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NEW APPLICATIONS OF MAPPING DEGREES TO MINIMAL SURFACE THEORY

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In [21], Tomi and Tromba showed how it was possible to use the degree theory of Smale [19] to solve the long open problem of proving that every smooth embedded curve in the boundary of a convex subset of $\mathbf{R}^3$ must bound an embedded minimal disk. Later Almgren and Simon [4] and Meeks and Yau [15] gave different proofs. In this paper we give other applications of degree theory to minimal surfaces. In particular, we show:

(1) If Φ is an even constant coefficient parametric elliptic functional in $\mathbf{R}^3$ and γ is a smooth embedded curve on the boundary of a strictly convex subset of $\mathbf{R}^3$, then γ bounds an embedded Φ-stationary and Φ-stable disk. Furthermore, a generic such curve bounds an odd number of embedded Φ-stationary disks and an even number of embedded Φ-stationary surfaces of each other topological type.

(2) Let N be a smooth Riemannian 3-manifold with strictly mean convex boundary diffeomorphic to the 2-sphere. Suppose either that N is not diffeomorphic to the 3-ball, or else that N contains a compact minimal surface without boundary. Then there exists a sequence D_i of embedded minimal disks in N such that $\partial D_i \subset \partial N$, ∂D_i converges to a smooth embedded curve γ, and the area of D_i tends to infinity.

(3) There exists a complete minimal hypersurface M in $\mathbf{R}^n$ such that M is singular, M is not a cone, and M is asymptotic at ∞ to an area minimizing cone C that is regular except at the origin.

(4) There exists a complete area minimizing hypersurface M in $\mathbf{R}^n$ such that M is asymptotic to an area minimizing cone C that is regular except at the origin, but M is not congruent to any leaf of the foliation of minimal hypersurfaces associated with C.

These results are proved in §§2, 3, 4, and 5, respectively. All depend on the preliminaries in §1, and §5 is a continuation of §4, but otherwise the sections are independent of each other. §6 discusses examples.

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Concerning (1), in 1961 Morrey [16] proved the existence of H^1 maps of disks that minimize such functionals Φ subject to prescribed boundary values. But still no regularity is known for such maps. On the other hand, there also exist surfaces that minimize Φ among all surfaces, of arbitrary topological type, having a prescribed boundary; such surfaces are known to be smooth away from the boundary [3].

Statement (2) shows that the method of proving (1), which requires an a priori bound on area, breaks down in a serious way in manifolds.

Statement (3) partially answers the question, raised by R. Hardt [6,1.6], of whether there exists a complete area minimizing hypersurface which is singular but not a cone. Note that such a hypersurface cannot be constructed by perturbing a cone by a small vectorfield since by monotonicity the tangent cone at infinity must be different from the tangent cone at the singularity.

Statements (3) and (4) also show (see §6) that the recent classification, due to Simon and Solomon [18], of complete minimal hypersurfaces asymptotic at infinity to quadratic cones fails for all other known cones.

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1. Preliminaries

In this section we summarize those results of [24] which we will use. Analogous results were proved earlier for the special case of two dimensional minimal disks in $\mathbf{R}^n$; cf. [5], [22], [21]. Let M be a compact connected m-dimensional Riemannian manifold with nonempty boundary, and let N be an $(m+1)$-dimensional Riemannian manifold with strictly mean-convex boundary. We regard two maps $f, g: M \to N$ as being equivalent if $f = g \circ u$ for some diffeomorphism $u: M \to M$ such that $u(x) = x$ for $x \in \partial M$. Let $[f]$ denote the equivalence class of f.

Theorem A [24, 3.3]. *Let $\mathscr{M} = \{[f]: f \in C^{2,\alpha}(M,N)$ is a minimal immersion with $f(\partial M) \subset \partial N\}$. Then $\mathscr{M}$ is a smooth Banach manifold and*

$$\Pi: \mathscr{M} \to C^{2,\alpha}(\partial M, \partial N)$$
$$\Pi([f]) = f \mid \partial M$$

is a smooth Fredholm map of Fredholm index 0.

In some situations it is possible to assign a mapping degree to Π:

Theorem B [24, 5]. *Let $\mathscr{M}'$ and W be open subsets of $\mathscr{M}$ and $C^{2,\alpha}(\partial M, \partial N)$, respectively, such that W is connected and $\Pi: \mathscr{M}' \to W$ is* *proper. Then there is an integer* $d = d(\mathscr{M}', W)$ *such that for generic* $\gamma \in W$,

$$\sum_{[f]\in\Pi^{-1}(\gamma)\cap\mathscr{M}'} (-1)^{\mathrm{index}[f]} = d.$$

In particular, this holds for each $\gamma \in W$ *such that every* $[f] \in \Pi^{-1}(\gamma) \cap \mathscr{M}'$ *has no nontrivial Jacobi fields that vanish on* ∂M. *Furthermore, if* $d \neq 0$, *then* $\Pi^{-1}(\gamma) \cap \mathscr{M}'$ *is nonempty for every* $\gamma \in W$.

Corollary 1. *For generic* $\gamma \in W$, *the number of elements of* $\Pi^{-1}(\gamma)\cap\mathscr{M}'$ *is congruent to* d *modulo* 2.

Corollary 2. *For generic* $\gamma \in W$, *the number of stable surfaces in* $\Pi^{-1}(\gamma) \cap \mathscr{M}'$ *is less than or equal to* d *plus the number of unstable surfaces.*

Whereas Theorem A is quite general. Theorem B is severely restricted by the hypothesis of properness. The following gives a useful criterion for properness.

Theorem C. *Let* $\mathscr{M}'$ *and* W *be open subsets of* $\mathscr{M}$ *and* $C^{2,\alpha}(\partial M, \partial N)$, *respectively, with* $\Pi(\mathscr{M}') \subset W$. *Then* $\Pi: \mathscr{M}' \to W$ *is proper if the following hold*:

(1) $\mathscr{M}'$ *is a closed subset of* $\Pi^{-1}(W)$.

(2) *If* $K \subset W$ *is compact and* $[f] \in \Pi^{-1}(K) \cap \mathscr{M}'$, *then the area and the curvatures of* $f(M)$ *are bounded above by a constant depending on* K.

Proof. Let $[f_i] \in \Pi^{-1}(K) \cap \mathscr{M}'$. Then by (2), it is fairly easy to show (cf. [23, 3]) that a subsequence $[f_{i(j)}]$ must converge to some regular minimal surface $[f]$. But by (1). $[f] \in \mathscr{M}'$. q.e.d.

In the applications to follow, we use Theorems B and C as follows. First we choose $\mathscr{M}'$ and W so that (1) of Theorem C holds. Then we either prove (2) and conclude that d exists, or else show that d does not exist and conclude that (2) is false.

2. Embedded stationary surfaces in $\mathbf{R}^3$

Let $\Phi: \partial B^3 \to (0, +\infty)$ be a smooth function. Then Φ defines a functional on C^1 surfaces in $\mathbf{R}^3$ by

$$\Phi(S) = \int_{x\in S} \Phi(\vec{n}(x))\, dx,$$

where $\vec{n}(x)$ is a unit normal to S at x, and the integration is with respect to surface area on S. We shall assume that Φ is elliptic, i.e., that for some $\lambda > 0$, the function

$$x \mapsto |x|(\Phi(x/|x|) - \lambda)$$

is a convex function of $x \in \mathbf{R}^3$. We shall also assume that Φ is even, i.e., that $\Phi(v) \equiv \Phi(-v)$. Then the theorems of §1 remain true if we replace "minimal" by "Φ-stationary". (If $\Phi(v) \equiv |v|$, then Φ defines the area functional, and S is Φ-stationary if and only if S is minimal.)

