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THE TOPOLOGY AND GEOMETRY OF EMBEDDED SURFACES OF CONSTANT MEAN CURVATURE

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The mean curvature function H on an oriented surface M in $\mathbf{R}^3$ is defined at a point p in M to be

$$H(p) = \tfrac{1}{2}(\lambda_1(p) + \lambda_2(p)),$$

where $\lambda_1(p)$ and $\lambda_2(p)$ are the principal curvatures of M at p. When H is constant, M is called a surface of constant mean curvature.

A surface is said to have finite type if it is homeomorphic to a closed surface with a finite number of points removed. An important problem in classical differential geometry is the classification of properly embedded finite type surfaces M of constant mean curvature in $\mathbf{R}^3$.

If M is a closed embedded surface of constant mean curvature, then it follows from Alexandrov [1] that M must be a round sphere. He proved this theorem with a technique which shows that any embedded closed hypersurface of $\mathbf{R}^n$ which has constant mean curvature is invariant under reflection in a large number of hyperplanes. The technique used in his proof is known as the Alexandrov reflection principle. The classical examples of properly embedded surfaces of finite type with zero mean curvature are the plane, the helicoid, and the catenoid. Surfaces of zero mean curvature are usually referred to as minimal surfaces. The remaining classical examples of properly embedded surfaces of nonzero constant mean curvature of finite type were found in 1841 by Delaunay [5]. The Delaunay surfaces are surfaces of revolution. Recently N. Kapouleas [11] has constructed several new examples of properly embedded surfaces of nonzero mean curvature which are homeomorphic to a sphere punctured in a finite number of points.

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Recently Hoffman and Meeks ([8], [9], [10]) have found examples of properly embedded minimal surfaces which are homeomorphic to closed surfaces of positive genus with three points removed. Besides finding new examples of properly embedded minimal surfaces of finite type, Hoffman and Meeks [8] have developed a theory to deal with global problems concerning the geometry of these surfaces. In particular, they obtained results concerning the geometry of the annular ends of a properly embedded minimal surface M in $\mathbf{R}^3$ even when M does not have finite type. An *annular end* of M is a properly embedded annulus A on M which is homeomorphic to $S^1 \times [0,1)$, where S^1 is a circle.

In this paper we shall study the geometry of the annular ends of a properly embedded surface M of nonzero constant mean curvature. As a consequence of this study we shall be able to prove the following theorems.

Theorem 1. *If M is a properly embedded surface in $\mathbf{R}^3$ with nonzero constant mean curvature, then M is not homeomorphic to a closed surface with a single point removed.*

Theorem 2. *Suppose M is a properly embedded surface in $\mathbf{R}^3$ with nonzero constant mean curvature. If M is homeomorphic to a closed surface with 2 points removed, then M is a bounded distance from some straight line.*

Theorem 3. *Suppose M is a properly embedded surface of nonzero mean curvature which is homeomorphic to a closed surface with three points removed. Then M stays a bounded distance from a plane.*

The above theorems are proved by studying stable minimal disks in one of the complements of M in $\mathbf{R}^3$. The curvature estimates for stable minimal surfaces given by R. Schoen [17] play an essential role in our analysis.

Recently, B. Palmer [16] and A. Silveira [19] independently proved that the only complete "stable" surfaces of nonzero constant mean curvature are the round spheres. A simple consequence of this stability theorem is that any foliation of an open subset of $\mathbf{R}^3$ by complete surfaces of a fixed constant mean curvature consists of parallel planes. This consequence was observed by Barbosa, Gomes, and Silveira [2]. We prove from their observation and Theorem 1 that the only foliation of $\mathbf{R}^3$ by surfaces, each of which has constant mean curvature, is by parallel planes.

The paper is divided into four sections. The first section deals with the geometry of a properly embedded annulus A with mean curvature function greater than or equal to 1, where A is homeomorphic to $S^1 \times [0,1)$. We prove a compactness theorem that shows that if P is a plane, and $P \cap A$ contains a connected component which is noncompact, then any parallel plane P' which is sufficiently far from P must intersect A in only compact connected components. In §2 we show that a compact surface of constant mean curvature

1 whose boundary is contained in the x_1x_2-plane has x_3 coordinate bounded in norm by 2. In §3 we apply the results of the previous sections to analyze the geometry of a properly embedded annulus of constant nonzero mean curvature. The main theorems follow directly from these geometric results. In §4 we apply Theorem 1 to study foliations of $\mathbf{R}^3$ where the leaves are surfaces of constant mean curvature.

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1. The geometry of properly embedded annuli with mean curvature function greater than 1

The following lemma is an immediate consequence of the curvature estimates for stable minimal disks given by R. Schoen [17].

