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MAPPINGS THAT MINIMIZE AREA IN THEIR HOMOTOPY CLASSES

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Let f: $M \to X$ be a continuous map from a compact connected oriented m-dimensional manifold M into a compact Riemannian manifold X. In this paper we consider the problem: does there exist a lipschitz map g: $M \to X$ that minimizes m-dimensional mapping area (or some other parametric elliptic functional) subject to the condition that g be homotopic to f? If so, what is the minimum area attained? And, if not, what is the infimum? It has long been known that in each *homology* class of X, there is an integral current that minimizes area (in that class). In this paper we show that, for $m \geqslant 3$, the homotopy problem reduces to the homology problem. For instance, if X is simply connected, the infimum area of mappings homotopic to f is equal to the minimum area among integral currents homologous to $f_{\#}([M])$ (where $[M] \in \mathscr{I}_m(M)$ is the m-dimensional integral cycle orienting M). Furthermore, if the current solution T is sufficiently regular, then the infimum is attained by a map whose image is the support of T together with a lower-dimensional singular set.

More generally, we allow M to be a compact manifold with (possibly empty) boundary. In this case, the homotopy problem is to minimize area among all maps g that are homotopic to f under homotopies H: $[0,1] \times M \to X$ that are fixed on ∂M (i.e., such that $H(t,x) = f(x)$ for $x \in \partial M$). Note that if $M = \mathbf{B}^m(0,1)$ and X is $\mathbf{R}^n$, this is the classical Plateau problem of minimizing area among maps g: $\mathbf{B}^m \to \mathbf{R}^n$ with boundary values $f|\partial\mathbf{B}^m$. Our main result is:

Theorem. *Suppose M is a compact connected oriented m-dimensional ($m \geqslant 3$) manifold with boundary, X is a Riemannian manifold (or more generally any local lipschitz neighborhood retract), and f: $M \to X$ is a lipschitz map. If X is simply connected (or if f_*: $\pi_1(M) \to \pi_1(X)$ is surjective), then*

$$\inf\{\mathrm{Area}(g)\colon g \text{ is homotopic to } f \text{ under a homotopy fixed on } \partial M\}$$
$$= \inf\{\mathrm{Area}(T)\colon T - f_*([M]) \text{ is an integral boundary in } X\}.$$

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Furthermore, if $f|\partial M$ is one-to-one and the current infimum is attained by an integral current T that is the image of a polyhedral chain under a one-to-one lipschitz map that is bilipschitz on ∂T, then the mapping infimum is attained by a map g whose image is T together with a singular set of dimension $\leqslant m-1$.

(See [11] for an example of a smooth embedding f: $\mathbf{B}^3 \to \mathbf{R}^4$ (with $f(\partial \mathbf{B}^3) \subset \partial \mathbf{B}^4$) for which the infimum is attained, but only by mappings g whose singular sets have positive two-dimensional area.)

Even if f_*: $\pi_1(M) \to \pi_1(X)$ is not surjective, the theorem still provides information because we can always lift f to a map $\tilde{f}$: $M \to \tilde{X}$ into a covering space $\tilde{X}$ of X so that $\tilde{f}_*$ is surjective. Analogous results are also true for nonorientable M (with flat chains modulo two replacing integral currents), but there are some surprising differences (see §4).

The case $m = 2$ is very different. For example our main result is false even when M is a two-dimensional disk and X is $\mathbf{R}^3$: the area minimizing integral current can have higher genus and less area than the area minimizing mapping (cf. [6, p. 56]). Of course much is known about the two-dimensional case. For instance Sacks and Uhlenbeck [7] and Schoen and Yau [9] independently showed that if f_*: $\pi_1(M) \to \pi_1(X)$ is injective and X is a compact manifold, then there is a branched minimal immersion that minimizes area among all maps g such that $g_* = f_*$ on $\pi_1(M)$. (In case X is 3-dimensional, the minimizing map has no branch points.) See [4] for an elegant unified treatment of the known results for 3-dimensional X.

In the special case $X = \mathbf{R}^n$, M orientable, the results of this paper were proved in [11].

The organization of the paper is as follows. §1 contains the simplicial version of the main theorem. This is the heart of the paper and the proofs are purely topological. The main tool is the homotopy extension theorem, which says that if $A \subset B$ are nice subsets of Euclidean space and if

$$H\colon ([0,1] \times A) \cup (\{0\} \times B) \to X$$

is lipschitz, then H may be extended to a lipschitz homotopy H: $[0,1] \times B \to X$ on all of B. Here "A is nice" means it is a lipschitz neighborhood retract, i.e., there are an open set U containing A and a lipschitz retraction of U onto A (see [5, p. 13] for a proof). In §2 (the only technical part of the paper), the general theorem is deduced from the results of §1. Here we use Federer's strong approximation theorem, according to which any integral current coincides (except on a set of arbitrarily small measure) with a curvilinear polyhedral chain. §§3 and 4 extend the results to nonsimply connected X and nonorientable M. §5 contains applications and open questions.

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Definitions and assumptions

1. M is a compact connected m-dimensional Riemannian manifold with (possibly empty) boundary.

2. $[M]$ is the m-dimensional integral current in $\mathscr{Z}_m(M, \partial M)$ orienting M. (In §4, where M is nonorientable, $[M]$ is the corresponding flat chain mod 2, i.e., the nonzero element of $\mathscr{Z}_m^2(M, \partial M)$.)