2.1 Theorem. *Let N be a compact subset of $\mathbf{R}^3$ with smooth, strictly convex boundary, M a surface such that ∂M is connected, and $\mathscr{M}$ the Banach manifold of Theorem A corresponding to Φ. Let W be the set of $C^{2,\alpha}$ embeddings of ∂M into ∂N and $\mathscr{M}' = \{[f] \in \Pi^{-1}(W) : f \text{ is an embedding}\}$. Then $\Pi : \mathscr{M}' \to C^{2,\alpha}(\partial M, \partial N)$ is proper and*

$$d = \begin{cases} 1 & \text{if } M \text{ is a disk,} \\ 0 & \text{if not.} \end{cases}$$

Corollaries. (1) *Every $\gamma \in W$ bounds an embedded Φ-stationary disk.*

(2) *A generic $\gamma \in W$ bounds an odd number of embedded Φ-stationary disks.*

(3) *If $g \neq 0$, then a generic $\gamma \in W$ bounds an even number of embedded Φ-stationary surfaces of genus g.*

Proof. Since Φ is even, the strong maximum principle implies that if $[f]$ is a limit of Φ-stationary embeddings such that $f \mid \partial M$ is an embedding, then $f(M)$ is embedded. Thus $\mathscr{M}'$ is closed in $\Pi^{-1}(W)$.

One can show with the first variation formula, applied to radial deformations, that the area of $f(M)$ is bounded in terms of the length of $f \mid \partial M$. Also, the principal curvatures of $f(M)$ are bounded in terms of $f \mid \partial M$, the area of $f(M)$, and the genus of M. (See [23] for a precise statement and proof.) Thus by Theorems B and C, $\Pi : \mathscr{M}' \to W$ is proper and has a degree d.

Now let $\gamma(\partial M)$ be the intersection of ∂N with a plane P. Then by the maximum principle, applied to the planes parallel to P, the only Φ-stationary surface bounded by γ is $P \cap N$. Since $P \cap N$ is strictly stable (and therefore has no nontrivial Jacobi fields which vanish on ∂M), this means that $d = 1$ if M is a disk and $d = 0$ if not.

2.2. Theorem. *Let N and Φ be as in Theorem 2.1. If γ_0 is a smooth embedded curve in ∂N, then γ_0 bounds an embedded Φ-stable disk.*

Remark. The author discovered this theorem by a different method. The proof here is a modification of a proof for the area functional shown by Bill Meeks. This argument was discovered independently by F. H. Lin [13].

Proof. Let γ_0 be a smooth embedded curve in ∂N. Let S be the union of all Φ-stationary surfaces bounded by γ_0, Ω one of the two components of $\partial N \sim (\gamma_0)$, and M the unit 2-disk. Then the set of $C^{2,\alpha}$ embeddings of ∂M into Ω has two connected components; let W be one of them. Let

$$\mathscr{M}' = \{[f] \in \Pi^{-1}(W) : f \text{ is an embedding and } f(M) \cap S = \varnothing\}.$$

Then $\mathscr{M}'$ is an open and closed subset of $\mathscr{M}$ by the maximum principle [29], and $\Pi: \mathscr{M}' \to W$ is proper as in Theorem 2.1. Thus $\Pi: \mathscr{M}' \to W$ has a degree d, and, as in Theorem 2.1, $d = 1$.

Now let $\gamma_i \in W$ be a sequence of generic curves such that $||\gamma_i - \gamma_0||_{2,\alpha} \to 0$. Then each γ_i bounds an embedded Φ-stationary disk D_i with $D_i \cap S = \varnothing$. By [23, 3], a sequence of D_i's converges to an embedded Φ-stationary disk D with $\partial D = \gamma_0$. Note that D lies on one side of S. We claim that D is "one-sided Φ-minimizing", i.e., that if V is a surface of any genus with $\partial V = \gamma_0$ and $\operatorname{int} V \subset$ the component of $N \sim D$ containing Ω, then $\Phi(V) \geq \Phi(D)$. For if not, then there is a surface (integral current) V which minimizes Φ subject to those conditions. Since $\Phi(V) < \Phi(D)$, $V \neq D$. In particular, D is between V and S. But by definition of S, $V \subset S$, a contradiction. Finally, note that the one-sided minimizing property implies stability.

Corollary. *There exist embedded Φ-stable disks D and D' such that $\partial D = \partial D' = \gamma_0$ and such that every Φ-stationary surface embedded or immersed and of any topological type lies between D and D'. In particular, if γ_0 bounds more than one Φ-stationary surface, then $D \neq D'$.*

2.3. Theorem. *Let M be a compact connected surface with more than one boundary component. Let $\mathscr{M}'$ and W be as in Theorem 2.1. Then $\Pi: \mathscr{M}' \to W$ is proper and the degree $d = 0$.*

Proof. Properness is exactly as before. To see that $d = 0$, for simplicity suppose that ∂M has exactly two boundary components, Γ_1 and Γ_2. Let $\gamma_i \in W$ be a sequence such that

$$(*) \qquad \begin{gathered} \operatorname{length} \gamma_i(\partial M) \to 0, \\ \gamma_i(\Gamma_1) \to p_1, \quad \gamma_i(\Gamma_2) \to p_2, \qquad p_1 \neq p_2. \end{gathered}$$

We claim that for sufficiently large i, $\Pi^{-1}(\gamma_i) \cap \mathscr{M}'$ is empty. For suppose $S_i \in \Pi^{-1}(\gamma_i) \cap \mathscr{M}'$. Since S_i is connected, there is a point x_i in S_i with

$$\operatorname{dist}(x_i, \partial S_i) \geq r = \tfrac{1}{3}|p_1 - p_2|.$$

By [22, Theorem 3], a subsequence of the S_i converges to a surface S with isolated singularities. But by $(*)$, $\operatorname{area}(S_i) \to 0$ as $i \to \infty$. The contradiction shows that for large i, S_i does not exist. Thus $d = 0$.

3. Disks of arbitrarily large area

Theorem. *Let N be a compact connected smooth Riemannian manifold whose boundary is strictly mean convex and diffeomorphic to the two-sphere S^2. Suppose that*

(1) *N is not diffeomorphic to the 3-ball B^3, or that*

(2) *N contains a compact minimal surface Σ without boundary.*

Then there exists a sequence of embedded minimal disks D_i in N such that

(3) $\partial D_i \subset \partial N$,

(4) *∂D_i converges to a smooth embedded curve Γ,*

(5) $\operatorname{area}(D_i) \to \infty$.

Proof. Case (1) reduces to case (2) as follows. If N is not diffeomorphic to B^3, then we can minimize area in the class of all embedded spheres in N that do not bound balls in N. The result is a compact minimal sphere $\Sigma \subset N$ [14].

Thus for simplicity let N be the unit ball in $\mathbf{R}^3$ equipped with a smooth Riemannian metric such that ∂N is strictly mean-convex and such that (2) holds. Now apply Theorems A and B with M = the unit 2-disk. Let W be the space of $C^{2,\alpha}$ embeddings of ∂M into ∂N, and let $\mathscr{M}'$ be the set of $[f] \in \Pi^{-1}(W)$ such that f is an embedding and such that for some open subset Ω of N:

$$\Omega \cap \Sigma = \varnothing,$$
$$\partial\Omega \cap \operatorname{int}(N) = f(\operatorname{int} M),$$
$$f(M) \text{ has the orientation induced by } \Omega.$$

By the maximum principle, $\mathscr{M}'$ is open and closed in $\Pi^{-1}(W)$.