Lemma 1.1. *There exists a universal constant C such that the following statement holds.*

Let $f\colon M \to \mathbf{R}^3$ be a stable compact orientable minimal immersion of a surface in $\mathbf{R}^3$ satisfying:

(i) *There exists a point $p \in M$ with $f(p) = (0,0,0)$.*

(ii) *The tangent space T_pM equals the x_1x_2-coordinate plane.*

(iii) *$f(\partial M)$ is contained in the complement of the ball $B_C = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 \leqslant C^2\}$.*

Then there exists a disk $D \subset M$ with $p \in D$ such that $f(D)$ is a graph of a function g defined on the disk $E = \{(x_1, x_2, 0) \mid x_1^2 + x_2^2 \leqslant 9\}$, and $|g| \leqslant 1$.

We use the above lemma to prove the following main result of this section.

Proposition 1.2. *Suppose A is a smooth properly embedded annulus in $\mathbf{R}^3$ which is homeomorphic to $S^1 \times [0,1)$ and which has mean curvature which is greater than or equal to 1 at each point on A. Let C be the universal constant given in Lemma 1.1. If the distance between parallel planes P_1 and P_2 is greater than $2C$, then either $P_1 \cap A$ or $P_2 \cap A$ contains connected components which are all compact.*

Proof. First we modify A inside a ball B to obtain a simply-connected properly embedded surface M such that $M \cap (\mathbf{R}^3 \setminus B)$ is contained in A. Let B be a round closed ball centered at the origin so that ∂B is transverse to A and ∂A is contained in the interior of B. Let $\mathscr{D}$ be a collection of pairwise disjoint embedded compact disks in B where the boundary curves of the disks in $\mathscr{D}$ correspond to the boundary curves of the noncompact component W of $A \cap (\mathbf{R}^3 \setminus \text{interior}(B))$. Suppose that $\mathscr{D}$ is chosen so that the surface $M = W \cup \mathscr{D}$ is smooth. Note that M is homeomorphic to $\mathbf{R}^2$. Alexander duality [7] implies that M separates $\mathbf{R}^3$ into two closed components with common boundary M. Let X be that component whose boundary outside of B has positive mean curvature with respect to the inward pointing normal.

Suppose, for simplicity, that P_1 and P_2 are transverse to A and that each contains connected components which are noncompact proper curves. Now choose a parametrization $f\colon \mathbf{R}^2 \to \mathbf{R}^3$ of M and note that there are properly embedded arcs $\alpha_1, \alpha_2\colon [0,1) \to \mathbf{R}^2$ so that $f(\alpha_1) \subset P_1 \cap A$ and $f(\alpha_2) \subset P_2 \cap A$. Choose a smooth embedded arc α_3 in $\mathbf{R}^2$ which joins $\alpha_1(0)$ and $\alpha_2(0)$ and so that $\alpha_1 \cup \alpha_2 \cup \alpha_3$ is a properly embedded arc (see Figure 1).

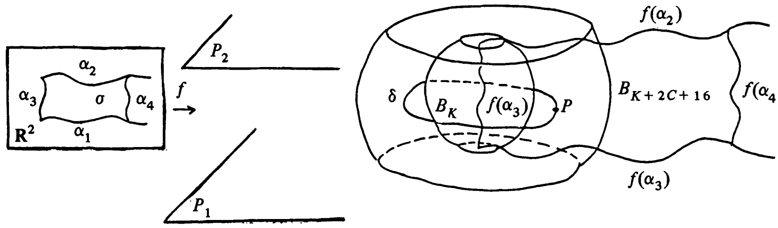

FIGURE 1

After a possible rigid motion of $M \cup P_1 \cup P_2$, we may assume that P_1 and P_2 are planes which are equidistant to the x_1x_2-plane. Now choose a ball $B_K = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 \leqslant K^2\}$ large enough so that B_K intersects $P_1 \cup P_2$, and $\alpha_3 \cup B \subset B_K$ (see Figure 1). Since $f^{-1}(B_{K+2C+16})$ is compact, there exists a smooth embedded arc α_4 in $\mathbf{R}^2 \setminus (f^{-1}(B_{K+2C+16}))$ such that $\mathbf{R}^2 \setminus (\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4)$ contains a unique component which is a bounded disk with a simple closed boundary curve σ (see Figure 1).

