*|^2$$

and compute from (8.1) that near q,

$$\omega_1 = 2i\sum_{\alpha=1}^{n}(u_\alpha^{-2}\partial u_\alpha \wedge \bar\partial u_\alpha - u_\alpha^{-1}\partial\bar\partial u_\alpha).$$

Write

$$u_\alpha = -\log\prod_{k\in I}|z_k|^{2a_{\alpha k}} - \log h_\alpha, \tag{8.2}$$

where h_α is a positive C^∞ function, and thus

$$u_\alpha \sim \left|\log\prod_{k\in I}|z_k|^2\right|.$$

Furthermore, Lemma 8.2(ii) implies that $-\partial\bar\partial u_\alpha$ is positive definite near q. Consequently the metric associated to ω_1 is quasi-isometric near q to

$$\begin{aligned}&\left|\log\prod_{k\in I}|z_k|^2\right|^{-2}\left(\sum_{\alpha=1}^{n}\partial u_\alpha\bar\partial u_\alpha\right)\\ &\quad+\left|\log\prod_{k\in I}|z_k|^2\right|^{-1}\left(\sum_{i\in I}dz_i d\bar z_i + \sum_{j=1}^{n-|I|}dw_j d\bar w_j\right).\end{aligned} \tag{8.3}$$

Thus it only remains to show that $\sum_{\alpha=1}^{n}\partial u_\alpha\bar\partial u_\alpha$ may be replaced in (8.3), up to quasi-isometry, by $\sum_{i\in I}\frac{1}{|z_i|^2}dz_i d\bar z_i$. We compute from (8.2) that

$$\partial u_\alpha = -\sum_{k\in I}a_{\alpha k}\frac{dz_k}{z_k} + \sum_A O(1)dz_A + \sum_C O(1)dw_C$$

and thus

$$\begin{aligned}\sum_{\alpha=1}^{n}\partial u_\alpha\bar\partial u_\alpha = &\sum_{k,l\in I}\left(\sum_{\alpha=1}^{n}a_{\alpha k}a_{\alpha l}\right)\frac{dz_k d\bar z_l}{z_k\bar z_l}\\ &+\sum_{A,B}O(|z_A|+|z_B|)\frac{dz_A d\bar z_B}{z_A\bar z_B}\\ &+\sum_{A,C}O(|z_A|)\frac{dz_A}{z_A}dw_C + \sum_{C,D}O(1)dw_C dw_D.\end{aligned} \tag{8.4}$$

Lemma 8.2(i) implies that the first term of (8.4) is quasi-isometric to what we want, whereas the remaining cross-terms are quasi-isometrically negligible near q by the arguments used to treat (6.7).

Theorem 8.4. *Let V be a complex analytic space with isolated singularities satisfying:*

(8.5) *There exists a Kähler form ω_0 on $V \setminus \mathrm{Sing}(V)$ which is dominated near $\mathrm{Sing}(V)$ by the Kähler form of an ambient metric, for some embedding of a neighborhood of $\mathrm{Sing}(V)$ in a domain of $\mathbb{C}^N$.*

Then for any resolution $\pi : \widetilde{V} \to V$, with π a projective morphism and with exceptional set D a divisor with normal crossings, there exists a Kähler metric on $\widetilde{V} \setminus D \cong V \setminus \mathrm{Sing}(V)$ which is distinguished relative to π.

Proof. For $K > 0$ large, $\omega_1 + K\omega_0$ is the desired Kähler metric on $V \setminus \mathrm{Sing}\, V$. By Proposition 8.3, we merely have to verify that locally near D, $\pi^*\omega_0$ is dominated by (6.1). However, (8.5) implies that

$$\pi^*\omega_0 \lesssim i \sum_{i \in I} \left(\prod_{k \in I \setminus \{i\}} |z_k|^2 \right) dz_i \wedge d\bar{z}_i + i \left(\prod_{k \in I} |z_k|^2 \right) \sum_{j=1}^{n-|I|} dw_j \wedge d\bar{w}_j ,$$

for coordinates satisfying (6.2), which easily yields our estimate.