For $-1 < t < 1$, let

$$\Gamma_t = \partial B^3 \cap \{(x, y, z) \colon z = t\}.$$

Now we claim (see below) that for t sufficiently near -1, Γ_t bounds a unique minimal surface S_t. This S_t is a strictly stable embedded disk which is or is not in $\mathscr{M}'$ according to which way we orient Γ_t. It follows that d, if it existed, would have to be both 1 and 0. Thus d does not exist, and Π is not proper by Theorem B. By Theorem C, this means there exists a sequence of embedded minimal disks D_i satisfying (3) and (4) and such that the area and/or the principal curvatures of the D_i tend to infinity. But the curvatures of such a disk D are bounded in terms of ∂D and the area of D [23]. Thus in fact the area of D_i must go to infinity.

To establish the claim, note that there is a neighborhood $U \subset N$ of $(0, 0, -1)$ that is foliated by strictly stable embedded minimal disks S_t with $\partial S_t = \Gamma_t$. (This is proved by the implicit function theorem as in, for example, the appendix to [23].) Now let R_i be a sequence of minimal surfaces in N with $\partial R_i = \Gamma_{t(i)}$, where $t(i) \to -1$ as $i \to \infty$. Let T_i be a disk that minimizes area subject to

$$\partial T_i = \Gamma_{t(i)},$$
$$T_i \subset R_i \cup \{\text{the component of } N \sim R_i \text{ not containing } (0,0,-1)\}.$$

Then T_i is an embedded minimal disk [15]. Clearly $\operatorname{area}(T_i) \leq \operatorname{area}(\partial N)$. Unless

(6) $\operatorname{area}(T_i) \to 0$,

thus by [23, 3(3)] the T_i (or a subsequence) would converge to a minimal surface T with $\partial T = (0,0,-1)$, and T is regular away from $(0,0,-1)$. But then (by [23, 2], for example) T would be regular everywhere, contradicting the maximum principle since ∂N is mean-convex and $\partial N \cap T$ is nonempty. Hence (6) holds. By the lower density bound for minimal surfaces, this means that for large i, T_i must be near $\partial T_i = \Gamma_{t(i)}$ and therefore in U. Thus R_i is also in U. But then by the maximum principle applied to R_i and the leaves S_t, $R_i = S_{t(i)}$. This completes the proof.

4. Complete minimal hypersurfaces.

Throughout this section and the next, B_r will denote the ball of radius r in $\mathbf{R}^{m+1}$ centered at the origin, and Σ will be regular $(m-1)$-dimensional minimal submanifold of the unit m-sphere ∂B_1 such that the cone $C = \{rx: x \in \Sigma, r \geq 0\}$ is area minimizing. Let Ω^+ and Ω^- be the two connected components of $\partial B_1 \sim \Sigma$, and for $x \in \Sigma$ let $n(x)$ be the unit vector that is normal to Σ, tangent to ∂B_1, and that points away from Ω^-. If $u: \Sigma \to \mathbf{R}$, we define $\tilde{u}: \Sigma \to \partial B_1$ by

$$\tilde{u}(x) = (x + u(x)n(x))/(1 + u(x)^2)^{1/2}.$$

We let $\|u\|_0$, $\|u\|_{k,\alpha}$, and $|u|$ denote the C^0, $C^{k,\alpha}$, and $\mathscr{L}^2$ norms, respectively, of u.

According to [10], $\mathbf{R}^{m+1}$ has a foliation $\mathscr{F}$ of area minimizing hypersurfaces, one of the leaves of $\mathscr{F}$ is C and the other leaves of $\mathscr{F}$ are all regular. If L_t is a leaf near C, then $L_t \cap \partial B_1 = \tilde{l}_t(\Sigma)$ for some $l_t: \Sigma \to \mathbf{R}$. Here L_t denotes the leaf such that

$$t = \begin{cases} \inf l_t & \text{if } t > 0, \\ \sup l_t & \text{if } t < 0. \end{cases}$$

In particular, $L_0 = C$.

We begin with an easy application of Theorem B that is interesting in its own right.

4.1. Theorem. *There is a $\delta = \delta(\Sigma) > 0$ such that if*

(1) *C is area minimizing,*

(2) *there is no isotopy in B_1 from $\overline{\Omega}^+$ to $\overline{\Omega}^-$ leaving Σ fixed,*

(3) *u_t $(0 \le t \le 1)$ is a path in $C^{2,\alpha}(\Sigma)$ such that for some $\varepsilon < \delta$, $u_0 = l_\varepsilon$, $u_1 = l_{-\varepsilon}$, and $||u_t||_{2,\alpha} < \delta$ for all t,*

then there is a $t \in (0,1)$ such that $\tilde{u}_t(\Sigma)$ bounds a singular minimal surface.

Remark. Note that if (2) does not hold, then ∂B_1 is topologically the double of $\overline{\Omega}^+$. This implies (by, for instance, the Meier-Vietoris sequence for $\partial B_1 = \overline{\Omega}^+ \cup \overline{\Omega}^-$) that Σ is a homology sphere. In other words, (2) holds unless Σ is a homology sphere.

Proof. Suppose not. Then there are a constant c and a neighborhood W of $\{u_t: 0 \le t \le 1\}$ in $C^{2,\alpha}(\Sigma)$ such that if $u \in W$ and T is a regular minimal surface with $\partial T = \tilde{u}(\Sigma)$, then the curvatures of T are bounded by c. (For if not, there would be a $t \in [0,1]$ and a sequence T_i of regular minimal surfaces such that

$$(*)\qquad \begin{gathered} \partial T_i \to \tilde{u}_t(\Sigma) \quad \text{in } C^{2,\alpha}, \\ \max_{x \in T_i}(\text{curvature of } T_i \text{ at } x) \to \infty. \end{gathered}$$

But a subsequence of the T_i would converge to some minimal surface T with $\partial T = \tilde{u}_t(\Sigma)$. By hypothesis, T is regular. But that contradicts Allard's regularity theorem [1], [2].)

Now let M be the closure of Ω^+, N the unit ball B_1, and $\mathscr{M}$ the Banach manifold of minimal surfaces given by Theorem A. Let $\mathscr{M}'$ be the connected component of $\Pi^{-1}(\tilde{W})$ that contains $L_\epsilon \cap B_1$, where $\tilde{W} = \{\tilde{u}: u \in W\}$. Then $\mathscr{M}'$ does not contain $L_{-\varepsilon} \cap B_1$, since any path in $\mathscr{M}'$ from $L_\epsilon \cap B_1$ to $L_{-\varepsilon} \cap B_1$ would be an isotopy violating (2). (By [10, 2.1], $(x,t) \to (1-t)x + t(x/|x|)$ $(0 \le t \le 1)$ defines isotopies from L_ϵ to $L_{-\varepsilon}$ to Ω^+ and Ω^-, respectively.)

