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# ON THE CUSPIDAL SPECTRUM FOR FINITE VOLUME SYMMETRIC SPACES

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## 1. Introduction

Let $K \setminus G/\Gamma$ be a noncompact locally symmetric space of finite volume. Here $G$ is a semisimple Lie group and $\Gamma$ is an arithmetic subgroup. Moreover, $K$ is a maximal compact subgroup.

If $\Delta$ is the Laplacian on $K \setminus G/\Gamma$, we consider $\Delta$ acting on the cuspidal functions $L^2_{\text{cusp}}(K \setminus G/\Gamma)$ in the sense of Langlands [14]. Our main result is the following:

**Theorem 1.1.** *Let $N(\lambda)$ be the number of linearly independent cuspidal eigenfunctions with eigenvalue less than $\lambda$. Then $N(\lambda)$ is finite for each fixed $\lambda > 0$.*

*Moreover, one has the asymptotic upper bound*:

$$\varlimsup_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} \leqslant (4\pi)^{-d/2} \frac{\text{vol}(K \setminus G/\Gamma)}{\Gamma(d/2+1)}. \tag{1.2}$$

*Here $d$ is the dimension of $K \setminus G/\Gamma$ and* vol *denotes the volume. Also, $\Gamma(d/2+1)$ is the ordinary Gamma function.*

The fact that $N(\lambda)$ is finite for fixed $\lambda > 0$ was announced by Borel and Garland [2], [10].

If $G = \text{SL}(2, R)$, then Theorem 1.1 has apparently been well known for some time. It certainly follows from the scattering theory of [15], although the explicit estimate is not stated there. Several authors [21] have given more detailed information for particular discrete subgroups $\Gamma$ of $\text{SL}(2, R)$. In the case $\Gamma = \text{SL}(2, Z)$, equality holds in (1.2) and the limit on the left-hand side exists [15], [20].

When $G$ is a real rank one, Gangolli and Warner [9] obtained the estimate $N(\lambda) \leqslant C\lambda^n$, for some $C$ and $n$. However, their method did not give a good estimate of $n$.

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Theorem 1.1 was proved for real rank one in the author's earlier paper [6]. The arguments given below are a natural development of the approach initiated in this earlier work. Note that for the present paper, $K \setminus G/\Gamma$ may have arbitrary rank.

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## 2. Basic facts concerning arithmetic groups

This section summarizes some standard facts concerning semisimple Lie groups $G$ and arithmetic subgroups $\Gamma$. For more details the reader is referred to [2] and [14].

Let $P = MAN$ be a parabolic subgroup of $G$. The parabolic subgroups $P_\theta$ belonging to $P$ are in one-one correspondence with subsets $\theta$ of the simple roots $\Psi$ of $\mathfrak{a}$, the Lie algebra of $A$. We may write $P_\theta = M_\theta A_\theta N_\theta$ where $N_\theta \subset N$, $A_\theta \subset A$, and $M_\theta \supset M$. The Lie algebra of $N_\theta$ consists of those positive roots containing at least one simple root not belonging to $\theta$. We denote $S_\theta = M_\theta N_\theta$ and $S = MN$.

We denote the simple roots of $\mathfrak{a}$ by $\alpha_1, \alpha_2, \cdots, \alpha_k$. Set $A_c = \exp\{v \in \mathfrak{a} \mid \alpha_i(v) \geqslant c$, for all $i\}$. Here $c$ is a real number and $\exp\colon \mathfrak{a} \to A$ is the diffeomorphism induced by the exponential map.

Suppose that $P$ is a percuspidal parabolic in the sense of Langlands [14]. In particular, $\Gamma \cap P \subseteq S$ and $S/\Gamma \cap S$ is compact. Moreover, for any parabolic $P_\theta$ belonging to $P$ one has $\Gamma \cap P_\theta \subseteq S_\theta$, $N_\theta/\Gamma \cap N_\theta$ is compact, and $S_\theta/\Gamma \cap S_\theta$ has finite volume. All percuspidal parabolics are conjugate in $G$.

If $P = MAN$ is any percuspidal parabolic, then set $\mathfrak{S}_c(P) = K \setminus MA_cN/\Gamma \cap P$, for any real number $c$. One may choose a finite set $\Omega$ of percuspidal subgroups $P$ so that $K \setminus G/\Gamma$ is covered by $\cup_{P \in \Omega} \mathfrak{S}_c(P)$, for some real number $c$,

## 3. The metric on the cusp

Let $P = MAN$ be a percuspidal parabolic. The manifold with boundary $\mathfrak{S}_c(P)$ will be referred to as the cusp.

By proper choice of base point, we may assume that $K \cap P = K \cap M$, as is done in [4, p. 246]. We denote $K \setminus P_c = K \setminus MA_cN = ZA_cN$, where $Z = K \setminus M$. Then $K \setminus P_c$ is contained in $K \setminus G$ and the Killing form of $G$ induces a right invariant metric on $K \setminus P_c$.

For each $(z, a) \in ZA_c$, the metric of $K \setminus P_c$ restricts to a metric on $N$. It is well known [4, p. 246] that this metric has uniformly bounded dependence on $z$, so the metric will be denoted by $g_a$. The crucial point is to understand the dependence of $g_a$ upon $a$. One obtains a flat metric $\hat{g}_a$, on the Lie algebra $\mathfrak{n}$ of $N$, by identifying $\mathfrak{n}$ with the tangent space of $N$ at the identity. Since $N$ is a simply connected nilpotent Lie group, the exponential map exp: $\mathfrak{n} \to N$ is a diffeomorphism. Here we mean the group exponential map of $N$, which does not depend upon a choice of metric. Pulling back the metric $\hat{g}_a$ by $(\exp)^{-1}$ one may define a metric $h_a$ on $N$.