Remark 8.5. (i) Condition (8.5) is certainly satisfied if V is a subvariety of a domain in projective space; one simply takes for ω_0 the restriction of the ambient Fubini-Study Kähler form. Now recall that a function ϕ on V is called *plurisubharmonic* (resp. C^∞ *psh, strictly psh,* etc.) if locally V can be embedded in a domain of $\mathbb{C}^N$ such that ϕ is the restriction of an ambient plurisubharmonic (resp. C^∞ psh, strictly psh, etc.) function. Then a complex analytic space V is called *Kähler* if it admits a covering by open subsets (V_α) and a system of C^∞ strictly plurisubharmonic functions ϕ_α on V_α with $\phi_\alpha - \phi_\beta$ pluriharmonic on $V_\alpha \cap V_\beta$. Condition (8.5) is again satisfied if V is Kähler; let $\omega_0 = i\partial\bar{\partial}\phi_\alpha$ on $V_\alpha \setminus (V_\alpha \cap \mathrm{Sing}(V))$. By a regularization result of Varouchas [50], a complex space V is Kähler if there exists (V_α) and ϕ_α as above, with ϕ_α only continuously strictly plurisubharmonic, so the theorem holds in this case as well.

(ii) We note that it is clear from the proof of Theorem 8.4 that condition (8.5) can be weakened, however we do not have an example where this is necessary.

8.2. L_2-cohomology and intersection cohomology. We need the following "semipurity" result which may be derived (see [17], [19], or [48]) from the decomposition theorem for intersection cohomology (see [4] or [38]–[40]); an alternate proof is given by Navarro Aznar [33].

Theorem 8.6 ([17], [19], [33], [48]). *Let (V, x) be an n-dimensional contractible germ of a complex analytic isolated singularity. Then the* *weight filtration W of the mixed Hodge structure on $H^{\cdot}(V\backslash\{x\}\,;\mathbb{C})$ satisfies*

$$W_kH^k(V\backslash\{x\}\,;\mathbb{C})=\begin{cases}0 & \text{for } k\geq n,\\ H^k(V\backslash\{x\}\,;\mathbb{C}) & \text{for } k\leq n-1.\end{cases}\tag{8.6}$$

Together with Theorems 7.7 and 7.8, this implies

Theorem 8.7. *Let V be a complex analytic space with isolated singularities and let $\pi:\widetilde{V}\to V$ be a resolution with exceptional divisor D having smooth irreducible components with normal crossings. Then for $c\in\mathbb{R}$ and for any distinguished metric relative to π, we have*

$$H^k_{(2)}(U^c)=0 \qquad \text{for } k\geq n,$$
$$H^k_{(2)}(U^c\,,\partial U^c)=0 \qquad \text{for } k\leq n,$$

where $U^c\subset\widetilde{V}\backslash D$ $(c\in\mathbb{R})$ is the cofinal system of punctured neighborhoods of D defined in §5.

Proof. Note that the convention for the weight filtration W on $H^{\cdot}(U^c)$ is such that $\mathrm{Gr}^W_{k+p}H^k(U^c)=E_\infty^{-p,k+p}$ for the weight spectral sequence. Thus Theorem 8.6 may be interpreted as saying that the sequence (7.14) is exact at $E_1^{-p,q}$ for $q-p\geq n$, $p\leq 0$ or $q-p\leq n-1$, $p>0$. Together with Theorems 7.7 and 7.8, this shows that the E_2 terms of the W-spectral sequences for $H^{\cdot}_{(2)}(U^c)$ and $H^{\cdot}_{(2)}(U^c\,,\partial U^c)$ vanish for $q-p\geq n$ and $q-p\leq n$, respectively. q.e.d.

Now let $\mathscr{L}^{\cdot}_{(2)}$ be the complex of sheaves on the space V associated to the presheaf $U\mapsto\mathrm{Dom}(\overline{d}_{U\backslash(U\cap\mathrm{Sing}(V))})$; when V is compact, $\Gamma(V,\mathscr{L}^{\cdot}_{(2)})=\mathrm{Dom}(\overline{d}_{V\backslash\mathrm{Sing}(V)})$. On $V\backslash\mathrm{Sing}(V)$, an L_2 version of the usual Poincaré lemma shows that $\mathscr{L}^{\cdot}_{(2)}$ is a resolution of the constant sheaf $\mathbb{C}_{V\backslash\mathrm{Sing}(V)}$, whereas for $p\in\mathrm{Sing}(V)$, Theorem 8.7 implies

$$\varinjlim_{U\ni p}\mathbb{H}^k(U\,,\mathscr{L}^{\cdot}_{(2)})=0 \qquad \text{for } k\geq n\,,$$

and

$$\varprojlim_{U\ni p}\mathbb{H}^k_c(U\,,\mathscr{L}^{\cdot}_{(2)})=0 \qquad \text{for } k\leq n\,.$$