The areas of surfaces in $\mathscr{M}'$ are bounded by the isoperimetric inequality, and we have already mentioned that their curvatures are bounded. Thus $\Pi: \mathscr{M}' \to \tilde{W}$ is proper and has a degree d. Now by the maximum principle applied to the leaves of $\mathscr{F}$, $L_\epsilon \cap B_1$ is the only minimal surface bounded by $\tilde{l}_\epsilon(\Sigma)$. Also, it is strictly stable since L_ϵ is minimizing and therefore stable. Thus $d = 1$. Likewise $L_{-\varepsilon} \cap B_1$ is the only minimal surface bounded by $\tilde{l}_{-\varepsilon}(\Sigma)$. Since $L_{-\varepsilon} \cap B_1 \notin \mathscr{M}'$, $d = 0$, a contradiction. q.e.d.

If $f: [a,b] \times \Sigma \to \mathbf{R}$, then f determines a surface $S(f) = S(f; a, b)$ by

$$S(f) = \{r(x + f(r,x)n(x))/(1 + f(r,x)^2)^{1/2}: r \in [a,b], x \in \Sigma\}.$$

Thus $S(f)$ is minimal if and only if f satisfies the appropriate Euler-Lagrange equation which we will call the "minimal surface equation". This equation is a divergence-form quasilinear elliptic equation whose linearization at 0 is

$$Ju(r,x) = r\left(\frac{\partial}{\partial r}\right)^2 u(r,x) + (m+1)\frac{\partial}{\partial r}u(r,x) + \frac{1}{r}J_\Sigma u(r,x),$$

where J_Σ is the Jacobi or second variation operator on $\Sigma \subset \partial B_1$:

$$J_\Sigma u(x) = \big(\Delta + |A(x)|^2 + (m-1)\big)u(x),$$

and $A(x)$ is the second fundamental form of Σ (as a submanifold of ∂B_1) at x. Let $\lambda_1 < \lambda_2 < \cdots$ be the eigenvalues of J_Σ, and V_1, $V_2, \cdots$ be the corresponding eigenspaces:

$$J_\Sigma u = -\lambda_1 u \quad \text{if and only if } u \in V_i.$$

4.2. Lemma. *If u is a solution of $Ju = 0$ on $(a,b) \times \Sigma$, then u has the form*

$$u(r,x) = \sum_{|i|>0} a_i \varphi_i(x) r^{(\delta(i)-m)/2},$$

where $\delta(i) = (i/|i|)(m^2 + 4\lambda_{|i|})^{1/2}$ and $\varphi_i \in V_{|i|}$. If u is a positive solution on $(0,b) \times \Sigma$, then $a_i = 0$ for $i < -1$. If u is a positive solution on $(0,\infty) \times \Sigma$, then $a_i = 0$ unless $|i| = 1$.

Remark. If for some i, $m^2 + 4\lambda_i = 0$, then $\delta(i) = 0$ and the term $a_{-i}\varphi_{-i}(x)r^{-m/2}$ in the above formula should be replaced by

$$a_{-i}\varphi_{-i}(x) r^{-m/2} \log(r).$$

The presence of the $\log(r)$ factor does not affect any of the arguments in this section.

Proof. The first statement is proved by separation of variables. To prove the second, note that by the Harnack inequality on $(r/2, 2r) \times \Sigma$,

$$\inf u(r,\cdot) \geq c_1 \sup u(r,\cdot) \geq c_2 ||u(r,\cdot)||_2,$$

(where c_1 and c_2 do not depend on r) and thus

$$\varphi_1(x) u(r,x) \geq c_2 \varphi_1(x) ||u(r,\cdot)||_2 \geq c_3 ||u(r,\cdot)||_2,$$

since $\varphi_1 > 0$. Integration over Σ gives

$$\sum_{|i|=1} a_i r^{(\delta(i)-m)/2} \geq c_4 \left(\sum_{|i|>0} a_i^2 r^{\delta(i)-m} \right)^{1/2},$$

or (after division by $r^{(\delta(-1)-m)/2}$)

$$(a_{-1} + a_1) r^{(\delta(1)-\delta(-1))/2} \geq c_5 \left(\sum_{|i|>0} a_i^2 r^{\delta(i)-\delta(-1)} \right)^{1/2}.$$

Letting $r \to 0$ shows that $a_i = 0$ for $i < -1$ by noting that $\delta(i) < \delta(j)$ for $i < j$. Similarly, letting $r \to \infty$, in case u is defined on $(0,\infty) \times \Sigma$, shows that $a_i = 0$ for $i > 1$.

4.3. Lemma. *Unless Σ is a totally geodesic $(m-1)$-sphere, $\lambda_i = 1-m$ and $(\delta(i)-m)/2 = -1$ for some $i \geq 2$, and $\lambda_j = (\delta(j)-m)/2 = 0$ for some $j \geq 3$.*

Proof. If $v \in \mathbf{R}^{m+1}$ and $v \neq 0$, then $x \mapsto v \cdot n(x)$ is an eigenfunction with eigenvalue $\lambda_i = 1-m$. Unless Σ is totally geodesic, this function changes sign, so $i \geq 2$. If P is an antisymmetric $(m+1)\times(m+1)$ matrix and $P \neq 0$, then $x \mapsto Px \cdot n(x)$ is an eigenfunction with eigenvalue $\lambda_j = 0$. Since $0 > 1-m$, $j > i \geq 2$, so $j \geq 3$.

4.4. Proposition. *For every $\delta \in (0, 1/4)$, there is a $\theta > 0$ such that if T is a (possibly singular) compact minimal surface with $\partial T = \tilde{u}(\Sigma)$, $||u||_{2,\alpha} \leq \theta$, then*

(1) *there is a function $f\colon [\delta, 1]\times\Sigma \to \mathbf{R}$ such that $f(1,x) = u(x)$, $||f||_{2,\alpha} < \delta$, and $T \sim B_\delta = S(f;\delta,1)$.*

Furthermore

(2) $||f|[3\delta, 1-\delta]\times\Sigma||_{3,\alpha} \leq C_\delta ||f(1,\cdot)||_0$,

(3) $||f|[3\delta, 1]\times\Sigma||_{2,\alpha} \leq C_\delta ||f(1,\cdot)||_{2,\alpha}$.

Proof. Suppose (1) is false. Then for every n there exist T_n and u_n satisfying the hypotheses with $\theta = 1/n$ but not satisfying (1). By the maximum principle, T_n lies between $L_{1/n}$ and $L_{-1/n}$. Hence as $n \to \infty$, $T_n \to C \cap B_1$. Also, the area of T_n is not greater than the area of the cone over $\tilde{u}_n(\Sigma)$, so $\text{area}(T_n) \to \text{area}(C \cap B_1)$. It follows from Allard's regularity theorem [1], [2] that (1) holds.

Now let l_t be the function f of (1) corresponding to $T = L_t \cap B_1$. Note in fact that l_t extends to a solution of the minimal surface equation on $[\delta,\infty)\times\Sigma$ so that

$$L_t \sim \partial B_\delta = S(l_t;\delta,\infty).$$

To prove (2) and (3), let $s = ||u||_0$. Then $\tilde{u}(\Sigma)$ and therefore, by the maximum principle, T lie between L_{-s} and L_s. Hence $l_{-s} \leq f \leq l_s$. Now l_s is a positive solution to the minimal surface equation on $[\delta,\infty)\times\Sigma$, so by the Harnack inequality

$$\sup(l_s|[2\delta,1]\times\Sigma) \leq C_\delta \inf(l_s|[2\delta,1]\times\Sigma) \leq C_\delta \inf l_s(1,\cdot) \leq C_\delta ||f(1,\cdot)||_0,$$

and likewise for l_{-s}. Thus

$$||f|[2\delta,1]\times\Sigma||_0 \leq C_\delta ||f(1,\cdot)||_0.$$

Now since f is a solution of the minimal surface equation, (2) and (3) follow from standard elliptic regularity, by using the fact [9,10.4] that $f = f - 0$ satisfies a homogeneous linear elliptic equation.