It will be useful to employ a comparison of the metric $g_a$ and $h_a$.

**Lemma 3.1.** *For $\varepsilon$ sufficiently small, one has, in a $g_a$ ball of radius $\varepsilon$ about the identity element, $g_a \geqslant C_1 h_a$. Here $C_1$ is independent of $a$.*

*Proof.* For a fixed value $a_0$ of $a$ one has, for some $\varepsilon > 0$, $g_{a_0} \geqslant C_1 h_{a_0}$, since exp is a diffeomorphism with differential the identity map. However, for any $a$, $zan = za_0(b^{-1}nb)b^{-1}$, where $b = a^{-1}a_0 \in A$. Since the Killing metric of $K \setminus P_c = ZA_cN$ is right invariant, $g_a = \mathrm{Ad}_b g_{a_0}$ and $\hat{g}_{a_0} = \mathrm{Ad}_b \hat{g}_{a_0}$. Notice that $A$ normalizes $N$. The lemma now follows from the commutative diagram:

$$\begin{tikzcd} \mathfrak{n} \arrow[r, "\exp"] \arrow[d, "\mathrm{Ad}_b"'] & N \arrow[d, "\mathrm{Ad}_b"] \\ \mathfrak{n} \arrow[r, "\exp"] & N \end{tikzcd}$$

The metric $(d\omega)^2$ on $K \setminus P_c$ is described very explicitly in [4, p. 247]. In fact, one may write:

$$(3.2) \qquad (d\omega)^2 = dz^2 + dr^2 + \sum_{\beta \in \Phi} e^{-2\beta(r)} \big(d\omega_\beta(z)\big)^2.$$

Here $r = (r_1, r_2, \cdots, r_k)$ are coordinates on $A_c$, obtained from the exponential map of $A$, exp: $\mathfrak{a} \to A$. In fact, $r_i(x) = \alpha_i(x)$, for $x \in \mathfrak{a}$, where $\alpha_i$ are the simple positive roots. Note that exp: $\mathfrak{a} \to A$ is a diffeomorphism, which allows us to identify $\mathfrak{a}$ with $A$. We may assume that $A_c$ is parameterized by $r_i \geqslant c$, for all $1 \leqslant i \leqslant k$. The $\beta$ belong to the set of positive roots $\Phi$ of $\mathfrak{a}$.

As given by (3.2), $g_a$ is the right invariant metric on $N$ which satisfies $g_a = \sum e^{-2\beta(r)}(d\omega_\beta(z))^2$ at the identity. It is difficult to obtain estimates on $g_a$ directly since the distributions defined by the root spaces, i.e. the $d\omega_\beta(z)$ are not integrable. Thus $g_a$ is not a product metric.

However, the metric $h_a$ is a product metric, along the root spaces in $\mathfrak{n}$, which agrees with $g_a$ at the identity. Of course, $h_a$ is not right invariant with respect to $N$. Nevertheless, it is easier to estimate geometric quantities in $h_a$. This explains the utility of Lemma 3.1.

A key technical lemma is:

**Lemma 3.3.** *Let $\rho(x, y)$ denote the geodesic distance in the metric $(d\omega)^2$. Then one has, for $\varepsilon$ sufficiently small, and any $x$, $y$, points in a fundamental domain for $\Gamma \cap N$:*

$$\sum_{\substack{\rho(x, y\gamma)<\varepsilon \\ 1\neq\gamma\in\Gamma\cap N}} \frac{1}{\rho(x, y\gamma)} \leqslant C_2\left(\max_{\substack{\alpha\in\Psi \\ n_\alpha=1}} \alpha(r)\right) \prod_{\beta\in\Phi} e^{n_\beta\beta(r)},$$

*where $n_\beta$ is the dimension of the root space corresponding to $\beta$. Here $\alpha$ runs over all simple positive roots of multiplicity one. The product in $\beta$ runs over all positive roots. Moreover, $r = r(x)$, or if desired $r = r(y)$.*

*Proof.* By Lemma 3.1 and formula (3.2), it suffices to obtain the analogous estimate for the Euclidean product metric $h_a$.

However, if $\rho$ is the geodesic distance in $h_a$, one has

$$\sum_{\substack{\rho(x, y\gamma)<\varepsilon \\ 1\neq\gamma\in\Gamma\cap N}} \frac{1}{\rho(x, y\gamma)} \leqslant C_3 \sum_{\substack{\rho_\beta(x, y\gamma)\leqslant C_4 e^{\beta(r)} \\ \beta\in\Phi}} \left(\sum_\beta e^{-\beta(r)}\rho_\beta(x, y\gamma)\right)^{-1}, \tag{3.4}$$

where $\beta$ are the positive roots of $\mathfrak{a}$ in $\mathfrak{n}$ and $\rho_\beta$ is a fixed Euclidean metric on the root space corresponding to $\beta$. Thus $\rho_\beta$ is independent of $r$.

A result of Moore [17, p. 155], states that the preimage of $\Gamma \cap N$ under exp: $\mathfrak{n} \to N$ is commensurable to a Euclidean lattice in the Lie algebra $\mathfrak{n}$. Using this fact, one obtains Lemma 3.3 after replacing the right sum in (3.4) by an integral:

$$\sum_{\substack{\rho(x, y\gamma)<\varepsilon \\ 1\neq\gamma\in\Gamma\cap N}} \frac{1}{\rho(x, y\gamma)} \leqslant C_5 \sum_{\alpha\in\Psi} \left(e^{\alpha(r)} \int_1^{C_4 e^{\alpha(r)}} t^{n_\alpha-2}\, dt\right) \times \prod_{\beta\in\Phi-\alpha} \int_1^{C_4 e^{\beta(r)}} t^{n_\beta-1}\, dt.$$

## 4. Neumann bracketing

Let $\phi \in L^2(K\setminus G/\Gamma)$ be a square integrable function. Suppose that $P$ is a percuspidal parabolic and $P_\theta = M_\theta A_\theta N_\theta$ is any associated parabolic.

