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VOLUMES OF HYPERBOLIC MANIFOLDS

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1. Introduction

In this paper we answer some questions of W. Thurston and A. Borel concerning the volumes of commensurable hyperbolic manifolds. Our main result implies that the set of covolumes of all arithmetic irreducible discrete subgroups Γ of $\mathrm{PGL}_2(\mathbf{R}^a) \times \mathrm{PGL}_2(\mathbf{C})^b$, as a and b range over all nonnegative integers such that $a + b \geqslant 1$, is discrete. We will show, in fact, that the same is true if one replaces the covolume of Γ in the above assertion by the G.C.D. of the covolumes of all subgroups which are commensurable to Γ.

We now formulate these results in precise terms.

Let H^2 denote the hyperbolic upper half-plane, and H^3 the hyperbolic upper half-space. Let a and b be nonnegative integers such that $a + b \geqslant 1$. The group $G_{a,b} = \mathrm{PGL}_2(\mathbf{R})^a \times \mathrm{PGL}_2(\mathbf{C})^b$ acts as a group of isometries of $H_{a,b} = (H^2)^a \times (H^3)^b$. We will let $\mathcal{C}_{a,b}$ be the set of discrete subgroups $\Gamma \subseteq G_{a,b}$ such that (i) Γ is irreducible in the sense that one cannot write $G_{a,b}$ as the direct product $H \cdot H'$ of nontrivial closed connected subgroups H and H' with $(\Gamma \cap H) \cdot (\Gamma \cap H')$ of finite index in Γ, and (ii) the volume $\mu(\Gamma)$ of $H_{a,b}/\Gamma$ is finite. Let $\mathcal{Q}_\Gamma$ be the set of subgroups $\Gamma' \subseteq G_{a,b}$ which are commensurable with Γ, i.e., for which $\Gamma \cap \Gamma'$ has finite index in Γ and in Γ'. It is shown by A. Borel in [1] that for each $\Gamma \in \mathcal{C}_{a,b}$ there is a largest number $g(\Gamma) > 0$ so that $\mu(\Gamma')$ is an integral multiple of $g(\Gamma)$ for all $\Gamma' \in \mathcal{Q}_\Gamma$. (An example given in [1, §5.6] shows that it is possible that $g(\Gamma) < \mu(\Gamma')$ for all $\Gamma' \in \mathcal{Q}_\Gamma$.)

We will prove the following theorem.

Theorem 1. *Let* $\mathcal{C} = \bigcup \{\mathcal{C}_{a,b}\colon a, b \in Z, a, b \geqslant 0$ *and* $a + b \geqslant 1\}$.

(i) *There is a smallest element of each set* $\{g(\Gamma)\colon \Gamma \in \mathcal{C}_{a,b}\}$ *and of* $\{g(\Gamma)\colon \Gamma \in \mathcal{C}\}$.

(ii) *The set* $\{g(\Gamma)\colon \Gamma \in \mathcal{C}$ *and* Γ *is arithmetic*$\}$ *is discrete.*

Corollary 1. *The set* $\{\mu(\Gamma)\colon \Gamma \in \mathcal{C}$ *and* Γ *is arithmetic*$\}$ *is discrete.*

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Theorem 1(i) answers affirmatively a question of W. Thurston [10, §8.8] as to whether there is a smallest element of $\{g(\Gamma)\colon \Gamma \in \mathcal{C}_{0,1}\}$. Corollary 1 implies the result proved by H. C. Wang in [11] when $a+b \geqslant 2$, and by A. Borel in [1] when $a+b=1$, that the set $\{\mu(\Gamma)\colon \Gamma \in \mathcal{C}_{a,b}$ *and* Γ *is arithmetic*$\}$ is discrete. The proof of Theorem 1(ii) is effective and number theoretic. This answers a question of A. Borel in [1] as to whether one can show the discreteness of the set of covolumes of arithmetic $\Gamma \in \mathcal{C}_{a,b}$ without the use of geometric arguments. We note finally that if $a+b \geqslant 2$, G. Margoulis has shown (cf. [4], [9]) that every $\Gamma \in \mathcal{C}_{a,b}$ is arithmetic.