Thus by the local characterization of intersection cohomology in [22] or [8], $\mathscr{L}^{\cdot}_{(2)}$ is quasi-isomorphic to the intersection cohomology sheaf $\mathscr{I}\mathscr{C}^{\cdot}$, and hence the hypercohomology $\mathbb{H}^{\cdot}(V\,,\mathscr{L}^{\cdot}_{(2)})$ is naturally isomorphic to intersection cohomology. However, $\mathscr{L}^{\cdot}_{(2)}$ is actually a fine resolution since the singularities are isolated, so we may replace hypercohomology by cohomology of global sections. Consequently we have

Corollary 8.8. $H^{\bullet}(\Gamma(V, \mathscr{L}^{\bullet}_{(2)})) \cong IH^{\bullet}(V; \mathbb{C})$. *In particular, if* V *is compact, then* $H^{\bullet}_{(2)}(V \setminus \operatorname{Sing}(V)) \cong IH^{\bullet}(V; \mathbb{C})$.

Theorem 0.1 in the introduction follows from Theorem 8.4 and Corollary 8.8.

9. L_2-Hodge structures

Let Y be a hermitian manifold (possibly with corners) and let $L_2^k(Y) = \bigoplus_{p+q=k} L_2^{p,q}(Y)$ be the decomposition into (p, q) type. If $\operatorname{Range}(\overline{d})$ is closed, then by (1.2) any decomposition of $\operatorname{Ker}(\Delta)$ induces a decomposition of L_2-cohomology. In particular, if $\operatorname{Ker}(\Delta)$ decomposes into (p, q) parts, i.e.,

$$\operatorname{Ker}(\Delta^k) = \bigoplus_{p+q=k} \operatorname{Ker}(\Delta^k) \cap L_2^{p,q}(Y), \tag{9.1}$$

then the L_2-cohomology acquires a *Hodge structure*

$$H^k_{(2)}(Y) \cong \bigoplus_{p+q=k} H^{p,q}, \qquad \overline{H^{p,q}} = H^{q,p}, \tag{9.2}$$

by setting $H^{p,q} = \operatorname{Ker}(\Delta^k) \cap L_2^{p,q}(Y)$. (Properly speaking, this is a *real* Hodge structure since we have not defined a rational structure on $H^k_{(2)}(Y)$.)

Define the filtration

$$F^p L_2^k(Y) = \bigoplus_{r \geq p} L_2^{r,k-r}(Y).$$

We have induced filtrations on $\operatorname{Dom}(\overline{d})$, $\operatorname{Ker}(\Delta^k)$, and $H^k_{(2)}(Y)$ in the usual manner.

Lemma 9.1. *Assume* $\operatorname{Range}(\overline{d})$ *is closed and* (9.1) *holds. Then the filtrations* F^p *on* $H^k_{(2)}(Y)$ *and* $\operatorname{Ker}(\Delta^k)$ *agree under the isomorphism induced by harmonic projection.*

Proof. It suffices to show harmonic projection preserves F^p since the inverse isomorphism $\operatorname{Ker}(\Delta^k) \xrightarrow{\sim} H^k_{(2)}(Y)$ clearly does. But this is obvious from (9.1). q.e.d.

This filtration on cohomology is called the *Hodge filtration.*

Remark 9.2. A sufficient condition for (9.1) to hold is that Y be complete (without boundary) and Kähler. In this case it is well known [1] that $\operatorname{Ker}(\Delta) = \operatorname{Ker}(\square)$, where $\square = \overline{\partial}^*\overline{\partial} + \overline{\partial}\,\overline{\partial}^*$ is the Hilbert space $\overline{\partial}$-Laplacian defined analogously to Δ in §1.1. In fact, we have [55, §1] the stronger result that

$$\Delta = 2\Box \qquad \text{(strict operator equality)}, \tag{9.3}$$

and, if $\mathrm{Range}(\overline{d})$ is closed, that the spectral sequence of F degenerates at E_1; in other words, $(\mathrm{Dom}(\overline{d}), F)$ is a *real Hodge complex.*

