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THE GEOMETRY OF LOOP GROUPS

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Abstract

The space ΩG of based loops on a compact Lie group admits a Kähler metric. Its curvature is expressed in terms of Toeplitz operators, and we define Chern classes by analogy with Chern-Weil theory in finite dimensions. In infinite dimensions extra geometric structure—a Fredholm structure—must be imposed before characteristic classes are defined. There is a natural Fredholm structure on ΩG induced from the family of Toeplitz operators. We use the index theorem for families of Fredholms parametrized by a group (proved in [20]) to show that the Chern classes of the Toeplitz family agree with the Chern classes defined by curvature. Explicit formulas for $\Omega \mathrm{SU}(n)$ are obtained. We also prove that the real characteristic classes of ΩG vanish for any group G. Extensions to more general groups of gauge transformations are considered.

Infinite dimensional geometry has received much attention recently, particularly due to motivations from physics. Rigorous consideration of infinite manifolds originated in the 1960's, when the foundations were carefully laid. Examples arising directly from variational problems in geometry and Lagrangian field theories in physics are manifolds of maps, and these can be modeled on Hilbert spaces. Exterior differential calculus, de Rham Theory, Riemannian connections, and all basic features of finite dimensional manifold theory generalize to Hilbert manifolds, but with one notable simplification: Hilbert manifolds are always parallelizable. This contrasts sharply with finite dimensionsm where twisted tangent bundles are ultimately due the nontrivial topology of $\mathrm{GL}(n;\mathbf{R})$. The structure group of a Hilbert manifold—the general linear group $\mathrm{GL}(\mathscr{H})$ on Hilbert space—is contractible, and the parallelizability of Hilbert manifolds is an immediate consequence.

Fredholm structures are reductions of the frame bundle to the topologically nontrivial group $\mathrm{GL}^{\mathrm{cpt}}(\mathscr{H})$ of invertible operators which differ from the identity by a compact operator [18]. This extra geometric structure was created to introduce twisting in the tangent bundle of a Hilbert manifold, but there seem to be few examples. Existence is not the issue, as a Hilbert manifold

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admits many Fredholm structures. Rather, one needs a geometric method to distinguish one [29]. Fredholm manifolds carry nontrivial characteristic classes, which in finite dimensions form a basic link between geometry and topology. Only with a Fredholm structure can one hope to generalize these deep connections which comprise the index theorem.

Roughly speaking, the study of the mapping space $\mathrm{Map}(M, N)$ involves the geometry of the target N and analysis on the source M. As the manifold with the simplest geometry is a compact Lie group G, since geometric quantities can be expressed in terms of its Lie algebra, and the manifold with the simplest analysis is the circle S^1, since Fourier series are available, the simplest manifold of maps is the loop group $LG = \mathrm{Map}(S^1, G)$. The *based* loop group ΩG admits a Kähler metric, which is our main object of study. In §2 we derive a formula for its curvature in terms of Toeplitz operators. Ricci curvature in infinite dimensions is computed by an infinite sum, which diverges in general. For the Kähler curvature of ΩG that sum is conditionally convergent (the absolute sum diverges logarithmically), and there is a natural order of summation. Furthermore, in finite dimensions Chern-Weil theory endows the Ricci curvature with topological meaning—it represents the first Chern class of the complex tangent bundle. By analogy, we define the first Chern class of ΩG to be the trace of the curvature. The second cohomology of ΩG is one dimensional, and this geometric first Chern class is $2n_G$ times the positive generator, where n_G is the *dual Coxeter number* of G. (The dual Coxeter number of $\mathrm{SU}(n)$ is n.) Higher Chern classes of ΩG are defined by traces of powers of the curvature, which converge absolutely, but direct calculation is too difficult.

Our main thesis is: The holonomy bundle of the Kähler connection provides a natural Fredholm structure on ΩG. This is the topological origin of the Chern classes which, a priori, are absent on a Hilbert manifold. Unfortunately, there are obstacles of a technical nature which prevent us from rigorously constructing the holonomy bundle in infinite dimensions. In finite dimensions the Ambrose-Singer Theorem determines the holonomy group from curvature, and this theorem also resisted our attempts at infinite dimensional generalization. We rigorously construct the Fredholm structure on ΩG by other means, in §5. The holonomy construction motivates our considerations in [20], where we study subgroups of $\mathrm{GL}(\mathscr{H})$. The structure group of a Fredholm manifold is $\mathrm{GL}^{\mathrm{cpt}}(\mathscr{H})$, but there are more delicate subgroups with summability properties, and for these we prove a Chern-Weil Theorem. If the holonomy bundle could be constructed, this theorem would identify its topological Chern classes with the geometric Chern classes calculated from curvature.

Although we consider the Fredholm structure abstractly as coming from the holonomy bundle, there is a more concrete approach that not only provides a rigorous construction of the Fredholm structure, but also gives greater insight into its geometry. Fredholm structures, which are reductions of the $\mathrm{GL}(\mathscr{H})$ frame bundle to the group $\mathrm{GL}^{\mathrm{cpt}}(\mathscr{H})$, are classified topologically by homotopy classes of maps to $\mathrm{GL}(\mathscr{H})/\mathrm{GL}^{\mathrm{cpt}}(\mathscr{H})$. Furthermore, there is a fibration $\mathrm{Fred}_0(\mathscr{H}) \to \mathrm{GL}^{\mathrm{cpt}}(\mathscr{H})$ with contractible fibers, so that $\mathrm{Fred}_0(\mathscr{H})$ also serves as a classifying space. Therefore, a Fredholm structure can always be specified, at least up to topological equivalence, by a family of index zero Fredholm operators. The problem of choosing a geometrically relevent Fredholm structure can be rephrased as the problem of choosing a family of index zero Fredholm operators relevant to the intrinsic geometry. Our choice of a particular family on ΩG is based on a new index theorem [20] for special families of Fredholms parametrized by a group $\mathfrak{G}$. Let $L^p(\mathscr{H})$ denote the pth Schatten ideal, which roughly consists of operators whose pth power is trace class. We consider families of operators $T\colon \mathfrak{G} \to \mathrm{Fred}_0(\mathscr{H})$ which are homomorphisms up to $L^p(\mathscr{H})$; i.e. $T(g)T(g') - T(gg') \in L^p(\mathscr{H})$ for all $g, g' \in \mathfrak{G}$. Then the Chern character classes $\mathrm{ch}_l(T)$ of the families index are represented by invariant differential forms on $\mathfrak{G}$ for $l \geq p$. (There is a version for *graded* Hilbert spaces which relates to work of Connes and Segal. However, our application to loop groups requires the full, ungraded version.) We review these ideas in §5.

The index theorem gives a formula for the Chern character of special families of Fredholms. On the other hand, the Chern-Weil Theorem gives a formula for the Chern character of a manifold in terms of curvature. In §5 we construct a family of operators parametrized by ΩG so that the index theorem formula for its Chern character matches exactly the Chern-Weil formula for the Chern character. Thus the Chern classes of this family are the Chern classes of ΩG that we define in §2 via curvature. Furthermore, since ΩG is a torsion-free space, the Chern classes uniquely determine the Fredholm structure. Therefore, the Fredholm structure that we rigorously construct from the family of Fredholms is the Fredholm structure abstractly determined by the holonomy construction. There is an obvious homotopy of the family we construct to the usual family of Toeplitz operators (in the adjoint representation). One of the Atiyah's proofs of Bott periodicity demonstrates that the stable version of this family is a homotopy equivalence, from which we determine the higher Chern classes of ΩG. We carry out the calculation for $G = \mathrm{SU}(n)$, and prove

$$\mathrm{Chern}(\Omega\mathrm{SU}(n)) = \exp(2n\{y_2 + y_6/3 + y_{10}/5 + \cdots + y_{4m-2}/(2m-1)\}),$$

where $y_{2l} \in H^{2l}(\Omega\mathrm{SU}(n); \mathbf{Z})$ are certain generators, and $m = [n/2]$.

The Fredholm structure which we distinguish on ΩG is one of many possible, and in §4 we present evidence which indicates that it is the correct geometric choice. The based loop group fits into the Kac-Moody theory as a flag manifold for a central extension of the free loop group. Many years ago Borel and Hirzebruch explored the relationship between characteristic classes and homogeneous spaces in finite dimensions, and they expressed the Chern classes of flag manifolds in terms of the roots of the Lie algebra. We verify that their group theoretic definition of the first Chern class formally agrees with the value we calculate from curvature. Our value of the first Chern class can also be tested in an index problem. For a finite dimensional complex manifold X, the space of based holomorphic maps $f\colon \mathbf{CP}^1 \to X$ is finite dimensional, and the Riemann-Roch Theorem expresses its (complex) dimension at f as $f^*(c_1(X))\,[\mathbf{CP}^1]$. Recently, Atiyah and Donaldson proved that the moduli space of based holomorphic maps $\mathbf{CP}^1 \to \Omega G$ is also finite dimensional and is diffeomorphic to the moduli space of based G-instantons on S^4. We formally apply the Riemann-Roch Theorem to the infinite dimensional based loop group, substituting our value of $c_1(\Omega G)$, to obtain $4kn_G - \dim(G)$ as the real dimension of the moduli space of unbased k-instantons. This is the correct dimension. The higher Chern classes of ΩG can be tested indirectly through the Pontrjagin and Stiefel-Whitney classes. On complex manifolds these real characteristic classes are derived from the Chern classes, and we prove in §5 that for ΩG they all vanish. As a real manifold ΩG is a Lie group, and the vanishing of the real characteristic classes is consistent with the geometry of finite dimensional Lie groups.

Mapping groups $\mathrm{Map}(M,G)$ for higher dimensional M are not complex manifolds, and we cannot define Chern classes. Still, they admit a natural family of Sobolev metrics, and in §1 we compute the curvature of these metrics. On the basis of these curvature formulas we define a real Fredholm structure via a family of real Fredholm operators, and in §6 we prove that the resulting reduced frame bundle is trivial. Again this fits finite dimensional theory, since $\mathrm{Map}(M,G)$ is a Lie group. At the end of §6 we speculate about a possible source of nontrivial characteristic classes for these more general groups of gauge transformations.

The curvature formulas of §2 can be applied to the homogeneous Kähler manifold $\mathrm{Diff}(S^1)/\mathbf{T}$, where $\mathbf{T}$ is the group of constant rotations of the circle. This was carried out recently in an intriguing paper of Bowick and Rajeev [15]. They propose that perturbations of the natural homogeneous Kähler metric on $\mathrm{Diff}(S^1)/\mathbf{T}$ define a field theory of closed strings.

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1. The curvature of $\mathrm{Map}(M, G)$

The space $\mathrm{Map}(M, G)$ of maps from a compact n-dimensional Riemannian manifold M into a compact Lie group G is an infinite dimensional manifold. As such there are foundational points in its Riemannian geometry which merit special attention, which we briefly administer before taking up curvature computations. These foundations were worked out in great detail by many mathematicians, and the reader may wish to read the expositions in [17], [28], [29].

We first recall the differentiable structure of mapping spaces. Infinite dimensional manifolds, like finite dimensional manifolds, are topological spaces which are locally homeomorphic to a model flat space, and on which a system of differentiably compatible coordinate charts is specified. But whereas the finite dimensional models are unique, once the dimension is fixed, infinite dimensional topological vector spaces exhibit a wide variety of behavior. Fortunately, mapping spaces admit differentiable structures modeled on Hilbert spaces, and we need not tangle with terrible topologies. Mapping groups enjoy an even richer structure—they are Hilbert Lie groups, that is, Hilbert manifolds which are groups and for which the group operations are smooth.

The local model for $\mathrm{Map}(M, G)$ is a completion of the Lie algebra of (smooth) maps $\mathrm{Map}(M, \mathfrak{g})$, where $\mathfrak{g}$ is the (finite dimensional) Lie algebra of G. Let Δ denote the Laplace operator d^*d on M, and $(\cdot, \cdot)_{\mathfrak{g}}$ the inner product on $\mathfrak{g}$ given by minus the Killing form. Then for any real number s the Sobolev H_s metric on $\mathrm{Map}(M, \mathfrak{g})$ is defined by

$$(1.1) \qquad (X, Y)_{H_s} = \int_M ((1+\Delta)^s X(x), Y(x))_{\mathfrak{g}}\, dx, \qquad X, Y \in \mathrm{Map}(M, \mathfrak{g}),$$

where dx is the Riemannian volume form on M. The Hilbert space completion of the smooth maps in this inner product is denoted $H_s(M, \mathfrak{g})$. The H_s maps are continuous for $s > n/2$. In this range there are corresponding completions $H_s(M, G)$ which are Hilbert manifolds modeled on $H_s(M, \mathfrak{g})$. To construct them, embed G smoothly in some $\mathbf{R}^N$, and define $H_s(M, G)$ to be the subspace of $H_s(M, \mathbf{R}^N)$ consisting of maps whose image lies in G. This makes sense since H_s maps are continuous. Furthermore, the space $H_s(M, G)$

is independent of the embedding $G \to \mathbf{R}^N$ for $s > n/2$. The exponential map $\exp\colon \mathfrak{g} \to G$ induces $\mathrm{Exp}: H_s(M, \mathfrak{g}) \to H_s(M, G)$, which gives a local chart near the identity. Left translation by smooth $f \in H_s(M, G)$ provides a system of coordinate charts covering the entire manifold; the Sobolev Composition Lemma ensures that the transition functions are smooth. Finally, the Sobolev Multiplication Theorem guarantees that the (pointwise) group operators are smooth. Altogether,

Theorem 1.2. *$H_s(M, G)$ is a Hilbert Lie group for $s >$* $\dim(M)/2$.

(See [22, Appendix A] for further details.)

There is a slight simplification if we treat the subgroup $\mathrm{Map}_0(M, G)$ of maps which take a fixed point on M into the identity element of G. Then the Laplacian Δ has no kernel on the corresponding Lie algebra, and we replace (1.1) by

$$(1.3) \qquad (X, Y)_{H_s} = \int_M (\Delta^s X(x), Y(x))_{\mathfrak{g}}\, dx, \qquad X, Y \in \mathrm{Map}_0(M, \mathfrak{g}).$$

The basepoint condition on $X \in \mathrm{Map}_0(M, \mathfrak{g})$ requires that X vanish at the basepoint of M.

A Riemannian metric $(\cdot, \cdot)$ on a Hilbert manifold $\mathscr{M}$ is a smooth choice of inner products in the tangent spaces. We do not require that the tangent spaces be complete in these inner products, but only that the inner products be continuous. In other words each tangent space is continuously embedded in its dual, but the embedding is not necessarily onto. For finite dimensional manifolds the Levi-Civita Theorem states that there is a unique torsion-free connection compatible with the Riemannian metric. It is determined by the formula

$$(1.4) \qquad \begin{aligned} 2(\nabla_X Y, Z) &= X(Y, Z) + Y(X, Z) - Z(X, Y) \\ &\quad + ([X, Y], Z) + ([Z, X], Y) - ([Y, Z], X) \end{aligned}$$

for vector fields X, Y, Z. This theorem persists for infinite dimensional Riemannian manifolds on which the tangent spaces are complete with respect to the Riemannian metric. Equation (1.4) defines a continuous linear functional on each tangent space mapping Z to the right-hand side, and for these metrics there is a unique tangent vector $\nabla_X Y$ which satisfies (1.4). For weak Riemannian metrics (the incomplete case) the existence of a covariant derivative satisfying (1.4) is not guaranteed; not all continuous functionals on a pre-Hilbert space are realized by the inner product, although any functional so realized has a unique such realization. In this case a torsion-free metric connection, if defined, is unique.

The space of maps $\mathrm{Map}(M, N)$ from a compact Riemannian manifold M into a Riemannian manifold N, completed as above using Sobolev space, inherits a natural family of Riemannian metrics—the Sobolev H_s metrics themselves. The H_s Riemannian metric is defined on $H_t(M,N)$ for any $t \geq s$, so long as $t > n/2$. Recall that the tangent space to $\mathrm{Map}(M,N)$ at a map f is the space of H_t sections of the pulled back tangent bundle $f^*(TN) \to M$. Define the H_s metric using the metric on M and the pulled back metric and connection on $f^*(TN)$. Of course, the H_s metric is strong in the H_s topology; that is, the tangent spaces are complete. For $t > s$ the H_t tangent spaces are incomplete with respect to the H_s metric.

The L^2 (or H_0) metric has the simplest geometry—it simply reflects the geometry of N pointwise. (This is a weak metric on any H_t completion, for t in the Sobolev range $t > n/2$.)

Proposition 1.5. *The L^2 curvature $R^{(0)}(X,Y)$ of $\mathrm{Map}(M,N)$ at f is the endomorphism of f^*TN given pointwise by the curvature $R_{(N)}(X,Y)$ of N.*

The easy proof can be found in [21, Appendix A]. In our situation $N = G$ is a group, and the L^2 metric plays the role of the Killing form.

When N is a group we replace the H_s metrics above *left invariant* metrics, i.e., with metrics defined first on the Lie algebra $\mathrm{Map}(M,\mathfrak{g})$ which are then extended by left translation. These metrics are given by (1.1) in the case of free maps and by (1.3) for based maps. We understand the H_s metric to live on the H_s completion for $s > n/2$, and on $H_{n/2+\varepsilon}$ for $0 \leq s \leq n/2$ and some $\varepsilon > 0$; usually we omit explicit reference to these completions. For $s > n/2$ the H_s metrics are strong, and the Levi-Civita connection is determined by (1.4). Even though the H_s metrics for $s \leq n/2$ are weak, and the existence of the Levi-Civita connection does not follow from general considerations, the formula in our next proposition makes clear its existence.

Proposition 1.6. *Let $(\cdot,\cdot)$ be a left invariant metric on a (Hilbert) Lie group. Then for left invariant vector fields X and Y we have*

$$\nabla_X Y = \tfrac{1}{2}\{\mathrm{ad}_X Y - \mathrm{ad}^*_X Y - \mathrm{ad}^*_Y X\}, \tag{1.7}$$

where ad^*_X is the adjoint of ad_X in the given metric.

The proposition follows directly by specializing (1.4).

For convenience we will calculate with the space of based maps $\mathrm{Map}_0(M,G)$, though our formulas remain valid for $\mathrm{Map}(M,G)$ with $(1+\Delta)$ replacing Δ. The transformation ad_X is defined by bracketing pointwise in the Lie algebra $\mathfrak{g}$. Then for smooth $X,Y,Z \in \mathrm{Map}_0(M,\mathfrak{g})$,

$$(\mathrm{ad}^*_X Y, Z)_{H_s} = (Y, \mathrm{ad}_X Z)_{H_s} = \int_M (\Delta^s Y, [X,Z])_{\mathfrak{g}} = -\int_M ([X,\Delta^s Y], Z)_{\mathfrak{g}},$$

from which

$$\mathrm{ad}^*_X = -(\Delta^{-s}\mathrm{ad}_X \Delta^s). \tag{1.8}$$

Equation (1.8) is also valid for Sobolev maps. The formula for the H_s connection now follows from Proposition 1.6:

$$\nabla_X^{(s)} = \tfrac{1}{2}\{\mathrm{ad}_X + \Delta^{-s}\mathrm{ad}_X\Delta^s - \Delta^{-s}\mathrm{ad}(\Delta^s X)\}. \tag{1.9}$$

The curvature of the H_s metric is given by

$$R^{(s)}(X,Y) = [\nabla_X^{(s)}, \nabla_Y^{(s)}] - \nabla_{[X,Y]}^{(s)}. \tag{1.10}$$

The exact formula for the curvature is not crucial at this stage. For now we are content to show that for smooth maps the curvature $R^{(s)}(X,Y)$ is a pseudodifferential operator on M, and we compute its order as a function of s.

Theorem 1.11. *For smooth $X, Y \in \mathrm{Map}(M,\mathfrak{g})$ and $s > 0$ the curvature $R^{(s)}(X,Y)$ is a pseudodifferential operator acting on the Sobolev completions of* $\mathrm{Map}(M,\mathfrak{g})$. *Its order is* $\mathrm{ord}\, R^{(s)}(X,Y) = \max(-1,-2s)$.