Let $\delta = \{(x_1, x_2, 0) \mid x_1^2 + x_2^2 = (K + C + 4)^2\}$ and note that δ and $f(\sigma)$ are linked in $\mathbf{R}^3$. Now modify the metric on $X \cap B$ to obtain a new Riemannian metric on X so that ∂X has positive mean curvature (see [14] for the construction of such a metric). With respect to this new metric, the work of Meeks and Yau ([13] and [14]) shows that $f(\sigma)$ is the boundary of a least area embedded disk E in X. (Actually to apply the theorem of Meeks-Yau one uses the fact that $f(\sigma)$ is the boundary of a disk in ∂X.) Since the curve δ links $f(\sigma)$, there exists a point $p \in \delta \cap E$.

Since $\Sigma = E \cap (X \setminus \text{interior}(B))$ is a stable orientable minimal surface in $\mathbf{R}^3$ in the flat metric whose boundary is a distance of at least C from p, Lemma 1.1 implies that Σ contains a subdisk F which is a graph over the disk centered at p in $T_p\Sigma$ of radius 3 and this graph is at most distance 1 from $T_p\Sigma$. Now choose an orthogonal coordinate system (y_1, y_2, y_3) for $\mathbf{R}^3$ centered at p so that $T_p\Sigma$ is the y_1y_2-coordinate plane. Let $V_t = \{(y_1, y_2, y_3) \mid y_1^2 + y_2^2 + (y_3 - t)^2 = 4\}$ and note that V_t is disjoint from B for $t \in [-4, 4]$ and V_t is disjoint from F for $t = [3, 4]$.

Assertion 1.3. *The sphere V_3 is contained in $X \setminus B$.*

Proof of Assertion 1.3. Throughout this proof refer to Figure 2. Consider the solid cylinder $C_3 = \{(y_1, y_2, y_3) \mid y_1^2 + y_2^2 \leqslant 9\}$ of radius 3. Suppose that the disk F is represented as a graph of a function $g(y_1, y_2)$ and recall that $|g| \leqslant 1$. Let $W = \{(y_1, y_2, y_3) \in C_3 \mid y_3 \geqslant g(y_1, y_2)\}$ and let $U_t = V_t \cap W$. Note that $U_t = \varnothing$ for $t < -2$ and $U_t = V_t$ for $t \geqslant 2$. If V_t is not contained in $X \setminus B$ for some $t \in [-3, 3]$, then U_t is not contained in $X \setminus B$ for some smallest $t_0 \in [-3, 3]$. Since ∂U_{t_0} is contained in F, F is disjoint from $\partial X \cup B$, and $U_{t_0+\varepsilon} \cap B = \varnothing$ for $\varepsilon \in (0, 1)$, there exists an interior point $q \in U_{t_0}$ such that $q \in U_{t_0} \cap (\partial X \setminus B)$ and U_{t_0} is contained in $X \setminus B$. This implies that the mean curvature of U_{t_0} at q must be greater than or equal to the mean curvature of ∂X at q. However, this statement contradicts the observation that the mean curvature of U_{t_0} at q is $1/2$ and the mean curvature of ∂X at q is at least 1. This proves the assertion.

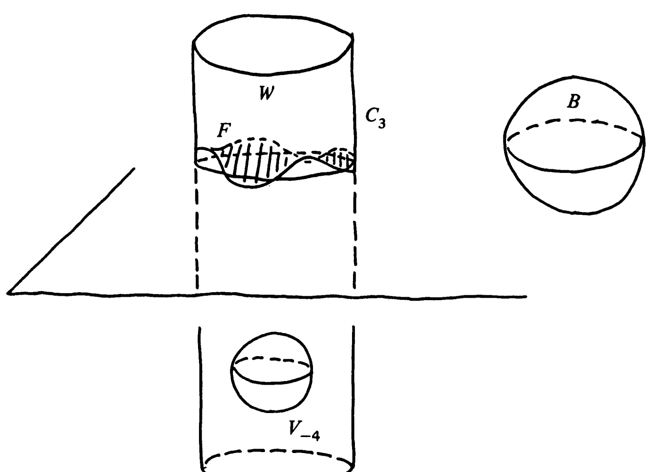

FIGURE 2

Assertion 1.3 implies that there exists a sphere S of radius 2 in $X \setminus B$. Now continuously translate S away from B and toward the boundary of X until the translated sphere intersects ∂X for the first time. As in the proof of Assertion 1.3, a simple comparison of mean curvatures of ∂X and of the translated sphere at the common point of contact gives the contradiction which proves the proposition.

Remark 1.4. With minor modifications in the proof of Proposition 1.2 one can prove the following more general result. Suppose that P_1 and P_2 are two properly embedded surfaces of distance greater than $2C$ outside some compact set. If M is a connected properly embedded annulus homeomorphic to $S^1 \times [0, 1)$ with mean curvature function at least 1, then $M \cap P_1$ and $M \cap P_2$ cannot both have noncompact connected components.