We may define

$$T(P, \theta)\phi(x) = \int_{N_\theta/\Gamma\cap N_\theta} \phi(xn)\, dn \tag{4.1}$$

for $x \in K\setminus G$. Here one has identified $\phi$ with a $\Gamma$ invariant function on $K\setminus G$. Notice that the integral in (4.1) is well defined since $N_\theta/\Gamma \cap N_\theta$ is compact. If

$T(P, \theta)\phi = 0$ for all $(P, \theta)$, then $\phi$ is said to be cuspidal. If in addition $\Delta\phi = \mu\phi$, for some $\mu \geqslant 0$, then $\phi$ is a cuspidal eigenfunction and $\mu$ belongs to the cuspidal spectrum.

Choose a finite set $P_1, P_2, \cdots, P_r$ of percuspidal parabolics so that the collection $\mathfrak{S}_c(P_i)$, $1 \leqslant i \leqslant r$, covers $K \backslash G/\Gamma$. A function $\psi$ on $\mathfrak{S}_c(P_i)$ is said to be cuspidal if $T(P_i, \theta)\psi = 0$, for the fixed parabolic $P_i$ and all $\theta$. Denote $\pi_i\colon \mathfrak{S}_c(P_i) \to K \backslash G/\Gamma$.

Now select a sequence of smooth compact manifolds with boundary $B_k \subset K \backslash G/\Gamma$ with $B_k \subset B_{k+1}$ and $UB_k = K \backslash G/\Gamma$. For each $i$ and $k$, let $X_{i,k} \subset \mathfrak{S}_c(P_i)$ be a smooth manifold with boundary which contains $\mathfrak{S}_c(P_i) - \pi_i^{-1}B_k$. Suppose that $X_{i,k}N = X_{i,k}$, to guarantee that the cuspidal condition still makes sense in $L^2(X_{i,k})$. Eventually, we will wish to choose $X_{i,k}$ so that the volume of $X_{i,k}$ is sufficiently close to the volume of $\mathfrak{S}_c(P_i) - \pi_i^{-1}B_k$.

Let $\Delta_{i,k}$ be the Laplacian $\Delta$ acting on the cuspidal functions in $L^2(X_{i,k})$ which satisfy Neumann boundary conditions. Denote $N_{i,k}(\lambda)$ to be the number of cuspidal eigenfunctions in $L^2(X_{i,k})$ with eigenvalue less than $\lambda$. Similarly, we define $N_k(\lambda)$ to be the number of eigenvalues less than $\lambda$ for the usual Neumann problem of the compact manifold with boundary $B_k$. It is not necessary to impose any cuspidal side condition in $B_k$.

The principle of modified Neumann bracketing developed in [6] and [15] now gives:

**Proposition 4.2.** *Let $k$ be a fixed integer and suppose that $\Delta_{i,k}$ has pure point spectrum for all $1 \leqslant i \leqslant r$. If $N(\lambda)$ is the number of linearly independent cuspidal eigenfunctions on $K \backslash G/\Gamma$ with eigenvalue less than $\lambda$, then, for any value of $\lambda$:*

$$N(\lambda) \leqslant N_k(\lambda) + \sum_{i=1}^{r} N_{i,k}(\lambda).$$

A priori, $\Delta_{i,k}$ might have nonempty essential spectrum so that Proposition 4.1 would not apply. However, we will show presently that $\Delta_{i,k}$ does indeed have pure point spectrum for all $i$ and $k$.

## 5. Interior parametrix

Let $P$ be a fixed percuspidal parabolic. If $P_\theta = M_\theta A_\theta N_\theta$ is a cuspidal parabolic associated to $P$, then denote $T_\theta = T(P, \theta)$, where $T(P, \theta)$ is the cuspidal projection given by (4.1). We will normalize Haar measure on $N_\theta$ so that $\int_{N_\theta/\Gamma \cap N_\theta} dn_\theta = 1$. Recall that $\theta$ is a subset of the positive roots $\Psi$. It is convenient to set $\mathfrak{L}_\theta = T_{\Psi-\theta}$.

The following algebraic lemma is well known [11, p. 12]:

**Lemma 5.1.** (i) *For any* $\theta \subset \Psi$, *one has* $\mathfrak{L}_\theta = \Pi_{\alpha\in\theta}\mathfrak{L}_\alpha$. *Here the product runs over simple positive roots contained in the subset* $\theta$.

(ii) *For any* $\theta \subset \Psi$ *one has* $\mathfrak{L}_\theta^2 = \mathfrak{L}_\theta$.

(iii) *For any two subsets* $\theta_1, \theta_2 \subset \Psi$, *the associate projections commute*, $\mathfrak{L}_{\theta_1}\mathfrak{L}_{\theta_2} = \mathfrak{L}_{\theta_2}\mathfrak{L}_{\theta_1}$.

Now let $X \supset \mathfrak{S}_c(P) - \pi^{-1}B$ be a smooth manifold with boundary as chosen in §4. Recall that $X$ depend upon integer parameters $i$, $k$. However, in the next two sections, both $P$ and $B$ are fixed so we will suppress the dependence upon $i$ and $k$. Our eventual goal is to construct the fundamental solution of the heat equation problem with cuspidal interior conditions and Neumann boundary conditions on $X$. In this section, a parametrix satisfying the interior cuspidal conditions will be obtained. Lemma 5.1 is vital for this purpose.