This paper is organized in the following way. In §2 we recall the definition of arithmetic Γ, and state some results of A. Borel concerning $g(\Gamma)$. In §3 we show how results of G. Margoulis and D. Kahzdan reduce the proof of Theorem 1 to that of Theorem 1(ii). We then prove Theorem 1(ii), using formulas for hyperbolic volumes due to A. Borel, upper bounds for class numbers due to R. Brauer, C. L. Siegel and R. Zimmert, and lower bounds for discriminants due to A. Odlyzko.

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2. Arithmetic subgroups

Let a and b be nonnegative integers such that $a+b \geqslant 1$. Let k be a number field having exactly b complex places and at least a real places. Let B be a quaternion algebra over k, which is unramified at a set of a real places, and which is ramified at all of the other real places of k.

If A is a k-algebra, let A^* be the multiplicative group of invertible elements of A. We have an injection

$$\gamma\colon B^* \to \prod_{v \text{ infinite}} (B \otimes_k k_v)^* = (\mathbf{H}^*)^{r_1 - a} \times \mathrm{GL}_2(\mathbf{R})^a \times \mathrm{GL}_2(\mathbf{C})^b,$$

where $\mathbf{H}$ denotes the real quaternions. Let

$$\pi\colon B^* \to G_{a,b} = \mathrm{PGL}_2(\mathbf{R})^a \times \mathrm{PGL}_2(\mathbf{C})^b$$

be the homomorphism induced by projecting onto the factors of $\prod_{v \text{ infinite}} (B \otimes k_v)^*$ which are not quaternionic.

Let $\mathfrak{D}$ be a maximal order in B, and $\mathfrak{D}^1$ the group of elements of reduced norm 1 to k. By [1, pp. 2, 3, 13, §5, §7.2], $\Gamma^1_{\mathfrak{D}} = \pi(\mathfrak{D}^1)$ is in $\mathcal{C}_{a,b}$. Following [1] we define $\mathcal{C}(k, B)$ to be the set of subgroups $\Gamma \subseteq G_{a,b}$ which are commensurable with $\Gamma^1_{\mathfrak{D}}$ for some maximal order $\mathfrak{D} \subseteq B$. A discrete subgroup Γ' of $G_{a,b}$ is definable arithmetically (cf. [1, p. 7]) if there is an isomorphism $i\colon G_{a,b} \to G_{a,b}$ mapping Γ' onto an element of $\mathcal{C}(k, B)$ for some k and B as above.

We now recall some results of Borel concerning $g(\Gamma)$ for $\Gamma \in \mathcal{C}(k, B)$. Let us fix the following notation.

R_f (resp. R_∞) = the set of finite (resp. infinite) places of k where B is ramified.

$\mathcal{P}_v$ = the prime ideal of k determined by the finite place v of k.

N_v = the absolute norm of $\mathcal{P}_v$.

$\mathcal{O}^*_{R_f}$ = the group of R_f-units of k, i.e., the multiplicative group of elements of k which are units at all finite places of k which are not in R_f.

$\mathcal{O}^*_{R_f, R_\infty}$ = the group of elements of $\mathcal{O}^*_{R_f}$ which are positive at all the places in R_∞.

$I(k)$ (resp. $P(k)$) = the group of fractional (resp. principal) ideals of k.

$P(k, R_\infty)$ = the group of principal ideals which have a generator which is positive at all the places in R_∞.

M_1 = the subgroup of $I(k)$ generated by $P(k, R_\infty)$ and the ideals $\mathcal{P}_v$ for $v \in R_f$.

$J_1 = I(k)/M_1$.

J_2 = the image of $P(k)$ in J_1.

${}_2J_1$ = the kernel of $y \to y^2$ in J_1.

e = the number of places over 2 in k which are not in R_f.

r_1 (resp. r_2) = $r_1(k)$ (resp. $r_2(k) = b$).

d_k = the absolute value of the discriminant of k.