2. Boundedness properties for constant mean curvature 1 surfaces

In this section we shall prove that a compact surface of constant mean curvature whose boundary is contained in a plane must stay close to that plane. A first step in proving this boundedness theorem is the following proposition which was proved by Serrin [18] in the case of surfaces.

Proposition 2.1. *Suppose $M^n \subset \mathbf{R}^{n+1}$ is a graph over a compact domain in the hyperplane $\{x_{n+1} = 0\}$. If M has constant mean curvature* 1, *then $\|x_{n+1} \mid M^n\| \leqslant 1$. Furthermore, this estimate is sharp for a hemisphere.*

Proof. If a graph in $\mathbf{R}^{n+1}$ lies above $\{x_{n+1} = 0\}$, has boundary in $\{x_{n+1} = 0\}$, and has constant mean curvature n (with the appropriate sign convention) and upper normal V^{n+1}, then one has

$$\Delta(x^{n+1} - v^{n+1}) = (-n + |A|^2)v^{n+1} \geqslant 0 \quad \text{on } M, \tag{2.1}$$

$$(x^{n+1} - v^{n+1}) \leqslant 0 \quad \text{on } \partial M. \tag{2.2}$$

Therefore one has by the maximum principle that $x^{n+1} \leqslant v^{n+1} \leqslant 1$ on the entire surface.

Proposition 2.2. *Suppose $M \subset \mathbf{R}^{n+1}$ is a compact embedded hypersurface of constant mean curvature. If the boundary of M is contained in a hyperplane P, then the maximum distance of M from P is* 2.

Proof. Let $q \in M$ be a point of maximum distance T from P and let K be the hyperplane of distance $T/2$ from P which separates P and the point q. Let Y be the component of $M \setminus K$ which contains the point q. A straightforward application of the Alexandrov reflection principle proves that Y is a graph over P with boundary in K. Proposition 2.1 implies that the graph Y which has height $T/2$ actually has height at most 1. Hence $T \leqslant 2$ which proves the proposition.

Corollary 2.3. *If M is a properly embedded connected hypersurface of constant mean curvature with some coordinate function being proper and bounded from below, then M is a round sphere.*

Proof. Recall that Alexandrov proved that if M is closed, then M is a round sphere. Suppose M is noncompact and $x_1 | M$: $M \to \mathbf{R}$ is proper and bounded from below. Since M is noncompact, we may assume, after a possible translation of M, that $x_1 | M$ is nonnegative on M, x_1 is not bounded from above, and there exists a point $p \in M$ with $x_1(p) = (0,0,0)$. Let $T \in (2, \infty)$ be a regular value for $x_1 | M$ and let M_T be the component of $x_1^{-1}[0, T]$ which contains p. Proposition 2.2 shows M_T cannot exist. This contradiction proves the corollary.

In the next lemma we have generalized the situation described for surfaces in Proposition 2.1 to the case of graphs over closed domains of the plane. Actually we do not need this more general result to prove our main theorems. However, we include it here because of some independently interesting ideas which occur in the proof and which, if suitably generalized, might give an elementary proof of Proposition 1.2 without the use of Schoen's curvature estimates for stable minimal disks. In this regard the interested reader might want to compare the proofs of Lemma 2.4 and Proposition 1.2.

Lemma 2.4. *Suppose R is a smooth connected closed region of $\mathbf{R}^2$ and M is the graph of a nonnegative function f: $R \to \mathbf{R}$ which is zero on ∂R. If M has constant mean curvature* 1, *then f is bounded in norm by* 2.

Proof. Let $P_t = \{(x_1, x_2, t) \mid x_1, x_2 \in \mathbf{R}\}$ and suppose, for simplicity, that P_1 is transverse to M. Let X be the region under the graph M and bounded by $R \cup M$.

Note that M has mean curvature $+1$ with respect to the inward pointing normal to X. To prove this observation assume that $(0,0,0)$ is a point in the interior of R. Consider the sphere $V_t = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + (x_3 - t)^2 = 4\}$. For large positive t, $V_t \cap M = \varnothing$. Since f is nonnegative, there is a largest $t_0 \in [1, \infty)$ such that $V_{t_0} \cap M \neq \varnothing$. A comparison of mean curvatures at a point of intersection shows that the mean curvature of V_{t_0} which is an absolute value $1/2$ must have an opposite sign from the sign of the mean curvature of M at a point in $V_{t_0} \cap M$ with respect to a common normal vector. This means that the mean curvature of M has the correct sign.