Proof. For convenience of notation we continue to consider $\mathrm{Map}_0(M,G)$ in place of $\mathrm{Map}(M,G)$; the results are the same. The transformation ad_X is essentially a multiplication operator, and so is a pseudodifferential operator of order 0. Furthermore, by Seeley's analysis [33], Δ^s and Δ^{-s} are pseudodifferential operators of order $2s$ and $-2s$, respectively. It follows from (1.9) that $\nabla_X^{(s)}$ is a pseudodifferential operator of order zero. Let $q = \min(1, 2s)$. We claim that

$$\nabla_X^{(s)} = \mathrm{ad}_X + (\text{order } -q). \tag{1.12}$$

To see this, simply observe that $[\mathrm{ad}_X, \Delta^s]$ has order $2s-1$, since Δ^s has scalar symbol. So

$$\Delta^{-s} - \mathrm{ad}_X\Delta^s = \mathrm{ad}_X + \Delta^{-s}[\mathrm{ad}_X, \Delta^s] = \mathrm{ad}_X + (\text{order } -1).$$

The last term in (1.9) is of order $-2s$, whence (1.12). The theorem now follows from the fact that $X \to \mathrm{ad}_X$ is a homomorphism of Lie algebras:

$$\begin{aligned} R^{(s)}(X,Y) &= [\mathrm{ad}_X + (\text{order } -q), \mathrm{ad}_Y + (\text{order } -q)] \\ &\quad - \{\mathrm{ad}_{[X,Y]} + (\text{order } -q)\} \\ &= (\text{order } -q). \end{aligned}$$

It is not hard to show that $R^{(s)}(X,Y)$ is a compact operator for any $X, Y \in H_t(M,G)$ if $s > 0$ and $t > n/2$, $t \geq s$.

The Ricci curvature of a Riemannian manifold is the symmetric bilinear form

$$\mathrm{Ric}(X,Y) = \mathrm{Trace}\{Z \to R^{(s)}(Z,X)Y\}. \tag{1.13}$$

In infinite dimensions this is the trace of an operator on Hilbert space, which makes sense only for operator of *trace class.*

Proposition 1.14. *For smooth* $X, Y \in \mathrm{Map}(M, g)$ *the operator* $Z \to R^{(s)}(Z, X)$ *is pseudodifferential of order* $\max(-1, -2s)$.

Proof. Rearranging (1.9) slightly, we see that $Z \to \nabla_Z^{(s)} Y$ is the operator

$$Z \to \tfrac{1}{2}\{-\mathrm{ad}_Y - \Delta^{-s}\mathrm{ad}(\Delta^s Y) + \Delta^{-s}\mathrm{ad}_Y \Delta^s\}. \tag{1.15}$$

The first and third terms sum to $\Delta^{-s}[\mathrm{ad}_Y, \Delta^s]$, which we saw above is pseudodifferential of order -1. Therefore, the operator (1.15) has order $-q = \max(-1, -2s)$. Now each term of

$$R^{(s)}(Z, X)Y = \nabla_Z^{(s)}(\nabla_X^{(s)} Y) - \nabla_X^{(s)} \nabla_Z^{(s)} Y - \nabla_{[Z,X]}^{(s)} Y \tag{1.16}$$

is easily seen to be of order $-q$ as a function of Z.

A pseudodifferential operator of order $-q$ in n dimensions is trace class only if $q > n$. Hence the Ricci curvature never exists without some modification.

Our main concern is a Kähler metric on $\mathrm{Map}_0(S^1, G)$, and for this metric we compute the Ricci curvature in the Kähler sense, that is, as $\mathrm{Trace}(R(X, Y))$. The next proposition will enable us to make sense of this trace. Observe that the Hilbert space $H_t(M, \mathfrak{g})$ on which the curvature operators can be written as the tensor product $H_t(M, \mathfrak{g}) = H_t(M, \mathbf{R}) \otimes \mathfrak{g}$. Then for any operator A on $H_t(M, \mathfrak{g})$ we can take the trace over the Lie algebra $\mathfrak{g}$ to obtain a new operator $\mathrm{Trace}_{\mathfrak{g}}(A)$ on $H_t(M, \mathbf{R})$.

Proposition 1.17. *For smooth* X, Y *the operator* $\mathrm{Trace}_{\mathfrak{g}}(R^{(s)}(X, Y))$ *is pseudodifferential of order* $-(q+1)$, *where* $q = \min(1, 2s)$.

Proof. We claim that $\mathrm{Trace}_{\mathfrak{g}}(\nabla_X^{(s)}) = 0$. Let $X = f \otimes a$, $f \in H_t(M, \mathbf{R})$, $a \in \mathfrak{g}$, be a decomposable vector; then $\nabla_X^{(s)} = Q_f \otimes \mathrm{ad}(a)$ for an operator Q_f on $h_t(M, \mathbf{R})$. Since $\mathrm{Trace}(\mathrm{ad}(a)) = 0$, it follows that $\mathrm{Trace}_{\mathfrak{g}}(\nabla_X^{(s)}) = 0$. The general element of $H_t(M, \mathfrak{g})$ is a finite sum of decomposable elements (over a basis of $\mathfrak{g}$), which proves the claim. If $Y = g \otimes b$ is also decomposable, then denoting by M_f the operator multiplication by f, we obtain

$$\begin{aligned}\mathrm{Trace}_{\mathfrak{g}}(R^{(s)}(X, Y)) &= [M_f + (\text{order } -q), M_g + (\text{order } -q)] \otimes -(a, b)_{\mathfrak{g}} \\ &= (\text{order } -(q+1)),\end{aligned}$$

since $[M_f, M_g] = 0$ and $[M_f, (\text{order } -q)] = (\text{order } -(q+1))$.

Proposition 1.14 implies that the Ricci curvature never exists in the strict sense. On the circle, however, operators of order -1 have logarithmically diverging trace norms, so are borderline trace class. The following proposition shows that the trace is conditionally convergent. By summing the Lie algebras indices first, we obtain a trace class operator on the circle, and so make sense of the Ricci curvature for any H_s metric, $s > 1/4$.

Proposition 1.18. *For smooth $X, Y \in \mathrm{Map}(M, \mathfrak{g})$ the operator*

$$\mathrm{Trace}_{\mathfrak{g}}\{Z \to R^{(s)}(Z, X)Y\}$$

is pseudodifferential of order $-2q$, where $q = \min(1, 2s)$.

Proof. We use the notation from the previous proof. By (1.15) and the discussion following we deduce that $\{Y \to \nabla_Z^{(s)} Y\}$ has the form $R_g \otimes \mathrm{ad}(b)$ for an operator R_g of order $-q$. The first term of (1.16) vanishes when we take $\mathrm{Trace}_{\mathfrak{g}}$. Using the face that $Q_f = M_f + (\text{order } -q)$, as in the proof of 1.17, we have

$$\begin{aligned}\mathrm{Trace}_{\mathfrak{g}}\{Z \to R^{(s)}(Z, X)Y\} &= \{Q_f R_g - R_g M_f\} \cdot (a, b)_{\mathfrak{g}} \\ &= \{[M_f, R_g] + (\text{order } -q) \cdot R_g\} \cdot (a, b)_{\mathfrak{g}} \\ &= (\text{order } -q-1) + (\text{order } -2q).\end{aligned}$$

2. The curvature of ΩG

To explore deeper properties of the geometry of $\mathrm{Map}(M, G)$, we specialize to the case where M is a circle. Set

$$LG = \mathrm{Map}(S^1, G), \qquad \Omega G = \mathrm{Map}_0(S^1, G).$$

LG is the loop group of G, which we now fix to be a compact, connected, simply connected, simple Lie group, and ΩG is the subgroup of based loops. The based loop groups exhibit more interesting geometry than their unbased counterparts, and we concentrate on them. In fact, the $H_{1/2}$ metric on ΩG is Kähler and will occupy most of our attention. Nevertheless, any mapping space $\mathrm{Map}(M, N)$ carries the natural family of H_s metrics, in general no single one is obviously distinguished, so that we are forced to treat all H_s metrics democratically. As the most accessible mapping spaces are loop groups, explicit computations being possible, we seize the opportunity to explore the Riemannian geometry of the whole family. Rather than report somewhat messy formulas for all H_s , we restrict ourselves to three distinguished cases: the L^2, $H_{1/2}$ and H_1 metrics. Toeplitz operators appear in the curvature of the Kähler $H_{1/2}$ metric. There Ricci curvature makes sense, and we define the first Chern class of ΩG to be the cohomology class represented by the Ricci form. Higher Chern classes are defined analogously by traces of powers of the curvature.

As we explained in §1, the groups LG and ΩG are Hilbert Lie groups with respect to certain Sobolev completions. When we discuss the H_s geometry in the continuous range $s > 1/2$, then we use the H_s completion. For these groups the inner product induced on each tangent space is complete, i.e., H_s ($s > 1/2$) is a strong metric. For $s \leq 1/2$ we use the $H_{1/2+\varepsilon}$ completions to study the H_s metrics, which are then weak metrics on the underlying Hilbert manifold. Here ε is a small positive number. We omit further reference to these completions.

The H_s metrics on ΩG are invariant under left translation by elements of the group ΩG, but there is a larger symmetry, and we are well advised to note this from the start. Let $\mathbf{T}$ denote the circle group; then $\mathbf{T}$ acts on the space of free loops LG by rotation. Form the semidirect product $\mathbf{T} \ltimes LG$. The centralizer of $\mathbf{T}$ in this larger group consists of $\mathbf{T}$ itself together with loops stable under the action of $\mathbf{T}$, that is, the point loops $G \subset LG$. The quotient is the based loop space

$$\Omega G = (\mathbf{T} \ltimes LG)/(\mathbf{T} \times G) \tag{2.1}$$

as can be seen via the map

$$\begin{gathered} T \ltimes LG \to \Omega G, \\ \langle e^{i\theta}, f(\cdot)\rangle \to f(0)^{-1} f(\cdot). \end{gathered}$$

(Of course, we could have written $\Omega G = LG/G$; the $\mathbf{T}$ factor plays a role below when we regard ΩG as a coadjoint orbit.) The Sobolev metrics are homogeneous metrics for $\mathbf{T} \ltimes \Omega G$. To see this more explicitly, we describe the tangent space to ΩG at the identity in this homogeneous representation. (The tangent space to ΩG regarded as a group is $\Omega\mathfrak{g}$.) There is a decomposition

$$\operatorname{Lie}(\mathbf{T} \ltimes LG) = \mathbf{R} \ltimes L\mathfrak{g} = (\mathbf{R} \oplus \mathfrak{g}) \oplus (L\mathfrak{g})_0 = \mathfrak{h} \oplus \mathfrak{m},$$

where $\mathfrak{m} = (L\mathfrak{g})_0$ consists of loops on $\mathfrak{g}$ whose integral over the circle is zero. In terms of Fourier series, the sum of the Fourier coefficients of a loop in $\Omega\mathfrak{g}$ vanishes, while a loop in $\mathfrak{m}$ has a vanishing constant term in its Fourier expansion. There is then an identification

$$\begin{gathered} \Omega\mathfrak{g} \leftrightarrow (L\mathfrak{g})_0, \\ X(\cdot) \to X(\cdot) - \frac{1}{2\pi}\int_{S^1} X(\theta)\, d\theta, \\ X(\cdot) - X(0) \leftrightarrow X(\cdot). \end{gathered} \tag{2.2}$$

We will often use these formulas to convert between the two representations of the tangent space.

Passing to the complexified Lie algebra of $\mathbf{T} \ltimes LG$, the complete decomposition under the action of $\mathbf{T} \times G$ is

$$\begin{aligned} (\mathbf{R} \ltimes L\mathfrak{g})_{\mathbf{C}} &= (\mathbf{R} \oplus \mathfrak{g})_{\mathbf{C}} \oplus \Big(\bigoplus_{n>0} z^n \mathfrak{g}_{\mathbf{C}}\Big) \oplus \Big(\bigoplus_{n>0} z^{-n} \mathfrak{g}_{\mathbf{C}}\Big) \\ &= \mathfrak{h}_{\mathbf{C}} \oplus \mathfrak{m}_+ \oplus \mathfrak{m}_-. \end{aligned} \tag{2.3}$$

Here we denote $z = e^{i\theta}$. Essentially this is the expansion of a loop into its complex Fourier series, with the positive and negative powers of z collected. The complexified tangent space to ΩG is identified with $\mathfrak{m}_+ \oplus \mathfrak{m}_-$. The L^2 metric pairs $z^n \mathfrak{g}_{\mathbf{C}}$ nontrivially only with $z^{-n}\mathfrak{g}_{\mathbf{C}}$ and is minus the Killing form on $\mathfrak{g}_{\mathbf{C}}$; the H_s metric restricted to these spaces is $|n|^{2s}$ times the L^2 metric.

Manifolds which are quotients of a group by the centralizer of a torus are called flag manifolds, and they can be embedded in the dual of the Lie algebra as an orbit of the coadjoint action. Our based loop group ΩG is an infinite dimensional example, and the *adjoint* embedding is

$$\begin{aligned} \Omega G &\to \mathbf{R} \ltimes L\mathfrak{g}, \\ f(\cdot) &\to \langle 1, f^{-1}(\cdot) f'(\theta)\rangle. \end{aligned} \tag{2.4}$$

Atiyah [4] and Pressley [31] observed that there are distinguished metrics on coadjoint orbits. The simplest is the homogeneous metric coming from the Killing form on the group; in our case this is the L^2 metric on ΩG. Then there is the submanifold metric induced by the embedding in the Lie algebra, which is given the flat metric defined by the Killing form. On $L\mathfrak{g}$ the L^2 metric plays the role of the Killing form, and the induced submanifold metric on ΩG is the H_1 metric. Finally, when the coadjoint orbit has a complex structure (e.g. for compact groups), it admits homogeneous Kähler metrics. On ΩG there is (up to a constant) a single homogeneous Kähler metric—the $H_{1/2}$ metric.

The Riemannian curvature of the L^2 metric on ΩG is quite easy to compute, for example directly from Proposition 1.5 once the curvature of the bi-invariant metric on G is known. Alternatively, the same argument which computes the curvature of G (from Proposition 1.6) applies to the bi-invariant L^2 metric on ΩG. Either way, we find

$$R^{(0)}(X,Y) = -\tfrac{1}{4}\mathrm{ad}_{[X,Y]}. \tag{2.5}$$

The sectional curvature of a Riemannian metric is defined by

$$K_{X,Y} = (R(X,Y)Y, X).$$

For the L^2 metric, omitting $d\theta$ from the notation for convenience,

$$K_{X,Y} = \frac{1}{2\pi}\int_{S^1} |[X,Y]|^2. \tag{2.6}$$

This sectional curvature is nonnegative. (Quite generally, for mapping spaces $\mathrm{Map}(M,N)$ the sectional curvature of the L^2 metric is given by integrating over M the sectional curvature of N (cf. Proposition 1.5). Thus if N is nonnegatively curved, then so is the mapping space.) The Ricci curvature, which we defined in (1.13), is the trace of an operator of order zero, by Proposition 1.18, which diverges. Therefore, the scalar curvature is also infinite. We summarize (2.5) and (2.6) in

Proposition 2.7. *For the L^2 metric on ΩG the Riemann curvature is $R^{(0)}(X,Y) = -\frac{1}{4}\mathrm{ad}_{[X,Y]}$. The sectional curvature is nonnegative and the Ricci tensor diverges.*

The L^2 metric is the only H_s metric which is bi-invariant. It follows from the next proposition that the L^2 metric is the only *symmetric* metric among the H_s metrics. Recall that a Riemannian manifold is said to be symmetric if the map $X \to -X$ in each tangent space extends to a global isometry.

Proposition 2.8. *A left invariant metric on a Lie group $\mathcal{G}$ which is symmetric is necessarily bi-invariant.*

Proof. The hypotheses imply that $g \to g^{-1}$ must be an isometry of $\mathcal{G}$, since this is the endomorphism on $\mathcal{G}$ whose induced endomorphism on the Lie algebra is $X \to X$. This, together with the fact that left translation is an isometry, implies that right translation is an isometry; i.e., the metric is bi-invariant.

Next we study the H_1 metric. The first point to verify is that the metric induced from the "Killing form" on $\mathbf{R} \ltimes L\mathfrak{g}$ via the embedding (2.4) is the H_1 metric. The constant metric

$$(\langle s, X\rangle, \langle t, Y\rangle)_{\mathbf{R}\ltimes L\mathfrak{g}} = \mathrm{st} + \frac{1}{2\pi}\int_{S^1}(X,Y)_{\mathfrak{g}}$$

plays the role of the Killing form. Then the pulled back metric on ΩG is left invariant, since this metric on $\mathbf{R} \ltimes L\mathfrak{g}$ is ad-invariant and the embedding (2.4) is induced by the adjoint action. Differentiation of (2.4) at the identity in ΩG yields

$$\begin{aligned} \Omega\mathfrak{g} &\to \mathbf{R} \ltimes L\mathfrak{g}, \\ X(\theta) &\to \langle 0, -X'(\theta)\rangle. \end{aligned}$$

So the induced metric on $\Omega\mathfrak{g}$ is indeed the H_1 metric

$$(X,Y) = \frac{1}{2\pi}\int_{S^1}(X'(\theta), Y'(\theta))_{\mathfrak{g}}.$$

The induced Riemannian covariant derivative ∇ is the orthogonal projection of the (flat) covariant derivative on $\mathbf{R}\ltimes L\mathfrak{g}$ to the tangent plane to ΩG, and the second fundamental form II is the projection onto the normal plane. We compute using the vector field on ΩG defined by the action of an element $Y \in L\mathfrak{g}$. At the point $(\mathrm{Ad}\exp tX)\langle 1,0\rangle$, $X \in L\mathfrak{g}$, it is given by $(\mathrm{Ad}\exp tX)(\langle 0, -Y'\rangle)$, and taking d/dt we obtain $D_X Y = \langle 0, -[X, Y']\rangle$ for the flat covariant derivative. Since the normal projection is given by integration over S^1 (this picks out the constant term in the Fourier expansion), we obtain

$$\begin{aligned} \nabla_X Y &= \left\langle 0, -[X,Y'] + \frac{1}{2\pi}\int_{S^1}[X,Y']\right\rangle, \\ \mathrm{II}(X,Y) &= \left\langle 0, \frac{-1}{2\pi}\int_{S^1}[X,Y']\right\rangle. \end{aligned} \tag{2.9}$$

The Gauss equation of Riemannian geometry, which reads

$$(R^{(1)}(X,Y)Z,W) + (\mathrm{II}(Y,Z),\mathrm{II}(X,W)) - (\mathrm{II}(X,Z),\mathrm{II}(Y,W))$$

since $\mathbf{R} \ltimes L\mathfrak{g}$ is flat, yields the formula for the H_1 curvature.

Proposition 2.10. *The curvature of the H_1 metric on ΩG is*

$$\begin{aligned} (R^{(1)}(X,Y)Z,W)_{H_1} = &\left(\frac{1}{2\pi}\int_{S^1}[Y,Z'], \frac{1}{2\pi}\int_{S^1}[X,W']\right)_{\mathfrak{g}} \\ &- \left(\frac{1}{2\pi}\int_{S^1}[X,Z'], \frac{1}{2\pi}\int_{S^1}[Y,W']\right)_{\mathfrak{g}} \end{aligned}$$

for $X,Y,Z,W \in \Omega\mathfrak{g}$.

It is interesting to observe that, in contrast to the L^2 case, the sectional curvature of the H_1 metric,

$$\begin{aligned} K^{(1)}_{X,Y} = &\left(\frac{1}{2\pi}\int_{S^1}[Y,Y'], \frac{1}{2\pi}\int_{S^1}[X,X']\right)_{\mathfrak{g}} \\ &- \left|\frac{1}{2\pi}\int_{S^1}[X,Y']\right|^2_{\mathfrak{g}'} \end{aligned} \tag{2.11}$$

takes both signs. Fix $a,b \in \mathfrak{g}$ with $[a,b] \neq 0$. Then for $X(\theta) = (\cos(\theta) - 1)a$ and $Y(\theta) = \sin(\theta)b$ the first term in (2.11) vanishes, and $K^{(1)}_{X,Y} < 0$. On the other hand, for

$$\begin{aligned} X(\theta) &= (\cos(\theta) - 1)a + \sin(\theta)b, \\ Y(\theta) &= (\cos(2\theta) - 1)a + \sin(2\theta)b, \end{aligned}$$

we compute $K^{(1)}_{X,Y} = 2|[a,b]|^2 > 0$.

It follows from Proposition 1.16 that the Ricci curvature of the H_1 metric is finite if we take a two-step trace, since operators of order -2 on the circle are of the trace class. Observe that the curvature formula in Proposition 2.10 is unchanged when we pass to the homogeneous representation (2.2). We calculate $\mathrm{Ricci}^{(1)}(X,Y)$ on decomposable complex basis elements $X = z^n a$ and $Y = z^m b$, $a,b \in \mathfrak{g}$. The trace vanishes unless $n = -m$, and in that case, using the Hermitian inner product on the complexification,

$$\begin{aligned}\mathrm{Ricci}^{(1)}(X,Y) &= \sum_{l,c}(R^{(1)}(z^l, c, z^n a)z^{-n}b, z^{-l}c)_{H_1} \\ &= \sum_c -([c,b],[a,c])_{\mathfrak{g}} = (a,b)_{\mathfrak{g}} = \frac{1}{n^2}(X,Y)_{H_1}.\end{aligned}$$

The scalar curvature of this metric is finite and positive.