3. X is a local lipschitz neighborhood retract. That is, X is a subset of an open U in $\mathbf{R}^n$ such that there exists a locally lipschitz retraction $r\colon U \to X$.

4. $\mathscr{B}_m(X)$ is the set of m-dimensional integral boundaries in X, i.e., $\{\partial R\colon R$ is an $(m+1)$-dimensional integral current supported in $X\}$.

5. $\mathbf{M}(T)$, $\operatorname{spt}(T)$, and $\|T\|$ denote the mass (i.e., m-dimensional area weighted by multiplicity), support, and associated Radon measure of the integral current (or flat chain) T. See [1] for definitions.

6. A *parametric integrand of degree m* on a set A in $\mathbf{R}^n$ is a continuous function $\Phi\colon A \times \Lambda_m \mathbf{R}^n \to \mathbf{R}$ such that $\Phi(x, tw) = t\Phi(x, w)$ for $t > 0$. If T is an m-dimensional integral current, we define

$$\langle \Phi, T \rangle = \int \Phi(x, \vec{T}x)\, d\|T\|x;$$

and if $f\colon M \to U$ is a lipschitz map, we define

$$\langle \Phi, f \rangle = \int \Phi\big(f(x), \langle w(x), \Lambda_m Df(x) \rangle\big)\, dx,$$

where $w(x)$ is a simple unit m-vectorfield orienting M. For nonorientable M and flat chains T mod 2, $\langle \Phi, f \rangle$ and $\langle \Phi, T \rangle$ can be defined in a similar way provided Φ is an even integrand, i.e., provided $\Phi(x, w) \equiv \Phi(x, -w)$.

1. Topological results

In this section we prove the simplicial version of the main theorem, namely:

Theorem 1. *If M is oriented with* $\dim M \geqslant 3$, *Y is a simply connected simplicial complex, $f\colon M \to Y$ is lipschitz with $f(\partial M) \subset Y^{(m-1)}$, and T is a simplicial m-chain in Y such that*

$$T - f_{\#}([M]) \in \mathscr{B}_m(Y),$$

then there is a lipschitz homotopy $H\colon [0,1] \times M \to Y$ from f to a map g such that

$$H(t, x) = f(x) \quad \text{for } x \in \partial M, \qquad g(M) \setminus \operatorname{spt}(T) \subset Y^{(m-1)},$$

and such that the interior of each m-dimensional simplex that occurs with multiplicity k in T is covered $\leqslant k$ times (with the proper orientation) by g.

(Here a *simplicial m-chain* in Y is an m-dimensional integral current T with $\operatorname{spt}(T) \subset Y^{(m)}$ and $\operatorname{spt}(\partial T) \subset Y^{(m-1)}$, where $Y^{(i)}$ is the i-skeleton of Y. Also, here "simplicial complex" means what is sometimes called a triangulated polyhedron [2]: in particular, the simplices of Y are affine simplices in Euclidean space, not merely the homeomorphic images of such.)

First we need several lemmas. Throughout this section Y will be a simplicial complex and M will be oriented.

Lemma 1. *If $f\colon M \to Y$ is lipschitz with $f(\partial M) \subset Y^{(m-1)}$, then there is a lipschitz homotopy H from f to a map g such that*

$$H(t, x) = f(x) \quad \text{for } x \in \partial M, \tag{1}$$

$$g(M) \subset Y^{(m)}, \tag{2}$$

$$g^{-1}(Y \setminus Y^{(m-1)}) = W_1 \cup \cdots \cup W_k, \tag{3}$$

where $\overline{W}_1, \overline{W}_2, \cdots, \overline{W}_k$ and ∂M are pairwise disjoint, and g maps each W_i diffeomorphically onto the interior of some n-dimensional simplex of Y.

Proof. Let $\dim M = m + j$; so $Y = Y^{(m+j)}$. If $j > 0$, then from the interior of each $(m + j)$-dimensional simplex of Y we can choose a point not in the image $f(M)$ (because the $m + j$-dimensional measure of $f(M)$ is zero). Let the set of points so chosen be P. Then there is a deformation retraction of $Y \setminus P$ onto $Y^{(m+j-1)}$ (map each $(m + j)$-simplex radially outward from the chosen point.) Thus f may be homotoped onto the $(m + j - 1)$-skeleton. Iterating j times, we get a map from M into $Y^{(m)}$. Hence we may assume that $f(M) \subset Y^{(m)}$. We may also assume that the restriction of f to $f^{-1}(Y \setminus Y^{(m-1)})$ is smooth since f can be homotoped to such a map. (This can be seen in a number of ways: for instance by smoothing f as in the proof of [8, Lemma 3.2].)