Let $W = P_1 \cap X$ and suppose that W contains a component Y and a pair of interior points r_0 and r_1 of distance greater than or equal to 1. Hence, there are two parallel lines L_0 and L_1 in P_1 of distance 1 and such that $r_0 \in L_0$ and $r_1 \in L_1$. Let $\hat{L}_0$ and $\hat{L}_1$ be the lines in the x_1x_2-plane obtained from L_0 and L_1, respectively, by a vertical x_3-translation. Furthermore, choose an embedded arc α_1: $[0,1] \to \text{interior}(Y)$ such that $\alpha_1 \cap L_0 = \alpha_1(0)$ and $\alpha_1 \cap L_1 = \alpha_1(1)$. For $t \in [0,1]$ let $\alpha_t(s) = (x_1(\alpha_1(s)), x_2(\alpha_1(s)), t)$ and let $E = \bigcup_{t \in [0,1]} \alpha_t$. Let Q be the convex hull of $L_0 \cup L_1 \cup \hat{L}_0 \cup \hat{L}_1$ (see Figure 3). Let J_1 and J_2 denote the closures of the two components of $Q \setminus E$.

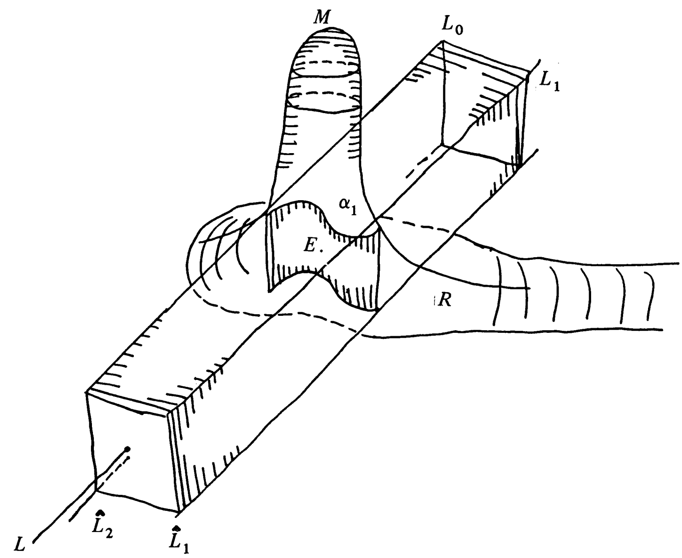

FIGURE 3

Let L be the line which is equidistant from the lines in $\{L_0, L_1, \hat{L}_0, \hat{L}_1\}$ and let V_t, $t \in (-\infty, \infty)$, be a continuous parametrization of the spheres of radius 1 with centers on L. For large negative values of t we may assume that $V_t \subset J_1$ and for large positive values of t that $V_t \subset J_2$. The argument given in the proof of Assertion 1.3 shows that $V_{t_0} \subset X$ for some t_0. Now translate V_{t_0} vertically upward until it touches M for the first time. The maximum principle [1] for surfaces of constant mean curvature applied to a point of contact between V_{t_0} and M shows that V_{t_0} must agree on an open set. The unique continuation property for surfaces of constant mean curvature shows that M is a sphere. This contradicts our assumption that M is a graph. We conclude that the component Y must be contained completely inside of a disk of radius 1 in the plane P_1. This proves that every component of W is compact. The lemma now follows directly from Proposition 2.1.

3. The main results

Proposition 3.1. *Suppose A is a properly embedded annulus of nonzero constant mean curvature where A is homeomorphic to $S^1 \times [0, 1)$. Then, after a possible rigid motion of A, the annulus A stays a bounded distance from the positive x_1-axis.*

Proof. After composing A with a homothety of $\mathbf{R}^3$, we shall assume that A has constant mean curvature 1. We first prove the following assertion concerning A.

Assertion 3.2. *Suppose* (y_1, y_2, y_3) *is an orthogonal coordinate system for* $\mathbf{R}^3$. *If* $y_1 | A$ *is unbounded from above, then the coordinate function* $y_1 | A$ *is a proper function on* A.

Proof of Assertion 3.2. Assume $y_1 | A$ is unbounded and let $\{s_1, s_2, \cdots, s_n, \cdots\}$ be a sequence of points on A with $y_1(s_i) \geqslant i$ for each integer i. Proposition 1.2 implies that, after a fixed vertical downward translation of A, we may assume that $y_1 | (\partial A)$ has negative values, every component of $(y_1 | A)^{-1}(t)$ is compact for $t \geqslant 0$, and $t = 0$ is a regular value for $y_1 | A$. Let $A_+ = (y_1 | A)^{-1}[0, \infty)$. Proposition 2.2 implies that A_+ must have at least 1 noncompact component. Suppose that Y is any smooth noncompact proper connected domain in A whose boundary consists of simple closed curves in A and $\partial Y \cap \partial A = \varnothing$. In this case Y has the property that $A \setminus (\text{interior}(Y))$ consists of a single component which is a compact annulus and a collection of compact disks. Furthermore, the domain Y has the property that it contains a unique boundary curve which is homotopically nontrivial. It follows from this description that A_+ contains a unique noncompact component E_0.