4.5. Proposition. *Let π_k and π'_k be the orthogonal projections of $\mathscr{L}^2(\Sigma)$ onto $V_1 + \cdots + V_k$ and $(V_1+\cdots+V_k)^\perp$, respectively. Suppose $(\delta(2)-m)/2 < p < q < (\delta(3)-m)/2$. Then there exist $R \in (0,1/2)$ and $\theta \in (0, R^2)$ with* *the following properties. If T is a compact minimal surface with $\partial T = \tilde{u}(\Sigma)$, $||u||_{2,\alpha} \leq \theta$, then there is an $f\colon [R^3, 1] \times \Sigma \to \mathbf{R}$ such that*

$$T \sim B_{R^3} = S(f; R^3, 1). \tag{1}$$

Furthermore, if

$$||\pi_2'(f(1,\cdot))||_{2,\alpha} \leq |\pi_2(f(1,\cdot))|,$$

then for $R^2 \leq t \leq R$,

$$|\pi_2(f(t,\cdot)| \geq t^p |\pi_2(f(1,\cdot))|, \tag{2}$$

$$||\pi_2'(f(t,\cdot))||_{2,\alpha} \leq t^q |\pi_2(f(1,\cdot))|. \tag{3}$$

Proof. The existence of f satisfying (1) is just Proposition 4.4. To prove (2), fix an R and suppose that it fails. Then there exist sequences T_n, f_n, and $t_n \in [R^2, R]$ satisfying (1) and

$$||f_n(1,\cdot)||_{2,\alpha} \leq 1/n, \tag{4}$$

$$||\pi_2'(f_n(1,\cdot))||_{2,\alpha} \leq |\pi_2(f(1,\cdot)|, \tag{5}$$

$$|\pi_2(f_n(t_n,\cdot))| < t_n^p |\pi_2(f(1,\cdot))|. \tag{6}$$

Let $s(n) = ||f_n(1,\cdot)||_0$. By the maximum principle, T_n lies between $L_{-s(n)}$ and $L_{s(n)}$, so

$$l_{-s(n)}/s(n) \leq f_n/s(n) \leq l_{s(n)}/s(n). \tag{7}$$

By the Harnack inequality and the estimates of Proposition 4.4, a subsequence of $l_{s(n)}/s(n)$ converges smoothly on compact subsets of $(0,\infty)\times\Sigma$ to a positive limit L which is a solution of the linearized minimal surface equation. By Lemma 4.2, L has the form

$$L = \varphi_1(x)(ar^{(\delta(1)-m)/2} + br^{(-\delta(1)-m)/2}) \qquad (a, b \geq 0). \tag{8}$$

Likewise (5) and (7) imply that (a further subsequence of) $f_n/s(n)$ converges uniformly in $C^{2,\alpha/2}$ on compact subsets of $(0,1] \times \Sigma$ and in $C^{2,\alpha}$ on compact subsets of $(0,1)\times\Sigma$ to a solution F of the linearized minimal surface equation. By (7), $L - F \geq 0$ on $(0,1] \times \Sigma$, so by Lemma 4.2,

$$L - F = \sum_{i \geq -1, i \neq 0} b_i \varphi_i(x) r^{(\delta(i)-m)/2}.$$

Combining this with (8), we see that F has the form

$$F = \sum_{i \geq -1, i \neq 0} a_i \varphi_i(x) r^{(\delta(i)-m)/2}. \tag{9}$$

By (6), $|\pi_2(F(t,\cdot)| \le t^p |\pi_2(F(1,\cdot))|$ for some $t \in [R^2, R]$, i.e.,

$$\Big(\sum_{i=-1,1,2} a_i^2 t^{(\delta(i)-m)}\Big)^{1/2} \le t^p \Big(\sum_{i=-1,1,2} a_i^2\Big)^{1/2}.$$

Since $(\delta(i)-m)/2 < p$ for $i \le 2$, this implies that

$$a_{-1} = a_1 = a_2 = 0. \tag{10}$$

But by (5),

$$\begin{aligned} ||\pi_2'(F(1,\cdot))||_{2,\alpha} &\le \liminf ||\pi_2'(f_n(1,\cdot)/s(n))||_{2,\alpha} \\ &\le \liminf |\pi_2(f_n(1,\cdot)/s(n))| = |\pi_2(F(1,\cdot)|. \end{aligned}$$

Thus by (10), $F(1,\cdot) = 0$. But by construction, $||F(1,\cdot)||_0 = 1$. This contradiction proves (2).

Now suppose (3) is false. Then there exist sequences T_n, f_n, and $t_n \in [R^2, R]$ satisfying (4), (5), and

$$||\pi_2'(f_n(t_n,\cdot))||_{2,\alpha} > t_n^q |\pi_2(f_n(1,\cdot))|.$$

Thus, exactly as above, we get a nonzero solution F to the linearized minimal surface equation of the form (9), and a $t \in [R^2, R]$ such that

$$||\pi_2'(F(1,\cdot))||_{2,\alpha} \le |\pi_2(F(1,\cdot)| \ne 0, \tag{11}$$

$$||\pi_2'(F(t,\cdot))||_{2,\alpha} \ge t^q |\pi_2(F(1,\cdot))|. \tag{12}$$

Now $F_3\colon (r,x) \mapsto (\pi_2'(F(r,\cdot))(x)$ is a solution of the linearized minimal surface equation, so by standard elliptic theory [9, Chapter 8],

$$\begin{aligned} ||F_e(t,\cdot)||_{2,\alpha} &\le C(t^{-1}\int_{t/2}^{2t}\int_\Sigma F_3(s,x)^2\,dx\,ds)^{1/2} \\ &\qquad\qquad \text{(where } C \text{ does not depend on } t \text{ or } F\text{)} \\ &= C\Big(t^{-1}\int_{t/2}^{2t}\sum_{i\ge 3} a_i^2 s^{\delta(i)-m}\Big)^{1/2} \\ &\le C\Big(t^{-1}\int_{t/2}^{2t}\sum_{i\ge 3} a_i^2 s^{\delta(3)-m}\,ds\Big)^{1/2} \\ &\qquad\qquad \text{(since } 2t \le 1\text{)} \\ &\le C'\Big(\sum_{i\ge 3} a_i^2 t^{\delta(3)-m}\Big)^{1/2} \\ &= t^q \cdot t^{(\delta(3)-m)/2-q} \cdot C'|\pi_2'(F(1,\cdot))| \\ &\le t^q \cdot (R^{(\delta(3)-m)/2-q} \cdot C'')|\pi_2(F(1,\cdot))| \end{aligned}$$

(by (11)). Now if R has been chosen small enough that the term in parentheses is < 1, then we get a contradiction with (12).

