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# THE CONFORMAL INVARIANCE OF HUYGEN'S PRINCIPLE

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## Introduction

In this paper we study certain hyperbolic equations satisfying the strong Huygen's principle (SHP) in the sense that the elementary solutions are supported on a hypersurface. Recently Lax and Phillips [7] have discussed the wave equation on odd-dimensional rank-one symmetric spaces and shown how certain coordinate transformations yield the SHP by reduction to the wave equation in Euclidean space. Also Helgason [3] has given examples of compact groups and symmetric spaces on which the SHP holds for the natural wave equations.

We will adopt the approach begun in [12] and indirectly suggested in [7] to show that, if there is a conformal transformation between two Lorentz manifolds $M_1$ and $M_2$ of constant scalar curvature, then the wave equation in $M_1$ satisfies the SHP if and only if the same is true in $M_2$. In this case the transformation maps characteristic cones to characteristic cones, so it is perhaps not surprising that it also provides a transformation between the elementary solutions. As a corollary we get new examples of Lorentz manifolds on which the wave equation satisfies the SHP.

More generally we derive similar principles for certain ultrahyperbolic equations on pseudo-Riemannian manifolds of constant scalar curvature, and we discuss the SHP for the Dirac and Maxwell equations. Finally we point out an elementary connection between causality-preserving transformations and automorphisms of complex domains.

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**1.** Let $M$ be a pseudo-Riemannian manifold of dimension $n$ with pseudo-metric $g$ and constant scalar curvature $K$. Consider the Laplace-Beltrami operator $\square$ on $M$, [12], [13], and the generalized wave operator

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$$L = \square + \frac{n-2}{4(n-1)} K. \tag{1}$$

As shown in [12] this operator is conformally quasi-invariant in the following sense: if $T\colon V \to W$ is a conformal diffeomorphism between two open sets in $M$, so that the pull-back metric $T^*g = \gamma g$ for some positive function $\gamma$ (the Jacobian of $T$ raised to the $(2/n)$th power), then

$$L\gamma^{(n-2)/4}T^* = \gamma^{(n+2)/4}T^*L.$$

In other words, for every smooth function $f$ on $W$ we have

$$L\big(\gamma(x)^{(n-2)/4}f(T(x))\big) = \gamma(x)^{(n+2)/4}(Lf)(T(x)), \tag{2}$$

which in particular says that the null-space of $L$ is invariant under the action

$$f(x) \to \gamma(x)^{(n-2)/4}f(T(x)). \tag{3}$$

The resulting representation of the conformal group of $M$ has in special cases been studied in detail [4], [12], [11]. Note that (3) fails to be unitary in $L^2(M)$.

Now suppose we are given two such manifolds $M_1$ and $M_2$ of the same dimension $n$ with data as above, i.e., pseudo-metrics $g_1$ and $g_2$, constant scalar curvatures $K_1$ and $K_2$ and "wave" operators $L_1$ and $L_2$ as in (1). We wish to generalize the covariance in (2) to mappings between $M_1$ and $M_2$: Let $V_1$ and $V_2$ be open sets in $M_1$ and $M_2$, and $T\colon V_1 \to V_2$ a conformal diffeomorphism, $T^*g_2 = \gamma g_1$. Then we have that $L_1$ and $L_2$ are intertwined via $T$ as follows:

**Proposition 1.** *Under the hypotheses above we have for any smooth function $f$ on $V_2$ that*

$$L_1\big(\gamma(x)^{(n-2)/4}f(T(x))\big) = \gamma(x)^{(n+2)/4}(L_2 f)(T(x)). \tag{4}$$