By far the most interesting metric on ΩG is the $H_{1/2}$ metric. It turns out to be homogeneous Kähler [31], and so its study is facilitated by exploiting the special properties of homogeneous manifolds and of Kähler manifolds. The Kähler structure on ΩG is most easily described by exhibiting its complex and symplectic structures, and then observing that the metric produced by combining these is the $H_{1/2}$ metric. The almost complex structure on ΩG is evident from the decomposition (2.3). The complexified tangent space is identified with $\mathfrak{m}_+ \oplus \mathfrak{m}_-$, with $\mathfrak{m}_+$ the holomorphic tangent space and $\mathfrak{m}_-$ the antiholomorphic tangent space. Alternatively, we can give a J operator on the real tangent space $\Omega\mathfrak{g}$, that is, an operator whose square is -1. From now on we adopt the notation

$$D = \frac{d}{d\theta} = iz\frac{d}{dz}.$$

Then D has no kernel on based loops $\Omega\mathfrak{g}$, since the kernel of D on all loops consists of constant loops, and the only loop based at zero which is constant is the zero loop. Noting that $|D|$ is the square root of the positive Laplacian $-d^2/d\theta^2$, we see that $J = D/|D|$ has square minus the identity. The torsion tensor defined by the almost complex structure vanishes since $\mathfrak{m}_+$ is closed under bracketing. We would like to conclude that ΩG is a complex manifold by applying an infinite dimensional Newlander-Nirenberg Theorem. The version stated by Penot [30] requires that the data be real analytic, which it is in our case (cf. the discussion in [31]). Alternatively, we can realize ΩG directly as a complex quotient of a complex group. Let $LG_{\mathbf{C}}$ denote the group $\mathrm{Map}(S^1, G_{\mathbf{C}})$ of loops in the complex Lie group $G_{\mathbf{C}}$ corresponding to the compact group G. Let $\mathscr{P}$ be the subgroup of loops which extend to holomorphic maps from $|z| \geq 1$ to $G_{\mathbf{C}}$; then

$$\Omega G = LG_{\mathbf{C}}/\mathscr{P}.$$

This representation of ΩG amounts to a factorization of loops in $LG_{\mathbf{C}}$, analogous to the factorization of a complex matrix as the product of a unitary matrix and an upper triangular matrix. In finite dimensions this is the Gram-Schmidt process. From a more sophisticated point of view it reflects the fact that the unitary group acts transitively on a certain Grassmannian, or flag manifold, and this approach generalizes to the loop group case [32, §8]. We emphasize that ΩG is not a complex Lie group.

The other aspect of the Kähler structure of ΩG is the symplectic form ω. Like the almost complex structure J, it is left invariant, and so can be described on the Lie algebra $\Omega\mathfrak{g}$:

$$\omega(X, Y) = \frac{1}{2\pi} \int_{S^1} (X', Y)_{\mathfrak{g}}. \tag{2.12}$$

Here X and Y are to be interpreted as elements of $\Omega\mathfrak{g}$. Alternatively, using the correspondence (2.2), we can take X and Y to belong to $\mathfrak{m}_+ \oplus \mathfrak{m}_-$; the formula for ω is unchanged. This shows that ω is invariant under the larger symmetry group $\mathbf{T} \ltimes \Omega G$. The form ω is nondegenerate since D has no kernel on based loops. A simple computation [31] shows that ω is closed. Alternatively, the form ω arises from the Kostant-Kirillov construction of symplectic structures on coadjoint orbits, and thus it is closed by general principles. Therefore, ΩG is an infinite dimensional Kähler manifold, and ω is the Kähler form for the Kähler metric

$$(X, Y) = \frac{1}{2\pi} \int_{S^1} \left(\left| \frac{d}{d\theta} \right| X(\theta), Y(\theta) \right)_{\mathfrak{g}} d\theta.$$

Comparing with (1.3) we see that we have recovered the $H_{1/2}$ metric.

The curvature for this metric can be obtained as a special case of a general formula for flag manifolds. We adapt the general argument in [19] to our particular situation.

Corresponding to the global action of $\mathbf{T} \ltimes LG$ on ΩG is an infinitesimal action which assigns a vector field ξ_Z to each element $Z \in \mathbf{R} \ltimes L\mathfrak{g}$. (Notice that the vector field ξ_Z is the *right* invariant extension of Z to $\mathbf{T} \ltimes LG$, in contrast to the left invariant extensions of §1.) Evaluation at the identity in ΩG (under complexification) gives the map which identifies $\mathfrak{m}_{\mathbf{C}} = \mathfrak{m}_+ \oplus \mathfrak{m}_-$ with the complexified tangent space to ΩG. Let ∇ denote the Kähler connection and $\mathscr{L}$ the Lie derivative. Then for any $Z \in \mathfrak{g}$ the difference $\nabla_{\xi_Z} - \mathscr{L}_{\xi_Z}$ is tensorial, and so defines a linear transformation on $\mathfrak{m}$. Since both ∇_{ξ_Z} and $\mathscr{L}_{\xi_Z}$ preserve the complex structure, under complexification this transformation separately preserves $\mathfrak{m}_+$ and $\mathfrak{m}_-$. Thus we obtain a map

$$\varphi\colon\ (\mathbf{R} \ltimes L\mathfrak{g})_{\mathbf{C}} \to \mathfrak{gl}(\mathfrak{m}_+) \tag{2.13}$$

as the $\mathbf{C}$-linear extension of

$$X \to (\nabla_{\xi_X} - \mathscr{L}_{\xi_X})\,|_{\mathfrak{m}_+}.$$

We express the $H_{1/2}$ connection by giving an explicit formula for φ, after introducing Toeplitz operators.

Classicially, Toeplitz operators are defined on the Hilbert space of L^2 holomorphic functions on the circle, that is, L^2 complex functions whose Fourier expansion consists entirely of nonnegative powers of $e^{i\theta}$. For a smooth function f on S^1 define

$$\begin{aligned} T_f\colon L^2(S^1;\mathbf{C})_{\mathrm{hol}} &\to L^2(S^1;\mathbf{C})_{\mathrm{hol}},\\ \phi &\to (f\cdot\phi)_+, \end{aligned}$$

where $(f\cdot\phi)_+$ is the holomorphic part of the product $f\cdot\phi$. In terms of Fourier series, we expand $f\cdot\phi = \sum_{n\in\mathbf{Z}} c_n z^n$; then $(f\cdot\phi)_+ = \sum_{n\geq 0} c_n z^n$. The Toeplitz operators we need are a generalization. Let $\mathfrak{m}_+ = H_t(S^1;\mathfrak{g}_{\mathbf{C}})_{\mathrm{shol}}$ $(t = n/2+\varepsilon)$ be the Hilbert space of *strictly* holomorphic Lie algebra valued functions, i.e., functions whose Fourier expansion consists entirely of positive powers of $e^{i\theta}$. For any $Z\colon\ S^1 \to \mathfrak{g}_{\mathbf{C}}$ we define the Toeplitz operator

$$\begin{aligned} T_Z\colon \mathfrak{m}_+ &\to \mathfrak{m}_+,\\ Y &\to [Z,Y]_+. \end{aligned} \tag{2.14}$$

The bracket is computed pointwise, and now "+" denotes projection onto the strictly positive components of the Fourier series.

We compute the Kähler connection in terms of Toeplitz operators.

Theorem 2.15. (a) $\varphi(H) = T_H$ *for* $H\in\mathfrak{h}_{\mathbf{C}}$;

(b) $\varphi(\overline{X}) = T_{\overline{X}}$ *for* $\overline{X}\in\mathfrak{m}_-$;

(c) $\varphi(X) = -T^*_{\overline{X}}$ *for* $X\in\mathfrak{m}_+$.

The Toeplitz operator in (a) is simply a multiplication operator—there is no projection. In (b) the identification $\mathfrak{m}_+ \simeq (\mathfrak{m}_-)^*$ displays Toeplitz operators as the coadjoint representation of $\mathfrak{m}_-$. Here the identification is via the L^2 metric. In (c) the adjoint is taken with respect to the $H_{1/2}$ metric on $\mathfrak{m}_+$. We compute

$$\varphi(X) = D^{-1}T_X D \quad \text{for } X\in\mathfrak{m}_+. \tag{2.16}$$

T_X is a multiplication operator since X is holomorphic.

Before giving the proof of Theorem 2.15 we recall some basic facts about the Kähler connection [25]. A Kähler manifold is first of all a Riemannian manifold, and so has a unique torsion-free metric connection—the Levi-Civita connection. On the other hand, the tangent bundle is holomorphic, and any holomorphic bundle with a Hermitian metric has a unique metric connection—the Hermitian connection—which agrees with the $\bar\partial$ operator in antiholomorphic directions. In particular, it vanishes on holomorphic vector fields in antiholomorphic directions. Finally, the Kähler condition is satisfied precisely when the Levi-Civita connection and the Hermitian connection coincide. These facts are valid in both finite and infinite dimensions.

Proof of Theorem 2.15. (a) This is the isotropy representation which defines the holomorphic tangent bundle. More explicitly, denoting the identity of ΩG by $\bar{e}$, $\xi_H(\bar{e}) = 0$ so that $\nabla_{\xi_H} = 0$ at $\bar{e}$. Also, the map $X \to \xi_H$ is an antihomomorphism of Lie algebras ($[\xi_X, \xi_Y] = -\xi_{[X,Y]}$), from which $\varphi(H) = \operatorname{ad} H$ is immediate.

(b) For $Y \in \mathfrak{m}_+$ let $(\xi_Y)_+$ denote the $(1,0)$ component of the vector field ξ_Y. It is a holomorphic vector field, and we use it to compute $\varphi(\overline{X})Y$. By the remarks preceding the proof,

$$\nabla_{\xi_{\bar{x}}}(\xi_Y)_+ = \bar{\partial}((\xi_Y)_+)(\xi_{\bar{x}}) = 0 \quad \text{at } \bar{e},$$

because $\xi_{\bar{x}}(\bar{e})$ is of type $(0,1)$ and $(\xi_Y)_+$ is holomorphic. As noted in the previous paragraph, we have $\mathscr{L}_{\xi_{\overline{X}}}\xi_Y = -\xi_{[\overline{X},Y]}$, and since $\xi_{\overline{X}}$ preserves holomorphic vector fields, $\mathscr{L}_{\xi_{\overline{X}}}(\xi_Y)_+(p) = -\xi_{[\overline{X},Y]_+}$ as desired.

(c) Both the Kähler covariant derivative ∇_{ξ_Z} and Lie derivative $\mathscr{L}_{\xi_Z}$ preserve the metric and complex structure for real Z. Therefore, φ maps the real Lie algebra $\mathbf{R} \ltimes L\mathfrak{g}$ into skew-Hermitian transformations, whence (c).

The curvature of the Kähler metric is

$$R(X,Y) = [\varphi(X), \varphi(Y)] - \varphi([X,Y]). \tag{2.17}$$

R is an invariant 2-form which, on real vectors, is a skew-Hermitian transformation of the holomorphic tangent space at each point; (2.17) is the expression for R at the basepoint $\bar{e}$. It is a quite general property of Kähler metrics that the curvature is of type $(1,1)$, a fact which also follows immediately in this case from 2.15.

Theorem 2.18. *For $X \in \mathfrak{m}_+$ and $\overline{Y} \in \mathfrak{m}_-$,*

$$R(X,\overline{Y}) = [D^{-1}T_X D, T_{\overline{Y}}] - T_{[X,\overline{Y}]_{\mathfrak{h}_{\mathbf{C}} \oplus \mathfrak{m}_-}} - D^{-1}T_{[X,\overline{Y}]_{\mathfrak{m}_+}} D.$$

On a Kähler manifold the Ricci tensor, which in Riemannian geometry is expressed as the symmetric bilinear form (1.13), is realized as the $(1,1)$ form

$$\sigma(X,\overline{Y}) = \operatorname{Trace} R(X,\overline{Y}). \tag{2.19}$$

Furthermore, by Chern-Weil theory $i\sigma/2\pi$ has cohomological significance *in finite dimensions* it is a representative of the first Chern class of the manifold in de Rham cohomology. In infinite dimensions there is no guarantee that the Ricci form (2.19) is defined, much less has topological significance, since the trace of a general operator in Hilbert space is undefined. For the $H_{1/2}$ metric on ΩG, though, not only does the Ricci curvature make sense, but it also has topological significance. To see that Ricci makes sense, we observe from Theorem 1.16 that $R(X,\overline{Y})$ is a pseudodifferential operator of order -1 on the circle if X, Y are smooth. (Theorem 1.16 was derived for the real curvature, that is, for $R(X,\overline{Y})$ acting on $\mathfrak{m}_+ \oplus \mathfrak{m}_-$. However, the curvature acts diagonally, and the operator on $\mathfrak{m}_-$ is minus the Hermitian conjugate of the operator on $\mathfrak{m}_+$; hence each operator separately is of order -1. Alternatively, the order of $R(X,\overline{Y})$ can be extracted directly from Theorem 2.18 by using basic properties of Toeplitz operators, which we exhibit in §5.) A pseudodifferential operator of order -1 on the circle is not quite trace class; its trace norm diverges logarithmically. But the operator $\mathrm{Trace}_{\mathfrak{g}}(R(X,\overline{Y}))$, obtained by contracting the Lie algebra indices, is of order -2 by Proposition 1.16, hence is of trace class. We use this two-step trace to make sense of the Ricci curvature of ΩG, and we denote it with a tilde.

Theorem 2.20.

$$\widetilde{\mathrm{Trace}}(R(X,\overline{Y})) = \frac{1}{2\pi i}\int_0^{2\pi} (X',\overline{Y})_{\mathfrak{g}_{\mathbf{C}}} = -i\omega(X,\overline{Y}).$$

We emphasize that the sum of the eigenvalues of $R(X,\overline{Y})$ is not absolutely convergent. Rather, it is conditionally convergent, and we specify the order of summation by first summing over a basis of $\mathfrak{g}$ and then over a basis of functions on S^1. Theorem 2.20 implies that ΩG is Kähler-Einstein. We remark that after removing the Lie algebra what is left is the trace of a Toeplitz operator and its adjoint. Such a trace is well known in the Operator Algebra literature (see [23], for example) when the adjoint is taken with respect to L^2. What we show is that the $H_{1/2}$ adjoint gives the same answer.

Proof. Clearly it suffices to verify 2.20 on basis elements $X = z^n a$ and $\overline{Y} = z^{-m}b$ for $a, b, \in \mathfrak{g}$. Then one easily sees that $\widetilde{\mathrm{Trace}}(R(X,\overline{Y}))$ vanishes unless $n = m$. In that case the third term of 2.18 is zero, and the second term

$$T_{[z^n a, z^{-n}b]_{\mathfrak{h}_{\mathbf{C}}\oplus\mathfrak{m}_-}} = \mathrm{ad}([a,b])$$

vanishes after taking $\mathrm{Trace}_{\mathfrak{g}}$, since ad maps into traceless operators. The only surviving term is

$$[D^{-1}T_{z^n a}D, T_{z^{-n}b}](z^l c) = \delta_{l>n}\left(\frac{l-n}{l}\right) z^l[a[bc]] - \left(\frac{l}{l+n}\right) z^l[b[ac]],$$

where $\delta_{l>n}$ indicates that the term appears only if $l > n$. Performing $\mathrm{Trace}_{\mathfrak{g}_{\mathbf{C}}}$ we obtain

$$-\left\{\delta_{l>n}\left(\frac{l-n}{l}\right) - \left(\frac{l}{l+n}\right)\right\} z^l(a,b)_{\mathfrak{g}_{\mathbf{C}}},$$

since $\mathrm{Trace}\{c \to [a[bc]]\} = \mathrm{Trace}\{c \to [b[ac]]\} = -(a,b)_{\mathfrak{g}_{\mathbf{C}}}$ is the Killing form. Finally, we sum over $l > 0$ to compute the Hilbert space trace:

$$\begin{aligned}\widetilde{\mathrm{Trace}}((R(z^n a, z^{-n}b))) &= -\sum_{l>0}\left\{\delta_{l>n}\left(\frac{l-n}{l}\right) - \left(\frac{l}{l+n}\right)\right\}(a,b)_{\mathfrak{g}_{\mathbf{C}}} \\ &= n(a,b)_{\mathfrak{g}_{\mathbf{C}}} = -i\omega(z^n a, z^{-n}b).\end{aligned}$$

The Ricci form $\sigma = \widetilde{\mathrm{Trace}}(R)$ is a closed 2-form on ΩG, and we now determine its cohomology class. By our assumption that G is simply connected and simple we have $\pi_1(G) = \pi_2(G) = 0$ and $\pi_3(G) = \mathbf{Z}$, whence the based loop group ΩG is connected, simply connected, and has $\pi_2(\Omega G) = \mathbf{Z}$. Hence $H_2(\Omega G) = \mathbf{Z}$ by the Hurewicz Theorem, and then $H^2(\Omega G; \mathbf{Z}) = \mathbf{Z}$ by the universal coefficient theorem in cohomology. We determine an explicit generator for this group as an invariant form on ΩG. Since $H^2(\Omega G; \mathbf{Z}) \simeq H^3(G; \mathbf{Z})$ by transgression, we must first produce a generator of the latter. Fortunately, Bott and Samelson [14] already solved this problem many years ago (cf. the discussion in [7, p. 453]). Any root space of a Lie algebra $\mathfrak{g}$ determines an inclusion $\mathfrak{su}(2) \hookrightarrow \mathfrak{g}$, hence a homomorphism $\mathrm{SU}(2) \to G$ by exponentiation. Bott and Samelson proved by Morse theoretic techniques that for a *highest* root space of $\mathfrak{g}$, this map represents a generator of $\pi_3(G)$. Now for $\mathrm{SU}(2)$ we can easily verify that the 3-form

$$\beta_{\mathrm{SU}(2)}(X, Y, Z) = \frac{1}{32\pi^2}([X, Y], Z)_{\mathfrak{su}(2)'}, \qquad X, Y, Z \in \mathfrak{su}(2),$$

represents a generator of $H^3(\mathrm{SU}(2); \mathbf{Z})$, where $(\cdot, \cdot)_{\mathfrak{su}(2)}$ is the killing form of $\mathfrak{su}(2)$. Comparing the Killing form of $\mathfrak{g}$ to that of $\mathfrak{su}(2)$, and denoting by n_G the reciprocal square length of the highest root (relative to the Killing form transferred to $\mathfrak{g}^*$), we conclude that

$$\beta_G(X, Y, Z) = \frac{1}{16\pi^2 n_G}([X, Y], Z)_{\mathfrak{g}}, \qquad X, Y, Z \in \mathfrak{g},$$

represents a generator of $H^3(G; \mathbf{Z})$. The integer n_G is termed the *dual Coxeter number* of G, and is given in the following table.

G	$\mathrm{SU}(n)$	$\mathrm{Spin}(n),\ n \geq 5$	$\mathrm{Sp}(n)$	G_2	F_4	E_6	E_7	E_8
n_G	n	$n-2$	$n+1$	4	9	12	18	30

The transgression of β_G to the based loop group is calculated from the evaluation map $\varepsilon\colon\ S^1 \times \Omega G \to G$ by pulling back β_G via ε and then integrating over S^1. The resulting form is not invariant, but is cohomologous to the invariant form [34, p. 328]

$$\gamma_{\Omega G}(X, Y) = \frac{1}{8\pi^2 n_G} \int_0^{2\pi} (X', Y)_{\mathfrak{g}_{\mathbf{C}}}, \qquad X, Y \in \Omega\mathfrak{g}. \tag{2.21}$$

We can also interpret $\gamma_{\Omega G}$ as an invariant form on $\mathfrak{m}$ (cf. the discussion following (2.12)). This form represents the desired positive generator of $H^2(\Omega G; \mathbf{Z})$.

We compare (2.20) and (2.21) to conclude

Proposition 2.22.

$$\frac{i}{2\pi}\widetilde{\mathrm{Trace}}(R)$$

represents $2n_G$ *times the generator of* $H^2(\Omega G;\mathbf{Z})$.

By analogy with finite dimensions we are led to the following definition.

Definition 2.23. The first Chern class of ΩG is defined to be $2n_G$ times the positive generator of $H^2(\Omega G:\mathbf{Z})$.

Powers of the curvature operator are trace class, whereby the usual Chern-Weil formulas define cohomology classes in every even dimension. We define these to be the higher Chern classes of ΩG. The most direct expression is for the Chern character classes $\mathrm{ch}_l(\Omega G)$, which we define to be the cohomology classes represented by

$$\left(\frac{i}{2\pi}\right)^l \frac{1}{l!}\mathrm{Trace}(R^l). \tag{2.24}$$

Direct computation of these classes form the curvature formula seems beyond reach. Even if explicit formulas are obtained, the identification of the cohomology classes represented would be quite difficult. Rather, we will use topological methods in §5 to identify these classes.