Of course, we are mainly interested in the case where Y is a smooth Zariski open dense subset of V, a compact complex space, and where

$$IH^{\cdot}(V;\mathbb{C}) \cong H^{\cdot}_{(2)}(Y) \tag{9.4}$$

holds. Since $IH^{\cdot}(V;\mathbb{C})$ is finite dimensional in this case, $\mathrm{Range}(\overline{d})$ is closed (although finite dimensionality often follows directly from the methods used to verify (9.4)). If (9.1) holds in addition, intersection cohomology acquires an *L_2-Hodge structure.*

Example 9.3. If $Y = V\setminus \mathrm{Sing}\, V$ is a given metric which is quasi-isometric to a piecewise flat metric relative to a triangulation of V, then Cheeger [11] has shown that the L_2-cohomology $H^{\cdot}_{(2)}(V\setminus \mathrm{Sing}\, V)$ is a finite-dimensional combinatorial invariant of V, represented by the L_2-harmonic forms and satisfying Poincaré duality. This invariant, he shows, turns out to be intersection cohomology; thus we have (9.4). If the singularities of V are isolated complex cones and the metric is Kähler (although incomplete), he has verified (9.1) by showing [12] that the almost complex structure J preserves $\mathrm{Dom}(\Delta)$ and hence $\mathrm{Ker}(\Delta)$.

Example 9.4. Let V be any Kähler variety (or, more generally, a complex analytic space satisfying (8.5)) with isolated singularities and let $Y = V\setminus \mathrm{Sing}\, V$ be equipped with a complete Kähler metric which is distinguished on V. Such metrics exist by Theorem 8.4 and Proposition 6.3. The conditions (9.1) and (9.4) follow from Remark 9.2 and Corollary 8.8, respectively.

Example 9.5. Let X be a bounded symmetric domain, and Γ an arithmetic group of automorphisms acting freely on X. Let V be the Baily-Borel-Satake compactification of the locally symmetric variety $\Gamma\backslash X$, and let $Y = \Gamma\backslash X$ be equipped with the complete Kähler metric induced by the Bergman metric on X. (Equivalently, give Y a metric induced from a metric on X invariant under the automorphism group G.) Assertion (9.4) is Zucker's conjecture [52] (more generally, one admits a metrized coefficient system induced from a representation of G). This was proven independently by Saper and Stern [44], [45] and by Looijenga [29]. Previously, special cases with $\mathbb{Q}$-rank 1, 2, and 3 were verified by Zucker [52], [54], the general $\mathbb{Q}$-rank 1 case was settled by Borel [7], and the general

$\mathbb{Q}$-rank 2 case by Borel and Casselman [9]. Again, (9.1) is automatic since the metric is complete and Kähler.

Example 9.6. Let V be a projective variety, and let $Y = V \setminus \operatorname{Sing} V$ have the induced Fubini-Study metric. Then (9.4) and (9.1) were conjectured by Cheeger, Goresky, and MacPherson [13]. In [27], Hsiang and Pati (see also Nagase [32]) have verified (9.4) for V a projective surface with isolated singularities. However the Fubini-Study metric on $V \setminus \operatorname{Sing} V$ is not complete, and (9.1) has not yet been verified, so the existence of the L_2-Hodge structure is open. Recently Ohsawa has shown that (9.4) holds for all projective varieties V with isolated singularities; the proof reduces the problem to (9.4) for a complete metric, and then applies Theorem 0.1.

The L_2-Hodge structure depends a priori on the metric on Y. Thus in the situation of Example 9.4, one may conceivably obtain different L_2-Hodge structures on $IH^{\cdot}(V;\mathbb{C})$ associated to different resolutions of V. However this is not the case; Zucker has proven the following theorem:

Theorem 9.7 (Zucker [55]). *Let V be a compact complex analytic space having only isolated singularities and satisfying* (8.5); *give V any Kähler distinguished metric. Then the corresponding L_2-Hodge structure on $IH^{\cdot}(V;\mathbb{C})$ coincides with the canonical Hodge structure* (*see below*). *The same result holds if V is the Baily-Borel-Satake compactification of an arithmetic quotient of a ball or a Hilbert modular surface, and $V \setminus \operatorname{Sing} V$ has the metric induced from the Bergman metric.*

Here the *canonical Hodge structure* on $IH^{\cdot}(V;\mathbb{C})$ (for V having isolated singularities) is defined as follows. We have a canonical isomorphism [21] (complex coefficients throughout)

$$IH^k(V) \cong \begin{cases} H^k(V \setminus \operatorname{Sing} V) & \text{for } k < n, \\ \operatorname{Im}(H^n(V) \to H^n(V \setminus \operatorname{Sing} V)) & \text{for } k = n, \\ H^k(V) & \text{for } k > n. \end{cases} \tag{9.5}$$

The terms on the right-hand side possess mixed Hodge structures constructed by Deligne [15], [16], which are pure [19], [33], [48].