Corollary. *If Σ is not a totally geodesic sphere, and T is a compact minimal surface with $\partial T = \tilde{u}(\Sigma)$, $0 < ||u||_{2,\alpha} \leq \theta$, and $||\pi_2'(u)||_{2,\alpha} \leq |\pi_2(u)|$, then there exist a $\rho < 1$ and a function $f: [\rho, 1] \times \Sigma \mapsto \mathbf{R}$ such that*

$$T \sim B_\rho = S(f; \rho, 1),$$
$$||f(\rho, \cdot)||_{2,\alpha} = \theta > ||f(r, \cdot)||_{2,\alpha} \qquad (\rho < r \leq 1),$$

and for $t \in [\rho/R, 1]$,

$$|\pi_2(f(Rt, \cdot))| \geq R^p |\pi_2(f(t, \cdot))|,$$
$$||\pi_2'(f(Rt, \cdot))||_{2,\alpha} \leq R^q |\pi_2(f(t, \cdot)|.$$

Proof. Note that if f is a solution of the minimal surface equation, then so is $(r, x) \mapsto f(\mu r, x)$. Now apply the proposition to $f(R^n r, x)$, $n = 0, 1, 2, \cdots$, until the first ρ such that $||f(\rho, \cdot)||_{2,\alpha} = \theta$.

(Note there must be such a $\rho > 0$, since otherwise $||f(r, \cdot)||_0$ grows like r^p as $r \to 0$, and by Lemma 4.3, $p < 0$.)

Theorem 4.6. *If*

(1) *C is area minimizing,*

(2) *there is no isotopy in B_1 from $\overline{\Omega}^+$ to $\overline{\Omega}^-$ leaving Σ fixed,*

(3) $\lambda_2 < (1 - m)$,

then there exists a complete singular minimal surface (without boundary) that is asymptotic to C at ∞ but is not a cone.

Remark. By Lemma 4.3, $\lambda_i = 1 - m$ and $(\delta(i) - m)/2 = -1$ for some i. Hypothesis (3) states that $i \geq 3$, which implies that the p and q of Proposition 4.5 are less than -1 since $\delta(j) < \delta(i)$ for $j < i$.

Proof. Let θ be as in the corollary to Proposition 4.5. Let

$$u_t(x) = l_{\cos(\pi t)\varepsilon}(x) + \varepsilon \cdot \varphi_2(x) \sin(\pi t),$$

where $\varepsilon < \theta$. Note that u_t $(0 \leq t \leq 1)$ is a path in $C^{2,\alpha}(\Sigma)$ from l_ε to $l_{-\varepsilon}$. By Theorem 4.1, there is a $t \in (0, 1)$ such that $\tilde{u}_t(\Sigma)$ bounds a singular minimal surface $T = T_\varepsilon$. By the corollary to Proposition 4.5, there are a $\rho = \rho(\varepsilon)$ and a function $f_\varepsilon: [\rho, 1] \times \Sigma \to \mathbf{R}$ such that

$$T_\varepsilon \sim B_\rho = S(f_\varepsilon; \rho, 1),$$

$$||f_\varepsilon(\rho, \cdot)||_{2,\alpha} = \theta > ||f_\varepsilon(r, \cdot)||_{2,\alpha} \qquad (\rho < r \leq 1), \tag{4}$$

$$|\pi_2(f_\varepsilon(r,\cdot))| \geq R^p |\pi_2(f(r/R,\cdot))| \qquad (\rho \leq r \leq R), \tag{5}$$

$$||\pi_2'(f(r,\cdot))||_{2,\alpha} \leq |\pi_2(f(r,\cdot))| \qquad (\rho \leq r \leq R). \tag{6}$$

Note that $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$. Now let V_ε be obtained by dilating T_ε by the factor $1/\rho$. Then $V_\varepsilon = S(g_\varepsilon; 1, 1/\rho)$, where $g_\varepsilon(r,x) = f_\varepsilon(\rho r, x)$. By (4) and Proposition 4.4, there is a sequence of ε's tending to 0 such that the g_ε converge smoothly on compact subsets of $[1,\infty) \times \Sigma$ to a function g. Thus the corresponding V_ε's converge to a minimal surface V with $V \sim B_1 = S(g; 1, \infty)$. Since the V_ε are all singular, so is V. By (4), $V \neq C$. Since $p < -1$, by (5) and (6) we have

$$||g(r,\cdot)||_{2,\alpha} = \mathscr{O}(r^p) = o(r^{-1}),$$

which implies that V is not a cone.

5. Complete area minimizing hypersurfaces

In the last section, we produced a complete minimal hypersurface V asymptotic to an area minimizing cone C at ∞ such that V is not congruent to any leaf of the foliation associated to C. This V is singular, but the proof does not tell us whether or not it is minimizing or even stable. In this section, under slightly different hypotheses on C, we prove that there is such a V which is minimizing, but we do not know whether or not it is singular.

5.1. Theorem. *Suppose Σ is a regular minimal hypersurface of the unit m-sphere $\partial B_1 \subset \mathbf{R}^{m+1}$ such that*

(1) $\lambda_2 < (1-m)$,

(2) *C is strictly stable and strictly minimizing.*

Then there exists a complete area minimizing hypersurface V that is asymptotic to C at ∞ but is not congruent to any leaf of the foliation associated with C.

Remark. Recall that C is stable if and only if $\lambda_1 \geq -m^2/4$, and strictly stable if and only if $\lambda_1 > -m^2/4$. By Lemma 4.2,

$$\lim_{t\to 0} t^{-1} l_t(r,x) = \left(ar^{(\delta(1)-m)/2} + br^{(-\delta(1)-m)/2}\right)\cdot \varphi_1(x).$$

The assumption of strict minimality means that $b = 0$.

Definition. If $f\colon [a,b]\times\Sigma \to \mathbf{R}$, let

$$Y(f,r) = r^m |\pi_1(f(r,\cdot))|^2 = r^m\left(\int_{x\in M} f(r,x)\varphi_1(x)\,dx\right)^2.$$

5.2. Proposition. *Suppose Σ is strictly stable. Then there is a $\mu > 0$ such that if $f_i\colon [1/16, 1]\times\Sigma \to \mathbf{R}$ $(i = 1,2)$ are solutions of the minimal surface* *equation with* $||f_i||_{2,\alpha} \leq \mu$ *and* $f_1 - f_2 = g \geq 0$, *and if* $Y(g, 1/4) \geq Y(g, 1/2)$, *then* $Y(g, 1/8) \geq Y(g, 1/4)$.

Furthermore, if $g(1, \cdot) \equiv 0$, *then* $Y(g, r/2) \geq Y(g, r)$ *for* $1/4 \leq r \leq 1$.