*In particular, solutions to $L_2 f = 0$ on $V_2$ are mapped to solutions of $L_1\tilde{f} = 0$ on $V_1$ via*

$$\tilde{f}(x) = \gamma(x)^{(n-2)/4}f(T(x)). \tag{5}$$

*Proof.* $\gamma(x)$ being related to the Jacobian of a conformal transformation between spaces of constant scalar curvature has to satisfy the following (nonlinear) differential equation (in [5] this is carried out for positive definite metric on a compact manifold, and the general case follows by similar differential geometric arguments)

$$\square_1 h = -\frac{n-2}{4(n-1)}\big(K_1 h - K_2 h^{(n+2)/(n-2)}\big), \tag{6}$$

where $h(x) = \gamma(x)^{(n-2)/4}$. On the other hand, by Lemma 3.1 in [12] we have

$$\square_1 hT^*f - \gamma hT^*\square_2 f = (\square_1 h)T^*f, \tag{7}$$

so that by combining (6) and (7) we get

$$\left(\square_1 + \frac{n-2}{4(n-1)}K_1\right)hT^*f = \gamma hT^*\left(\square_2 + \frac{n-2}{4(n-1)}K_2\right)f,$$

which is exactly (4). Note that one can prove that (6) is also a sufficient condition for a positive function $h(x)$ to be equal to $\gamma(x)^{(n-2)/4}$ corresponding to some conformal transformation of $M_1$ into a space of scalar curvature $K_2$.

**Remark 2.** (a) This proposition was established in [12] in many special cases and has been a natural conjecture.

(b) Equation (6) is actually invariant under conformal transformations in $M_1$ of the form (3) by virtue of the quasi-invariance (2).

It is clear that (5) also sets up a correspondence between distributions in $V_1$ and $V_2$ and that we still have (4) and a correspondence between solutions to $L_2 f = 0$ and $L_1 h = 0$. (5) preserves singular support, so if $f$ is supported along a hypersurface, so is $h$. This is the well-known use of coordinate transformations as in (5) to arrive at a SHP in one manifold by knowing it in another. The virtue of conformal transformations is that they preserve characteristic cones (in the case of Lorentz manifolds, i.e., metrics of signature $(+-\cdots-)$) for the Laplace-Beltrami operators, so that the local behavior of solutions is the same in conformally equivalent regions. Hence in any manifold locally conformally equivalent to the examples given in [3] (e.g., $\mathbf{R} \times SU(2n)$) we have the SHP.

Now let $M_0$ be $\mathbf{R}^n$ ($n$ even $\geqslant 4$) with its standard Lorentz metric

$$ds^2 = dx_1^2 - dx_2^2 - \cdots - dx_n^2$$

and wave operator $L_0 = \partial^2/\partial x_1^2 - \partial^2/\partial x_2^2 - \cdots - \partial^2/\partial x_n^2$. This has retarded and advanced fundamental solutions supported on the boundary of the forward (resp. backward) light cone. As a first example of our techniques, let us construct a conformal transformation of $M_0$ into the de Sitter space

$$H = \left\{(y_0, y_1, \cdots, y_n) \in \mathbf{R}^{n+1} | y_0^2 + y_1^2 - y_2^2 - \cdots - y_n^2 = 1\right\},$$

which is a homogeneous space for $0(2, n-1)$ with isotropy group $0(1, n-1)$ (the Lorentz group). Let

$$T(x) = \frac{\left(1 - \frac{1}{4}x^2, x\right)}{1 + \frac{1}{4}x^2},$$

where $x^2 = x_1^2 - x_2^2 - \cdots - x_n^2$. Then [11] $T$ is a conformal diffeomorphism from $\{x | 1 + \frac{1}{4}x^2 \neq 0\}$ onto $\{y | y_0 \neq -1\}$ (in particular from a neighborhood of the origin of $M_0$). On $H$ we have $L_H = \square_H - ((n-2)/2)^2$ and $\gamma(x) = (1 + \frac{1}{4}x^2)^{-2}$ so that

$$L_0 \gamma^{(n-2)/4} T^* = \gamma^{(n+2)/4} T^* L_H. \tag{8}$$

Since a $\delta$-signal in $M_0$ propagates along the boundary of the light-cone, i.e., along light-rays, also in $H$ sharp signals will remain confined to light-rays. This is true locally by (8) and hence eveywhere by the homogeneity of $H$. In particular spacelike Cauchy data of small support will propagate to form lacunae where the solution is zero. Also in $\tilde{H}$, the universal covering of $H$, homogeneous for the universal covering group of $0(2, n-1)$, the SHP is satisfied.