10. L_2-$\bar\partial$-cohomology and MacPherson's conjecture

We remain in the setting of §9. The *L_2-$\bar\partial$-cohomology* $H^{p,q}_{(2),\bar\partial}(Y)$ is defined analogously to ordinary L_2-cohomology. Namely, let $\bar\partial^{p,q}$ denote the operator closure of $\bar\partial$ acting on the domain $\{\phi \in A^{p+q}(Y) \cap L_2^{p,q}(Y) \mid \bar\partial\phi \in L_2^{p,q+1}(Y)\}$. Define $H^{p,q}_{(2),\bar\partial}(Y) = H^q(\operatorname{Dom}(\bar\partial^{p,\cdot}))$. As for ordinary

L_2-cohomology (1.2),

$$(10.1) \qquad H^{p,q}_{(2),\bar\partial} \cong \mathrm{Ker}(\square) \oplus (\overline{\mathrm{Range}(\bar\partial)}/\,\mathrm{Range}(\bar\partial)),$$

where the second factor is either infinite dimensional or 0 (the latter case occurring when $\mathrm{Range}(\bar\partial^{p,q-1})$ is closed).

Proposition 10.1. *If Y is a complete Kähler manifold without boundary and* $\mathrm{Range}(\bar d)$ *is closed, the Hodge structure on L_2-cohomology* (9.2) *can be represented as*

$$H^k_{(2)}(Y) \cong \bigoplus_{p+q=k} H^{p,q}_{(2),\bar\partial}(Y).$$

Proof. The closure of $\mathrm{Range}(\bar d)$ (resp. $\mathrm{Range}(\bar\partial)$) is equivalent to the closure of $\mathrm{Range}(\Delta)$ (resp. $\mathrm{Range}(\square)$). Thus by (9.3), $\mathrm{Range}(\bar d)$ closed implies $\mathrm{Range}(\bar\partial)$ closed and $\mathrm{Ker}(\square) = \mathrm{Ker}(\Delta)$. Thus

$$H^{p,q}_{(2),\bar\partial}(Y) \cong \mathrm{Ker}(\square^{p,q}) = \mathrm{Ker}(\Delta^k) \cap L_2^{p,q}(Y) = H^{p,q}. \qquad \text{q.e.d.}$$

If $H^{0,q}_{(2),\bar\partial}(Y)$ is finite dimensional for all q, define the *L_2-$\bar\partial$-index* to be

$$(10.2) \qquad \chi_{(2)}(Y) = \sum_{q=0}^{\dim Y} (-1)^q \dim H^{0,q}_{(2),\bar\partial}(Y).$$

Let V be a compact analytic variety and define the *arithmetic genus* to be

$$\chi(V) = \sum_{q=0}^{\dim V} (-1)^q \dim H^q(V, \mathcal{O}_V).$$

MacPherson [30] has conjectured that the L_2-$\bar\partial$-index of $V\setminus \mathrm{Sing}\, V$ equals the arithmetic genus of any resolution $\widetilde V$ of V, in the contexts:

(I) Where $V\setminus \mathrm{Sing}\, V$ has a Kähler metric induced from an embedding of V in a Kähler manifold (e.g., V a projective variety with the Fubini-Study metric on $V\setminus \mathrm{Sing}\, V$).

(II) Where V is the Baily-Borel-Satake compactification of a locally symmetric variety with the Bergman metric.

In other words, the conjecture states that $\chi(V)$, which is a birational invariant for smooth V, may be replaced by $\chi_{(2)}(V\setminus \mathrm{Sing}\, V)$ to yield a birational invariant for possibly singular V. For the context we considered in §8 (which was motivated by (II)), the following theorem proves the conjecture; in fact, there is even equality term by term.