Proof. Suppose the first conclusion of the proposition is not true. Then there exist sequences of solutions f_1^n, f_2^n such that

$$||f_i^n||_{2,\alpha} \leq 1/n, \tag{1}$$

$$f_1^n - f_2^n = g^n \geq 0, \tag{2}$$

$$Y(g^n, 1/4) \geq Y(g^n, 1/2), \tag{3}$$

$$Y(g^n, 1/8) < Y(g^n, 1/4). \tag{4}$$

Because the minimal surface equation is quasilinear, g^n satisfies a homogeneous *linear* elliptic equation, the coefficients of which are expressions involving f_1^n and f_2^n (cf. the proof of [9, 10.4] or the appendix of [23]). Let

$$s(n) = \sup g^n \big| [1/8, 1/2] \times \Sigma.$$

Then it is standard (by the Harnack inequality, for example) that a subsequence of $g^n/s(n)$ converges uniformly on compact subsets of $(1/16, 1) \times \Sigma$ to a function G that is a solution of the linearized minimal surface equation. Thus by Lemma 4.2

$$\pi_1(G(r, \cdot)) = (ar^{(-m+\delta(1))/2} + br^{(-m-\delta(1))/2}) \cdot \varphi_1(\cdot), \tag{5}$$

so that

$$Y(G, r) = a^2 r^{\delta(1)} + 2ab + b^2 r^{-\delta(1)}.$$

Taking the limit of (3) and (4) we have

$$a^2 r^{\delta(1)} + b^2 r^{-\delta(1)} \geq a^2 (2r)^{\delta(1)} + b^2 (2r)^{-\delta(1)},$$
$$a^2 r^{\delta(1)} + b^2 r^{-\delta(1)} \geq a^2 (r/2)^{\delta(1)} + b^2 (r/2)^{-\delta(1)},$$

where $r = 1/4$. Adding these inequalities gives

$$2(a^2 r^{\delta(1)} + b^2 r^{-\delta(1)}) \geq (a^2 r^{\delta(1)} + b^2 r^{-\delta(1)})(2^{\delta(1)} + 2^{-\delta(1)}),$$

which implies that $a = b = 0$. This is impossible since $G \geq 0$ and $\sup G | [1/8, 1/2] \times \Sigma = 1$. Hence the first conclusion is proved.

To prove the second conclusion, suppose it fails. Then there exist sequences f_1^n and f_2^n of solutions satisfying (1), (2), and

$$g^n(1, \cdot) \equiv 0, \qquad Y(g^n, r_n/2) < Y(g^n, r_n) \tag{6}$$

for some $r_n \in [1/4, 1]$. Thus, as above, a subsequence of $g^n/s(n)$ converges to a nonzero limit G satisfying (5). Since $G(1,\cdot) = 0$, $b = -a$. Furthermore, G is nonnegative and not identically zero, so $|a| > 0$. Letting $n \to \infty$ in (6) gives $Y(G, r/2) \le Y(G, r)$ or

$$a^2((r/2)^{\delta(1)} + (r/2)^{-\delta(1)} - 2) \le a^2(r^{\delta(1)} + r^{-\delta(1)} - 2),$$

which is false (since $r \le 1$ and $\delta(1) > 0$ by strict stability).

Proof of Theorem 5.1. Fix a small $\varepsilon > 0$ and let

$$u_t \colon \Sigma \to \mathbf{R}, \qquad u_t(x) = \varepsilon(\varphi_1(x)\cos(\pi t) + \varphi_2(x)\sin(\pi t)).$$

Note that (by the maximum principle) the area minimizing surfaces bounded by $\tilde{u}_0(\Sigma)$ lie on one side of C, and those bounded by $\tilde{u}_1(\Sigma)$ lie on the opposite side of C. Thus there exists some $t \in (0,1)$ such that either

(1) $\tilde{u}_t(\Sigma)$ bounds an area minimizing surface T_ε which passes through the origin,

or

(2) $\tilde{u}_t(\Sigma)$ bounds two area minimizing surfaces T_ε^1 and T_ε^2 such that the origin lies in the region between T_ε^1 and T_ε^2.

We consider only case (2), since case (1) becomes a special case of case (2) by allowing $T_\varepsilon^1 = T_\varepsilon^2$ in (2).

Now apply the corollary to Proposition 4.5 to get $\rho = \rho(\varepsilon)$, $R \in (0, 1/2)$, and functions $f_\varepsilon^1 \le f_\varepsilon^2$ on $[\rho(\varepsilon), 1] \times (\Sigma)$ such that

$$\begin{gathered} T_\varepsilon^i \sim B_\rho = S(f_\varepsilon^i, \rho, 1), \\ |\pi_2(f_\varepsilon^i(Rt,\cdot)| \ge R^p |\pi_2(f_\varepsilon^i(t,\cdot)| \qquad (\rho/R \le t \le 1), \\ ||\pi_2'(f_\varepsilon^i(t,\cdot)||_{2,\alpha} \le |\pi_2(f_\varepsilon^i(t,\cdot)| \qquad (\rho \le t \le R), \\ \sup_{i=1,2} ||f_\varepsilon^i(\rho,\cdot)||_{2,\alpha} = \theta > \sup_{i=1,2} ||f_\varepsilon^i(t,\cdot)||_{2,\alpha}. \end{gathered}$$

Furthermore, by Proposition 5.2 applied inductively to $f_\varepsilon^i(2^n\cdot,\cdot)$, $n = 0, 1, 2, \cdots$,

$$Y(f_\varepsilon^2 - f_\varepsilon^1, t) \ge Y(f_\varepsilon^2 - f_\varepsilon^1, 2t) \qquad (\rho \le t \le 1/2).$$

Now scale T_ε^i by $1/\rho(\varepsilon)$ and pass to a subsequence of $\varepsilon \to 0$ to get limits T^i $(i = 1, 2)$ with

$$T^i \sim B_1 = S(f^i; 1, \infty), \qquad f^i(r, x) = \lim_{\varepsilon \to 0} f_\varepsilon^i(\rho(\varepsilon)r, x).$$

Then for $1 \le r < \infty$

$$Y(f^1 - f^2, r) \ge Y(f^1 - f^2, 2r), \tag{3}$$

$$(4) \qquad \begin{gathered} \|\pi_2'(f^i(r,\cdot)\|_{2,\alpha} \leq |\pi_2(f^i(r,\cdot)|, \\ \sup_{i=1,2} \|f^i(1,\cdot)\|_{2,\alpha} = \theta \geq \sup_{i=1,2} \|f^i(r,\cdot)\|_{2,\alpha}. \end{gathered}$$

Without loss of generality,

$$(5) \qquad \|f^2(1,\cdot)\|_{2,\alpha} = \theta \geq \sup_{i=1,2} \|f^i(r,\cdot)\|_{2,\alpha}.$$

Also, for $R^{-1} \leq r < \infty$,

$$(6) \qquad |\pi_2(f^i(1,\cdot)| \geq r^p |\pi_2(f^i(r,\cdot)|.$$

Now we claim that either f^1 or f^2 must change sign (i.e., take on both positive and negative values). For suppose not, since the origin lies between T^1 and T^2 we then have

$$f^1(r,x) \leq 0 < f^2(r,x) \qquad (1 \leq r < \infty).$$

Thus T^1 and T^2 must be leaves of the foliation $\mathscr{F}$ [10, 2.1], so by the strict minimality of C,

$$\lim_{n\to\infty} f^i(nr,x)/s(n) = \varphi_1(x) \cdot c_i r^{(\delta(1)-m)/2}, \qquad c_1 \leq 0 < c_2,$$

where $s(n) = \max\{\|f^i(n\cdot,\cdot)\|_0 : i = 1,2\}$. Hence by (3),

$$(c_2 - c_1)^2 \cdot 1^{\delta(1)} \geq (c_2 - c_1)^2 \cdot 2^{\delta(1)},$$

which is a contradiction.