3. Characteristic classes in finite dimensions

The characteristic classes of a finite dimensional manifold M are topological invariants of its tangent bundle. There are many ways to express them, and in this section we briefly review the facts relevant to our study of ΩG. The topology implicit in the tangent bundle is carried by its bundle of frames, whose classifying map induces cohomology classes on M—the *topological* characteristic classes. We discuss the geometry of the frame bundle at some length, as it provides motivation for our infinite dimensional considerations. *Geometrically*, there is a definition of characteristic classes (over the reals) through the curvature of a linear connection. The Chern-Weil Theorem states that the geometric characteristic classes agree with those defined by topology. On homogeneous manifolds there is a third, *group theoretic* definition of characteristic classes due to Borel and Hirzebruch, which coincides with the previous two.

This section is largely expository. Our purpose in collecting these known finite dimensional results is to provide the proper perspective for the discussion in §5.

The frame bundle of a smooth real n-dimensional manifold is constructed as follows. At each point $x \in M$ consider the set F_x of frames of the tangent space

T_xM. A frame at x is an invertible map $f\colon \mathbf{R}^n \to T_xM$, and any two frames f_1, f_2 are related by $f_2 = f_1 \cdot g$ for the invertible map $g = f_1^{-1} f_2\colon \mathbf{R}^n \to \mathbf{R}^n$. Thus the group $\mathrm{GL}(\mathbf{R}^n)$ of invertible transformations of $\mathbf{R}^n$ acts simply transitively on F_x. We denote the collection $\{F_x\}_{x\in M}$ by $\mathrm{GL}(M)$. There is an obvious projection $\mathrm{GL}(M) \xrightarrow{\pi} M$ sending a frame at x to the point x. A cover of M by coordinate charts U_α identifies $\pi^{-1}(U_\alpha)$ with $U_\alpha \times \mathrm{GL}(\mathbf{R}^n)$, and transition functions for $\mathrm{GL}(M)$ can be constructed from those on M. Hence $\mathrm{GL}(M)$ inherits a smooth structure, and the fibration $\mathrm{GL}(M) \to M$ is a principal $\mathrm{GL}(\mathbf{R}^n)$-bundle, the frame bundle of M. The same construction works for complex manifolds, only the group $\mathrm{GL}(\mathbf{C}^n)$ of invertible transformations of the modeling space $\mathbf{C}^n$ replaces $\mathrm{GL}(\mathbf{R}^n)$.

Extra intrinsic geometric structure on a manifold is encoded as a reduction of the structure group of the frame bundle. For example, a Riemannian metric also reduces the structure group. The subgroup of $\mathrm{GL}(\mathbf{R}^n)$ which fixes the standard metric on $\mathbf{R}^n$ is the orthogonal group $\mathrm{O}(\mathbf{R}^n) = \mathrm{O}(n)$, and the reduced $\mathrm{O}(n)$ bundle of frames is formed by the orthonormal bases in each tangent space. The salient feature of the orthonormal frame bundle $\mathrm{O}(M)$ is that it admits a unique torsion-free connection. We already discussed the infinite dimensional version of this Levi-Civita Theorem in §1. A Hermitian structure on a complex n-manifold M is a Hermitian metric on each tangent space. There is a corresponding bundle of unitary frames $\mathrm{U}(M)$ with structure group $\mathrm{U}(\mathbf{C}^n) = \mathrm{U}(n)$. The Hermitian metric is Kähler if this frame bundle $\mathrm{U}(M)$ admits a torsion-free connection.

Arbitrary linear connections, that is, connections on $\mathrm{GL}(M)$, provide a different kind of geometric structure on M. Here the corresponding reduced bundle of frames does not have a direct local description. Rather, observe in general that if Q is a subbundle of a principal bundle P, then a connection on Q always extends uniquely to a connection on P, whereas a connection on P does not necessarily restrict to a connection on Q. Given a connection on P it makes sense to ask for the smallest bundle Q to which the connection restricts. Let $p \in P$ be a fixed basepoint, and consider the union Q' of all curves starting at p whose tangents are horizontal relative to the given connection. Clearly $Q' \subseteq Q$. On the other hand, it is possible to show that Q' is a principal bundle, called the *holonomy bundle*, to which the given connection reduces. Thus $Q = Q'$ is the bundle we seek. The holonomy bundle depends on the basepoint p only up to equivalence, so we are justified in referring to "the" holonomy bundle of a connection. For a linear connection on the frame bundle $\mathrm{GL}(M)$, the holonomy bundle is the reduced bundle of frames which best describes the associated intrinsic geometry. The holonomy construction, so to speak, points to the relevant geometry. For example, if we choose a Riemannian connection on $\mathrm{GL}(M)$, the holonomy bundle is contained in the orthonormal frame bundle $O(M)$ of the Riemannian metric. If the metric happened to be Kähler, then the holonomy bundle would lead us to the unitary frame bundle $\mathrm{U}(M)$. There is even a finite list, due to Berger, of the possible geometries that can arise from a Riemannian connection in finite dimensions.

Let Q be the holonomy bundle of a connection on a G-bundle $P \xrightarrow{\pi} M$, and denote the structure group of Q by H. This *holonomy group* can be identified with the set of points in the fiber at p which are hit by horizontal curves through p, i.e., by horizontal lifts of loops based at $\pi(p)$. The Ambrose-Singer Theorem identifies the Lie algebra of H in terms of the curvature R of the connection, which is a $\mathfrak{g}$-valued 2-form on P.

Theorem 3.1 (*Ambrose-Singer* [1]). *For a finite dimensional principal G-bundle $P \to M$ with connection, the holonomy algebra of the holonomy bundle Q is the subspace $\mathfrak{h}$ of $\mathfrak{g}$ spanned by the curvature $R_q(X, Y)$ as q ranges over Q and X, Y over T_qQ.*

Notice that if the holonomy algebra is an *ideal* in $\mathfrak{g}$, then we can let q range over all of P, since curvature changes by conjugation as we move in a fiber. The standard proof of 3.1 runs roughly as follows. We may as well assume that $P = Q$, since we can always replace P with its holonomy bundle. Consider the distribution on P given by all horizontal vectors together with vertical vectors belonging to $\mathfrak{h} \subseteq \mathfrak{g}$. An elementary computation shows that this distribution is integrable, and the Frobenius Theorem constructs an integral manifold P'. It is not hard to show that P' contains all horizontal curves through p, and now the construction above implies that $P = P'$. Then $\mathfrak{h} = \mathfrak{g}$ is immediate.

We turn now to the topological definition of Chern classes. Let M be an n-dimensional complex manifold, and for simplicity of exposition assume that M is Kähler. Then the bundle of unitary frames $\mathrm{U}(M)$ is classified (up to homotopy) by a map $f\colon M \to B\mathrm{U}(n)$. The characteristic classes of M are the elements in $f^*(H^*(B\mathrm{U}(n))$. Recall that the space $B\mathrm{U}(n)$ is torsion-free, and its integral cohomology is

$$H^*(B\mathrm{U}(n); \mathbf{Z}) = \mathbf{Z}[c_1, c_2, \cdots, c_n], \qquad \deg c_l = 2l. \tag{3.2}$$

The c_l are the universal *Chern classes.* So the lth Chern class of M is $f^*(c_l)$. We remark that for many purposes it is better, when working over the *reals*, to replace the generators c_l in (3.2) with certain generators ch_l whose sum is the *Chern character*. There are integral classes σ_l whose image in real cohomology is $l! \cdot \mathrm{ch}_{l'}$ and the σ_l are related to the c_l by Newton's formulas:

$$\begin{aligned} \sigma_1 &= c_1, \\ \sigma_2 &= c_1^2 - 2c_2, \\ &\cdots \\ \sigma_l - \sigma_{l-1}c_1 + \sigma_{l-2}c_2 - \cdots + (-1)^{l-1}lc_l &= 0. \end{aligned} \tag{3.3}$$

There is an inclusion $\mathrm{U}(n) \to \mathrm{O}(2n)$, reflecting the fact that any complex manifold may be viewed as a real manifold, and the pullback $H^*(B\mathrm{O}(2n)) \to H^*(B\mathrm{U}(n))$ on cohomology, induced from the resulting map $B\mathrm{U}(n) \to B\mathrm{O}(2n)$, expresses the Stiefel-Whitney and Pontrjagin classes of a complex manifold in terms of its Chern classes. As a result,

$$\begin{aligned} w_{2l} &= c_l \pmod 2, \\ w_{2l-1} &= 0, \\ p_1 &= c_1^2 - 2c_2, \end{aligned} \tag{3.4}$$

etc. The first relation states in particular that $w_2 = c_1 \pmod 2$. It is well known that a real orientable manifold admits a spin structure if and only if its second Stiefel-Whitney class vanishes, which for complex manifolds then translates to the condition that the first Chern class be divisible by two. A glance at 2.23 shows this to be true for our defined value of $c_1(\Omega G)$, and we are led to state that ΩG is a spin manifold.

A second approach to Chern classes is geometric, rather than topological. For convenience we again restrict ourselves to Kähler manifolds, although the theory applies in much greater generality. Recall that the curvature of a Kähler manifold M is a differential form of type (1,1) whose value at each point is a skew-Hermitian transformation of the tangent space. Ordinary differential forms are constructed by taking higher traces of the curvature.

Theorem 3.5 (*Chern-Weil*). *Let M be a Kähler manifold with curvature R. Then the inhomogeneous differential form* $\det(1 + (iR/2\pi))$ *is closed, and its de Rham cohomology class* $[\det(1 + (iR/2\pi))]$ *is the image of* $1 + c_1(M) + c_2(M) + \cdots$ *in real cohomology.*

Equating the two dimensional cohomology classes in 3.4, we find in particular

$$c_1(M) = \left[\frac{i}{2\pi}\mathrm{Trace}(R)\right]. \tag{3.6}$$

Equation (3.6) was the motivation behind our definition of $c_1(\Omega G)$ in §2. Also, equation (2.24) is the Chern-Weil form for the Chern character.

There is a third approach to characteristic classes in terms of group theory, which applies to homogeneous manifolds [10]. Our interest is in *flag manifolds* (cf. [19]). A flag manifold is the quotient of a compact Lie group G by the centralizer of a torus in G. It can be realized as a coadjoint orbit of G. We first study the generic orbit—the full flag manifold. Let $T \subset G$ be a maximal torus of G, and consider the adjoint action of T on the Lie algebra $\mathfrak{g}$. Since T is abelian, under complexification there is a decomposition

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus \left(\bigoplus_{\alpha>0} \mathfrak{g}_\alpha\right) \oplus \left(\bigoplus_{\alpha>0} \mathfrak{g}_\alpha\right) = \mathfrak{t}_{\mathbf{C}} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_- \tag{3.7}$$

into one dimensional root space $\mathfrak{g}_\alpha$, collected into positive and negative roots (relative to a fixed Weyl chamber). The complexified tangent space to the quotient G/T at a fixed basepoint can be identified with $\mathfrak{n}_+ \oplus \mathfrak{n}_-$, which gives an integrable almost complex structure. Because $\mathfrak{n}_+$ splits into a direct sum of one dimensional spaces under T, the holomorphic tangent bundle to G/T splits into a direct sum of line bundles. The first Chern class of G/T is then the sum of the Chern classes of these line bundles. Consider the fibration $G \to G/T$ with fiber T. If we assume that G is simply connected, then transgression gives an isomorphism $H^1(T\colon \mathbf{Z}) \simeq H^2(G/T; \mathbf{Z})$, which can be described explicitly using differential forms. More relevant to us is the identification of $H^1(T;\mathbf{Z})$ with the weight lattice in $\mathfrak{t}^* = H^1(T;\mathbf{R})$. Therefore, $c_1(G/T) \in H^2(G/T;\mathbf{Z})$ is identified with an element of the weight lattice. The same holds for the line bundle associated to the root space $\mathfrak{g}_\alpha$, and it is practically a tautology that its first Chern class is $\alpha \in \mathfrak{t}^*$. Therefore,

Theorem 3.8. *The first chern class of the flag manifold G/T is the sum of the positive roots $2\rho_G = \sum_{\alpha>0} \alpha$.*

The factor of 2 is inserted because ρ_G, defined to be half the sum of the positive roots of G, is also the sum of the *fundamental weights*. This alternative characterization of $c_1(G/T)$, as twice the sum of the fundamental weights, makes sense for the Kac-Moody algebra and is our link with the infinite dimensional situation.

The topology of the real frame bundle of G/T is essentially trivial. G/T is the *full flag manifold* of G, and is realized as the principal coadjoint orbit of G in $\mathfrak{g}^*$. We computed $c_1(G/T)$ above by expressing the holomorphic tangent bundle as the homogeneous bundle associated to $G \to G/T$ via the representation of T on $\mathfrak{n}_+$. Now we observe that the (real) normal bundle to G/T in $\mathfrak{g}^*$ can be identified with the homogeneous bundle associated to the adjoint representation of T on $\mathfrak{t}^*$. But this bundle is trivial, since T is abelian. Furthermore, the sum of the (real) tangent bundle and normal bundle to G/T is the restriction of the tangent bundle of $\mathfrak{g}^*$, which is also trivial. This proves

Proposition 3.9. *The real tangent bundle to G/T is stably trivial.*

As a consequence, all of the Pontrjagin and Stiefel-Whitney classes of G/T vanish. The simplest case is $G = \mathrm{SU}(2)$. Then G/T is the 2-sphere, and our argument is the standard one (for all spheres) which proves that the tangent bundle is stably trivial. Nongeneric coadjoint orbits do not have stably trivial tangent bundles in general. The first example is $G = \mathrm{SU}(3)$, where $\mathbf{CP}^2$ occurs as a coadjoint orbit. The second Stiefel-Whitney class $w_2(\mathbf{CP}^2) \neq 0$ since $\mathbf{CP}^2$ is not a spin manifold. Also, the first Pontrjagin class $p_1(\mathbf{CP}^2) \neq 0$ since $\mathbf{CP}^2$ has nonzero signature.

The Chern classes of an *intermediate flag manifold*, these nongeneric coadjoint orbits, can also be computed in terms of roots. Such manifolds are homogeneous spaces G/H for H the centralizer of some torus in G. The generic orbit occurs when H is the maximal torus, but more general orbits are obtained from centralizers of subtori. Under the action of H the complexified Lie algebra decomposes as

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{h}_{\mathbf{C}} \oplus \mathfrak{m}_+ \oplus \mathfrak{m}_-, \tag{3.10}$$

where $\mathfrak{m}_+$ is the sum of the positive *complementary* root spaces, and $\mathfrak{m}_-$ is the sum of the negative complementary root spaces. The maximal torus T is contained in H, and the complementary roots are those roots of $\mathfrak{g}$ which do not occur in the decomposition of $\mathfrak{h}_{\mathbf{C}}$ under T. The submanifold G/H is complex, as is evident from (3.10), and now

Proposition 3.11. *The first Chern class of an intermediate flag manifold G/H is the sum of the positive complementary roots.*

In the proposition $H^2(G/H)$ is identified with a subgroup of $H^1(T)$ via its image in $H^2(G/T)$ under the pullback from $G/T \to G/H$ using the transgression alluded to above. The holomorphic tangent bundle to G/H splits when pulled up to G/T (this is the "splitting principle" in the theory of characteristic classes), and exactly the positive complementary roots occur in the splitting.

We summarize the various definitions of Chern classes.

Theorem 3.12. *Let M be a finite dimensional complex manifold. The following definitions for the Chern classes of M are equivalent*:

Chern I: *The Chern classes of M are the topological characteristic classes of its frame bundle, obtained by transgressing certain cohomology classes in* $\mathrm{GL}(\mathbf{C}^n)$.

Chern II: *If M admits a Kähler metric with curvature R, then the sum of the Chern classes of M is represented by the differential form* $\det(1+(iR/2\pi))$.

Chern III: *The first Chern class of a flag manifold G/H is the sum of the positive complementary roots. For the full flag manifold G/T the first Chern class equals twice the sum of the fundamental weights.*

4. The first Chern class of ΩG

The smooth structure of an infinite dimensional manifold does not carry nontrivial topological information. In finite dimensions nontrivial topology in the frame bundle determines characteristic classes, but a theorem of Kuiper asserts that for Hilbert manifolds the frame bundle is always trivial. Extra geometric structure must be imposed before nonzero characteristic classes appear. The based loop group ΩG carries extra structure—it is a Kähler manifold. We compute its curvature in §2. Chern classes for finite dimensional Kähler manifolds can be defined in terms of curvature 3.12 (II), and this definition makes sense on ΩG. We calculated that according to this definition $c_1(\Omega G)$ is $2n_G$ times the generator of $H^2(\Omega G)$. Now we check this value against the group theory definition of Chern classes 3.12(III). The based loop group is an intermediate flag manifold for an affine Kac-Moody group. The full flag manifold $\mathscr{F}$ fibers over ΩG with fiber G/T, and our curvature formulas lead to its (geometric) first Chern class, which turns out to be the sum of $c_1(\Omega G)$ and $c_1(G/T)$. This agrees with twice the sum of the fundamental weights of the corresponding Kac-Moody group, which defines the group theoretic first Chern class. The first Chern class of the full flag manifold in finite dimensions is also the sum of the positive roots, and from this point of view we have "regularized" the sum of the positive integers to be $2n_G/\dim G$. Applied to the complementary roots associated with ΩG, this regularization computes the correct value of $c_1(\Omega G)$. Pontrjagin and Stiefel-Whitney classes for ΩG and $\mathscr{F}$ are derived from the Chern classes by standard formulas (3.4), and we prove in §5 that these (geometric) real characteristic classes vanish. The fact that ΩG as a real manifold is a Lie group suggests that these real classes on ΩG *should* vanish, and so provides some verification of the higher Chern classes. Furthermore, the triviality of these classes for finite dimensional full flag manifold G/T is consistent with their vanishing for $\mathscr{F}$.

Since characteristic classes in infinite dimensions depend on extra geometric structure, the defined values could conceivably vary with the geometry; there is no underlying topology which remains fixed, as there is in finite dimensions. Therefore, these classes should be checked in geometric problems. Kac-Moody groups provide one geometric setting in which to verify the Chern classes of ΩG; instantons on the 4-sphere provide another. The first Chern class of ΩG is related to the instanton equations through their algebro-geometric interpretation. By a formal argument our value of $c_1(\Omega G)$ predicts the correct dimension of the moduli space.

Fundamentally, Chern classes in finite dimensions arise from the nontrivial topology of $\mathrm{GL}(\mathbf{C}^n)$, reflected in the twisting of the frame bundle 3.12(I). Kuiper's Theorem states that the structure group of Hilbert manifold, the group $\mathrm{GL}(\mathscr{H}_{\mathbf{C}})$ of all invertible transformations on a complex Hilbert space, is contractible. This leads to the triviality of the frame bundle, as noted above. On ΩG we have a Kähler metric, which reduces to the group of unitaries $U(\mathscr{H}_{\mathbf{C}})$, but this group is still contractible. The extra geometric structure that yields a nontrivial reduced frame bundle is the Levi-Civita connection. In §3 we argued that the holonomy bundle of a linear connection picks out the reduction of the frame bundle relevant to the geometry of the connection. Although we are unable to generalize the holonomy bundle construction and the Ambrose-Singer Theorem to infinite dimensions, our intuition still derives from their application to ΩG. We conclude this section with a discussion of these ideas (cf. [19, §3]).

Kac-Moody algebras are defined from a general algebraic point of view by Cartan matrices, generators and relations, etc. The first examples are of *finite type*, and are the finite dimensional (simple) Lie algebras. The next class is the set of *affine* algebras. Algebraically one constructs these affine algebras from a finite dimensional simple algebra $\mathfrak{g}_{\mathbf{C}}$ by considering the $\mathfrak{g}_{\mathbf{C}}$-valued Laurant series $\mathfrak{g}_{\mathbf{C}} \otimes \mathbf{C}[z, z^{-1}]$. The operator $D = z\,d/dz$ operates as a derivation, and leads to the semidirect sum

$$\mathbf{C}D \ltimes \{\mathfrak{g}_{\mathbf{C}} \otimes \mathbf{C}[z, z^{-1}]\}. \tag{4.1}$$

The complex Lie algebra $(\mathbf{R} \ltimes L\mathfrak{g})_{\mathbf{C}}$ which we studied in §2 is the completion of (4.1) in a Hilbert space topology. There is also a one dimensional central extension, represented by the Lie algebra cocycle (2.12). The corresponding group extension is a circle bundle over $\mathbf{T} \ltimes LG$. This group is a "compact form" for the affine Kac-Moody algebra. "Compact form" should be taken quite seriously. Not only loop groups, but all loop spaces $\mathrm{Map}(S^1, N)$ behave in most respects like compact, finite dimensional manifolds. We will see some manifestations of this phenomenon shortly.