In the interior of each m-dimensional simplex Δ_i of Y, choose a point p_i that is regular for f, and choose an $\varepsilon > 0$ small enough so that

$$f^{-1}\Big(\bigcup_i \mathbf{U}(p_i, \varepsilon)\Big) = W_1 \cup \cdots \cup W_k,$$

where f maps each W_i diffeomorphically onto some $\mathbf{U}(p_j, \varepsilon)$, and so that $\overline{W}_1, \overline{W}_2, \cdots, \overline{W}_k$ and ∂M are disjoint ($\mathbf{U}(p, \varepsilon)$ is the open ball of radius ε centered at p). Now define $\Phi\colon Y^{(m)} \to Y^{(m)}$ as follows. First let Φ map each $\Delta_i \setminus \mathbf{U}(p_i, \varepsilon)$ radially outward from p_i onto $\partial\Delta_i$. (Thus if $x \in \Delta_i \setminus \mathbf{U}(p_i, \varepsilon)$, then x lies on the line segment joining p_i to $\phi(x)$.) Then extend ϕ to all of $Y^{(m)}$ in such a way that each $\mathbf{U}(p_i, \varepsilon)$ is mapped diffeomorphically onto Δ_i. Note that the extended map ϕ is homotopic to the identity (on $Y^{(m)}$). Thus $g = \phi \circ f$ is the desired map.

Lemma 2. *Suppose, in addition to the hypotheses of Lemma* 1, *that T has no null-homologous components. Then there is a lipschitz homotopy H*; $[0,1] \times M \to Y$ *from f to a map g such that*

$$H(t,x) = f(x) \quad \text{for } x \in \partial M, \; g_\#([M]) = T, \qquad g(M) \subset Y^{(m)}.$$

(A current S is said to be a *null-homologous component of T if $S \in \mathscr{B}_m(Y)$, $S \neq 0$, and* $\mathbf{M}(T) = \mathbf{M}(S) + \mathbf{M}(T - S)$.)

Proof. We may assume that f satisfies conclusions (2) and (3) of Lemma 1 (otherwise it could be homotoped to a map that did). By hypothesis, $f_\#([M]) - T = \partial R$ for some R supported in Y. Write $R = \sum_{i=1}^N S_i$, where each S_i is (the current associated with) one of the $(m+1)$-dimensional simplices of Y. Choose R so that N is as small as possible. If $N = 0$, we are done. If not, choose some S_i, say S_N, such that one of the m-dimensional faces of ∂S_N is $f_\#([W_j])$ for one of the W_j of Lemma 1. (This is possible since otherwise ∂R would be a null-homologous component of T.) Clearly there is a lipschitz homotopy H; $[0,1] \times M \to Y$ such that

$$H(0,x) = f(x) \quad \text{for } x \in M,$$
$$H(t,x) = f(x) \quad \text{for } x \in M \setminus W_j,$$

and such that H sweeps out S_N once, i.e.,

$$H_\#([0,1] \times [M]) = H_\#\big([0,1] \times [W_j]\big) = S_N. \tag{4}$$

Write $f_1(\cdot) = H(1,\cdot)$. Then by (4),

$$f_{1\#}([M]) - T = \partial\left(\sum_{i=1}^{N-1} S_i\right).$$

Iterating the process we get $f_2, f_3, \cdots$ with

$$f_{k\#}([M]) - T = \partial\left(\sum_{i=1}^{N-k} S_i\right).$$

Then $g = f_N$ is the desired map.

Lemma 3. *Let h*: $(\mathbf{B}^m, \partial\mathbf{B}^m) \to (\mathbf{B}^m, \partial\mathbf{B}^m)$ *be a lipschitz map such that the induced map of homology groups $h_\#$*: $H_m(\mathbf{B}^m, \partial\mathbf{B}^m) \to H_m(\mathbf{B}^m, \partial\mathbf{B}^m)$ *is* 0, *i.e., such that the current $h_\#([\mathbf{B}^m])$ is* 0. *Then there is a lipschitz homotopy H*: $[0,1] \times \mathbf{B}^m \to \mathbf{B}^m$ *such that*

$$H(0,\cdot) = h(\cdot), \qquad H(t,x) = h(x) \quad \text{for } x \in \partial\mathbf{B}^m,$$
$$H(1,x) \in \partial\mathbf{B}^m \quad \text{for } x \in \mathbf{B}^m.$$

Proof. The hypothesis implies (by the long exact homology sequence for $(\mathbf{B}^m, \partial\mathbf{B}^m)$) that the restriction of h to $\partial\mathbf{B}^m$ has degree 0 (as a map from the

$(m-1)$-sphere to itself). Hence there is a map ϕ: $\mathbf{B}^m \to \partial\mathbf{B}^m$ such that $\phi(x) = h(x)$ for $x \in \partial\mathbf{B}^m$. Indeed there is a lipschitz map ψ with the same property. For by the Stone-Weierstrass theorem there is a smooth map ϕ': $\mathbf{B}^m \to \mathbf{R}^m$ such that $|\phi' - \phi| < 1 = \text{radius } \mathbf{B}^m$. Let $\psi(x) = y/|y|$ where

$$y(x) = (2|x| - 1)\phi(x/|x|) + 2(1 - |x|)\phi'(x/|x|)$$

when $1/2 \leqslant |x| \leqslant 1$, and $y(x) = \phi'(2x)$ when $|x| \leqslant 1/2$. Now let $H(t, x) = (1-t)h(x) + t\phi(x)$.

Proof of Theorem 1. First we may assume that T has no null-homologous components (since if it had such a component ∂Q, then we could replace T by $T - \partial Q$). By Lemma 2, we may assume $f(M) \subset Y^{(m)}$ and $f_\#([M]) = T$. Having gotten $f(M)$ onto $Y^{(m)}$, we can ignore the rest of Y. In other words, from now on we assume $Y = Y^{(m)}$.