The above discussion implies that for any positive regular value t for $Y_1 | A$, $(Y_1 | A)^{-1}([t, \infty))$ has a unique noncompact component E_t. Let A_t denote the noncompact subdomain of A whose boundary is the homotopically nontrivial boundary curve of E_t. Since every component of $A_t \setminus (\text{interior}(E_t))$ is compact, Proposition 2.2 implies that $Y_1 | A_t$ is bounded from below by the constant $t - 2$. Hence $Y_1 | A$ is proper, which completes the proof of the assertion.

Assume now that the boundary of A is contained in a round ball B centered at the origin. Choose a sequence of points $\{p_1, \cdots, p_n, \cdots\}$ on A where $\|p_i\| \geqslant i$. Let $Q = \{p_i/\|p_i\|\}$ be the related set of normalized vectors and let q be a limit point of the set Q. After a possible rotation of A, we may assume that $q = (1, 0, 0)$.

Suppose that A is not a bounded distance from the x_1-axis. Since A has one end and the x_1-coordinate is unbounded from above on A, Assertion 3.2 implies that the x_1-coordinate is proper. Hence $x_1 | A$ must be bounded from below. Since A is not a bounded distance from the positive x_1-axis, we now conclude that after a possible rotation of A around the x_1-axis, $x_3 | A$ is proper and goes to $+\infty$ at the end of A.

Let R be the radius of a ball B centered at the origin which contains ∂A. Since $x_3 | A$ is proper and bounded from below, there is a regular value $T \geqslant R + 10$ and a Jordan curve γ_T in $(x_3 | A)^{-1}(T)$ which is homotopically nontrivial in A. From this information it is straightforward to find a plane P

which satisfies the following properties:

(3.3) P is transverse to A.

(3.4) P is disjoint from $\gamma_T \cup B$ and the positive x_1-axis.

(3.5) The normal vector to P is of the form $v/\|v\|$ where $v = (-\varepsilon, 0, 1)$, $\varepsilon > 0$.

(3.6) P separates γ_T and B.

(3.7) The distance from γ_T to P is greater than 2.

Let (y_1, y_2, y_3) be the orthogonal coordinate system on $\mathbf{R}^3$ generated by the ordered set of vectors $\{v_1 = v, v_2 = (0,1,0), v_3 = v_2 \times v_2\}$ and with $y_3^{-1}(0) = P$. By construction of the limit point q of Q, it follows that the set of numbers $\{y_1(p_1), \cdots, y_1(p_n), \cdots\}$ is unbounded from below. Assertion 3.2 implies that $y_1 | A$ is proper and goes to $-\infty$ at the end of A. It follows that there is a homotopically nontrivial Jordan curve δ in $x_3^{-1}([T, \infty))$ such that $y_1(\delta)$ is negative (see Figure 4).

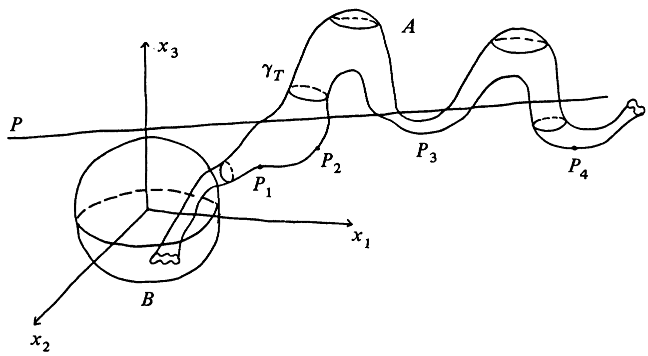

FIGURE 4

The curves ∂A and δ form the boundary of a compact annulus $\hat{A}$ in A and the boundary of $\hat{A}$ has negative y_1-coordinate. On the other hand $(y_1 | \hat{A})^{-1}([0, \infty))$ is compact, has its boundary contained in the plane $y_1^{-1}(0) = P$, and contains points on γ_T which have distance greater than 2 from the plane $y_1^{-1}(0)$. This contradicts Proposition 2.2 and shows that $x_2 | A$ and $x_3 | A$ must be bounded. Since $x_1 | A$ is bounded from below, the proposition is proved.