To apply (3.8) to the Kac-Moody group $\mathbf{T} \ltimes LG$ (we ignore the central extension temporarily), we must first specify a group which plays the role of the maximal torus. This turns out to be $\mathbf{T} \times T$, the torus $T \subset LG$ sitting inside as point loops. Under the action of $\mathbf{T} \times T$ the complex Kac-Moody algebra decomposes into a direct sum of one dimensional spaces:

$$(\mathbf{R} \ltimes L\mathfrak{g})_{\mathbf{C}} = (\mathbf{R} \oplus \mathfrak{t})_{\mathbf{C}} \oplus \mathfrak{m}_+ \oplus \mathfrak{m}_-, \tag{4.2}$$

where

$$\mathfrak{m}_+ = \left\{ \bigoplus_{\alpha>0} \mathfrak{g}_\alpha \right\} \oplus \left\{ \bigoplus_{n>0} z^n \left[\mathfrak{t}_{\mathbf{C}} \oplus \left(\bigoplus_\alpha \mathfrak{g}_\alpha \right) \right] \right\},$$

$$\mathfrak{m}_- = \left\{ \bigoplus_{\alpha>0} \mathfrak{g}_{-\alpha} \right\} \oplus \left\{ \bigoplus_{n>0} z^{-n} \left[\mathfrak{t}_{\mathbf{C}} \oplus \left(\bigoplus_\alpha \mathfrak{g}_\alpha \right) \right] \right\}.$$

We read off the positive roots of the Kac-Moody algebra from $\mathfrak{m}_+$. It is the set

$$\begin{aligned} &\{\langle 0,\alpha\rangle : \alpha > 0 \text{ is a positive root of } \mathfrak{g}\} \\ &\quad \cup \dim T \cdot \{\langle n, 0\rangle : n = 1,2,3\cdots\} \\ &\quad \cup \{\langle n,\alpha\rangle : n = 1,2,3,\cdots \text{ and } \alpha \text{ is a root of } \mathfrak{g}\} \end{aligned} \tag{4.3}$$

sitting inside $H^1(\mathbf{T} \times T) = \mathbf{R} \times \mathfrak{t}^*$. The quotient manifold

$$\mathscr{F} = (\mathbf{T} \ltimes LG)/(\mathbf{T} \times T)$$

is the full flag manifold of the Kac-Moody group. Our based loop group ΩG is an intermediate flag manifold, as is clear from (2.1), since $\mathbf{T} \times G$ is the centralizer of $\mathbf{T}$. The set of positive complementary roots is the union of the second and third sets in (4.3); this follows from the decomposition (2.3). If Proposition 3.11 were true in infinite dimensions, we could compute the first Chern class of ΩG as the sum

$$\sum_{n=1}^{\infty} \left(\dim T \cdot \langle n, 0\rangle + \sum_\alpha \langle n,\alpha\rangle \right) = \left\langle \dim G \cdot \sum_{n=1}^{\infty} n, 0 \right\rangle. \tag{4.4}$$

Of course, this sum diverges, and the easy computation of the first Chern class of flag manifolds in finite dimensions fails here. Similarly, to compute the first Chern class of the full flag manifold $\mathscr{F}$ we sum all of the positive roots, and an analogous computation yields

$$\left\langle \dim G \cdot \sum_{n=1}^{\infty} n, 2\rho \right\rangle, \tag{4.5}$$

which is also infinite. Now in finite dimensions the first Chern class of the full flag manifold is also twice the sum of the fundamental weights. Fundamental weights do make sense for the Kac-Moody situation, there is a finite number of them, and their sum is readily computable.

Proposition 4.6. *The sum of the fundamental weights of the affine Kac-Moody Lie algebra* $\mathbf{R} \ltimes L\mathfrak{g}$ *is* $\langle n_G, \rho\rangle$.

This is [24, Exercise 7.16]. The *weights* of the Kac-Moody algebra lie in $H^1(\mathbf{T} \times T)$, where $\mathbf{T}$ is the circle of the central extension. The *roots* lie in $H^1(\mathbf{T} \times T)$, where $\mathbf{T}$ is the circle of the derivation D. For the purposes of our heuristic arguments these circles must be identified. Combining (3.8) and (4.6) we conclude that the group theory definition of Chern classes (3.12(III)) yields $c_1(\mathscr{F}) = \langle 2n_G, 2\rho\rangle$. Now we can make sense of the divergent sum in (4.4). For comparing (4.6) with (4.5), we see that we have set

$$\dim G \cdot \sum_{n=1}^{\infty} n = 2n_G. \tag{4.7}$$

Plug (4.7) into (4.4); then the group theory definition of Chern classes formally gives $c_1(\Omega G) = 2n_G$, which agrees with the geometric definition from curvature.

This argument is somewhat convoluted, so we repeat it for clarity. The first Chern class of a finite dimensional manifold can be computed in terms of positive roots, and for the *full* flag manifold G/T it can be expressed as twice the sum of the fundamental weights; this is the content of (3.8) and (3.11). The sum of the positive roots of the Kac-Moody algebra diverges, whereas twice the sum of the fundamental weights makes sense. Thus we defined the (regularized) sum of positive roots to be twice the sum of the fundamental weights. Then the sum of the complementary roots for the *intermediate* flag manifold ΩG is also regularized, and its regularized value agrees with our curvature computations in §2. Therefore, definitions 3.12(II) and 3.12(III) make sense for ΩG, and they coincide for c_1.

To see that the geometric and group theoretic definitions of the first Chern class agree on $\mathscr{F}$, we compute the Ricci curvature of $\mathscr{F}$ in a homogeneous Kähler metric. A geometric definition for $c_1(\mathscr{F})$ follows, as in §2, and it agrees with the group theoretic definition above. The full flag manifold $\mathscr{F}$ fibers holomorphically over ΩG with fiber G/T. Furthermore, since $LG \sim \Omega G \times G$ topologically, it follows that $\mathscr{F} \sim \Omega G \times G/T$ as topological spaces. This product decomposition does not hold in the holomorphic category. Nevertheless, as (4.5) suggests, the first Chern class of LG is the sum of the first Chern class of ΩG and of G/T, where we mean the Chern classes defined by curvature. This is hardly obvious and requires calculation. Now we remarked earlier that our curvature formula (2.18) holds for any (Kähler) flag manifold [19]. For $\mathscr{F}$ we merely need to substitute the decomposition (4.2) for (2.3). Then Theorem 2.15 remains valid, the operators T_Z still defined by (2.14), but now with projection onto the $\mathfrak{m}_+$ of (4.2). In fact, with respect to the splitting $\mathfrak{m}_\pm = \mathfrak{m}_\pm^{(0)} \oplus \mathfrak{m}_\pm^{(1)}$ given by (4.2), the operators $\varphi(Z)$ are block triangular. If the curvature were block diagonal, then $\mathscr{F}$ would be a Riemannian product. The situation is not quite so simple, but the trace of the curvature still behaves like a product.

Proposition 4.8. *For the full flag manifold* $\mathscr{F}$ *the Ricci curvature, written as an invariant* $(1,1)$ *form on* $\mathfrak{m}_+ \oplus \mathfrak{m}_-$ (*cf.* (4.2)), *is*

$$\begin{aligned}\operatorname{Trace}(R(X^{(0)}+X^{(1)},\overline{Y}^{(0)}+\overline{Y}^{(1)}))\\ = -\left\{\sum_{\alpha>0}\alpha([X^{(0)},\overline{Y}^{(0)}])+i\omega(X^{(1)},\overline{Y}^{(1)})\right\}.\end{aligned}$$

Proof. First we compute the curvature of G/T (cf. [19]). Homogeneous Kähler metrics are parametrized by the interior of a Weyl chamber, and we fix a choice $\mu \in \mathfrak{t}^*$. Let H_μ be the dual element in $\mathfrak{t}$. Then $D_{G/T} = \operatorname{ad}(H_\mu)$ replaces $D_{\Omega G} = iz\,d/dz$ in (2.16) and (2.18). The curvature formula (2.18) applies, but the "Toeplitz operators" are defined with respect to the decomposition (3.7). Only the second term of (2.18) contributes to the trace, which is

$$\operatorname{Trace}(R_{(G/T)}(X,\overline{Y})) = -\sum_{\alpha>0}\alpha([X,\overline{Y}]_{\mathfrak{t}_{\mathbf{C}}}) \tag{4.9}$$

for $X \in \mathfrak{n}_+$ and $\overline{Y} \in \mathfrak{n}_-$. Notice that (4.9) is independent of μ.

On $\mathscr{F}$ we use the homogeneous Kähler metric determined by μ and the fixed scale factor of the inner product on ΩG. The Ricci curvature is again independent of μ. Using (2.15) we calculate the φ operators for $\mathscr{F}$. Thus

$$\varphi(Z) = A_Z + E_Z, \tag{4.10}$$

where E_Z is the diagonal matrix

$$A_Z = \begin{pmatrix}\varphi_{G/T}(Z) & 0\\ 0 & \varphi_{\Omega G}(Z)\end{pmatrix} \tag{4.11}$$

of the φ operators for G/T and ΩG, and E_Z is the error term. We write $P_+\colon \mathfrak{g}_{\mathbf{C}} \to \mathfrak{n}_+$ for the projection onto the positive roots and $\pi_0\colon \bigoplus_{n\in\mathbf{Z}} z^n\mathfrak{g}_{\mathbf{C}} \to \mathfrak{g}_{\mathbf{C}}$ for projection onto the zero component. Then using the decomposition $\mathfrak{m}_\pm = \mathfrak{m}_\pm^{(0)} \oplus \mathfrak{m}_\pm^{(1)}$ as above,

$$\begin{aligned}E_H &= 0,\\ E_{\overline{X}} &= \begin{pmatrix}0 & P_+\pi_0\operatorname{ad}(\overline{X}^{(1)})\\ 0 & \operatorname{ad}(\overline{X}^{(0)})\end{pmatrix},\\ E_X &= \begin{pmatrix}0 & 0\\ D_{\Omega G}^{-1}\operatorname{ad}(X^{(1)})D_{G/T} & \operatorname{ad}(X^{(0)})\end{pmatrix},\end{aligned} \tag{4.12}$$

where $H, X, \overline{X}$ are in $\mathfrak{h}_{\mathbf{C}}, \mathfrak{m}_+, \mathfrak{m}_-$ respectively, relative to (4.2). Now plug (4.10) into the curvature formula (2.18). The terms involving A_Z are diagonal, and taking Trace yields the sum of the Ricci curvatures of G/T and ΩG. This is the result stated in the theorem, so we must prove that the terms involving E do not contribute.

Observe that because $\mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}(a)) = 0$ for $a \in \mathfrak{g}$,

$$\mathrm{Trace}_{\mathfrak{g}}(E_{\overline{X}}) = \mathrm{Trace}_{\mathfrak{g}}(E_X) = 0. \tag{4.13}$$

Thus the second and third terms in (2.18) do not contribute to the error term. There are three error terms which arise from the first term in (2.18):

$$[A_X, E_{\overline{Y}}], \quad [E_X, A_{\overline{Y}}], \quad [E_X, E_{\overline{Y}}].$$

The diagonal part of the first term is

$$[A_X, E_{\overline{Y}}] \,|\, _{\mathfrak{m}_+^{(1)}} = [D_{\Omega G}^{-1} T_{X^{(1)}} D_{\Omega G}, \mathrm{ad}(\overline{Y}^{(0)})] \,|\, _{\mathfrak{m}_+^{(1)}}.$$

Its trace is zero, since there are no diagonal entries relative to the usual basis of $\mathfrak{m}_+^{(1)}$. The second term $[E_X, A_{\overline{Y}}]$ behaves similarly. Only the third term $[E_X, E_{\overline{Y}}]$ requires computation. Explicitly,

(4.14)

$$[E_X, E_{\overline{Y}}] = \begin{pmatrix} -P_+\pi_0 \mathrm{ad}(\overline{Y}^{(1)}) D_{\Omega G}^{-1} \mathrm{ad}(X^{(1)}) D_{G/T} & * \\ * & \begin{array}{c} D_{\Omega G}^{-1} \mathrm{ad}(X^{(1)}) D_{G/T} P_+ \pi_0 \mathrm{ad}(\overline{Y}^{(1)}) \\ +[\mathrm{ad}(X^{(0)}), \mathrm{ad}(\overline{Y}^{(0)})] \end{array} \end{pmatrix}.$$

Now

$$\mathrm{Trace}_{\mathfrak{g}} \left([\mathrm{ad}(X^{(0)}), \mathrm{ad}(\overline{Y}^{(0)})] \Big|_{\mathfrak{m}_+^{(1)}} \right) = 0$$

is obvious. Fix basis vectors $X^{(1)} = z^n a$ and $Y^{(1)} = z^{-n} b$. Then the upper left square in (4.14) operates on $\mathfrak{m}_+^{(0)} = \mathfrak{n}_+ \subset \mathfrak{g}_{\mathbb{C}}$, and for $c \in \mathfrak{n}_+$

$$-P_+\pi_0 \mathrm{ad}(\overline{Y}^{(1)}) D_{\Omega G}^{-1} \mathrm{ad}(X^{(1)}) D_{G/T}(c) = -P_+ \mathrm{ad}(b) \frac{1}{n} \mathrm{ad}(a) \mathrm{ad}(H_\mu)(c).$$

The trace over $\mathfrak{n}_+$ equals

$$-\frac{1}{n} \mathrm{Trace}_{\mathfrak{g}_{\mathbb{C}}}(P_+ \mathrm{ad}(b) \mathrm{ad}(a) \mathrm{ad}(H_\mu)). \tag{4.15}$$

The lower right square in (4.14) operates on $\mathfrak{m}_+^{(1)}$, and only contributes to the trace on the finite dimensional space $z^n \mathfrak{g}_{\mathbb{C}} \subset \mathfrak{m}_+^{(1)}$. Then for $c \in \mathfrak{g}_{\mathbb{C}}$,

$$D_{\Omega G}^{-1} \mathrm{ad}(X^{(1)}) D_{G/T} P_+ \pi_0 \mathrm{ad}(\overline{Y}^{(1)})(z^n c) = \frac{1}{n} \mathrm{ad}(a) \mathrm{ad}(H_\mu) P_+ \mathrm{ad}(b)(z^n c).$$

$\mathrm{Trace}_{\mathfrak{g}_{\mathbb{C}}}$ of this expression is

$$\frac{1}{n} \mathrm{Trace}_{\mathfrak{g}_{\mathbb{C}}}(\mathrm{ad}(a) \mathrm{ad}(H_\mu) P_+ \mathrm{ad}(b)). \tag{4.16}$$

Hence (4.15) and (4.16) cancel.

Therefore, the error terms E_Z do not contribute to the Ricci curvature, and the proposition is proved.

We have now checked the first Chern class of both ΩG and $\mathscr{F}$, calculated from curvature, against the value predicted by the positive roots in the Kac-Moody algebra. The first Chern class of the full flag manifold also enters the representation theory of compact groups, particularly through the Weyl character formula. There is an analogous Kac character formula for representations of Kac-Moody groups, and our value of $c_1(\mathscr{F})$ fits in here as well, as we explain in [19]. There seems to be no geometric problem against which to directly check the higher geometric Chern classes. However, we prove in §§5 and 6 that the Pontrjagin classes, which are certain combinations of Chern classes, and the Stiefel-Whitney classes, which are mod 2 reductions of the Chern classes, all vanish for both ΩG and $\mathscr{F}$. The vanishing of these real characteristic classes of $\mathscr{F}$ fits the facts in finite dimensions; the real classes for the corresponding full flag manifold G/T in finite dimensions vanish by Proposition 3.9. The based loop ΩG is a real Hilbert Lie group, and by analogy with finite dimensional groups we expect its real characteristic classes to be zero. Interestingly, ΩG plays two roles as a flag manifold—it is a factor in the full flag manifold $\mathscr{F}$, and by itself is an intermediate flag manifold. In finite dimensions the real tangent bundle to intermediate flag manifolds (like projective spaces) tends not to be stably trivial, as we observed in §3. Therefore, in contrast to its complex geometry, with respect to its real geometry ΩG behaves more like a group, or a factor of the full flag manifold, than it does like an intermediate flag manifold. Notice, too, that flag manifolds of compact Lie groups are never themselves groups, so the based loop groups is quite special in this regard.

The first Chern class of ΩG can be checked in another branch of geometry—the instanton equations on S^4. The setting for Yang-Mills is a principal bundle $P \to S^4$ with group G. Such bundles are classified by an *instanton number* k, which is the four dimensional characteristic class of P. (For $\mathrm{SU}(n)$ bundles k is minus the second Chern class of the associated complex vector bundle.) *Instantons* are connections A whose curvature F_A satisfies the *self-dual Yang-Mills equations*

$$F_A = *F_A. \tag{4.17}$$

Recall that F_A is a 2-form on S^4 with values in the adjoint bundle associated to P, and $*F_A$ is the dual 2-form given by the Hodge star operator. There is an infinite dimensional symmetry group of these equations, the group $\mathscr{G}_0$ of based gauge transformations, i.e., automorphisms of P covering the identity map on S^4, which we normalize to be the identity on the fiber at the north pole. Then the moduli space $\mathscr{M}_{\mathrm{SD}}(G,k)$ of self-dual connections is the space of instantons modulo $\mathscr{G}_0$. The self-dual equations (4.17) are elliptic transverse to the $\mathscr{G}_0$ action, so the moduli space is finite dimensional. For the standard metric on S^4, $\mathscr{M}_{\mathrm{SD}}(G,k)$ is a smooth manifold, and its dimension is computed in [7] as the index of a certain Dirac operator on S^4. The result is

$$\dim_{\mathbf{R}} \mathscr{M}_{\mathrm{SD}}(G,k) = 4kn_G, \tag{4.18}$$

where n_G is again the dual Coxeter number of G. (The dimension of the moduli space of *unbased* instantons differs by $\dim(G)$.)

Many mathematicians have studied the Yang-Mills equations, starting in the mid 70's, and it was quickly realized that these equations have an interpretation in algebraic geometry. The state of the art in that development is a recent theorem of Atiyah [5] and Donaldson [16], which they prove only for classical groups. Consider $\mathbf{CP}^1$ with a basepoint, and let $\mathscr{M}_{\mathrm{hol}}(G,k)$ denote the space of all *based* holomorphic maps $\mathbf{CP}^1 \to \Omega G$ of degree k, that is, holomorphic maps which send the basepoint to the constant loop at the identity and induce multiplication by k on second homology.

Theorem 4.19 (*Atiyah-Donaldson*). *The moduli space $\mathscr{M}_{\mathrm{SD}}(G,k)$ of k-instantons modulo based gauge transformations is diffeomorphic to the moduli space $\mathscr{M}_{\mathrm{hol}}(G,k)$ of degree k based holomorphic maps $\mathbf{CP}^1 \to \Omega G$.*

We next compute a formula for $\dim \mathscr{M}_{\mathrm{hol}}(G,k)$ and compare it with (4.18). Consider first a *finite dimensional* complex manifold X and the space of based holomorphic maps $f\colon\ \mathbf{CP}^1 \to X$. This space may not be a smooth manifold, but we restrict our attention to regular points f.

Proposition 4.20. *At a smooth point $f\colon\ \mathbf{CP}^1 \to X$ the complex dimension of the* (*tangent space to the*) *moduli space of based holomorphic maps $\mathbf{CP}^1 \to X$ is $f^*(c_1(X))[\mathbf{CP}^1]$.*

Proof. The tangent space to the moduli space at f consists of holomorphic sections of $f^*(TX) \to \mathbf{CP}^1$. By Grothendieck's Theorem $f^*(TX)$ splits holomorphically into a direct sum of line bundles $\bigoplus \mathscr{O}(d_i)$, with $\mathscr{O}(d)$ the dth power of the (positive) hyperplane bundle. Our regularity assumption is $d_i \geq 0$. (This is the generic case if X has positive first Chern class.) Holomorphic sections of $\mathscr{O}(d)$ are homogeneous polynomials on $\mathbf{C}^2$ of degree d, and we require our sections to vanish at a fixed point, since we consider deformations of *based* maps. The dimension of the space of based degree d polynomials is d, so the tangent space to the moduli space at f has complex dimension

$$\sum d_i = c_1(f^*(TX))[\mathbf{CP}^1] = f^*(c_1(X))[\mathbf{CP}^1].$$

This calculation only describes the tangent space to the moduli space. A separate argument, which by now is standard (see [7], [22, §3] for the argument in the instanton setting) is needed to show that these deformations are integrable, which leads to the manifold structure of the moduli space. Since we are only interested in the formal aspects of Proposition 4.20, we do not pursue this analysis.