$\zeta_k(z)$ = the Dedekind zeta function of k.

The following result is shown in [1, Corollary 5.4, Theorem 7.3, §§8.4–8.6].

Theorem 2.1 (*Borel*). *If $\Gamma \in \mathcal{C}(k, B)$, then $g(\Gamma)$ is a positive integral multiple of*

$$2^{-e} \prod_{v \in R_f} (Nv - 1) \frac{2 d_k^{3/2} \zeta_k(2) \left[\mathcal{O}^*_{R_f, R_\infty} : \mathcal{O}^{*2}_{R_f}\right]^{-1}}{2^{2r_1 + 3r_2 - 2a} \pi^{2r_1 + 2r_2 - a} \left[{}_2J_1 : J_2\right]}.$$

We now make some preliminary simplifications.

Lemma 2.1. *Let t be the number of primes over 2 in k. Let $\mathcal{O}^*_k$ be the group of units in k, and let $\mathcal{O}^*_{k,+}$ be the group of units which are positive at all of the real places of k. Then $2^{-e}[\mathcal{O}^*_{R_f, R_\infty} \colon \mathcal{O}^{*2}_{R_f}]^{-1} \prod_{v \in R_f} (Nv - 1)$ is an integral multiple of $2^{-(r_1 + r_2 + t + a)} [\mathcal{O}^*_k \colon \mathcal{O}^*_{k,+}]$.*

Proof. Let $\mathcal{O}^*_{\varnothing, R_\infty}$ be the group of units of k which are positive at the places in R_∞. We have an injection

$$\mathcal{O}^*_k / \mathcal{O}^*_{\varnothing, R_\infty} \hookrightarrow \mathcal{O}^*_{R_f} / \mathcal{O}^*_{R_f, R_\infty}$$

because $\mathcal{O}_k^* \cap \mathcal{O}_{R_f,R_\infty}^* = \mathcal{O}_{\varnothing,R_\infty}^*$. Therefore

$$a_0\left[\mathcal{O}_k^*: \mathcal{O}_{\varnothing,R_\infty}^*\right] = \left[\mathcal{O}_{R_f}^*: \mathcal{O}_{R_f,R_\infty}^*\right] \tag{2.1}$$

for some integer a_0.

Let r_f be the number of places in R_f. By the Dirichlet unit theorem, $\mathcal{O}_{R_f}^*$ is an abelian group of rank $r_1 + r_2 + r_f - 1$ which has a finite cyclic torsion subgroup of even order. Therefore

$$\left[\mathcal{O}_{R_f}^*: \mathcal{O}_{R_f,R_\infty}^*\right]\left[\mathcal{O}_{R_f,R_\infty}^*: \mathcal{O}_{R_f}^{*2}\right] = \left[\mathcal{O}_{R_f}^*: \mathcal{O}_{R_f}^{*2}\right] = 2^{r_1+r_2+r_f}. \tag{2.2}$$

From (2.1) and (2.2) we have

$$\begin{aligned}\left[\mathcal{O}_{R_f,R_\infty}^*: \mathcal{O}_{R_f}^{*2}\right]^{-1} &= \left[\mathcal{O}_{R_f}^*: \mathcal{O}_{R_f,R_\infty}^*\right]2^{-(r_1+r_2+r_f)}\\ &= a_0\left[\mathcal{O}_k^*: \mathcal{O}_{\varnothing,R_\infty}^*\right]2^{-(r_1+r_2+r_f)}.\end{aligned} \tag{2.3}$$

Let r_f' be the number of $v \in R_f$ which lie over 2. If $v \in R_f$ does not lie over 2, 2 divides $(Nv - 1)$. Therefore (2.3) shows

$$\begin{aligned}&2^{-e}\left[\mathcal{O}_{R_f,R_\infty}^*: \mathcal{O}_{R_f}^{*2}\right]^{-1} \prod_{v\in R_f} (Nv - 1)\\ &= 2^{-(e+r_1+r_2)}a_0\left[\mathcal{O}_k^*: \mathcal{O}_{\varnothing,R_\infty}^*\right] \prod_{v\in R_f} ((Nv - 1)/2)\\ &= a_1 2^{-(e+r_1+r_2+r_f')}\left[\mathcal{O}_k^*: \mathcal{O}_{\varnothing,R_\infty}^*\right]\\ &= a_1 2^{-(r_1+r_2+t)}\left[\mathcal{O}_k^*: \mathcal{O}_{\varnothing,R_\infty}^*\right]\end{aligned} \tag{2.4}$$

for some integer a_1, where the last equality results from the fact that $t = r_f' + e$ is the total number of places over 2 in k.