Theorems 1 and 2 follow immediately from Proposition 3.1 and Corollary 2.3. The proof is as follows:

Proof of Theorems 1 *and* 2. Suppose M is a properly embedded surface of nonzero constant mean curvature. If M is homeomorphic to a closed surface with one point removed, then there exists an annular end A of M homeomorphic to $S^1 \times [0,1)$ such that $M \setminus (\text{interior} A)$ is compact. It follows from Proposition 3.1 that some coordinate function on M is proper and bounded from below. However, Corollary 2.3 shows M cannot have a proper coordinate function bounded from below which proves Theorem 1. If M is homeomorphic to a closed surface with two points removed, then there exist two pairwise disjoint annular ends A_1 and A_2 on M where each annulus is homeomorphic to $S^1 \times [0,1)$ and $M \setminus (\text{interior}(A_1 \cup A_2))$ is compact. Proposition 3.1 implies that M stays a bounded distance from two rays. If these rays are parallel, then M stays a bounded distance from a line. However, if the rays are not parallel, then clearly there is a coordinate function (in some orthogonal coordinate system) of M which is proper and bounded from below but no such proper coordinate function exists by Corollary 2.3. This concludes the proof of Theorem 2.

Using an argument similar to the proof of Theorem 2, we prove Theorem 3.

Proof of Theorem 3. Let A_1, A_2, A_3 be three pairwise disjoint annular ends and note that $M \setminus (\text{interior}(A_1 \cup A_2 \cup A_3))$ is compact. Proposition 3.1 implies that there exist rays L_1, L_2, and L_3 such that M stays a bounded distance from $L_1 \cup L_2 \cup L_3$. Let $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ be the inward pointing unit normals at the end points of L_1, L_2, L_3, respectively. The surface M will stay a bounded distance from a plane if and only if the vector parts $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ of $v_1, \tilde{v}_2, \tilde{v}_3$ are coplanar. If $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ are not coplanar, then there exists a vector w such that the inner product of w with each of the vectors in $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ is positive. It follows immediately from this description that orthogonal projection of M onto a line parallel to w is a proper function that is bounded from below. The existence of such a proper function contradicts Corollary 3.2, which proves the theorem.

Remark 3.9. The known surfaces which satisfy the hypotheses of Theorem 2 are the surfaces of revolution found by Delaunay. It is likely that the Delaunay surfaces are the only possible examples of properly embedded surfaces of constant mean curvature which are homeomorphic to a closed surface with two points removed. We conjecture that a noncompact connected surface of constant mean curvature which is disjoint from two nonparallel planes must be a surface of revolution. This conjecture together with Theorem 2 would demonstrate the uniqueness of the Delaunay surfaces. More generally, we conjecture that if M is a properly embedded connected surface of constant mean curvature which is contained in a half-space of $\mathbf{R}^3$, then M will be invariant under reflection through some plane which is parallel to the boundary of the half-space. If this conjecture is true, then all surfaces satisfying the hypotheses of Theorem 3 must be invariant under reflection through a plane. An example of properly embedded surface with genus 0 and three ends was recently given by N. Kapouleas. It has the conjectured reflectional symmetry.

4. Foliations of constant mean curvature

As mentioned in the introduction we shall prove that one cannot foliate $\mathbf{R}^3$ with surfaces of (possible varying) constant mean curvature except by parallel planes. In their theses B. Palmer [16] and A. Silveira [19] proved that the only stable complete surfaces of constant nonzero mean curvature are spheres. It follows immediately from their result [2] that one cannot foliate an open set of $\mathbf{R}^3$ by complete surfaces of a fixed nonzero constant mean curvature. We use this observation to prove that the only foliation of $\mathbf{R}^3$ by "soap bubbles" is by parallel planes.

Theorem 4.1. *Suppose F is a C^2-foliation of $\mathbf{R}^3$ where each leaf of F is a surface of constant mean curvature. Then F consists entirely of parallel planes.*

Proof. Let L be a leaf of F. If L is not proper, then the foliation must have nontrivial holonomy along some nonsimply-connected leaf $\tilde{L}$. Clearly, if the foliation F has nontrivial holonomy, then there is an open set $\mathscr{L}$ of leaves of F such that each leaf of this open set contains the leaf $\tilde{L}$ in its closure. Since F is a C^2-foliation, every leaf in $\tilde{L}$ must have the same constant mean curvature as the leaf L. However, the result in [2] shows that each leaf in $\mathscr{L}$ is a flat plane. Since the limit of flat planes is also a flat plane, the leaf $\tilde{L}$ is simply connected, which gives a contradiction. Hence, every leaf of F must be proper.

If every leaf of F is simply-connected, then the main theorem in [15] also implies that every leaf of F is proper. Thus, if every leaf of F is simply-connected, Theorem 1 shows that every leaf of F must have zero mean curvature. Since the leaves of a minimal orientable foliation have least area, the foliation F must consist entirely of planes by [3] or [6].