Take X to be the infinite dimensional based loop group ΩG, and apply (4.20), arguing now by analogy, to conclude that at a degree k map f,

$$\dim_{\mathbf{C}} \mathscr{M}_{\mathrm{hol}}(G,k) = f^*(c_1(\Omega G))[\mathbf{CP}^1] = 2kn_G. \tag{4.21}$$

This agrees with $\dim_{\mathbf{C}} \mathscr{M}_{\mathrm{SD}}(G,k)$ by (4.18), which we expect from the Atiyah-Donaldson Theorem, and provides additional confirmation for our value of $c_1(\Omega G)$. Incidentally, it gives a small bit of evidence that Atiyah and Donaldson's result is valid for the exceptional groups.

The dimensions we have equated were both calculated by the index theorem. For the instanton space we computed the index of a certain Dirac operator on S^4, and for the space of holomorphic maps we used a $\bar{\partial}$ operator on $\mathbf{CP}^1$. In fact, there is a further index problem, a family of Toeplitz operators parametrized by ΩG, which defines the Chern classes of ΩG topologically, and ultimately it is Bott periodicity that relates the three indices. That the dimensions of the two moduli spaces are identical is a linearized version of Atiyah-Donaldson, an isomorphism of the tangent spaces. Therefore, the Atiyah-Donaldson Theorem, which identifies the solution space of a nonlinear $\bar{\partial}$ operator on the 2-sphere with the solution space of a nonlinear Dirac operator on the 4-sphere, should be regarded as a *nonlinear Bott Periodicity Theorem.*

We may hope to define Chern classes for other infinite dimensional complex manifolds, say for $\mathbf{CP}^\infty$. However, there are several arguments which suggest that $c_1(\mathbf{CP}^\infty) = \infty$. In other words, there seems to be no finite value which makes sense geometrically. For example, we may view $\mathbf{CP}^\infty$ as a limit of finite dimensional projective spaces, behave as a bad analyst, exchanging limits with anything in sight, and thereby conclude that

$$c_1(\mathbf{CP}^\infty) = c_1\left(\lim_{n\to\infty} \mathbf{CP}^n\right) = \lim_{n\to\infty} c_1(\mathbf{CP}^n) = \lim_{n\to\infty} (n+1)x = \infty,$$

where x is the generator of H^2. Alternatively, we may believe that Proposition 4.20 holds for $\mathbf{CP}^\infty$, so that if m is the dimension of degree one based holomorphic maps $\mathbf{CP}^1 \to \mathbf{CP}^\infty$, then $c_1(\mathbf{CP}^\infty) = mx$. But there is an infinite dimensional space of such maps, since it at least contains the infinite dimensional space of complex 2-planes through a fixed line in an infinite complex Hilbert space, the whole situation viewed projectively. Again we are led to believe $c_1(\mathbf{CP}^1) = \infty$. The behavior of infinite projective space reveals its true infinite dimensional nature, whereas the loop space exhibits finite dimensional features. The difference lies in the groups controlling the geometry. For $\mathbf{CP}^\infty$ it is the full group of unitary operators on Hilbert space which plays a crucial role. This very large group has many associated infinities. But the intrinsic geometry of the based loop group is tied up with a much smaller group of operators for which corresponding quantities are finite. These remarks refer to the topological interpretation of Chern classes in infinite dimensions, to which we now turn.

Suppose that $\mathscr{M}$ is an infinite dimensional smooth Hilbert manifold. As in finite dimensions there is a tangent space $T_x\mathscr{M}$ at each $x \in \mathscr{M}$, now modeled by a separable Hilbert space $\mathscr{H}$, which we allow to be either real or complex. The notion of a basis makes sense for Hilbert spaces, and we let F_x denote the space of bases of $T_x\mathscr{M}$. Formally, F_x is the set of topological linear isomorphisms $f\colon\ \mathscr{H} \to T_x\mathscr{M}$, and the group of bounded invertible transformations on $\mathscr{H}$ acts simply transitively on F_x by composition. Let $\mathrm{GL}(\mathscr{H})$ denote this group. We collect $\{F_x\}_{x\in\mathscr{M}}$ into a space $\mathrm{GL}(\mathscr{M})$, and as before $\mathrm{GL}(\mathscr{M})$ is in a natural way a principal bundle over $\mathscr{M}$ with structure group $\mathrm{GL}(\mathscr{H})$.

The starting point for our topological discussion of characteristic classes in infinite dimensions is a theorem of Kuiper, which states that the group $\mathrm{GL}(\mathscr{H})$ is contractible. Kuiper's Theorem holds for both real and complex (even quaternionic) Hilbert spaces. One immediate consequence of Kuiper's Theorem is the parallelizability of Hilbert manifolds. Kuiper's Theorem seems to preclude the possibility of characteristic classes, at least in a straightforward way from the topology of the frame bundle.

Let us return now to the Kähler metric on ΩG. The Kähler metric reduces the general linear frame bundle $\mathrm{GL}(\mathscr{H})$ to the bundle $\mathrm{U}(\Omega G)$ of unitary frames. An easy corollary of Kuiper's Theorem states that its structure group $\mathrm{U}(\mathscr{H})$ is contractible, so we still lack characteristic classes from a topological viewpoint. The crucial observation at this stage is Theorem 1.11 (cf. Proposition 5.16), which implies that the curvature operators of the Kähler metric span a subspace of compact operators in $\mathfrak{u}(\mathscr{H})$. (In fact, at smooth loops the operators are almost trace class. This summability is crucial to our considerations.) We expounded at length about finite dimensional frame bundles to convince the reader that the corresponding proper subgroup in $\mathrm{U}(\mathscr{H})$ is the key to the intrinsic geometry and topology of ΩG. If we could apply the holonomy bundle construction and the Ambrose-Singer Theorem in this context, then we would conclude that the holonomy group of the Levi-Civita connection is contained in this subgroup. Furthermore, this subgroup has nontrivial topology, so the reduced frame bundle could carry nontrivial characteristic classes.

Unfortunately, we are unable at present to extend the construction of the holonomy bundle and the Ambrose-Singer Theorem to our infinite dimensional situation. The crucial ingredient for both is the Frobenius Theorem, but the usual version of the Frobenius Theorem for Banach manifolds stated in the literature requires that the integrable distribution consist of *closed* subspaces (of the tangent spaces) which have *closed complements* [27], [30]. These restrictive hypotheses confine us to closed subgroups of the structure group $\mathrm{U}(\mathscr{H})$. Even if we were willing to compromise on this point, which we could conceivably do, the Lie algebra of the closed subgroup we are interested in is the closed ideal of compact skew-Hermitian operators, and it does not have a closed complement in $\mathfrak{u}(\mathscr{H})$. So the standard machinery of infinite dimensional manifold theory does not apply. Still, we are confident that there is an extension of Ambrose-Singer powerful enough to accomodate our situation.

5. The geometric frame bundle of ΩG

At the end of §4 we argued that the holonomy bundle of the Kähler connection is a reduction of the $\mathrm{GL}(\mathscr{H})$ frame bundle to the group $\mathrm{GL}^{\mathrm{cpt}}(\mathscr{H})$, the group of invertible operators on an infinite dimensional complex Hilbert space $\mathscr{H}$ which differ from the identity by a compact operator. (The reduced group is actually smaller—it consists of unitary operators with summability properties—but for now $\mathrm{GL}^{\mathrm{cpt}}$ is good enough.) However, our argument was formal, since we could not construct the holonomy bundle rigorously. Reductions to $\mathrm{GL}^{\mathrm{cpt}}(\mathscr{H})$ are classified topologically by homotopy classes of maps $\Omega G \to \mathrm{GL}(\mathscr{H})/\mathrm{GL}^{\mathrm{cpt}}(\mathscr{H}) = \mathscr{G}(\mathscr{H})$. Since there is a homotopy equivalence $\mathrm{Fred}_0(\mathscr{H}) \to \mathscr{G}(\mathscr{H})$, where $\mathrm{Fred}_0(\mathscr{H})$ is the space of Fredholm operators of index zero, these reductions are also classified by families of index zero Fredholm operators on ΩG. Reductions of the frame bundle to $\mathrm{GL}^{\mathrm{cpt}}(\mathscr{H})$ are termed *Fredholm structures* [18]. As a finite dimensional analogy, consider a real n-manifold M. If the frame bundle $\mathrm{GL}(M)$ is endowed with a Riemannian connection, then the holonomy bundle is an *abstract* reduction of $\mathrm{GL}(M)$ to $\mathrm{O}(n)$. A *concrete* reduction is provided by the Riemannian metric: the reduced bundle consists of bases with respect to which the metric is the standard metric on $\mathbf{R}^n$. Similarly, a Fredholm family $f \to T(f)$ on ΩG gives a concrete reduction of $\mathrm{GL}(\Omega G)$ to $\mathrm{GL}^{\mathrm{cpt}}$, once a trivialization of $\mathrm{GL}(\Omega G)$ is fixed by choosing a distinguished basis b_f at each $f \in \Omega G$: the reduced frame bundle consists of frames b'_f such that the matrix of $T(f)$ with respect to b_f in the domain and b'_f in the range is in $\mathrm{GL}^{\mathrm{cpt}}$.

In this section we construct a particular Fredholm structure on ΩG, and we call the resulting reduced bundle the *geometric frame bundle.* Our choice of a family of Fredholms is motivated by the curvature formula (2.18) and by the index theorem proved in [20]. The latter provides an explicit formula for Chern character forms of families of Fredholm operators parametrized by a group of homogeneous space. We first review this index theorem. We then construct an explicit family so that the Chern character forms for the index agree with the curvature forms of §2. Because these (left-invariant) forms are identical, the Chern classes defined topologically by the Fredholm structure are the Chern classes defined geometrically by curvature. We calculate the higher Chern classes via a simple homotopy and Bott periodicity. We carry out computations for $\Omega\mathrm{SU}(n)$. Finally, we prove that the real characteristic classes of ΩG vanish for any G.

To describe the index theorem we introduce $L^1(\mathscr{H})$, the space of trace class operators, and $L^2(\mathscr{H})$, the space of Hilbert-Schmidt operators. (From now on we often delete "$\mathscr{H}$" from the notation.) There are corresponding groups GL^1 (resp. GL^2) which consist of invertible operators A such that $A - 1$ is trace class (resp. Hilbert-Schmidt). The GL^p are Banach Lie groups; they can be defined for any $p \geq 1$ using Schatten ideals of operators. Recall that GL denotes the group of all invertibles. Set $\mathscr{G}^p = \mathrm{GL}/\mathrm{GL}^p$. Since GL^p is normal in GL, the quotient $\mathscr{G}^p$ is a group. However, GL^p is not closed in GL—its closure is $\mathrm{GL}^{\mathrm{cpt}}$—so that the induced topology on $\mathscr{G}^p$ is not Hausdorff. Hence we consider $\mathscr{G}^p$ as an abstract group.

Let $\mathfrak{gl}$ denote the Lie algebra of all bounded operators on $\mathscr{H}$. There is an isomorphism from $\mathscr{G}^p$ onto the identity component of the invertibles in $\mathfrak{gl}/L^p$. Furthermore, the inverse image of the invertibles in $\mathfrak{gl}/L^p$ under the quotient map $\mathfrak{gl} \to \mathfrak{gl}/L^p$ is the space of Fredholm operators. Thus we obtain a surjection $\pi\colon\ \mathrm{Fred}_0 \to \mathscr{G}^p$, where Fred_0 is the set of Fredholms of index zero. Now Fred_0 is homotopy equivalent to $B\mathrm{GL}(\infty)$, so its real cohomology has primitive generators ch_l in dimension $2l$. These are the universal Chern character classes. For special families of Fredholms there is an explicit formula.

Theorem 5.1 [20]. *Let $\mathfrak{G}$ be a Banach Lie group. Suppose that $T\colon\ \mathfrak{G} \to \mathrm{Fred}_0$ is a smooth family of index zero Fredholms such that*

$$\mathfrak{G} \xrightarrow{T} \mathrm{Fred}_0 \xrightarrow{\pi} \mathscr{G}^p$$

is a homomorphism of (abstract) groups; i.e.,

$$T(g)T(g') - T(gg') \in L^p, \qquad g, g' \in \mathfrak{G}.$$

Assume further that $\langle g, g'\rangle \to T(g)T(g') - T(gg')$ is a smooth map into L^p. Let $\dot{T}$: $\mathrm{Lie}(\mathfrak{G}) \to \mathfrak{gl}$ be the differential of T at the identity, and define the left *invariant L^p-valued 2-form*

$$\Omega(X,Y) = [\dot{T}(X), \dot{T}(Y)] - \dot{T}([X,Y]), \qquad X,Y \in \mathrm{Lie}(\mathfrak{G}) \tag{5.2}$$

on $\mathfrak{G}$. Then for $l \geq p$ the cohomology class $T^*\mathrm{ch}_l$ is represented invariantly by the form

$$\gamma_{2l} = -\left(\frac{i}{2\pi}\right)^l \frac{1}{l!}\mathrm{Trace}(\Omega^l). \tag{5.3}$$

We understand elements of Lie $(\mathfrak{g})$ to be left invariant vector fields which, in the definition of Ω, are evaluated at the identity. Our hypotheses imply that the trace in (5.3) exists.

There is a universal GL^p bundle over Fred_0, which is given explicitly as the semi-direct product $\mathrm{GL} \ltimes L^p \to \mathrm{Fred}_0$, where GL acts on L^p by left multiplication. By pullback T induces a GL^p bundle $\tilde{\mathfrak{G}} \to \mathfrak{G}$. Furthermore, since $\pi \circ T$ is a group homomorphism, $\tilde{\mathfrak{G}}$ is a group. Now the minus sign in (5.3) is unfortunate, but it is eradicated by replacing $\tilde{\mathfrak{G}} \to \mathfrak{G}$ with the "opposite GL^p bundle", obtained by exchanging left and right multiplication in $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$.

There is a corollary of Theorem 5.1 for homogeneous spaces, which applies to $\Omega G = LG/G$.

Corollary 5.4. *In the situation of Theorem* 5.1, *suppose that $\mathfrak{G}$ is a Lie subgroup of $\mathfrak{G}$ such that T restricted to $\mathfrak{H}$ is a homomorphism into* GL. *Assume also that*

$$T(hgh^{-1}) = T(h)T(g)T(h)^{-1}, \qquad g \in \mathfrak{G},\ h \in \mathfrak{H}. \tag{5.5}$$

Then the invariant forms γ_{2l} of (5.3) *are pullbacks via $\sigma\colon\ \mathfrak{G} \to \mathfrak{G}/\mathfrak{H}$ of invariant forms $\bar{\gamma}_{2l}$ on $\mathfrak{G}/\mathfrak{H}$, and γ_{2l} represents the lth Chern character of the induced GL^p-bundle over $\mathfrak{G}/\mathfrak{H}$.*

The GL^p bundle over $\mathfrak{G}/\mathfrak{H}$ is constructed using the splitting of

$$1 \to \mathrm{GL}^p \to \tilde{\mathfrak{G}} \to \mathfrak{G} \to 1$$

over $\mathfrak{H}$—divide out by the image of $\mathfrak{H}$ in $\mathfrak{G}$.

We exploit the similarity between (5.2) and (2.17). Recall that the curvature of $\Omega G = LG/G$ is expressed in terms of the decomposition (2.3)

$$\mathrm{Lie}(LG)_{\mathbf{C}} = L\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}_{\mathbf{C}} \oplus \mathfrak{m}_+ \oplus \mathfrak{m}_-.$$

These spaces are completed in appropriate Sobolev metrics, so that $\mathfrak{m}_+$ and $\mathfrak{m}_-$ are Hilbert spaces. The Kähler connection was described in Theorem 2.15 by a map $\varphi\colon\ L\mathfrak{g}_{\mathbf{C}} \to \mathfrak{gl}(\mathfrak{m}_+)$ given explicitly in terms of Toeplitz operators

(2.15), (2.16). Furthermore, the curvature (2.17) is the $\mathfrak{gl}(\mathfrak{m}_+)$-valued left-invariant 2-form (2.17)

$$R(X,Y) = [\varphi(X), \varphi(Y)] - \varphi([X,Y]).$$

Although the curvature is not quite trace class, we made sense of its trace, by a two-step procedure. Powers of the curvature are trace class, and we defined the geometric Chern character classes of ΩG by the formula (2.24)

$$\mathrm{ch}_l(\Omega G) = \left(\frac{i}{2\pi}\right)^l k\frac{1}{l!}\mathrm{Trace}(R^l).$$

(We have equated a cohomology class with a representative differential form.)

Both the curvature and index theorem describe invariant forms on ΩG. Below we construct a smooth family of index zero Fredholms $T\colon LG \to \mathrm{Fred}_0(\mathfrak{m}_+)$ parametrized by the loop group satisfying:

$$(5.6)\qquad \begin{array}{rl} \text{(i)} & LG \xrightarrow{T} \mathrm{Fred}_0(\mathfrak{m}_+) \to \mathscr{G}^1(\mathfrak{m}_+) \text{ is a homomorphism;} \\ \text{(ii)} & T \text{ restricted to the constant loops } G \subset LG \text{ is a} \\ & \qquad \text{homomorphism into } \mathrm{GL}(\mathfrak{m}_+); \\ \text{(iii)} & T(g_0 f g_0^{-1}) = T(g_0)T(f)T(g_0)^{-1}, \qquad g_0 \in G, f \in LG. \end{array}$$

As in (5.2) define the L^1-valued 2-form

$$(5.7)\qquad \Omega(X,Y) = [\dot T(X), \dot T(Y)] - \dot T([X,Y])$$

on $L\mathfrak{g}$. Then by Theorem 5.1 and Corollary 5.4, the lth Chern character class of the induced GL^1-bundle $\mathscr{Q}$ over ΩG is represented by the invariant differential form

$$(5.8)\qquad \mathrm{ch}_l(\mathscr{Q}) = \left(\frac{i}{2\pi}\right)^l \frac{1}{l!}\mathrm{Trace}(\Omega^l).$$

(We use the opposite bundle to get the correct sign.)

Comparing (2.15)-(2.17) with (5.7)-(5.8) we arrive at the following conclusion.

Proposition 5.9. *Suppose that we construct a family of Fredholms* $T^{(1)}$ *satisfying* (5.6) *and also*

$$(5.10)\qquad \dot T^{(1)} = \varphi,$$

for φ *given in* (2.15), (2.16). *Then the Chern classes of the induced bundle* $\mathscr{Q} \to \Omega G$ *agree with the Chern classes defined by curvature.*

If the holonomy bundle $\mathscr{Q}' \to \Omega G$ could be constructed, then by the infinite dimensional Chern-Weil Theorem proved in [20] its Chern character would be represented by the forms (2.24). This is enough to identify the putative bundle

$\mathscr{Q}'$ with $\mathscr{Q}$. First we recall that the cohomology of ΩG is torsion-free and is concentrated in even dimensions [11].

Proposition 5.11. *If the* GL^p*-bundles* $\mathscr{Q}, \mathscr{Q}'$ *over* ΩG *have the same Chern character, then* $\mathscr{Q}$ *and* $\mathscr{Q}'$ *are topologically equivalent bundles.*

Proof. GL^p bundles over finite complexes can be thought of as stable complex vector bundles, i.e., elements of K-theory. Then the proposition follows from the fact that the Chern character homomorphism from K-theory to rational cohomology is injective for torsion-free spaces having the homotopy type of a CW complex. (The proof for finite complexes is [6, Corollary 2.5]. The extension to infinite complexes in [2, Lemma 4.9].)