Let $\mathscr{F}$ be the class of lipschitz maps g: $M \to Y$ such that

(5) $\quad g$ is homotopic to f with ∂M fixed (i.e., as in (1)),

(6) $$g^{-1}(Y \setminus Y^{(m-1)}) = W_1 \cup \cdots \cup W_k,$$

where the $\overline{W}_i$'s and ∂M are pairwise disjoint, and g maps each W_i diffeomorphically onto the interior of some m-dimensional simplex S_j of Y.

By Lemma 1, $\mathscr{F}$ is nonempty. Let $g \in \mathscr{F}$ be such that k (i.e., the number of W_i's in (6)) is as small as possible. We claim that g satisfies the conclusions of the theorem. First observe that, by (5),

$$g_\#([M]) - f_\#([M]) \in \mathscr{B}_m(Y).$$

But $\mathscr{B}_m(Y) = 0$ since $\dim Y = m$. Thus $g_\#([M]) = T$.

Suppose g does not satisfy the conclusion of the theorem. Then there exist two W_i's, say W_1 and W_2, that get mapped to the same S_j, say S_1, but with opposite orientations. Let $q \in \operatorname{spt}(\partial S_1) \setminus Y^{(m-2)}$ and let Γ be a path in M from $\overline{W}_1$ and $\overline{W}_2$ such that the two endpoints of Γ are mapped to q, and such that Γ does not intersect ∂M or any of the $\overline{W}_i$'s except at its endpoints. Note that this implies $g(\Gamma) \subset Y^{(m-1)}$.

Since $Y^{(m-1)}$ is simply-connected, $g|\Gamma$ can be homotoped to the point q in $Y^{(m-1)}$. (It is here, and nowhere else in the paper, that we use the condition $m \geqslant 3$.) That is, there is a lipschitz homotopy H: $[0,1] \times \Gamma \to Y^{(m-1)}$ such that $H(0, x) = g(x)$, $H(1, x) = q$ and $H(t, x) = q$ for $x \in \partial\Gamma$. Extend H to $\partial M \cup \Gamma \cup \partial\overline{W}_1 \cup \cdots \cup \partial\overline{W}_k$ by setting

$$H(t, x) = g(x) \quad \text{for } x \in \partial M \cup \partial\overline{W}_1 \cup \cdots \cup \partial\overline{W}_k.$$

By the homotopy extension theorem (cf. [5, p. 13]), we can further extend this H to a lipschitz homotopy H: $[0,1] \times (M \setminus \cup W_i) \to Y^{(m-1)}$. Finally, extend H to all of M by setting

$$H(t, x) = g(x) \quad \text{for } x \in \bigcup W_i.$$

Now let $h(x) = H(1, x)$. Then $h \in \mathscr{F}$, $h(x) = g(x)$ for $x \in W_1 \cup \cdots \cup W_k$, $h(x) \in Y^{(m-1)}$ for $x \notin W_1 \cup \cdots \cup W_k$ and $h(x) = q$ for $x \in \Gamma$. Now the closed set

$$K = \partial M \cup \left(\bigcup_{i>2} \overline{W}_i \right) \cup h^{-1}(Y^{(m-2)})$$

does not intersect $W_1 \cup W_2 \cup \Gamma$. Hence there is an open set U containing $W_1 \cup W_2 \cup \Gamma$ such that U is disjoint from K, $\overline{U}$ is diffeomorphic to $\mathbf{B}^m$, and U is diffeomorphic to the interior of $\mathbf{B}^m$. Note that

$$h\colon (\overline{U}, \partial\overline{U}) \to (\operatorname{spt}(S_1), \operatorname{spt}(S_1))$$

satisfies the hypotheses of Lemma 3. Hence there is a lipschitz homotopy H': $[0,1] \times \overline{U} \to \operatorname{spt}(S_1)$ such that $H'(0, x) = h(x)$, $H'(t, x) = h(x)$ for $x \in \partial\overline{U}$ and $H'(1, x) \in \operatorname{spt}(\partial S_1)$ for $x \in \overline{U}$. Now extend H' to all of M by setting $H'(t, x) = h(x)$ if $x \notin \overline{U}$. Then the map $g'(\cdot) = H'(1, \cdot)$ is in $\mathscr{F}$ and satisfies

$$g'(x) = g(x) \; (= f(x)) \quad \text{for } x \in \bigcup_{i>2} W_i,$$

$$g'\left(M \Big\backslash \left(\bigcup_{i>2} W_i \right) \right) \subset Y^{(m-1)}.$$

But this contradicts the choice of g since g' satisfies (6) with fewer W_i's than g does.

2. The Main results

Theorem 2. *Suppose M is oriented with* $\dim M \geqslant 3$, *X is simply connected, f: $M \to X$ is lipschitz with $f|\partial M$ one-to-one, and*

$$(*) \qquad T - f_{\#}([M]) \in \mathscr{B}_m(X),$$

where T is the image of an integral polyhedral chain P under a one-to-one lipschitz map ψ that is bilipschitz on ∂P. Then there is a lipschitz homotopy H: $[0,1] \times M \to X$ from f to a map g such that $H(t, x) = f(x)$ for $x \in \partial M$, $g(M) \setminus \operatorname{spt}(T)$ has dimension $\leqslant m - 1$ and each (curvilinear) m-dimensional simplex that occurs with multiplicity k in T is covered $\leqslant k$ times (with the proper orientation) by g.