Suppose $E(t, x, y)$ is the fundamental solution for the heat equation on the simply connected space $K\backslash G$. Then $E$ is smooth on $(0,\infty)\times K\backslash G\times K\backslash G$ and satisfies the estimates [5]:

$$
\begin{aligned}
&|E(t,x,y)| \leq C_1 t^{-d/2}\exp\left(\frac{-\rho^2(x,y)}{4t}\right),\\
(5.2)\qquad &\left|\frac{\partial E}{\partial \rho}(t,x,y)\right| \leq C_2 t^{-d/2}(\rho/t)\exp\left(\frac{-\rho^2(x,y)}{4t}\right)\\
&\qquad + C_3 t^{-d/2}\exp\left(\frac{-\rho^2(x,y)}{4t}\right)
\end{aligned}
$$

uniformly for $0 < t \leq \tau$, any $\tau > 0$. Here $\rho(x, y)$ is the geodesic distance from $x$ to $y$ in $K\backslash G$ and $d$ is the dimension of $K\backslash G$.

Let $P = MAN$. Then $G = KMAN$, and by proper choice of base point one has $K\backslash G = (K\cap M\backslash M)AN = ZAN$. Set $Y = K\backslash G/\Gamma\cap P = ZAN/\Gamma\cap P$. Then $Y$ is a complete Riemannian manifold. Moreover, $Y$ contains $\mathfrak{S}_cP = ZA_cN/\Gamma\cap P$, and therefore $Y$ also contains the manifold with boundary $X$. In fact, $X$ is an open set in $Y$.

Consider the infinite sum:

$$
(5.3)\qquad F(t,x,y) = \sum_{\gamma\in\Gamma\cap P} E(t,x,y\gamma).
$$

By the results of [5], this sum converges uniformly on compact sets in $(0,\infty)\times K\backslash G\times K\backslash G$. Moreover, $F(t, x, y)$ represents the fundamental solution of the heat equation problem on $Y$.

Of course, $F(t, x, y)$ must be modified by projection onto the cuspidal conditions (4.1). Set

$$\bar{F}(t, x, y) = \prod_{\alpha \in \Psi} (1 - \mathfrak{L}_\alpha(y))F(t, x, y) \tag{5.4}$$
$$= \sum_{\theta \subset \Psi} (-1)^{|\theta|} \mathfrak{L}_\theta(y) F(t, x, y).$$

Here the product runs over all simple roots and the sum runs over subsets $\theta$ of the simple roots. The projectors $\mathfrak{L}_\theta(y)$ act on the third argument $y$ of $F(t, x, y)$. It is immediate, from Lemma 5.1, that for all subsets $\psi \subset \Psi$, one has $\mathfrak{L}_\psi(y)\bar{F}(t, x, y) = 0$. Thus $F$ satisfies the cuspidal condition (4.1) and is suitable for an interior parametrix. By symmetry and isometry invariance of the heat kernel, one also has $\mathfrak{L}_\psi(x)\bar{F}(t, x, y) = 0$, for all $\psi \subset \Psi$.

It is crucial to estimate the parametrix $\bar{F}(t, x, y)$ as a function of $x$ and $y$ for small $0 < t \leqslant \tau$, any fixed $\tau$. For this purpose, we identify $x, y \in Y$ with points $x, y$ in the universal cover $K \setminus G$, which realize the geodesic distance from $x$ to $y$ in $Y$.

Our basic technical estimate is:

**Lemma 5.5.** *For any fixed simple root $\alpha$, let $F_\alpha(t, x, y) = (1 - \mathfrak{L}_\alpha(y))F(t, x, y)$. Suppose that $0 < t \leqslant \tau$, where $\tau$ is fixed. One has the inequality*:

$$|F_\alpha(t, x, y)| \leqslant B_1 t^{-d/2} \min(e^{-\alpha r(x)}, e^{-\alpha r(y)}),$$
$$\times \max\left( \prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))}, \prod_{\beta \in \Phi} e^{n_\beta \beta(r(y))} \right)$$
$$\times \max_{\substack{\sigma \in \Psi \\ n_\sigma = 1}} \max_{x, y} (\sigma(r(x)), \sigma(r(y))) \exp(-\rho^2(x, y)/32t)$$

*uniformly for $x, y \in \mathfrak{S}_c(P)$, any given $c$. Here $r$ are the coordinates given by* (3.2) *and one uses the notation of Lemma* 3.3.

*Proof.* Let $\psi = \Psi - \alpha$ be the complement of $\alpha$ in $\Psi$. Then by definition:

$$F_\alpha(t, x, y) = \sum_{\gamma \in \Gamma \cap P} E(t, x, y\gamma) - \int_{N_\psi / N_\psi \cap \Gamma} E(t, xn, y\gamma)\, dn.$$

Using (5.2), we estimate the term coming from the identity element $\gamma = 1$:

$$F_\alpha(t, x, y) = \sum_{\substack{\gamma \in \Gamma \cap P \\ \gamma \neq 1}} E(t, x, y\gamma) - \int_{N_\psi / N_\psi \cap \Gamma} E(t, xn, y\gamma)$$
$$+ O\left(t^{-d/2} \exp(-\rho^2(x, y)/4t)\right).$$

The mean value theorem combined with (5.2) yields

$$|F_\alpha(t,x,y)| \leq B_2 t^{-d/2} \sum_{\substack{\gamma\in\Gamma\cap P\\ \gamma\neq 1}} \left[\frac{\rho(x,y\gamma)}{t}\exp\left(\frac{-\rho^2(x,y\gamma)}{8t}\right) + \exp\left(\frac{-\rho^2(x,y\gamma)}{8t}\right)\right]$$
$$\times \min(\operatorname{diam}(x),\operatorname{diam}(y)) + O\left(t^{-d/2}\exp(-\rho^2(x,y)/4t)\right).$$

Here $\operatorname{diam}(x)$ is the diameter of $N_\psi/N_\psi \cap \Gamma$ at $x$. By formula (3.2), one has $\operatorname{diam}(x) = O(e^{-\alpha(r)})$, where $r = r(x)$ are the coordinates on $A_c$ used in (3.2).