We next define a one-parameter family of maps $T^{(s)}\colon\ LG \to \mathrm{Fred}_0(\mathfrak{m}_+)$, $0 \le s \le 1$. Each $T^{(s)}$ satisfies (5.6), but with $\mathscr{S}^2$ replacing $\mathscr{S}$. (If we restrict to smooth loops, then $T^{(s)}$ is a homomorphism modulo $L^{1+\varepsilon}$ for any $\varepsilon > 0$.) Only $T^{(1)}$ satisfies (5.10). The homotopically equivalent $T^{(0)}$ is simpler than $T^{(1)}$, and we will be able to recognize it in terms of Bott periodicity. Also, $T^{(0)}$ does define a homomorphism into $\mathscr{S}^1$. To construct $T^{(s)}$ consider the adjoint embedding $G \to \mathrm{Ad}(G) \subset \mathrm{GL}(\mathfrak{g})$ which induces

$$\begin{aligned} LG &\to L(\mathrm{Ad}\,G) \subset L\{\mathrm{GL}\,(\mathfrak{g})\}, \\ f &\to \mathrm{Ad}\,f. \end{aligned}$$

Since $\mathrm{GL}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is a group of matrices, there is a Fourier expansion

$$\mathrm{Ad}\,f = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}, \qquad f_n \in \mathfrak{gl}(\mathfrak{g}_{\mathbf{C}}).$$

Set

$$\begin{aligned} (\mathrm{Ad}\,f)_+ &= \sum_{n=1}^{\infty} f_n e^{in\theta} \in L\{\mathfrak{gl}(\mathfrak{g}_{\mathbf{C}})\}, \\ (\mathrm{Ad}\,f)_- &= \sum_{n=-\infty}^{0} f_n e^{in\theta} \in L\{\mathfrak{gl}(\mathfrak{g}_{\mathbf{C}})\}. \end{aligned} \tag{5.12}$$

Now $\mathfrak{m}_+ \subset L\mathfrak{g}_{\mathbf{C}}$ is the subspace of strictly holomorphic loops, and $L\{\mathfrak{gl}(\mathfrak{g}_{\mathbf{C}})\}$ acts on $\mathfrak{m}_+$ by the Toeplitz construction, combined with the natural action of $\mathfrak{gl}(\mathfrak{g}_{\mathbf{C}})$ on $\mathfrak{g}_{\mathbf{C}}$. In other words, if $\pi_+\colon\ L\mathfrak{g}_{\mathbf{C}} \to \mathfrak{m}_+$ denotes the projection onto the strictly holomorphic loops, then

$$T_f = \pi_+(\mathrm{Ad}\,f) \tag{5.13}$$

is the Toeplitz operator for f. Let T_{f_+} and T_{f_-} denote the Toeplitz action of $(\mathrm{Ad}\,f)_+$ and $(\mathrm{Ad}\,f)-$, respectively. Set

$$T^{(s)}(f) = T_{f_-} + D^{-s} T_{f_+} D^s, \tag{5.14}$$

where $D = d/d\theta$ acting on $\mathfrak{m}_+$, as usual. Note that $T^{(0)}(f) = T_f$ is the usual Toeplitz operator associated to f (via the adjoint representation).

To verify (5.6)(i) we must use all Sobolev loops, not just smooth loops. For the case $s = 0$ we have (cf. [32]).

Proposition 5.15. *Let $\mathscr{H}_+$ denote the Hilbert space of strictly holomorphic H_t functions on S^1, $t \geq 1/2$. Let T_f denote the Toeplitz operator on $\mathscr{H}_+$ corresponding to f. Then for $f, g \in H_t(S^1)$, the operator $T_f T_g - T_{fg}$ is trace class.*

We have stated this proposition for *functions* f. In our application to maps $f\colon\ S^1 \to G$ we require $t > 1/2$ in order to define $H_t(S^1, G)$.

Proof. Write $H_t(S^1) = \mathscr{H}_+ \oplus \mathscr{H}_-$ according to positive and nonpositive Fourier series. Then multiplication by$f$ is the matrix

$$M_f = \begin{pmatrix} A & b \\ c & D \end{pmatrix},$$

and we prove that b and c are Hilbert-Schmidt. It follows easily that $T\colon\ H_t(S^1) \to \mathscr{G}^1(\mathscr{H}_+)$ is a homomorphism.

Let

$$f = \sum a_n z^n \in H_t.$$

We use the orthonormal basis $\{z^l/(l^2+1)^{t/2}\}$ for H_t. Then

$$M_f\left(\frac{z^l}{(l^2+1)^{t/2}}\right) = \left(a_n \frac{((n+l)^2+1)^{t/2}}{(l^2+1)^{t/2}}\right) \frac{z^{n+l}}{((n+l)^2+1)^{t/2}}.$$

The expression in brackets is the $l, n+l$ entry of M_f, say $M_{l,n+l}$. Thus

$$\begin{aligned} \|b\|^2_{L^2(H_t)} &= \sum_{\substack{l>0 \\ n+l\leq 0}} |M_{l,n+l}|^2 \\ &= \sum_{n<0} |a_n|^2 \sum_{0<l\leq -n} \frac{((n+l)^2+1)^{t/2}}{(l^2+1)^{t/2}} \\ &\leq \sum_{n<0} n|a_n|^2 \leq \|f\|_{H_{1/2}} \leq \|f\|_{H_t} < \infty. \end{aligned}$$

Similarly

$$\begin{aligned} \|c\|^2_{L^2(H_t)} &= \sum_{\substack{n+l>0 \\ l\leq 0}} |M_{l,n+l}|^2 = \sum_{n>0} |a_n|^2 \sum_{-n<l\leq 0} \frac{((n+l)^2+1)^{t/2}}{(l^2+1)^{t/2}} \\ &\leq \sum_{n>0} n|a_n|^2 \leq \|f\|_{H_{1/2}} \leq \|f\|_{H_t} < \infty. \end{aligned}$$

Our next result treats $T^{(s)}$ for $s > 0$. Now we must replace trace class by Hilbert Schmidt. (We thank David Jerison for pointing out that our Hilbert-Schmidt estimate for H_t functions, $t = 1/2 + \varepsilon$, is best possible, in the sense that no smaller Schatten class can be used.)

Proposition 5.16. *Let $\mathscr{H}_+$ denote the Hilbert space of strictly holomorphic H_t functions on S^1, $t > 1/2$. Set $D = d/d\theta$. Then for $s \le 1$ and any $f \in \mathscr{H}_+$, the operator $D^{-s}[M_f, D^s]$ is Hilbert-Schmidt on $\mathscr{H}_+$. Consequently, for H_t loops $f, g \in LG$ the operator $T^{(s)}(f)T^{(s)}(g) - T^{(s)}(fg)$ is Hilbert-Schmidt.*

Proof. We use the orthonormal basis $\{z^l/l^t\}_{l=1}^{\infty}$ for $\mathscr{H}_+$. Let $f = \sum_{n>0} a_n z^n$ and denote $A = D^{-s}[M_f, D^s]$. Then

$$\begin{aligned} A\left(\frac{z^l}{l^t}\right) &= \frac{1}{(l+n)^s} a_n (l^s - (l+n)^s)\frac{1}{l^t} z^{l+n} \\ &= \left[a_n \frac{(l^s - (l+n)^s)}{(l+n)^s}\frac{(l+n)^t}{l^t}\right]\frac{z^{l+n}}{(l+n)^t}. \end{aligned}$$

So the expression in brackets is $A_{l+n,l}$. The Hilbert-Schmidt norm squared is

$$\sum_{\substack{l>0\\ n>0}} |A_{l+n,l}|^2 = \sum_{n>0} |a_n|^2 \sum_{l>0}\left(1 - \left(\frac{l}{l+n}\right)^s\right)^2 \left(\frac{l+n}{l}\right)^{2t}.$$

This increases in s, and it suffices to treat the case $s = 1$. Then the sum over l is

$$\begin{aligned} \sum_{l=1}^{\infty} \frac{1}{(1+l/n)^{2-2t}(l/n)^{2t}} &\sim \int_1^{\infty} \frac{dx}{(1+x/n)^{2-2t}(x/n)^{2t}} \\ &= n\int_{1/n}^{\infty} \frac{dy}{(1+y)^{2-2t}y^{2t}} \le n\left[C + \int_{1/n}^{1}\frac{dy}{y^{2t}}\right] \\ &\le n[C + C(1/n)^{1-2t}] \le Cn^{2t}. \end{aligned}$$

Therefore,

$$\|A\|^2_{L^2(H_t)} \le C\sum_{n=1}^{\infty} |a_n|^2 n^{2t} \le C\|f\|^2_{H_t},$$

which proves the first assertion in the proposition. Now

$$\begin{aligned} T^{(s)}(f) &= T_{f_-} + D^{-s}T_{f_+}D^s \\ &= T_f + D^{-s}[T_{f_+}, D^s] = T_f + (\text{order } -1). \end{aligned} \tag{5.17}$$

The lemma implies

$$T^{(s)}(f)T^{(s)}(g) - T^{(s)}(fg) = T_fT_gT_{fg} + (\text{order } -1) = (\text{order } -1),$$

as desired.

Proposition 5.16, together with the fact that $T^{(s)}$ maps the constant loop at the identity into the identity operator on $\mathfrak{m}_+$, proves the homomorphism property (5.6)(i), for $\mathscr{G}^2$ replacing $\mathscr{G}^1$. The Toeplitz family $T^{(0)}$ defines a homomorphism into $\mathscr{G}^1(\mathfrak{m}_+)$, by Proposition 5.15. Fix a constant loop $g_0 \in G \subset LG$. Then $T^{(s)}(g_0) = (\operatorname{Ad} g_0)$ defines a homomorphism $G \to \mathrm{GL}(\mathfrak{m}_+)$, whence (5.6)(ii). Condition (5.6)(iii) holds because $(\operatorname{Ad} g_0)$ commutes with π_+ and with D:

$$\begin{aligned} T^{(s)}(g_0 f g_0^{-1}) &= T_{(g_0 f g_0^{-1})_-} + D^{-s} T_{(g_0 f g_0^{-1})_+} D^s \\ &= T_{g_0 f_- g_0^{-1}} + D^{-s} T_{g_0 f_+ g_0^{-1}} D^s \\ &= (\operatorname{Ad} g_0) T_{f_-} (\operatorname{Ad} g_0^{-1}) + D^{-s} (\operatorname{Ad} g_0) T_{f_+} (\operatorname{Ad} g_0^{-2}) D^s \\ &= (\operatorname{Ad} g_0) \{ T_{f_-} + D^{-s} T_{f_+} D^s \} (\operatorname{Ad} g_0^{-1}) \\ &= T^{(s)}(g_0) T^{(s)}(f) T^{(s)}(g_0^{-1}). \end{aligned}$$

Finally, the Fredholm family $T^{(1)}$ was chosen precisely to satisfy (5.10).

We summarize the discussion in

Theorem 5.18. *For each loop $f \in LG$ define the operator $T^{(s)}(f) = T_{f_-} + D^{-s} T_{f_+} D^s$ on $\mathfrak{m}_+$. Then $T^{(s)}(f)$ is Fredholm of index zero, and the composition*

$$LG \xrightarrow{T^{(s)}} \mathrm{Fred}_0 \to \mathscr{G}^2$$

is a homomorphism. In addition, the Toeplitz family $T^{(0)}$ defines a homomorphism into $\mathscr{G}^1$. The GL^2-bundle $\mathscr{Q}^{(s)}$ over ΩG induced by these families are all isomorphic, and their Chern classes agrees with the Chern classes defined by curvature.

These results apply to the full flag manifold $\mathscr{F} = LG/T$ with one small modification—we must replace the decomposition (2.3) with (4.2). We can use the Toeplitz family, now projecting to the new $\mathfrak{m}_+$, to compute the Chern classes of $\mathscr{F}$, and this simplifies the calculation of Proposition 4.8 somewhat.

Returning to the based loop group ΩG, we recall the relationship between the Toeplitz family $T^{(0)}$ and Bott periodicity. For this we introduce a stable Toeplitz map as follows. Let $\mathscr{H}_N$ denote the Hilbert space of strictly holomorphic maps $S^1 \to \mathbf{C}^N$ in some Sobolev completion. The usual Toeplitz construction defines a map

$$\alpha_N \colon \Omega\mathrm{GL}(N) \to \mathrm{Fred}_0(\mathscr{H}_N).$$

Furthermore, α_N and α_{N+1} are compatible with the obvious inclusions. Since $\mathrm{Fred}_0(\mathscr{H}_N) \sim B\mathrm{GL}(\infty)$, and the inclusion $\mathrm{Fred}_0(\mathscr{H}_N) \to \mathrm{Fred}_0(\mathscr{H}_{N+1})$ is a homotopy equivalence, there is an induced limiting map

$$\alpha \colon \Omega\mathrm{GL}(\infty) \to B\mathrm{GL}(\infty). \tag{5.19}$$

Proposition 5.16 implies that each α_N determines a homomorphism $\Omega\mathrm{GL}(N) \to \mathscr{G}(\mathscr{H}_N) = \mathrm{GL}(\mathscr{H}_N)/\mathrm{GL}^{\mathrm{cpt}}(\mathscr{H}_N)$ by composing with the projection $\mathrm{Fred}_0(\mathscr{H}_n) \to \mathscr{G}(\mathscr{H}_N)$. It follows that α_N, hence α, is a homomorphism of H-spaces. Atiyah [3] proves that (5.19) is a homotopy equivalence. It follows that α induces an isomorphism $\pi_l(\mathrm{GL}(\infty)) \to \pi_{l-2}(\mathrm{GL}(\infty))$. This is Bott periodicity.

We can factor the Toeplitz family $T^{(0)}$ on LG through α. To pass from free loops to based loops we observe that the composition of the inclusion $\Omega G \to LG$ with the projection $LG \to LG/G = \Omega G$ is a homotopy equivalence. This means that the $\mathrm{GL}(\infty)$-bundle over LG/G induced by $T^{(0)}$ is equivalent to the bundle induced by restricting $T^{(0)}$ to $\Omega G \subset LG$. That said, identifying $\mathrm{Fred}_0(\mathfrak{m}_+)$ with $B\mathrm{GL}(\infty)$, and setting $N = \dim(G)$, we see that $T^{(0)}$ is the composition

$$\Omega G \xrightarrow{\Omega(\mathrm{Ad})} \Omega\mathrm{GL}(N;\mathbf{C}) \xrightarrow{\iota} \Omega\mathrm{GL}(\infty;\mathbf{C}) \xrightarrow{\alpha} B\mathrm{GL}(\infty;\mathbf{C}). \tag{5.20}$$

Here $\Omega(\mathrm{Ad})$ is the map on based loops obtained from the adjoint representation $G \to \mathrm{Ad}(G) \subset \mathrm{GL}(N;\mathbf{C})$, and we have inserted "$\mathbf{C}$" in the notation for emphasis. Theorem 5.18 states that the Chern classes of ΩG, as defined by curvature, can be calculated from (5.20).

It suffices to calculate over the reals, in view of the fact that ΩG is torsion-free. Since ΩG is a group, $H^*(\Omega G;\mathbf{R})$ is a Hopf algebra with primitive generators given by $\pi_*(\Omega G)\otimes\mathbf{R}$. Furthermore, $\pi_n(\Omega G) = \pi_{n+1}(G)$, and the real homotopy groups $\pi_{n+1}(G)\otimes\mathbf{R}$ are well known. Namely, attached to each compact simple group G are certain odd integers $2m_i - 1$, the number of which is the rank of G, and $\pi_{n+1}(G)\otimes\mathbf{R} = \mathbf{R}$ for $n+1 = 2m_i - 1$ and is zero otherwise. The m_i are called the *exponents* of G. For simple groups, $m_1 = 2$. Therefore, the real cohomology of ΩG is a symmetric algebra on generators Y_{2m_i-2} in dimension $2m_i - 2$, where m_i are the exponents of G.

For $\mathrm{SU}(n)$ the exponents are $2, 3, \cdots, n$, and

$$H^*(\mathrm{SU}(n);\mathbf{R}) = \mathbf{R}[y_2, y_4, \cdots, y_{2n-2}]. \tag{5.21}$$

Let

$$e\colon\ S^1 \times \Omega\mathrm{SU}(n) \to \mathrm{SU}(n) \tag{5.22}$$

be the evaluation map. We fix the generators of $H^*(\Omega\mathrm{SU}(n);\mathbf{R})$ by setting

$$y_{2l-2} = \int_{S^1} e^*\omega_{2l-1'} \tag{5.23}$$

where ω_{2l-1} is defined in (3.8). The integral in (5.23) is to be interpreted as the slant product of $e^*\omega_{2l-1}$ with the homology class $[S^1]$. We remark that the multiplicative structure of the integral cohomology is somewhat subtle [12]. For example, the integral generator of $H^{2l}(\Omega\mathrm{SU}(2);\mathbf{Z})$ is $(y_2)^l/l!$.

The Toeplitz family (5.20) defines a homomorphism of H-spaces by Proposition 5.21, and so the induced map on cohomology takes primitive generators to primitive generators. Since $\{\mathrm{ch}_l\}$ is a basis for the primitive cohomology of $B\mathrm{GL}(\infty)$ (cf. [9]), for any group G we conclude

Proposition 5.24. *If $l+1$ is an exponent of G, then* $\mathrm{ch}_l(\Omega G) = n_l(G)\cdot y_{2l}$ *for some integer $n_l(G)$. If $l+1$ is not an exponent of G, then* $\mathrm{ch}_l(\Omega G) = 0$.

We will prove later (Corollary 5.32) that $\mathrm{ch}_l(\Omega G) = 0$ for l even, no matter what the exponents of G.

The integers $n_l(G)$ can be computed from (5.20), at least in principle. For example, we compute $n_1(G) = 2n_G$ by understanding the primitive generator of $H^3(G)$, as in §2. Of course, this agrees with the result from the curvature computation. We carry out the computation of higher $\mathrm{ch}_l(G)$ only for the unitary group. Now

$$\begin{aligned} H^*(\Omega\mathrm{GL}(N);\mathbf{R}) &= \mathbf{R}[x_2, x_4, \cdots, x_{2N-2}], \\ H^*(\Omega\mathrm{GL}(\infty);\mathbf{R}) &= \mathbf{R}[x_2, x_4, \cdots], \end{aligned} \tag{5.25}$$

where as in (5.23) we specify x_{2l} to be the transgression of ω_{2l+1} in the path fibration. Recall that we defined $\sigma_l = l!\cdot\mathrm{ch}_l$ in $H^*(B\mathrm{GL}(\infty);\mathbf{R})$.

Proposition 5.26. $\alpha^*((-1)^{l-1}\sigma_l) = x_{2l}$.

Proof. The inverse periodicity map $\beta\colon B\mathrm{GL}(\infty) \to \Omega\mathrm{GL}(\infty)$ is defined in K-theory by tensoring with the Bott class $1 - H$. Furthermore, $\Omega\mathrm{GL}(\infty) \sim \Omega^2 B\mathrm{GL}(\infty)$, and there is a commutative diagram

$$\begin{tikzcd} S^2 \times B\mathrm{GL}(\infty) \arrow[r, "1\otimes\beta"] \arrow[dr, "\phi"'] & S^2 \times \Omega^2 B\mathrm{GL}(\infty) \arrow[d, "\varepsilon"] \\ & B\mathrm{GL}(\infty) \end{tikzcd}$$

Here ε is the evaluation map, and ϕ is the classifying map for the universal bundle tensored with $1 - H$. Let $z \in H^2(S^2)$ denote the generator; then $\mathrm{ch}(1 - H) = -z$. The homomorphism property of the Chern character yields

$$\phi^*(\mathrm{ch}_{l+1}) = -z \otimes \mathrm{ch}_l.$$

On the other hand, ε^* is essentially transgression in the double fibration

$$\begin{tikzcd} \Omega^2 B\mathrm{GL}(\infty) \sim \Omega\mathrm{GL}(\infty) \arrow[r] & * \arrow[d] & \\ & \Omega\mathrm{GL}(\infty) \sim \mathrm{GL}(\infty) \arrow[r] & * \arrow[d] \\ & & B\mathrm{GL}(\infty) \end{tikzcd}$$

Since ch_{l+1} transgresses to $(-1)^l \omega_{2l+1}/l!$ in the universal fibration, and ω_{2l+1} transgresses to x_{2l} in the path fibration (cf. (5.23)), we obtain

$$\varepsilon^*(\mathrm{ch}_{l+1}) = -z \otimes (-1)^{l-1} x_{2l}/l!.$$

Recalling that $\sigma_l = l! \cdot \mathrm{ch}_l$, the previous equations give the desired result.

At the next stage in (5.20) we have trivially

$$\iota^*(x_{2l}) = \begin{cases} x_{2l} & \text{if } \leq N-1; \\ 0 & \text{if } l \geq N. \end{cases} \tag{5.27}$$

The remaining step in the computation of the Chern classes is the action of $\Omega(\mathrm{Ad})^*$. This depends on more detailed knowledge of $H^*(G)$, and we illustrate with $G = \mathrm{SU}(n)$.