Proof. Let P be contained in the Euclidean space E. By the lipschitz extension theorem [1, 2.10.43], we can extend ψ: $\operatorname{spt}(P) \to X \subset \mathbf{R}^n$ to a lipschitz map ψ: $E \to \mathbf{R}^n$. Define Ψ: $E \to E \times \mathbf{R}^n$ by $\Psi(x) = \langle x, \psi(x)\rangle$ and Π: $E \times \mathbf{R}^n \to \mathbf{R}^n$ by $\Pi(x, y) = y$.

By $(*)$, $f_\#([\partial M]) = \partial T = \psi_\#(\partial P)$. Since f and ψ are one-to-one on ∂M and on $\operatorname{spt}(\partial P)$, this means $f(\partial M) = \psi(\operatorname{spt}(\partial P))$. So $\psi^{-1} \circ f$: $\partial M \to \operatorname{spt}(\partial P) \subset E$ is lipschitz and can be extended to a lipschitz map u: $M \to E$. Then the lipschitz map F: $M \to E \times X$ defined by $F(x) = \langle u(x), f(x)\rangle$ satisfies $\Pi \circ F = f$ and $F_\#([\partial M]) = \Psi_\#(\partial P)$. Hence

$$\Psi_\#(P) - F_\#([M]) \in \mathscr{Z}_m(E \times X).$$

Since Π: $E \times X \to X$ is a homotopy equivalence and since

$$\Pi_\#(\Psi_\#(P) - F_\#([M])) = T - f_\#([M]) = 0 \in \mathscr{B}_m(X),$$

it follows that $\Psi_\#(P) - F_\#([M]) \in \mathscr{B}_m(E \times X)$, and therefore

$$\Psi_\#(P) - F_\#([M]) = \partial R$$

for some $(m+1)$-dimensional integral current R in $E \times X$.

Let L: $E \times \mathbf{R}^n \to E \times \mathbf{R}^n$ be the bilipschitz homeomorphism defined by $L(x, y) = \langle x, y - \psi(x)\rangle$ and let r: $U \to X$ be a locally lipschitz retraction from the open set $U \subset \mathbf{R}^n$ to $X \subset U$. Finally, let Y be a simply-connected compact polyhedron in $L(E \times U)$ such that

$$L(F(M) \cup \Psi(\operatorname{spt}(P)) \cup \operatorname{spt}(R)) \subset \operatorname{interior}(Y).$$

Then $L \circ F$, $T' = (L \circ \Psi)_\#(P)$ and Y satisfy the hypotheses of Theorem 1; let H: $[0,1] \times M \to Y$ be the homotopy given in the conclusion. Then

$$r \circ L^{-1} \circ H\colon [0,1] \times M \to X$$

is the desired homotopy.

Theorem 3. *Suppose M is oriented with* $\dim(M) \geqslant 3$, *X is simply connected, f: $M \to X$ is lipschitz, and Φ is a parametric integrand of degree m on $U \supset X$. Write $\mathscr{F} = \{g \mid g$ is homotopic to f in X under a lipschitz homotopy fixed on $\partial M\}$. Then*

$$(*) \qquad \inf\{\langle \Phi, g\rangle : g \in \mathscr{F}\} = \inf\{\langle \Phi, T\rangle : T - f_\#([M]) \in \mathscr{B}_m(X)\}.$$

Proof. First note that if $g \in \mathscr{F}$, then

$$g_\#([M]) - f_\#([M]) \in \mathscr{B}_m(X).$$

Since $\langle \Phi, g\rangle \geqslant \langle \Phi, g_\#([M])\rangle$ for every g: $M \to X$, this implies that the left side of $(*)$ is greater than or equal to the right side.

To prove the reverse inequality, let T be an integral current in X with $T - f_\#([M]) \in \mathscr{B}_m(X)$. Let K be a compact simply-connected subset of U with $\operatorname{spt}(T) \cup f(M) \subset \operatorname{interior}(K)$ and let $\varepsilon > 0$. Write $M' = M \setminus W$, where W is an open collaring of ∂M in M. We may assume that $n = \dim(U)$ is much larger than m (otherwise replace X, U, $\mathbf{R}^n$ and f by $X \times \{0\}$, $U \times \mathbf{R}^N$, $\mathbf{R}^n \times \mathbf{R}^N$ and $x \to \langle f(x), 0\rangle$). Then we can find a map f': $M \to \operatorname{int}(K)$ such that

$$\begin{aligned} &f'_{\#}(\partial M') \text{ is a polyhedral chain,} \\ &f'|\partial M' \text{ is one-to-one,} \\ &f'|W \text{ has mapping area } < \varepsilon \end{aligned} \tag{1}$$

and such that f' is homotopic (in K) to f under a homotopy that leaves ∂M fixed. By the strong approximation theorem [1, 4.2.19], there is a curvilinear polyhedral chain P such that

$$\begin{aligned} &\partial P = f'_{\#}([\partial M']), \\ &P - \big(T - f'_{\#}([W])\big) \in \mathscr{B}_m(\operatorname{int} K), \\ &\mathbf{M}\big(P - \big(T - f'_{\#}([W])\big)\big) < \varepsilon. \end{aligned} \tag{2}$$