We know that R_∞ contains all but a of the real places of k. Therefore

$$\left[\mathcal{O}_{\varnothing,R_\infty}^*: \mathcal{O}_{k,+}^*\right] \quad \text{divides } 2^a, \tag{2.5}$$

and

$$\left[\mathcal{O}_k^*: \mathcal{O}_{\varnothing,R_\infty}^*\right] = \left[\mathcal{O}_k^*: \mathcal{O}_{k,+}^*\right]\left[\mathcal{O}_{\varnothing,R_\infty}^*: \mathcal{O}_{k,+}^*\right]^{-1} = a_2 2^{-a}\left[\mathcal{O}_k^*: \mathcal{O}_{k,+}^*\right] \tag{2.6}$$

for some integer a_2. One combines (2.4) and (2.6) to finish the proof of Lemma 2.1.

Corollary 2.1. *Let h_k be the class number of k. If $\Gamma \in \mathcal{C}(k, B)$, then $g(\Gamma)$ is a positive integral multiple of*

$$g_a(k) = \frac{d_k^{3/2}\left[\mathcal{O}_k^*: \mathcal{O}_{k,+}^*\right]\zeta_k(2)}{2^{3r_1+4r_2+t-a-1}\pi^{2r_1+2r_2-a}h_k}.$$

Proof. With the notation of Theorem 2.1, the group J_1/J_2 is quotient of the ideal class group $I(k)/P(k)$ of k. Since ${}_2J_1/J_2$ is a subgroup of J_1/J_2, $[{}_2J_1 \colon J_2]$ divides h_k. Corollary 2.1 now follows from Theorem 2.1 and Lemma 2.1.

3. Proof of Theorem 1

Suppose $\Gamma \in \mathcal{C}$ is not arithmetic. By the results of G. Margoulis [4], [9], Γ is in $\mathcal{C}_{0,1}$ or $\mathcal{C}_{1,0}$. As A. Borel observes in [1, §1], a result announced by G. Margoulis in [4] implies that $\mathcal{C}_\Gamma$ has a unique maximal element Γ_0, for which $g(\Gamma) = \mu(\Gamma_0)$. D. Kahzdan and G. Margoulis have shown (cf. [7, XI, 11.9]) that for each fixed pair (a, b), in particular for $(a, b) = (0, 1)$ or $(1, 0)$, there is a smallest element of $\{\mu(\Gamma') \colon \Gamma' \in \mathcal{C}_{a,b}\}$. Hence Theorem 1(i) will follow from Theorem 1(ii).

We now find a lower bound for the number $g_a(k)$ of Corollary 2.1. With the notation of §2, let w_k be the number of roots of unity in k, and let $\mathrm{Reg}(k)$ be the regulator of k. The Brauer-Siegel Theorem (cf. [3, pp. 322, 300]) shows

$$\begin{aligned}(3.1)\quad h_k \mathrm{Reg}(k) \leqslant\ & 2^{-r_1} w_k s(s-1)\Gamma(s/2)^{r_1}\Gamma(s)^{r_2} \\ & \cdot \left(2^{-2r_2}\pi^{-(r_1+2r_2)}d_k\right)^{s/2}\zeta_k(s) \quad \text{for real } s > 1.\end{aligned}$$