More generally, suppose $M_1$ and $M_2$ are globally hyperbolic [1] Lorentzian manifolds of constant scalar curvature (e.g., $\tilde{H}$ and $M_0$ above) so that each has defined on it unique global retarded and advanced fundamental solutions. For example $E_1^+(x, x')$ is the distribution on $M_1 \times M_1$ satisfying

$$L_1 E_1^+(x, x') = \delta(x, x')$$

(acting on second coordinate) where $\delta(x, \cdot)$ is the Dirac $\delta$-distribution at the point $x$ and for a fixed $x$, $E_1^+(x, x')$ has forward time-like support. Similarly for $E_1^-$, $E_2^+$, $E_2^-$. Then [12]

**Proposition 3.** *Let $T$: $M_1 \to M_2$ be conformal with $M_1$ and $M_2$ as above. Then*

$$\gamma(x)^{(n-2)/4} E_2^+(T(x), T(x'))\gamma(x)^{(n-2)/4} = E_1^+(x, x'), \tag{9}$$

(*and the same for $E^-$*).

Considering the conformal compactification of $M_0$ ($n = 4$) we get as a corollary the advanced fundamental solution on $\mathbf{R} \times S^3$ to be (at $\tau = \rho = 0$)

$$E^+(\tau, \rho) = \frac{1}{4\pi} \frac{\delta(\tau - \rho),}{\sin \rho},$$

where $\rho$ is the polar angle on $S^3$ from the north pole.

In the setting of Proposition 3 the Greens function

$$G_1(x, x') = E_1^+(x, x') - E_1^-(x, x')$$

will satisfy a relation similar to (9). Define for a test function $\varphi(x)$ on $M_1$ (and similarly in $M_2$)

$$(\varphi, \varphi)_1 = \int_{M_1}\int_{M_1} G_1(x, x')\varphi(x')\overline{\varphi(x)}\, dx' dx. \tag{10}$$

Then by virtue of the covariance of $G$ we have for any test function $\psi$ on $M_2$ and $\varphi = \gamma^{(n+2)/4} T^* \psi$

$$
\begin{aligned}
(\varphi, \varphi)_1 &= \int_{M_1}\int_{M_2} G_1(x, x')\gamma(x')^{(n+2)/4}\psi(T(x'))\,\overline{\psi(T(x))}\,\gamma(x)^{(n+2)/4}\,dx'\,dx \\
&= \int_{M_1}\int_{M_2} \gamma(x)^{(n-2)/4} G_2(T(x), y')\psi(y')\,\overline{\psi(T(x))}\,\gamma(x)^{(n+2)/4}\,dy'\,dx \\
&= \int_{M_2}\int_{M_2} G_2(y, y')\psi(y')\,\overline{\psi(y)}\,dy'\,dy \\
&= (\psi, \psi)_2.
\end{aligned}
$$

Hence the Hermitian form (10) is conformally invariant, and if positive (as on $M_0$, see also [8]) it defines an invariant unitary structure on the space of test functions. Actually this invariant form is defined on the space of solutions $f$ to the wave equation via

$$f(x) = \int_M G(x, x')\varphi(x')\,dx',$$

where the action of the conformal group of $f$ is the one given by (3). This action then leaves invariant $(f, f) = (\varphi, \varphi)$. In some cases [11], [4] there results an irreducible unitary continuous representation of the conformal group of the manifold on the space of solutions to $Lf = 0$.

Finally let us give a list of (conformally flat) locally symmetric Lorentzian manifolds of constant scalar curvature which have open dense subsets which are conformal images of open sets in the linear space $M_0$. In particular they all satisfy the SHP for the wave operator [12]. The cases are ($n$ even $\geqslant 4$):

(a) $\mathbf{R} \times S^{n-1}$,

(b) $O(2, n-1)/O(1, n-1)$,

(c) $O(1, n)/O(1, n-1)$,

(d) $O(1, q)/O(q) \times O(1, n-q)/O(1, n-q-1)$ $(q = 0, 1, \cdots, n-1)$,

(e) $O(2, q)/O(1, q) \times O(n-q)/O(n-q-1)$, $(q = 1, 2, \cdots, n-2)$,

and any covering of these.