By (1) and (2),

$$\mathbf{M}(P - T) < 2\varepsilon. \tag{3}$$

Now $f'|M'$, P and $\operatorname{int}(K)$ satisfy the hypotheses of Theorem 2. Hence there is a homotopy H: $[0,1] \times M' \to \operatorname{int}(K)$ from f' to a map g': $M' \to \operatorname{int}(K)$ such that $g'_{\#}([M]) = P$, $\langle \Phi, g'\rangle = \langle \Phi, P\rangle$ and $H(t, x) = f'(x)$ for $x \in \partial M'$. Now define g: $M \to X$ by

$$g(x) = f'(x) \quad \text{if } x \in W, \qquad g(x) = g'(x) \quad \text{if } x \in M \setminus W.$$

Then $r \circ g \in \mathscr{F}$(where r is the locally lipschitz retraction from U onto X) and

$$\begin{aligned} \langle \Phi, r \circ g\rangle &= \langle \Phi, r \circ g|M'\rangle + \langle \Phi, r \circ g|W\rangle \\ &< \langle \Phi, r \circ g|M'\rangle + C\varepsilon \quad (\text{by } (1), \text{ where } C \text{ depends on } K) \\ &= \big\langle \Phi, r \circ g|M' \cap g^{-1}(X)\big\rangle + \big\langle \Phi, r \circ g|M' \setminus g^{-1}(X)\big\rangle + C\varepsilon \\ &= \big\langle \Phi, g|M' \cap g^{-1}(X)\big\rangle + \big\langle \Phi, P \llcorner (U \setminus X)\big\rangle + C\varepsilon \\ &\leqslant \langle \Phi, g|M'\rangle + \langle \Phi, P - T\rangle + C\varepsilon \quad (\text{since } \operatorname{spt}(T) \in X) \\ &= \langle \Phi, P\rangle + \langle \phi, P - T\rangle + C\varepsilon \\ &\leqslant \langle \Phi, T\rangle + 2\langle \Phi, P - T\rangle + C\varepsilon \\ &\leqslant \langle \Phi, T\rangle + C'\varepsilon \quad (\text{by } (3)). \end{aligned}$$

Since we can find such an $r \circ g \in \mathscr{F}$ for any $\varepsilon > 0$, the left side of $(*)$ is $\leqslant$ the right side.

3. Nonsimply-connected X

So far we have been assuming that X (or Y in §1) is simply connected. But we only used simple connectivity once: in the proof of Theorem 1, the closed curve $f \circ \Gamma$ was homotoped to a point in Y. However, even if Y is not simply-connected, it is possible to choose the path Γ so that $f \circ \Gamma$ is null-homotopic in Y, provided $f_*\colon \pi_1(M) \to \pi_1(Y)$ is surjective. *Thus Theorems* 1, 2, *and* 3 *remain true* (*without simple connectivity*) *if the induced map* f_* *of fundamental groups is surjective.*

Now suppose X is a compact Riemann manifold. Even if $f_*\colon \pi_1(M) \to \pi_1(X)$ is not surjective, it is always possible to lift f to a map $\tilde{f}\colon M \to \tilde{X}$ into a covering space $\tilde{X}$ of X so that $\tilde{f}_*$ is surjective. We then have:

Theorem 4. *Suppose M is oriented with* $\dim M \geqslant 3$, *X is a compact Riemannian manifold, Φ is a parametric integrand of degree m on X, and $f\colon M \to X$ is lipschitz. Let $\pi\colon \tilde{X} \to X$ be a covering space of X, and let $\tilde{f}\colon M \to \tilde{X}$ be a map such that $f = \pi \circ \tilde{f}$ and such that $\tilde{f}_*\colon \pi_1(M) \to \pi_1(\tilde{X})$ is surjective. Then*

$$\inf\{\langle \Phi, g\rangle : g \in \mathscr{F}\} = \inf\{\langle \Phi, T\rangle : T - \tilde{f}_\#([M]) \in \mathscr{B}_m(\tilde{X})\},$$

where $\mathscr{F}$ is the set of maps $g\colon M \to X$ that are homotopic to f under lipschitz homotopies where $\mathscr{F}$ is the set of maps $g\colon M \to X$ that are homotopic to f under lipschitz homotopies $H\colon [0,1] \times M \to X$ such that $H(t,x) = f(x)$ for $x \in \partial M$.

Furthermore, if $f|\partial M$ is one-to-one and if the infimum is attained by a current T that is the one-to-one image of a polyhedral chain P under a lipschitz map that is bilipschitz on ∂P, then it is also attained by a map $\pi \circ g \in \mathscr{F}$, where $g\colon M \to \tilde{X}$, $g_\#([M]) = T$, and each m-dimensional (*curvilinear*) *simplex that occurs with multiplicity k in T is covered $\leqslant k$ times by g.*

(Here we use Φ to denote both the integrand on X and its lift to $\tilde{X}$.)