It is an elementary lemma that $we^{-w}$ is uniformly bounded for real $w \geq 0$. Consequently,

$$|F_\alpha(t,x,y)| \leq B_3 \min(e^{-\alpha(r(x))}, e^{-\alpha(r(y))})t^{-d/2}$$
$$\times \sum_{\substack{\gamma\in\Gamma\cap P\\ \gamma\neq 1}} \left[\frac{1}{\rho(x,y\gamma)}\exp\left(\frac{-\rho^2(x,y\gamma)}{16t}\right) + \exp\left(\frac{-\rho^2(x,y\gamma)}{8t}\right)\right]$$
$$+ O\left(t^{-d/2}\exp(-\rho^2(x,y)/4t)\right).$$

For any fixed $\varepsilon > 0$, we employ the estimate of [5, p. 491] to obtain:

$$|F_\alpha(t,x,y)| \leq B_4 t^{-d/2} \min(e^{-\alpha(r(x))}, e^{-\alpha(r(y))})$$
$$\times \left[\sum_{\substack{\gamma\in\Gamma\cap P\\ \gamma\neq 1\\ \rho(x,y\gamma)<\varepsilon}} \frac{1}{\rho(x,y\gamma)}\exp\left(\frac{-\rho^2(x,y\gamma)}{16t}\right) + \max(\operatorname{Vol}^{-1}(x),\operatorname{Vol}^{-1}(y))\exp\left(\frac{-\rho^2(x,y)}{32t}\right)\right]$$
$$+ O\left(t^{-d/2}\exp(-\rho^2(x,y)/4t)\right).$$

Here $\operatorname{Vol}^{-1}(x) = 1/\operatorname{Vol}(x)$, and $\operatorname{Vol}(x)$ is the volume of $N/\Gamma \cap N$ at $x$.

By formula (3.2), one has $\mathrm{Vol}^{-1}(x) = O(\prod_{\beta\in\Phi} e^{n_\beta \beta(r(x))})$. Here $\Phi$ is the collection of positive roots of $\mathfrak{a}$ in $\mathfrak{n}$. Thus

$$|F_\alpha(t, x, y)| \leqslant B_5 t^{-d/2} \min(e^{-\alpha(r(x))}, e^{-\alpha(r(y))})$$
$$\times \Bigg[ \sum_{\substack{\gamma\in\Gamma\cap P\\ \gamma\neq 1\\ \rho(x, y\gamma)<\varepsilon}} \frac{1}{\rho(x, y\gamma)} \exp\left(\frac{-\rho^2(x, y\gamma)}{16t}\right)$$
$$+\max\left(\prod_{\beta\in\Phi} e^{n_\beta\beta(r(x))}, \prod_{\beta\in\Phi} e^{n_\beta\beta(r(y))}\right)\exp\left(\frac{-\rho^2(x, y)}{32t}\right)\Bigg].$$

If $\varepsilon$ is sufficiently small, then referring to (3.2) we see that for $\gamma \in \Gamma \cap P$ and $\rho(x, y\gamma) < \varepsilon$, one must have $\gamma \in \Gamma \cap N$. Therefore Lemma 3.3 applies to yield:

$$|F_\alpha(t, x, y)| \leqslant B_1 t^{-d/2} \min(e^{-\alpha(r(x))}, e^{-\alpha(r(y))})$$
$$\times \max\left(\prod_{\beta\in\Phi} e^{n_\beta\beta(r(x))}, \prod_{\beta\in\Phi} e^{n_\beta\beta(r(y))}\right)$$
$$\times \max_{\substack{\sigma\in\Psi\\ n_\sigma=1}} \max_{x, y} (\sigma(r(x)), \sigma(r(y))) \exp(-\rho^2(x, y)/32t).$$

Here $\sigma$ runs over the simple roots of multiplicity one.

Using Lemma 5.5, it is easy to deduce:

**Proposition 5.6.** *If $\bar{F}(t, x, y)$ is the cuspidal parametrix defined by* (5.4), *then one has the estimate*:

$$|\bar{F}(t, x, y)| \leqslant B_7 t^{-d/2} \min_{\alpha\in\Psi} \min_{x, y} (e^{-\alpha(r(x))/2}, e^{-\alpha(r(y))/2}),$$
$$\times \max\left(\prod_{\beta\in\Phi} e^{n_\beta\beta(r(x))}, \prod_{\beta\in\Phi} e^{n_\beta\beta(r(y))}\right) \cdot \exp(-\rho^2(x, y)/32t)$$

*uniformly for $x, y \in \mathfrak{S}_c(P)$, $0 < t \leqslant \tau$, for any given $c$ and $\tau > 0$.*

*Proof.* For any simple root $\alpha$, we have

$$\bar{F}(t, x, y) = \prod_{\substack{\beta\in\Psi\\ \beta\neq\alpha}} (1 - \mathfrak{L}_\beta(y)) F_\alpha(t, x, y).$$

Moreover, the projections $\mathfrak{L}_\beta(y)$, defined by (4.1), are $L^\infty$-bounded.

Using the definition (4.1) and Lemma 5.5, one obtains immediately:

$$|\bar{F}(t,x,y)| \leq B_6 t^{-d/2}\min\left(e^{-\alpha(r(x))}, e^{-\alpha(r(y))}\right)$$
$$\times \max\left(\prod_{\beta\in\Phi} e^{n_\beta\beta(r(x))}, \prod_{\beta\in\Phi} e^{n_\beta\beta(r(y))}\right)$$
$$\times \max_{\substack{\sigma\in\Psi\\ n_\sigma=1}} \max_{x,y}\left(\sigma(r(x)), \sigma(r(y))\right)\exp\left(-\rho^2(x,y)/32t\right).$$

Here $\alpha$ is arbitrary.