If in (d) $q = n - 1$, then the manifold is

$$O(1, n-1)/O(n-1) \times O(1, 1)/O(1),$$

where the first part is the "space" part, and the second the "time" part. By an extra coordinate transformation (nonconformal) the "time" part can be made $S^1$ or its covering $\mathbf{R}$ so that we also get the SHP on $\mathbf{R} \times O(1, n-1)/O(n-1)$ as in [3], [7].

**Remark 4.** Suppose $\mathbf{R}$ is equipped with the metric $h(s)^2 ds^2$ where $h$ is positive; the corresponding Laplace-operator is $h^{-1}(\partial/\partial s)h^{-1}(\partial/\partial s)$. On the other hand if $a\colon \mathbf{R} \to \mathbf{R}$ is monotone and bijective, we have $\partial^2/\partial s^2 f(a(s)) = a'(s)^2 f''(a(s)) + a''(s) f'(a(s))$. Hence the change of coordinate via $a$ maps

$\partial^2/\partial s^2$ into $h^{-1}(\partial/\partial s)h^{-1}(\partial/\partial s)$ if and only if $a'(s)^2 = h(s)^{-2}$ and $a''(s) = h(s)^{-1}(h^{-1})'(s) = -h'(s)h(s)^{-3}$. But the second equation is implied by the first which can simply be solved by $a(s) = \int_0^s h(t)^{-1}dt$.

As another special case of the above we mention $\mathbf{R} \times \widetilde{SL}(2, \mathbf{R})$ (universal covering) where minus the Killing form on $sl(2, \mathbf{R})$ induces a pseudometric of signature $(+--)$ which together with the (negative) metric $-ds^2$ on $\mathbf{R}$ makes this space into a Lorentz manifold on which $L = C - \partial/\partial s^2 + 1$ satisfies the SHP. Here $C$ is the Casimir operator of $SL(2, \mathbf{R})$. In fact, this case arises in (e) when $n = 4$ and $q = 2$.

**Remark 5.** The spectrum of $L$ on $(SL(2, \mathbf{R}) \times S^1)/\mathbf{Z}_2 \simeq U(1, 1)$ plays a role in the decomposition of holomorphic discrete series representations of $SU(2, 2)$ when restricted to $S(U(1, 1) \times U(1, 1))$, quite analogous to the case of restrictions to the maximal compact subgroup $S(U(2) \times U(2))$.

**2.** Both the Dirac equation (for zero mass) and Maxwell's equations (in vacuum) are known to satisfy the SHP in $n$-dimensional Minkowski space ($n$ even $\geqslant 4$). This is readily seen from the fact that the components of the fields satisfy the wave equation.

The conformal invariance of Dirac's equation has been studied in [6]. Let us summarize what we shall need: Assume $M$ is a Lorentz manifold of dimension $2m$ with $H^2(M, \mathbf{Z}_2) = 0$ (so that the spin bundle is well-defined). Denote by $S$ the spin bundle (with complex fiber dimension $2^m$) with structure group $\mathrm{Spin}(1, 2m-1)$, a double covering of $SO(1, 2m-1)$. The Dirac operator $D$ acts on sections of $S$ by composing the covariant differentiation in $S$ with Clifford multiplication, in local coordinates $D = \sum_{i=1}^{2m} \gamma_i \nabla_{e_i}$ where $\{e_i\}$ is an orthonormal Lorentz frame at $x \in M$ and $\gamma_i = \gamma(e_i)$, the $\gamma$-matrices corresponding to $\gamma\colon T_xM \to S_x \otimes S_x^*$.

Suppose $T\colon M \to M$ is conformal (maybe only in an open set), then the differential $T_*\colon T_xM \to T_{T(x)}M$ preserves the inner product up to a scalar, in particular it has an action on the spin fiber (via the spin covering $\mathrm{Spin}(1, 2m-1) \to SO(1, 2m-1)$)

$$\tau(T_*)\colon S_x \to S_{T(x)}.$$

Now the action on a section $\psi$ of $S$ is going to be

$$V(T)\colon \psi(x) \to \gamma_{T^{-1}}(x)^{(2m-1)/4}\tau(T_*)\psi(T^{-1}(x)).$$

We then have (the integrated form of the corresponding infinitesimal formula in [6], see also [4] for the flat case)

$$DV(T) = (\gamma_{T^{-1}})^{1/2}V(T)D. \tag{11}$$

(11) also holds when $T$ maps from one manifold into another. Clearly then,

$$D\psi = 0 \tag{12}$$

is conformally invariant, and the SHP holds for the hyperbolic equation (12) if and only if it holds on a conformally equivalent manifold. In particular the SHP holds for (12) on, e.g., $\mathbf{R} \times S^3$.