Proof. Let $\tilde{\mathscr{F}}$ be the set of maps $g\colon M \to \tilde{X}$ that are homotopic to $\tilde{f}$ under lipschitz homotopies $H\colon [0,1] \times M \to \tilde{X}$ such that $H(t,x) = \tilde{f}(x)$ for $x \in \partial M$. Then by the homotopy lifting theorem,

$$\mathscr{F} = \{\pi \circ g : g \in \tilde{\mathscr{F}}\}.$$

But $\langle \Phi, \pi \circ g\rangle = \langle \Phi, g\rangle$ for every $g\colon M \to \tilde{X}$, so Theorem 4 follows immediately by applying Theorems 2 and 3 (and the first paragraph of this section) to $\tilde{f}$.

Example. Let M be S^3, X be the connected sum of two copies of $S^1 \times S^3$, and let $f\colon M \to X$ be an embedding whose image is the 3-sphere along which the two copies are joined. (In other words, the two components of $X \setminus f(S^3)$ are each diffeomorphic to $(S^1 \times S^3) \setminus \{\text{a point}\}$.) Then $f_\#([M]) \in \mathscr{B}_m(X)$, but $\tilde{f}_\#([M]) \notin \mathscr{B}_m(\tilde{X})$ (where $\tilde{X}$ is the universal cover). Thus the infimum mapping area is positive.

4. Nonorientable domains

In the foregoing we have been assuming that M is orientable. The results for nonorientable M are analogous, but there are a few surprising differences.

Theorem 5. *Suppose M is nonorientable with* $\dim M \geqslant 3$, *Y is a simplicial complex, $f\colon M \to Y$ is lipschitz with $f(\partial M) \subset Y^{(m-1)}$, and T is a simplicial m-chain* mod 2 *in Y such that $T - f_{\#}([M]) \in \mathscr{B}_m^2(X)$. Suppose also that f_* maps the orientation-preserving subgroup $\pi_1^+(M)$ of $\pi_1(M)$ onto $\pi_1(Y)$. Then there is a lipschitz homotopy $H\colon [0,1] \times M \to Y$ from f to a map g such that*

$$H(t,x) = f(x) \quad \textit{for } x \in \partial M, \qquad g(M) \setminus \operatorname{spt}(T) \subset Y^{(m-1)},$$

and such that the interior of each m-dimensional simplex in $\operatorname{spt}(T)$ *is covered at most once by g.*

Proof. Lemmas 1 and 3 of §1 do not use orientability of M. Also Lemma 2 remains true (with the same proof) if we replace $\mathscr{B}_m(Y)$ by $\mathscr{B}_m^2(Y)$. The proof of the present theorem is, for the most part, the same as the proof of Theorem 1. As in that proof, we get open sets W_1 and W_2 in M that are mapped to the same m-dimensional simplex S_1 of Y. Now give W_1 and W_2 orientations so that $g|W_1$ and $g|W_2$ determine opposite orientations on S_1. Then the assumption about f_* guarantees that we can find an orientation-preserving path Γ joining W_1 to W_2 (as in the proof of Theorem 1) such that $f|\Gamma$ is contractible in X. The rest of the proof is as before.

It follows that Theorems 2 and 3 remain true for nonorientable M and nonsimply-connected X (with integral currents replaced by flat chains mod 2) provided f_* maps $\pi_1^+(M)$ onto $\pi_1(X)$. There is, however, a new twist. Let X be a compact Riemannian manifold. Then we can always lift $f\colon M \to X$ to a map $\tilde{f}\colon M \to \tilde{X}$ into a covering space $\tilde{X}$ so that $\tilde{f}_*$ is surjective. But $\tilde{f}_*$ does not necessarily map $\pi_1^+(M)$ onto $\pi_1(\tilde{X})$. If it does, then the conclusions of Theorem 4 (with flat chains mod 2 instead of integral currents) follow. If not, then we must lift again.

Theorem 6. *Suppose M is nonorientable with* $\dim M \geqslant 3$, *X is a Riemannian manifold, Φ is an even parametric integrand of degree m on X, $f\colon M \to X$ is lipschitz; and f_* maps $\pi_1(M)$ onto $\pi_1(X)$ but $f_*(\pi_1^+(M)) \neq \pi_1(X)$. Let $\pi\colon \hat{M} \to M$ be the oriented double cover of M and let $\pi\colon \hat{X} \to X$ be the covering space of X corresponding to the subgroup $f_*(\pi_1^+(M))$ of $\pi_1(X)$. Let γ denote the nontrivial covering transformation of both $\hat{M}$ and $\hat{X}$ (so that $\pi(\gamma(x)) = \pi(x)$ and $\gamma(x) \neq x$ for $x \in \hat{M}$ or $x \in \hat{X}$). Then f lifts to a map $\hat{f}\colon \hat{M} \to \hat{X}$ so that*

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{f}} & \hat{X} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

is commutative, $\hat{f}_*$ *is surjective and*

$$(1)\qquad \inf\{\langle \Phi, g\rangle : g \text{ is homotopic to } f \text{ in } X \text{ with } \partial M \text{ fixed}\}$$

is equal to

$$(2)\qquad \tfrac{1}{2}\inf\{\langle \Phi, T\rangle : T - \hat{f}_\#([\hat{M}]) = \partial R,\ R \text{ an integral current in } \hat{X} \text{ with } \gamma_\#(R) = -R\}.$$

Furthermore, if $f|\partial M$ *is one-to-one, and the infimum in* (2) *is attained by a sufficiently regular* T (*i.e., a* T *as in Theorem* 2), *then the infimum in* (1) *is attained by a map whose image is* $\pi(\operatorname{spt}(T))$ *together with a set of dimension* $\leqslant (m-1)$.