Proposition 5.6 now follows by taking a minimum over $\alpha$.

The same method gives estimates for the higher order derivatives of $F(t, x, y)$:

**Proposition 5.7.** *If $\bar{F}(t, x, y)$ is the cuspidal parametrix defined by* (5.4), *then one has the estimate*:

$$\left|\left(\frac{\partial}{\partial t}\right)^i \nabla_x^j \nabla_y^k \bar{F}(t,x,y)\right|$$
$$\leq B_8 t^{-d/2-i-j-k}\min_{\alpha\in\Psi}\min_{x,y}\left(e^{-\alpha(r(x))/2}, e^{-\alpha(r(y))/2}\right)$$
$$\times \max\left(\prod_{\beta\in\Phi} e^{n_\beta\beta(r(x))}, \prod_{\beta\in\Phi} e^{n_\beta\beta(r(y))}\right)\exp\left(-\rho^2(x,y)/32t\right).$$

*Proof.* First observe that the averaging process (4.1) commutes with covariant differentiation, since $N$ acts isometrically. One then follows routinely through the proof of Proposition 5.6 and Lemma 5.5 by using the higher order estimates of [5]

## 6. Boundary corrections

Let $W_1$ be a complete Riemannian manifold without boundary containing a submanifold $W_2$ with boundary $\partial W_2$. We assume that $W_1$ and $W_2$ have the same dimension, i.e. the interior of $W_1$ is an open set in $W_2$. If $W_1$ and $W_2$ are compact, then given a fundamental solution $F$ of the heat equation on $W_1$, the method of single layer potentials [19, pp. 175–194] allows one to modify $F$ to obtain a fundamental solution of the heat equation on $W_2$ with Neumann boundary conditions. If the universal cover of $W_1$ has bounded geometry, i.e. the curvature is absolutely bounded and the injectivity radius is bounded below, and if $\partial W_2$ is compact, one can employ [5] to generalize the single layer potential construction given in [19]. However, when $\partial W_2$ is noncompact, further hypotheses are required.

We will use the single layer potential construction to modify $\bar{F}(t, x, y)$, given by (5.4), yielding a fundamental solution to the heat equation problem with Neumann boundary conditions on $\partial X$ and cuspidal conditions on the interior of $X$. Here $X$ is defined as in §5. Even though $\partial X$ may be noncompact, its topology and geometry are precisely known outside a compact set. Thus, no serious difficulty arises when applying the methods of [19].

The basic estimates are the following:

**Proposition 6.1.** *Let $\bar{F}(t, x, y)$ be given by* (5.4). *Then set*

$$Q^{(0)}(t, x, y) = \bar{F}(t, x, y),$$

$$Q^{(m+1)}(t, x, y) = \int_0^t ds \int_{\partial X} \bar{F}(x, u, s) \frac{\partial}{\partial \nu} Q^{(m)}(u, y, t-s)\, du.$$

*Here, the unit normal derivative $\partial/\partial\nu$ is applied to the argument $u$ of $Q^{(m)}$.*

*One has the estimates, for $m \geq 1$:*

$$\left| \left(\frac{\partial}{\partial t}\right)^i \nabla_x^j \nabla_y^k Q^{(m)}(t, x, y) \right| \leq C_1^m (\Gamma(m/2))^{-1} t^{-d/2-i-j-k}$$
$$\times \exp\big(-C_2(\sigma^2(x) + \sigma^2(y))/t\big) \exp\big(-C_3 \rho^2(x, y)/t\big)$$
$$\times \min_{\alpha \in \Psi} \min_{x, y} \left(e^{-\alpha(r(x))/2}, e^{-\alpha(r(y))/2}\right)$$
$$\times \max\left( \prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))} \prod_{\beta \in \Phi} e^{n_\beta \beta(r(y))} \right).$$

*The notation is that of Proposition* 5.7. *Moreover, $\sigma(x)$ is the distance from $x$ to $\partial X$.*

*Proof.* The argument proceeds by induction starting from Proposition 5.7. One uses the method of [19] combined with the precise description of the metric on $X$ given in (3.2). Since $\partial X$ is given outside a compact set by $r_i = c$, for some $i$, in the coordinates of §3, the details are quite straightforward.

The fundamental solution is obtained as in [19].

**Theorem 6.2.** *Let $\bar{E}(t, x, y) = \sum_{m=0}^{\infty} (-2)^m Q^{(m)}(t, x, y)$, where $Q^{(m)}$ are given by Proposition* 6.1. *Then $\bar{E}$ is the fundamental solution of the heat equation with Neumann boundary conditions and cuspidal interior conditions on $X$.*

*One has the estimate*:

$$|\bar{E}(t, x, y)| \leq C_4 t^{-d/2} \min_{\alpha \in \Psi} \min_{x, y} \left(e^{-\alpha(r(x))/2}, e^{-\alpha(r(y))/2}\right)$$
$$\times \max\left( \prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))}, \prod_{\beta \in \Phi} e^{n_\beta \beta(r(y))} \right) \exp\left( \frac{-C_3 \rho^2(x, y)}{t} \right)$$
$$\times \left(1 + C_5 \exp\big(-C_2(\sigma^2(x) + \sigma^2(y))/t\big)\right).$$

*Here $C_3$ and $C_4$ depend only upon $Y = K \setminus G/\Gamma \cap P$. However, $C_2$ and $C_5$ are dependent upon the submanifold $X$.*

*Proof.* From §5 and Proposition 6.1, it is clear that $\bar{E}(t, x, y)$ satisfies the heat equation and cuspidal interior conditions on $X$.

To show that $\bar{E}(t, x, y)$ satisfies Neumann boundary conditions one establishes the jump relations [19, p. 187] for the $Q^{(m)}$. This is primarily a local computation, which is undisturbed by the noncompactness of $X$.