Theorem 10.2. *Let V be a compact complex analytic space with only isolated singularities and satisfying* (8.5); *consider any Kähler distinguished* *metric on* V. *Then*

$$H^{0,q}_{(2),\bar\partial}(V\setminus \operatorname{Sing} V) \cong H^q(\widetilde V, \mathscr{O}_{\widetilde V}),$$

where $\widetilde V$ *is any resolution of* V *and* $0 \le q \le \dim V$. *The same result holds for the cases of context* (II) *mentioned in Theorem* 9.7.

Proof. Since $H^q(\widetilde V, \mathscr{O}_{\widetilde V})$ is a birational invariant for smooth varieties, we can assume $\widetilde V \to V$ has an exceptional divisor D with smooth components intersecting in normal crossings. Then from the weight spectral sequence for $V\setminus\operatorname{Sing} V = \widetilde V\setminus D$ as in [15] or [23], we have an isomorphism of Hodge structures

$$\begin{aligned} \operatorname{Gr}^W_k H^k(V\setminus \operatorname{Sing} V) &\cong \operatorname{Im}(H^k(\widetilde V) \to H^k(\widetilde V\setminus D)) \\ &\cong \operatorname{Coker}(H^{k-2}(D^1) \xrightarrow{G} H^k(\widetilde V)), \end{aligned} \tag{10.3}$$

where D^1 is the disjoint union of the components of D, and G is an alternating sum of Gysin morphisms. We will also need

Lemma 10.3. *For* $k \le n$, *there is an isomorphism of Hodge structures* $IH^k(V) \cong \operatorname{Gr}^W_k H^k(V\setminus\operatorname{Sing} V)$, *where* $IH^q(V)$ *is given the canonical Hodge structure* (*see* (9.5)).

Proof. For $k < n$, this is clear from (9.5) since $H^k(V\setminus\operatorname{Sing} V)$ is pure [19], [33], [48, §1]. For $k = n$, we simply need to show

$$\operatorname{Im}(H^n(V) \to H^n(V\setminus\operatorname{Sing} V)) = \operatorname{Im}(H^n(\widetilde V) \to H^n(\widetilde V\setminus D)). \tag{10.4}$$

Let $\widetilde U$ be a regular neighborhood of D, and U its image in V. Then $H^n(\widetilde U) \to H^n(\widetilde U\setminus D)$ is the zero map [48, (1.11)]. (10.4) and hence the lemma now follow from the Mayer-Vietoris diagram:

$$\begin{array}{ccccccc} H^n(V) & \to & H^n(V\setminus\operatorname{Sing} V) & \oplus & H^n(U) & \to & H^n(U\setminus\operatorname{Sing} U) \\ \downarrow & & \| & & \downarrow & & \| \\ H^n(\widetilde V) & \to & H^n(\widetilde V\setminus D) & \oplus & H^n(\widetilde U) & \to & H^n(\widetilde U\setminus D). \end{array}$$ q.e.d.

We now apply Proposition 10.1, Theorem 9.7, Lemma 10.3, and (10.3), respectively, to see that

$$\begin{aligned} &H^{0,q}_{(2),\bar\partial}(V\setminus\operatorname{Sing} V) \\ &\quad\cong (0, q) \text{ part of the } L_2\text{-Hodge structure on } IH^q(V) \\ &\quad\cong (0, q) \text{ part of the canonical Hodge structure on } IH^q(V) \\ &\quad\cong (0, q) \text{ part of } \operatorname{Coker}(H^q(D^1) \xrightarrow{G} H^q(\widetilde V)). \end{aligned}$$

But the Gysin map is a morphism of Hodge structures of type $(1, 1)$ [23], that is, it maps type (p, q) into type $(p+1, q+1)$, and thus in particular its image lies in $\bigoplus_{r\geq 1} H_{\bar\partial}^{r,q-r}(\widetilde{V})$. Hence $H_{(2),\bar\partial}^{0,q}(V\setminus \operatorname{Sing} V) \cong H_{\bar\partial}^{0,q}(\widetilde{V}) \cong H^q(\widetilde{V}, \mathcal{O}_{\widetilde{V}})$ as desired.

Corollary 10.4. *In the situation of the theorem,* $\chi_{(2)}(V\setminus \operatorname{Sing} V) = \chi(\widetilde{V})$. This proves Theorem 0.3 of the introduction.

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