Proof. Observe that any homotopy H: $[0,1]\times M \to X$ lifts to a homotopy $\hat{H}$: $[0,1]\times\hat{M}\to\hat{X}$ that is γ-equivariant in the sense that $\gamma(\hat{H}(t,x)) = \hat{H}(t,\gamma(x))$. Thus the infimum (1) is equal to

$$(3)\qquad \tfrac{1}{2}\inf\{\langle \Phi, \hat{g}\rangle : \hat{g} \text{ is } \gamma\text{-homotopic to } \hat{f} \text{ in } \hat{X} \text{ with } \partial M \text{ fixed}\},$$

where "γ-homotopic" means "homotopic by a γ-equivariant homotopy". But now Lemmas 1 and 2 and Theorems 1, 2 and 3 (and their proofs) remain true if we replace X, M and f by $\hat{X}$, $\hat{M}$ and $\hat{f}$; homotopies by γ-equivariant homotopies; and integral currents by integral currents Q such that $\gamma_\#(Q) = -Q$. Then (2) and (3) are equal, and the rest of the theorem follows immediately.

5. Applications and open questions

As immediate consequences of the preceding sections we have:

Corollary 1. *If*

(1) *M is orientable or ∂M is empty*;

(2) $3 \leqslant \dim M = \dim(X) - 1 \leqslant 7$;

(3) *X is a smooth compact Riemannian manifold or* $X = \mathbf{R}^n$;

(4) *f: $M\to X$ is lipschitz, and $f|\partial M$ is smooth and one-to-one*;

(5) *$f_*(\pi_1(M))$ is of finite index in $\pi_1(X)$,*

then there exists a map g: $M\to X$ of least mapping area in the (∂M *fixed*) *homotopy class of f, and the image of g is a smooth submanifold of X together with a singular set of dimension* $\leqslant (m-1)$.

Proof. Theorems 2–6 reduce the existence and regularity of g to existence and regularity of solutions to a homological minimization problem. Conditions (3) and (5) guarantee existence for the homology problem (by the compactness theorem for integral currents and flat chains mod 2 [1, 5.1.6]). Conditions (1) and (2) guarantee that the homological minimizer is sufficiently regular (by [1, 5.4.15, 16], [3] and [10]).

Corollary 2. *Let M be a connected compact manifold with boundary $\partial M = N$. Let $f: N \to \mathbf{R}^n$ be lipschitz, and Φ be a parametric integrand of degree m on $\mathbf{R}^n$. Then*

$$\inf\{\langle\Phi, g\rangle | g: M \to \mathbf{R}^n \textit{ is lipschitz and } g|\partial M = f\}$$

is equal to the infimum of $\langle\Phi, T\rangle$ among integral currents (or flat chains mod 2 *if M is not orientable) T such that $\partial T = f_{\#}([N])$.*

Open questions

1. Must every homologically area-minimizing integral current (or flat chain mod 2) be a curvilinear polyhedral chain? If so, hypotheses (1) and (2) in Corollary 1 are unnecessary.

2. Let $\tilde{X}$ be a covering space of a compact Riemannian manifold X, and let S be an integral current (or flat chain mod 2) in $\tilde{X}$. Does there exist a T that minimizes area subject to the condition $T - S \in \mathscr{B}_m(\tilde{X})$ (or $\mathscr{B}_m^2(\tilde{X})$)? If so, we could drop hypothesis (5) from Corollary 1. The difficulty is that if $\tilde{X}$ is not compact, a minimizing sequence T_i might tend to wander off to infinity. If $f_*(\pi_1(M))$ is a normal subgroup of $\pi_1(X)$, then $\tilde{X}$ will have deck transformations with which we can translate the T_i back to some compact region. Using this idea one can prove that it suffices to assume, in place of (5), that $\pi_1(X)$ has a subgroup K of finite index such that $f_*(\pi_1(M))$ is normal in K. But can (5) be eliminated altogether?

3. What if, instead of minimizing the mapping area of g, we try to minimize the area (i.e., Hausdorff measure) $\mathscr{H}^m(g(M))$ of the image? The same proofs given above show that for simply connected X,

$$\inf\{\mathscr{H}^m(g(M)): g \text{ is homotopic to } f \text{ with } \partial M \text{ fixed}\}$$
$$= \inf\{\mathscr{H}^m(\operatorname{spt}(T)): T - f_{\#}([M]) \in \mathscr{B}_m(X)\}.$$

But the proofs of Theorems 4, 5 and 6 break down, since when we lift $g: M \to X$ to $\tilde{g}: M \to \tilde{X}$, $\mathscr{H}^m(g(M))$ and $\mathscr{H}^m(\tilde{g}(M))$ need not be equal.

Now suppose M is not connected but is the union of finitely many connected components M_i. If we are minimizing the mapping area of g, we can apply Theorems 2–6 to the components separately since

$$\text{mapping area of } g = \sum_i (\text{mapping area of } g|M_i).$$

(Note, however, we are not claiming that Theorems 2–6 are true as stated for nonconnected M; Theorem 3, for example, is not.) But this reduction to the connected case is not possible if we are minimizing $\mathscr{H}^m(g(M))$, since in general it is not equal to $\sum_i(\mathscr{H}^m(g(M_i)))$.

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