R. Zimmert [12, p. 375] has proved

$$(3.2)\qquad \mathrm{Reg}(k) \geqslant (.02) w_k \exp(.46 r_1 + .1 r_2).$$

Since t is the number of primes over 2 in k, we have the trivial bounds

$$(3.3)\qquad t \leqslant r_1 + 2r_2 \quad \text{and} \quad \left[\mathcal{O}_k^* \colon \mathcal{O}_{k,+}^*\right]\zeta_k(2) \geqslant 1.$$

From the Euler product of $\zeta_k(s)$, one has $1 \leqslant \zeta_k(s) \leqslant \zeta_Q(s)^{r_1+2r_2}$. Letting $s = 2$ in (3.1), we deduce an upper bound on h_k from (3.1) and (3.2). This bound, Corollary 2.1, and (3.3) together show that there is an absolute constant $c_0 > 0$, independent of a and k, for which

$$(3.4)\qquad g_a(k) > c_0^{r_1+r_2} d_k^{1/2} 2^{a+1}\pi^a.$$

For real $s > 1$, define

$$Z_k(s) = -\zeta_k'(s)/\zeta_k(s) = \sum_{\mathcal{P}} \log N\mathcal{P}/\left((N\mathcal{P})^s - 1\right),$$

where the sum is over the prime ideals $\mathcal{P}$ of k. The following lower bounds are implied by those of A. Odlyzko in [5, Theorem 1 and Lemma 2]. There exist absolute positive constants c_1, c_2, c_3 so

$$(3.5)\qquad d_k \geqslant (50.6)^{r_1}(19.9)^{2r_2}\exp\left(2Z_k(s) - 2(s-1)^{-1} - c_1\right) \quad \text{for } s \in (1, 1 + c_2),$$

(3.6) $\zeta_k(s) \leqslant \exp(Z_k(s) + c_3(s-1)(r_1+r_2))$ for all $s > 1$.

There is an absolute constant $c_4 > 0$ so $\Gamma(s/2) \leqslant \pi^{1/2}\exp(c_4(s-1))$ and $\Gamma(s) \leqslant \exp(c_4(s-1))$ for $s \in (1, 1+c_2)$. Thus (3.1), (3.2) and (3.6) show

$$h_k \leqslant 2^{-r_1}s(s-1)(2\pi)^{-r_2}d_k^{s/2}\cdot 50$$

(3.7) $$\cdot\exp(Z_k(s) - .46r_1 - .1r_2 + (s-1)(c_3+c_4)(r_1+r_2))$$

for $s \in (1, 1+c_2)$.

We use (3.7), (3.5) and (3.3) to find a lower bound for $g_a(k)$ in Corollary 2.1. One finds

(3.8) $$g_a(k) \geqslant f(s)^{r_1}h(s)^{r_2}\exp((2-s)Z_k(s))j(s)2^{a+1}\pi^a$$

for real $s \in (1, 1+c_2)$,

where

(3.9) $$f(s) = (50.6)^{(3-s)/2}\exp(.46 - (c_4+c_3)(s-1))2^{-3}\pi^{-2},$$

(3.10) $$h(s) = (19.9)^{3-s}\exp(.1 - (c_4+c_3)(s-1))2^{-5}\pi^{-1},$$

(3.11) $$j(s) = .02\exp\left(-(3-s)(s-1)^{-1} - (3-s)c_1/2\right)/(s(s-1)).$$

We have $\lim_{s\to 1} f(s) = 1.01^+$, $\lim_{s\to 1} h(s) = 4.35^+$ and $e^{(2-s)Z_k(s)} \geqslant 1$ if $s < 2$. Thus for $s > 1$ sufficiently close to 1, (3.8) yields a lower bound

(3.12) $$g_a(k) \geqslant (1.01)^{r_1}(4.35)^{r_2}c_5 2^{a+1}\pi^a,$$

where $c_5 > 0$ is an absolute constant independent of a and k.

We conclude from (3.12) and (3.4) that $\lim_{(a,k)} g_a(k) = \infty$, where the limit is over all pairs (a, k) of nonnegative integers a and number fields k for which $r_1(k) \geqslant a$. This and Corollary 2.1 complete the proof of Theorem 1(ii), and hence also of Theorem 1.

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