Thus one of the f^i, say f^2, changes sign, so that T^2 is neither C nor any leaf of the foliation $\mathscr{F}$. Now suppose C' is a cone with $C' \neq C$ (C' could be a translate of C, for example). Then

$$(7) \qquad \lim_{r\to\infty} \operatorname{Dist}(C' \cap \partial B_r, C' \cap \partial B_r) > 0,$$

where $\operatorname{Dist}(\cdot,\cdot)$ is the Hausdorff distance. Furthermore, if L is a leaf of the minimal foliation associated with C', then

$$\lim_{r\to\infty} \operatorname{Dist}(L \cap \partial B_r, C' \cap \partial B_r) = 0,$$

by [10, 2.1], so

$$(8) \qquad \lim_{r\to\infty} \operatorname{Dist}(L \cap \partial B_r) > 0.$$

On the other hand, (4) and (6) imply that

$$\operatorname{Dist}(T^2 \cap \partial B_r, C \cap \partial B_r) = \mathscr{O}(r^{p+1}).$$

The hypothesis on λ_2 implies that $p < -1$ (see the remark after Theorem 4.6), so that

$$\lim_{r\to\infty} \mathrm{Dist}(T^2 \cap \partial B_r, C \cap \partial B_r) = 0.$$

Thus (by (7) and (8)) T^2 is neither a cone nor any leaf of the foliation associated with a cone.

6. Concluding remarks

In this section we show that the hypotheses of §§4 and 5 are satisfied for most of the known examples of area minimizing cones. Every Σ for which C is known to be area minimizing is *isoparametric*, that is, the set of principal curvatures $\kappa_1(x), \cdots, \kappa_{m-1}(x)$ of Σ at x does not depend on x. In particular, this is the case for the examples in Lawson's list [12] and for the examples constructed by Ferus, Karcher, and Münzner from Clifford algebras [8]. For such cones Σ, the functions $x \mapsto v \cdot x$ ($v \in \mathbf{R}^{n+1}$) are eigenfunctions of J_Σ with eigenvalue λ_i where $\lambda_1 < \lambda_i \le 1-m$. Furthermore, $\lambda_i = 1-m$ if and only if Σ is S^{m-1} or $S^p \times S^{m-1-p}$. (See the last section of [18] for a discussion of these facts about isoparametric Σ.) Also, the only isoparametric Σ that is a homology sphere is the totally geodesic S^{m-1} (cf. [11, 6.4(2)] or [17]). *Thus except for S^{m-1} and $S^p \times S^{m-1-p}$, every isoparametric Σ such that C is minimizing satisfies the hypotheses of Theorem* 4.6.

Strict stability and strict minimality are not well understood in general, but they hold for all the (minimizing) examples in Lawson's list [12] except S^2 (see [10, 3.3]). Also, for every isoparametric Σ except S^2, if C is stable, then it is strictly stable. (This follows from the fact that $\lambda_1 = g(1-m)$, where g is the number of distinct principal curvatures of Σ [18].) Bruce Solomon has observed that the examples of Ferus, Karcher, and Munzner [8] are all strictly minimizing. (The proof in [7], [6] that they are minimizing actually shows that they are strictly minimizing, because the inequalities there are strict.) *Thus except for S^{m-1} and $S^p \times S^{m-1-p}$, the hypotheses in §5 are satisfied for every Σ in Lawson's list* [12] *such that C is minimizing and for all the examples of Ferus, Karcher, and Munzner* [8].

On the other hand, if $\Sigma = S^{m-1} \subset S^m$, then by monotonicity every minimal hypersurface asymptotic to C at ∞ is congruent to C. And if $\Sigma = S^p \times S^{m-1-p}$ and C is minimizing, Leon Simon and Bruce Solomon [18] have shown that every minimal hypersurface asymptotic to C at ∞ is congruent to C or to a leaf of the foliation associated with C.

We conclude this paper with two open questions. Is there a complete hypersurface V asymptotic to C at ∞ but not congruent to C, such that V

is both singular and area minimizing? Can one classify all complete minimal (or minimizing) hypersurfaces asymptotic to C? The first question would be settled affirmatively if one could show that for small $||u||_{2,\alpha}$, $\tilde{u}(\Sigma)$ bounds a *unique* area minimizing surface. The second question seems very difficult.

References

[1] W. K. Allard, *First variation of a varifold*, Ann. of Math. (2) **95** (1972) 417–491.

[2] ——, *First variation of a varifold: boundary behavior*, Ann. of Math. (2) **101** (1975) 418–446.

[3] F. J. Almgren, Jr., R. Schoen & L. Simon, *Regularity and singularity estimates on hypersurfaces minimizing parametric elliptic variation integrals*, Acta Math. **139** (1977) 217–265.

[4] F. J. Almgren, Jr. & L. Simon, *Existence of embedded solutions to Plateau's problem*, Ann. Scuola Norm. Sup. Pisa **6** (1979) 447–495.

[5] R. Böhme & A. J. Tromba, *The index theorem for classical minimal surfaces*, Ann. of Math. (2) **113** (1981) 447–499.

[6] J. Brothers (ed.), *Some open problems in geometric measure theory*, Proc. Sympos. Pure Math., Vol. 44, Amer. Math. Soc., Providence, RI, 1985, 441–464.

[7] D. Ferus & H. Karcher, *Non-rotational minimal spheres and minimizing cones*, Comment. Math. Helv. **60** (1985) 247–269.

[8] D. Ferus, H. Karcher & H.-F. Münzner, *Cliffordalgerbren und neue isoparametrische Hyperflachen*, Math. Z. **177** (1981) 479–502.

[9] D. Gilbarg & N. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer, Berlin, 1983.

[10] R. Hardt & L. Simon, *Area minimizing hypersurfaces with isolated singularities*, J. Reine Angew. Math. **362** (1985) 102–129.

[11] W. Y. Hsiang, R. S. Palais & C. L. Terng, *The topology of isoparametric submanifolds*, J. Differential Geometry **27** (1988) 423–460.

[12] H. B. Lawson, Jr., *The equivariant Plateau problem and interior regularity*, Trans. Amer. Math. Soc. **173** (1972) 231–249.

[13] F. H. Lin, *Existence and regularity of embedded disks*, J. Differential Geometry **27** (1988) 553–560.

[14] W. H. Meeks, III, L. Simon & S. T. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive ricci curvature*, Ann. of Math. (2) **116** (1982) 621–659.

[15] W. H. Meeks, III & S. T. Yau, *The classical Plateau problem and the topology of 3-manifolds*, Topology **21** (1982) 409–442.

[16] C. B. Morrey, *The parametric variational problem for double integrals*, Comm. Pure Appl. Math. **14** (1961) 569–575.

[17] H.-F. Münzner, *Isoparametrische Hyperflachen in Spharen.* I, II, Math. Ann. **251** (1980) 57–71: **256** (1981) 215–232.

[18] L. Simon & B. Solomon, *Minimal hypersurfaces asymptotic to quadratic cones in* $\mathbf{R}^{n+1}$, Invent. Math. **86** (1986) 535–551.

[19] S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965) 861–866.

[20] B. Solomon & B. White, *A strong maximum principal for hypersurfaces that are minimal with respect to parametric elliptic functionals*, in preparation.

[21] F. Tomi & A. J. Tromba, *Extreme curves bound embedded minimal surfaces of the type of the disk*, Math. Z. **158** (1978) 137–145.

[22] A. J. Tromba, *Degree theory on oriented infinite dimensional varieties and the Morse number of minimal surfaces spanning a curve in* $\mathbf{R}^n$. *Part* I: $n \geq 4$. Trans. Amer. Math. Soc. **290** (1985) 385–413.

[23] B. White, *Curvature estimates and compactness theorems in* 3*-manifolds for surfaces that are stationary for parametric elliptic functionals*, Invent. Math. **88** (1987) 243–257.

[24] ——, *The space of* m*-dimensional surfaces that are stationary for a parametric elliptic functional*, Indiana Univ. Math. J. **36** (1987) 567–602.

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