The upper bound for $\bar{E}$ follows by writing $\bar{E} = Q^{(0)} + (\bar{E} - Q^{(0)})$ and quoting the estimates of Propositions 5.6 and 6.1.

## 7. Spectral function on the cusp

In this section we give an asymptotic upper bound for $N_X(\lambda)$. Here $N_X(\lambda)$ denotes the number of eigenvalues less than $\lambda$ for the Laplacian with cuspidal interior conditions and Neumann boundary conditions on $X$, defined as in §6.

We begin with the following elementary lemma [7]:

**Lemma 7.1.** *Let $B$ denote a nonnegative self adjoint operator acting on a Hilbert space. Suppose that the associated heat operator* $\exp(-tB)$ *is trace class, for all $t > 0$. Then $B$ has pure point spectrum, so we may define $N_B(\lambda)$ as the number of eigenvalues of $B$ less than $\lambda$. If, for some positive integer $d$,*

$$\overline{\lim_{t\to 0}}\, t^{d/2}\operatorname{Tr}(e^{-tB}) \leq D_1$$

*then*

$$\overline{\lim_{\lambda\to \infty}}\, \lambda^{-d/2}N_B(\lambda) \leq D_1 e.$$

Consider the Laplacian $\Delta_X$ acting on $L^2X$ with Neumann boundary conditions and cuspidal interior conditions. The associated heat kernel $\bar{E}(t, x, y)$ for $\exp(-t\Delta_X)$ is estimated in Theorem 6.2. One may deduce:

**Theorem 7.2.** *The heat kernel $\bar{E}(t, x, y)$ defines a trace class operator* $\exp(-t\Delta_X)$. *Moreover, one has the estimate:*

$$\operatorname{Tr}(e^{-t\Delta_X}) \leq D_2 t^{-d/2}\int_{r(X)} \min_{\alpha\in\Psi}(e^{-\alpha(r)/2})\,dr + O(t^{-d/2+1/2}).$$

*The constant $D_2$ depends only upon $Y = K \setminus G/\Gamma \cap P$. Here $r(X)$ is the set of $r$ coordinates, as in* (3.2), *for points in $X$.*

*Proof.* By the spectral theory of self adjoint operators, $\bar{E}(t, x, y)$ satisfies the semigroup property:

$$\bar{E}(t, x, y) = \int_X \bar{E}(t, x, z)\bar{E}(t, z, y)\,dz \tag{7.3}$$

and symmetry $\bar{E}(t, x, y) = \bar{E}(t, y, x)$.

Setting $x = y$, and integrating we find that

$$\int_X |\bar{E}(t, x, y)|^2 dx dy = \int_X \bar{E}(t, x, x)\, dx. \tag{7.4}$$

The key estimate of Theorem 6.2 now gives, for small $t > 0$:

$$\int_X \bar{E}(t, x, x)\, dx \leq D_3 t^{-d/2} \int_{r(X)} \min_{\alpha \in \Psi} (e^{-\alpha(r)/2})\, dr + O(t^{-d/2+1/2}). \tag{7.5}$$

The integral on the right-hand side of (7.5) converges, so $\bar{E}$ is Hilbert-Schmidt by (7.4). However, the semigroup property (7.3) now shows that $\bar{E}$ is trace class. Then (7.5) gives the required upper bound for $\mathrm{Tr}(e^{-t\Delta_X})$.

It is convenient to denote $\mathfrak{M}(X) = \int_{r(X)} \min_{\alpha \in \Psi} (e^{-\alpha(r(x))/2})\, dr$.

From Lemma 7.1 and Theorem 7.2, one has immediately:

**Corollary 7.6.** *Let $X$ be as in the first paragraph of this section. Then*

$$\overline{\lim_{\lambda \to \infty}} \lambda^{-d/2} N_X(\lambda) \leq D_4 \mathfrak{M}(X).$$

*The constant $D_4$ depends only upon $Y = K \setminus G/\Gamma \cap P$. Otherwise, $D_4$ is independent of the particular choice of submanifold $X$.*

## 8. Proof of the main theorem

It is now a straightforward matter to complete the proof of Theorem 1.1 of the introduction. Let $B_k$ denote an exhaustion of $K \setminus G/\Gamma$ as in §4. Suppose $X_{i,k}$ are smooth manifolds in $\mathcal{S}_c(P_i)$ as chosen there.

One has the asymptotic estimate of Minakshisundaram-Pleijel [1]:

**Proposition 8.1.** *Let $W$ be a compact Riemannian manifold with boundary. If $N_W(\lambda)$ denotes the number of eigenvalues less than $\lambda$ for the Neumann problem on $W$, then*

$$\lim_{\lambda \to \infty} \frac{N_W(\lambda)}{\lambda^{d/2}} = (4\pi)^{-d/2} \frac{\mathrm{Vol}(W)}{\Gamma(d/2 + 1)}.$$

*Here $d$ is the dimension of $W$ and* Vol$(W)$ *is the volume of $W$.*

If $N(\lambda)$ is the number of cuspidal eigenvalues on $K \setminus G/\Gamma$ which are less than $\lambda$, then by Proposition 4.2, for any $k$:

$$\overline{\lim_{\lambda \to \infty}} \frac{N(\lambda)}{\lambda^{d/2}} \leq \overline{\lim_{\lambda \to \infty}} \frac{N_k(\lambda)}{\lambda^{d/2}} + \sum_{i=1}^{r} \overline{\lim_{\lambda \to \infty}} \frac{N_{i,k}(\lambda)}{\lambda^{d/2}}.$$

Here $N_k(\lambda)$ is the number of eigenvalues less than $\lambda$ for the Neumann problem on the compact Riemannian manifold $B_k$.