Proposition 5.28.

$$\Omega(\mathrm{Ad})^*(x_{2l}) = \begin{cases} 2n y_{2l} & 1 \leq l \leq n-1, \ l \ \mathit{odd}\,; \\ 0 & \mathit{otherwise}. \end{cases}$$

Proof. The vanishing of $\Omega(\mathrm{Ad}) * (x_{2l})$ for $l \geq n$ is a consequence of (5.24). For $l \leq n-1$ we have

$$\Omega(\mathrm{Ad}) * (x_{2l}) = A_l y_{2l}$$

for some integer A_l, since $\Omega(\mathrm{Ad})^*$ preserves primitivity. The evaluation map (5.22) and the slant product commute with $\mathrm{Ad}\colon\ \mathrm{SU}(n) \to \mathrm{U}(n^2-1)$, from which

$$\mathrm{Ad}^*(\omega_{2l-1}) = A_{l-1}\omega_{2l-1}.$$

Then since a fixed multiple of ch_l represents a transgression of ω_{2l-1} in the classifying spaces $B\mathrm{SU}(n)$ and $B\mathrm{U}(n^2-1)$,

$$B(\mathrm{Ad})^*(\mathrm{ch}_l) = A_{l-1}\mathrm{ch}_l + (\text{terms involving lower order } \mathrm{ch}_j).$$

It will be convenient to replace $\mathrm{SU}(n)$ by $\mathrm{U}(n)$. Then relative to the usual basis $\{\varepsilon_i\}_{i=1}^n$ of $H^1(T)$, T the maximal torus of $\mathrm{U}(n)$, the roots are $\varepsilon_i - \varepsilon_j$, $i \neq j$. The cohomology of $B\mathrm{U}(n)$ is identified with the symmetric algebra on ε_i, and ch_l is represented by a multiple of the power sum

$$\sum_{i=1}^n (\varepsilon_i)^l = \sum \varepsilon^l.$$

The pullback under the adjoint representation is

$$\begin{aligned} \sum_{i\neq j}(\varepsilon_i - \varepsilon_j)^l &= \sum_{i \neq j}\sum_{\alpha=0}^{l}(-1)^\alpha \binom{l}{\alpha}(\varepsilon_j)^\alpha(\varepsilon_i)^{l-\alpha} \\ &= \sum_{\alpha=0}^{l}(-1)^\alpha\binom{l}{\alpha}\left\{\sum \varepsilon^\alpha \sum \varepsilon^{l-\alpha} - \sum \varepsilon^l\right\}. \end{aligned}$$

This vanishes for l odd, and for l even the coefficient of $\sum \varepsilon^l$ is

$$2n - \sum_{\alpha=0}^{l} (-1)^\alpha \binom{l}{\alpha} = 2n.$$

Therefore, $A_l = 2n$ for l odd and 0 for l even, $l \leq n-1$, as claimed.

For any manifold M we set

$$\mathrm{Chern}(M) = 1 + c_1(M) + c_2(M) + \cdots$$

to be the total Chern class. Something (5.26), (5.27), and (5.28) we obtain

Theorem 5.29. *The total Chern class of* $\Omega\mathrm{SU}(n)$ *is*

$$\mathrm{Chern}(\Omega\mathrm{SU}(n)) = \exp(2n\{y_2 + y_6/3 + y_{10}/5 + \cdots + y_{4m-2}/(2m-1)\}),$$

where $m = [n/2]$.

Proof. The Newton formulas (3.4) can be written consisely as

$$\mathrm{Chern} = \exp\left(\sum (-1)^{l-1}\sigma_l/l\right).$$

The theorem is now immediate.

We next examine the real geometric frame bundle of ΩG. It is, of course, the realification of the complex $\mathrm{GL}(\infty; \mathbf{C})$-bundle we have been discussing.

Theorem 5.30. *The real geometric frame bundle of* ΩG *is trivial.*

Corollary 5.31. *The Pontrjagin and Stiefel-Whitney classes of* ΩG *vanish.*

Corollary 5.32. *The even Chern character classes* $\mathrm{ch}_{2k}(\Omega G)$ *vanish.*

We prove Theorem 5.30 below. Corollary 5.31 is immediate.

Proof of Corollary 5.32. The relationship between the Chern and Pontrjagin classes (cf. (3.4)) is universally expressed by the map

$$r\colon\ B\mathrm{GL}(\infty; \mathbf{C}) \to B\mathrm{GL}(\infty; \mathbf{R}), \tag{5.33}$$

obtained from the inclusions $\mathrm{GL}(N; \mathbf{C}) \to \mathrm{GL}(2N : \mathbf{R})$ by stabilizing and passing to the classifying spaces. This is a map of H-spaces, and so r^* maps the primitive real cohomology of $B\mathrm{GL}(\infty; \mathbf{R})$ into the primitive real cohomology of $B\mathrm{GL}(\infty; \mathbf{C})$. Now over the reals $B\mathrm{GL}(\infty; \mathbf{R})$ has primitive generators in dimensions $4, 8, \cdots$—which comprise a "Pontrjagin character"—and these pull back via r^* to the even Chern character classes $\mathrm{ch}_2, \mathrm{ch}_4, \cdots$. By the previous corollary the Pontrjagin character of ΩG is trivial, whence $\mathrm{ch}_{2k}(\Omega G) = 0$.

The topology of the real geometric frame bundle is determined by the Toeplitz family (5.20), after composing with (5.33). The key to the proof of Theorem 5.30 is the observation that the adjoint representation is real. This leads to a factorization of (5.20):

(5.34)

$$\begin{tikzcd} & \Omega\mathrm{GL}(N;\mathbf{R}) \arrow[r, "\iota"] \arrow[d, "c"] & \Omega\mathrm{GL}(\infty;\mathbf{R}) \arrow[d, "c"] & & \\ \Omega G \arrow[ur, "\Omega(\mathrm{Ad})"] \arrow[r, "\Omega(\mathrm{Ad})"] & \Omega\mathrm{GL}(N;\mathbf{C}) \arrow[r, "\iota"] & \Omega\mathrm{GL}(\infty;\mathbf{C}) \arrow[r, "\alpha"] & B\mathrm{GL}(\infty;\mathbf{C}) \arrow[r, "r"] & B\mathrm{GL}(\infty;\mathbf{R}). \end{tikzcd}$$

Theorem 5.30 follows from

Proposition 5.35. *The composition*

$$\Omega\mathrm{GL}(\infty;\mathbf{R}) \xrightarrow{c} \Omega\mathrm{GL}(\infty;\mathbf{C}) \xrightarrow{\alpha} B\mathrm{GL}(\infty;\mathbf{C}) \xrightarrow{r} B\mathrm{GL}(\infty;\mathbf{R})$$

is homotopically trivial.

Proof. This is a corollary of Bott's original proof of the Periodicity Theorem [13]. Of course, we can replace $\mathrm{GL}(\infty;\mathbf{R})$ and $\mathrm{GL}(\infty;\mathbf{C})$ in (5.36) by $\mathrm{O}(\infty)$ and $\mathrm{U}(\infty)$, respectively. Furthermore, the homotopy equivalence $\alpha\colon\ \Omega\mathrm{U}(\infty) \to B\mathrm{U}(\infty)$ may be replaced by any homotopy equivalence. One step in the proof of periodicity for the orthogonal group is the equivalence $B\mathrm{O}(\infty) \sim \Omega(\mathrm{U}(\infty)/\mathrm{O}(\infty))$. We assert that the diagram

(5.37)
$$\begin{tikzcd} B\mathrm{U}(\infty) \arrow[r, "\beta_1"] \arrow[d, "r"] & \Omega\mathrm{U}(\infty) \arrow[d, "q"] \\ B\mathrm{O}(\infty \arrow[r, "\beta_2"] & \Omega\left(\frac{\mathrm{U}(\infty)}{\mathrm{O}(\infty)}\right) \end{tikzcd}$$

is homotopy commutative, where q is the natural quotient map, and the horizontal arrows realize periodicity. It will then follow that (5.36) is equivalent to

$$\Omega\mathrm{O}(\infty) \xrightarrow{c} \Omega\mathrm{U}(\infty) \xrightarrow{q} \Omega\left(\frac{\mathrm{U}(\infty)}{\mathrm{O}(\infty)}\right).$$

This composition maps $\Omega\mathrm{O}(\infty)$ to a point, and therefore is homotopically trivial.

Bott constructs the periodicity maps in (5.37) by analyzing the Morse Theory of the energy function. Consider the space of loops homotopic to the closed geodesic

(5.38)
$$\left(\begin{array}{c|c} e^{i\theta}I_n & 0 \\ \hline 0 & e^{-i\theta}I_n \end{array}\right), \qquad 0 \le \theta \le 2\pi,$$

in $\mathrm{U}(2N) = (\mathrm{U}(2N)\times\mathrm{U}(2N))/\mathrm{U}(2N)$. Then the little group $\mathrm{U}(2N)$ acts transitively on the set of all closed geodesics in this space, the stabilizer is $\mathrm{U}(N)\times\mathrm{U}(N)$, and the periodicity map β_1 is the stable version of the inclusion

$$\frac{\mathrm{U}(2N)}{\mathrm{U}(N)\times\mathrm{U}(N)} \to \Omega\left(\frac{\mathrm{U}(2N)\times\mathrm{U}(2N)}{\mathrm{U}(2N)}\right).$$

We can lift (5.38) to $\mathrm{U}(2N)\times\mathrm{U}(2N)$, and then project to a closed geodesic in $\mathrm{U}(4N)/\mathrm{O}(4N)$. Again the little group acts transitively on all closed geodesics in its homotopy class, and the inclusion

$$\frac{\mathrm{O}(4N)}{\mathrm{O}(2N)\times\mathrm{O}(2N)}\to\Omega\left(\frac{\mathrm{U}(4N)}{\mathrm{O}(4N)}\right)$$

stabilizes to the homotopy equivalence β_2. From this description, we see that there is a commutative diagram

$$\begin{array}{ccc}\dfrac{\mathrm{U}(2N)}{\mathrm{U}(N)\times\mathrm{U}(N)} & \longrightarrow & \Omega\left(\dfrac{\mathrm{U}(2N)\times\mathrm{U}(2N)}{\mathrm{U}(2N)}\right)\\ \downarrow & & \downarrow\\ \dfrac{\mathrm{O}(4N)}{\mathrm{O}(2N)\times\mathrm{O}(2N)} & \longrightarrow & \Omega\left(\dfrac{\mathrm{U}(4N)}{\mathrm{O}(4N)}\right)\end{array}\tag{5.39}$$

where the vertical map on the left is induced by realification and that on the right by inclusion. Diagram (5.37) is the stabilization of (5.39), which proves our assertion that (5.37) commutes.

We give an alternative proof of Theorem 5.30 in §6.

6. The real geometric frame bundle of map (M,G)

We return to the general situation of §1 and apply our techniques to the real frame bundle of $\mathrm{Map}(M,G)$ for any compact Riemannian manifold M. Here we can take either based or unbased maps. Theorem 1.11 implies that the curvature of any H_s metric, $s>0$, on $\mathrm{Map}(M,G)$ is a compact operator. Our analogy to finite dimensions suggests that the holonomy group of the Levi-Civita connection consists of orthogonal operators which differ from the identity by a compact operator. This group, $\mathrm{O}^{\mathrm{cpt}}$, has nontrivial topology, which potentially gives rise to nonzero real characteristic classes on $\mathrm{Map}(M,G)$. As in §5, we introduce a family of Fredholm operators to rigorously construct this real Fredholm structure. There is a homotopy to a family of invertible operators, which is automatically null homotopic. We conclude that the real characteristic classes of $\mathrm{Map}(M,G)$ vanish.

In §1 we computed the formula

$$\nabla_X^{(s)}=\tfrac{1}{2}\{\mathrm{ad}_X+\Delta^{-s}\mathrm{ad}_X\Delta^s-\Delta^{-s}\mathrm{ad}(\Delta^s X)\}\tag{6.1}$$

for the covariant derivative associated to the H_s metric, operating on left invariant vector fields. These vector fields are identified with the tangent space at the identity, $\mathscr{H}_{\mathbf{R}}=H_t(M,\mathfrak{g})$. We work with based maps $\mathrm{Map}_0(M,G)$ so that Δ is invertible; our considerations remain valid for unbased maps by replacing Δ with $(\Delta+1)$. The H_s curvature is given by the usual formula

$$R^{(s)}(X,Y)=[\nabla_X^{(s)},\nabla_Y^{(s)}]-\nabla_{[X,Y]}^{(s)}.\tag{6.2}$$

Motivated by §5 we define a family of operators on $\mathscr{H}_{\mathbf{R}}$, parametrized by $\mathrm{Map}_0(M,G)$ as follows. For $f \in \mathrm{Map}_0(M,G)$ let M_f denote the adjoint action on f on $\mathscr{H}_{\mathbf{R}}$, defined pointwise. Also, since $\mathrm{Ad}\, f \in \mathrm{Map}_0(M, \mathrm{Ad}\, G) \subset \mathrm{Map}_0(M, \mathfrak{gl}(\mathfrak{g}))$ maps into matrices, it makes sense to take its Laplacian, and so we define $M_{\Delta^s f}$ to be the multiplication operator corresponding to $\Delta^s(\mathrm{Ad}\, f)$. Set

$$T^{(s)}(f) = \tfrac{1}{2}\{M_f + \Delta^{-s} M_f \Delta^s - \Delta^{-s} M_{\Delta^s f}\}. \tag{6.3}$$

Proposition 6.4. *For smooth maps $f, g \in \mathrm{Map}_0(M,G)$ the operator $T^{(s)}(f)T^{(s)}(g) - T^{(s)}(fg)$ is pseudodifferential of order $\max(-1,-2s)$. In particular, for $s > 0$ it is compact.*

Proof. Let $-q = \max(-1,-2s)$. Then as in §1

$$T^{(s)}(f) = M_f + (\text{order } -q).$$

The proposition is immediate from the fact that $M_f M_g = M_{fg}$.

It is not hard to see that $T^{(s)}(f)T^{(s)}(g) - T^{(s)}(fg)$ is compact for all $f, g \in H_t(M,G)$. Since $T^{(s)}$ sends the constant map at the identity to the identity operator, we conclude that $T^{(s)}$ maps into $\mathrm{Fred}_0(\mathscr{H}_{\mathbf{R}})$, and defines a homomorphism

$$\mathrm{Map}_0(M,G) \xrightarrow{T^{(s)}} \mathrm{Fred}_0(\mathscr{H}_{\mathbf{R}}) \to \mathscr{G}(\mathscr{H}_{\mathbf{R}}).$$

Note that $T^{(0)}$ also defines a family of Fredholm operators. The families $T^{(s)}$ for different values of s are homotopic. In the case $M = S^1$ and $s = 1/2$ we recover the realification of the family used in §5 to define the reduced complex frame bundle of ΩG.

The family $T^{(s)}$ determines a reduction of the real $\mathrm{GL}(\mathscr{H}_{\mathbf{R}})$ frame bundle to a $\mathrm{GL}^{\mathrm{cpt}}(\mathscr{H}_{\mathbf{R}})$ frame bundle. We assert that this reduction faithfully replaces the holonomy construction. Our argument in §5 relied on the formula for the Chern classes and the fact that the Chern classes completely characterize a bundle over ΩG. That argument breaks down here for several reasons. First, in the real category there are $\mathbf{Z}/2\mathbf{Z}$ characteristic classes—the Stiefel-Whitney classes—which are not accessible by curvature. Also, even when s is large the curvature only has order -1, and so is in $L^{n+\varepsilon}$ for $n = \dim(M)$. This means that the lower Pontrjagin character forms, defined by traces of low powers of the curvature, will diverge. (This is not a serious problem, though, since the curvature could be regulated using the Laplacian on M.) Finally, it is no longer true in general that the characteristic classes classify $\mathrm{GL}(\infty; \mathbf{R})$-bundles over $\mathrm{Map}_0(M,G)$.

In spite of these negative considerations, we use (6.3) to introduce a real Fredholm structure. The *real geometric frame bundle* of $\mathrm{Map}_0(M,G)$ is the $\mathrm{GL}^{\mathrm{cpt}}(\mathscr{H}_{\mathbf{R}})$-bundle induced from $T^{(s)}$. This reduced frame bundle is trivial.

Proposition 6.5. *The maps $T^{(s)}$ are null homotopic.*

Proof. As all $T^{(s)}$ are homotopy equivalent, it suffices to prove the assertion for $T^{(0)}$. But $T^{(0)}(f) = \frac{1}{2}M_f$ maps into invertible operators $\mathrm{GL}(\mathscr{H}_{\mathbf{R}})$. Since $\mathrm{GL}(\mathscr{H}_{\mathbf{R}}) \subset \mathrm{Fred}_0(\mathscr{H}_{\mathbf{R}})$ is contractible, $T^{(0)}$ is homotopically trivial.

Proposition 6.5 is consistent with finite dimensional theory. As a real manifold $\mathrm{Map}(M, G)$ is a Lie group, and in finite dimensions Lie groups are always parallelizable. If we are to define a reduced frame bundle in infinite dimensions with geometric significance, then we expect that the reduced bundle will still be trivial.

These methods also apply to the full flag manifold $\mathscr{F} = LG/T$ of the loop group.

Proposition 6.6. *The real geometric frame bundle of $\mathscr{F}$, as defined by a homogeneous Kähler metric, is trivial.*

Proof. Write $\mathrm{Lie}(LG) = \mathfrak{t} \oplus \mathfrak{m}$, where $\mathfrak{t}$ is the Lie algebra of T and $\mathfrak{m} = \mathfrak{m}^{(0)} \oplus \mathfrak{m}^{(1)}$ with

$$\begin{aligned} \mathfrak{m}^{(0)} &= \text{ sum of root spaces of } G; \\ \mathfrak{m}^{(1)} &= \bigoplus_{n \neq 0} \{\sin(n\theta)\mathfrak{g} \oplus \cos(n\theta)\mathfrak{g}\}. \end{aligned}$$

(This is the real version of the decomposition (4.2).) Recall that Kähler metrics on $\mathscr{F}$ are parametrized by elements μ in the interior of a Weyl chamber (together with a scale factor on $d/d\theta'$ which we fix), and $H_\mu \in \mathfrak{t}$ is the element dual to μ. Let $D = D^{(0)} + D^{(1)}$ be the diagonal operator

$$D = \left(\begin{array}{c|c} \mathrm{ad}\, H_\mu & 0 \\ \hline 0 & d/d\theta \end{array}\right)$$

on $\mathfrak{m}$. D is invertible on $\mathfrak{m}$. For $f \in LG$ and $0 \le s \le 1$ define

$$T^{(s)}(f) = \tfrac{1}{2}\{M_f + D^{-s}M_f D^s - D^{-s}M_{d^s f}\}. \tag{6.7}$$

The families (6.7) are the realifications of the families used in §5 to describe the complex geometric frame bundle. The real version of Corollary 5.4 gives an induced $\mathrm{GL}^{\mathrm{cpt}}(\mathscr{H}_{\mathbf{R}})$-bundle over LG/T. The $T^{(s)}$ are all homotopic, and $T^{(0)}$ is a homomorphism into the invertibles, hence is null homotopic. Therefore, the reduced real frame bundle is trivial. We reiterate that the complex geometric frame bundle of $\mathscr{F}$, which is pinned down by its Chern classes, determines the real geometric frame bundle. Therefore, there is no problem interpreting Proposition 6.6 in terms of our original Kähler curvature computation (for $\mathscr{F}$). It states that the even Chern classes $\mathrm{ch}_{2k}(\mathscr{F})$, as defined by curvature, vanish mod 2, and that those combinations of Chern classes which define the Pontrjagin classes also vanish (cf. (3.4)).

As the real geometric frame bundle of $\mathrm{Map}(M, G)$ is topologically trivial, we are led to speculate about where to find nontrivial topology in the case where$M$ is no longer a circle. Philosophically, whereas loop groups behave like compact Lie groups, the groups $\mathrm{Map}(M, G)$ for $\dim(M) > 1$ behave like noncompact Lie groups. For these groups spin geometry replaces complex geometry, for example in the construction of representations [8]. Assume now that M is an odd dimensional spin manifold. Let $\mathscr{H}$ denote the space of (Sobolev) $\mathfrak{g}$-valued spinor fields on M, and decompose $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ according to the positive and nonpositive spectrum of the Dirac operator. Let $\pi_+ \colon \mathscr{H} \to \mathscr{H}_+$ denote the projection. An element $f \in \mathrm{Map}(M, G)$ defines an operator $M_f \in \mathfrak{gl}(\mathscr{H})$ which acts by the adjoint action on the Lie algebra indices of the spinor fields. Set

$$T_f = \pi_+ M_f \pi_+ \in \mathfrak{gl}(\mathscr{H}_+).$$

Then T_f is Fredholm of index zero, and there is a family

$$T \colon \mathrm{Map}_0(M, G) \to \mathrm{Fred}_0(\mathscr{H}_+). \tag{6.8}$$

Note that (6.8) reduces to the Toeplitz family on Lg when $M = S^1$. When $M = S^{2n-1}$, (6.8) again factors through Bott periodicity; the homotopy class for T for general M is determined by the Atiyah-Singer Index Theorem for Families. Segal [35] studied this family in the context of "anomalies." We suggest it here as being relevant to the spin geometry of $\mathrm{Map}(M, G)$, possibly in the supermanifold framework.

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