Suppose now that every leaf of F is proper but some leaf L of F is not simply-connected. Since L is proper it divides $\mathbf{R}^3$ into two components, X and Y. Since $\mathbf{R}^3$ is simply-connected, Van Kampen's theorem [7] implies that one of these components, say X, has compressible boundary. In other words, the inclusion of the fundamental group of the boundary of X into the fundamental group of X has a nontrivial kernel. In particular, Dehn's lemma [13] implies that there is a Jordan curve γ on L which is the boundary of a disk D in X and where γ is homotopically nontrivial in L.

Suppose that such a D is chosen to be in general position with F. Such a general position disk will have an induced foliation by curves with a finite number of singularities. A simple innermost circle argument shows, after a possible new choice of L and γ, that the disk D intersects F in a collection of concentric Jordan curves $\Gamma = \{\gamma_t \mid t \in (0,1]\}$ where $\gamma_1 = \gamma$, the γ_t converge to

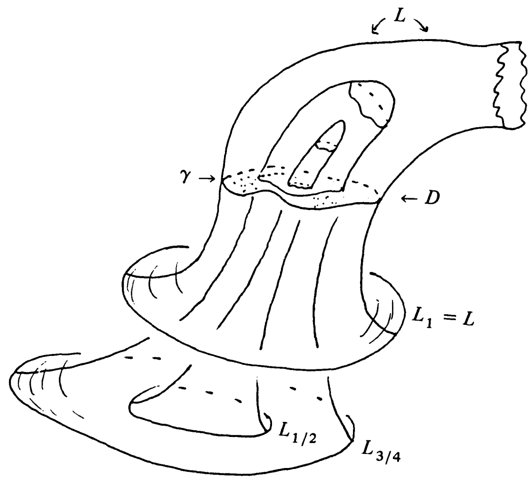

FIGURE 5

a point as $t \to 0$, and each γ_t bounds a unique disk D_t on a leaf L_t of F when $t \neq 1$ (see Figure 5).

Since $\mathbf{R}^3$ is simply-connected, the foliation F can be oriented by a unit normal vector field N to F. Once the vector field N has been chosen, the mean curvature of the leaves of F induce a function $H: \mathbf{R}^3 \to \mathbf{R}$. Using N it makes sense to ask whether or not H is increasing or monotonic at a given point in space. If H is not monotonic at a given point p, then a simple argument shows the leaves that pass near p are stable and one easily deduces from [16] and [19] that the leaves near p are parallel planes. (Note, we are using the fact that the leaves of F are proper and noncompact and hence no leaf is a round sphere.) From this discussion one easily deduces that if the earlier chosen leaf L is minimal (recall L is not a flat plane), then a nearly leaf $\hat{L}$, on either side of L, will have positive mean curvature when considered to be the boundary of the component R of $\mathbf{R}^3 \setminus \hat{L}$ which is disjoint from L. Fix such a leaf $\hat{L}$. Since ∂R has positive mean curvature, the remark at the end of §4 in [12] shows that R is a handlebody. The fact that R is a handlebody implies that there is a homotopically nontrivial Jordan curve γ in ∂R which is the boundary of a disk in R. It follows that the earlier chosen disk D can be chosen to be contained in R. Since the leaf $\tilde{L}$ which contains ∂D is disjoint from L, the strong half-space theorem [10] implies that $\tilde{L}$ is not minimal. Thus, after a possible replacement of L by $\tilde{L}$ we may assume that L has positive mean curvature. After a possible homothety of $\mathbf{R}^3$ we can further assume that the mean curvature of the disks D_t, $t \in (0, \varepsilon)$, is greater than 1 for some fixed positive number ε.

Let K be the closure of $\bigcup_{t\leqslant 1} D_t$. Since F has no holonomy, each disk D_t intersects D only in the point set γ_t. It follows that K must be simply-connected. If K is compact, then γ must bound the disk $\tilde{D} = (\partial K)\setminus \mathring{D}$ which is contained on L. However, our choice of γ was such that γ does not bound a disk on L. It follows that K is a closed subset of $\mathbf{R}^3$ which is not a bounded distance from a round ball B containing D. In particular there exist $T \in (0, \varepsilon)$ and a point $p_T \in D_T$ such that the distance from p_T to B is greater than 2. Let P be a plane which separates p_T and B and such that the distance from P to p_T is greater than 2. Let C be the closure of the component of $(\mathbf{R}^3 \setminus P) \cap D_T$ which contains the point p_T. The proof of Proposition 2.2 shows that p_T cannot exist. This contradiction proves the theorem.

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