Using Proposition 8.1 one obtains

$$\varlimsup_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} \leqslant (4\pi)^{-d/2} \frac{\operatorname{Vol}(B_k)}{\Gamma(d/2+1)} + \sum_{i=1}^{r} \varlimsup_{\lambda \to \infty} \frac{N_{i,k}(\lambda)}{\lambda^{d/2}}.$$

Applying Corollary 7.6, one may deduce:

$$\varlimsup_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} \leqslant (4\pi)^{-d/2} \frac{\operatorname{Vol}(B_k)}{\Gamma(d/2+1)} + \sum_{i=1}^{r} C_i \mathfrak{M}_i(X_{i,k}). \tag{8.2}$$

However, $\lim_{k \to \infty} \operatorname{Vol}(B_k) = \operatorname{Vol}(K \backslash G/\Gamma)$. Moreover, with a sensible choice of $X_{i,k}$, $\lim_{k \to \infty} \mathfrak{M}_i(X_{i,k}) = 0$, for all $i$.

Theorem 1.1 of the introduction follows by letting $k \to \infty$ in (8.2).

## 9. Coefficients in a bundle

The results derived above may be extended in a routine way to suitable differential operators acting on sections of equivariant vector bundles. In fact, the constructions of [5] are valid for any second order operator, which is $G$-invariant, and has leading symbol given by the metric tensor. Consequently, one may follow the previous sections of the present paper line by line to obtain:

**Theorem 9.1.** *Let $\rho$ be any irreducible unitary representation of $K$, acting on a finite dimensional space of dimension* $\dim(\rho)$. *Suppose that $N(\lambda)$ is the number of cuspidal eigenfunctions less than $\lambda$ for the Casimir operator acting on sections of the associated vector bundle $V_\rho \to G/K$. Then one has the asymptotic upper bound*:

$$\varlimsup_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} \leqslant (4\pi)^{-d/2} \frac{\operatorname{Vol}(K \backslash G/\Gamma)}{\Gamma(d/2+1)} \dim(\rho).$$

By the argument of Matsushima-Murakami [16, p. 385], we may identify the Hodge Laplacian on $p$-forms with the Casimir operator on the bundle associated to the $p$th exterior power of the isotropy representation of $K$. Thus, a special case of Theorem 9.1 is:

**Corollary 9.2.** *Let $N(\lambda)$ be the number of cuspidal eigenfunctions with eigenvalue less than $\lambda$ for the Hodge Laplacian acting on differential p-forms. Then one has the asymptotic upper bound*:

$$\varlimsup \frac{N(\lambda)}{\lambda^{d/2}} \leqslant (4\pi)^{-d/2} \frac{\operatorname{Vol}(K \backslash G/\Gamma)}{\Gamma(d/2+1)} \binom{d}{p}.$$

## References


[1] M. Berger, P. Gauduchon & E. Mazet, *Le spectre d'une varieté Riemannienne*, Lecture Notes in Math., Vol. 194, Springer, Berlin, Heidelberg and New York, 1971.

[2] A. Borel, *Cohomology of arithmetic groups*, Proc. Internat. Congr. Math., Vancouver, 1974, Vol. 1, pp. 435–442.

[3] ______, *Introduction aux groupes arithmetiques*, Hermann, Paris, 1969.

[4] ______, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. (4) **7** (1974) 235–272.

[5] H. Donnelly, *Asymptotic expansions for the compact quotients of properly discontinuous group actions*, Illinois J. Math. **23** (1979) 485–496.

[6] ______, *On the point spectrum for finite volume symmetric spaces of negative curvature*, Comm. Partial Differential Equation **6** (1981) 963–992.

[7] H. Donnelly & P. Li, *Lower bounds for the eigenvalues of Riemannian manifolds*, Michigan Math. J., to appear.

[8] H. Donnelly, *On the essential spectrum of a complete Riemannian manifold*, Topology **20** (1981) 1–14.

[9] R. Gangolli & F. Warner, *Zeta functions of Selberg's type for some noncompact quotients of symmetric spaces of rank one*, Nagoya Math. J. **78** (1980) 1–44.

[10] H. Garland, *The spectrum of noncompact $G/\Gamma$ and the cohomology of arithmetic groups*, Bull. Amer. Math. Soc. **75** (1969) 807–811.

[11] Harish-Chandra, *Automorphic forms on semisimple lie groups*, Lecture Notes in Math., Vol. 62, Springer, Berlin, Heidelberg and New York, 1968.

[12] T. Kubota, *Elementary theory of Eisenstein series*, Wiley, New York, 1973.

[13] S. Lang, $SL_2(R)$, Addison-Wesley, Reading, MA, 1975.

[14] R. P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Math., Vol. 544, Springer, Berlin, Heidelberg and New York, 1976.

[15] P. Lax & R. Phillips, *Scattering theory for automorphic functions*, Ann. of Math. Studies, No. 87, Princeton Univ. Press, Princeton, NJ, 1976.

[16] Y. Matsushima & S. Murakami, *On vector valued harmonic forms and automorphic forms on symmetric Riemannian manifolds*, Ann. of Math. (2) **78** (1973) 365–416.

[17] C. C. Moore, *Decomposition of unitary representations defined by discrete subgroups of nilpotent Lie groups*, Ann. of Math. **82** (1965) 146–82.

[18] W. Müller, *Spectral theory of non-compact Riemannian manifolds with cusps and a related trace formula*, preprint.

[19] D. B. Ray & I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Math. **7** (1971) 145–210.

[20] S. Tanaka, *Selberg's trace formula and spectrum*, Osaka J. Math. **3** (1966) 205–216.

[21] A. B. Venkov, *The spectral theory of automorphic functions, the Selberg zeta function, and some problems of analytic number theory and mathematical physics*, Russian Math. Surveys **34** (1979) 79–153.


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