**Remark 6.** If $M$ is a reductive coset space $G/H$, we can parallelize $S$ as follows [9]: Let $\mathbf{p}$ be the complement of the Lie algebra of $H$ and assume the actions

$$\alpha: H \to SO(\mathbf{p}),$$
$$\tilde{\alpha}: H \to \mathrm{Spin}(\mathbf{p}) \to \mathrm{Aut}\, L,$$

where $L$ is the fiber of the spin bundle

$$S = G \times_H L.$$

Let $\Gamma(S)$ be the smooth sections and

$$\eta: \Gamma(S) \to L \otimes C^\infty(G),$$

where $\eta$ is (well) defined by

$$\{g, \eta(\psi)(g)\} = \psi(gH)$$

for $\psi \in \Gamma(S)$. $\eta(\psi)$ satisfies for $h \in H$ and $g \in G$

$$h \cdot \eta(\psi)(gh) = \eta(\psi)(g).$$

Then the action of $D$ is

$$\eta(D\psi) = \sum_{i=1}^{2m} (\gamma(X_i) \otimes \nu(X_i))\eta(\psi),$$

where $\{X_i\}$ is a Lorentz frame at the origin, and $\nu$ is the usual identification of $\mathbf{p}$ with vector fields on $G/H$. As proven in [9] in the positive definite case we also here have

$$\eta(D^2\psi) = -(I \times \nu(C) + c)\eta(\psi),$$

where $C$ is the Casimir operator (so that $\nu(C)$ is the Laplace-Beltrami operator on $G/H$), and $c$ is the constant from before. Hence the square of the Dirac operator in this way becomes the wave operator. We conclude that if the wave equation in $G/H$ satisfies the SHP, so does Dirac's equation (this is the case [3] e.g. on $\mathbf{R} \times K$, $K$ a compact semi-simple Lie group of odd dimension).

For Maxwell's equations on forms $\omega$ of degree $m$, $\dim M = 2m$,

$$d\omega = 0, \quad \delta\omega = 0,$$

where $\delta = {}^*d^*$ is the adjoint to $d: \Lambda^k M \to \Lambda^{k+1} M$, we have for $T$ conformal, $dT^* = T^*d$ (always) and $\delta T^* = \gamma T^*\delta$ (when acting on forms of degree $m$).

Hence $\omega \to T^*\omega$ preserves solutions to Maxwell's equations, even when going from one manifold to another. We conclude that Maxwell's equations satisfy the SHP on all manifolds locally conformally equivalent to even-dimensional Minkowski space.

**3.** The role of conformal transformations as seen above is to preserve the characteristic cones, in other words, they are causality preserving. A manifold $M$ endowed with a smooth field of proper closed convex cones (one cone in each tangent space) is said to be causal [10] with the natural induced time-like and space-like ordering. The group preserving the causal structure is called the causal group of $M$, and in the cases above it essentially coincides with the conformal group. The cone-field however need not be derived from a pseudo-metric (in which case the cone is of rank 2) but can be of higher rank, as e.g. in $H(n)$, all Hermitian $n \times n$ matrices, where the cone of all positive definite matrices has rank $n$. Another example is $U(n)$ with the left-invariant cone field induced from $H(n)$, the Lie algebra (up to multiplication by $i$). There is in some cases a close connection between causal manifolds and their causal groups on the one hand, and open subdomains of $\mathbf{C}^n$ and their automorphism groups on the other.

Suppose $D$ is a Hermitian symmetric space of tube type, and $M$ its Shilov boundary, $\dim_{\mathbf{R}} M = \dim_{\mathbf{C}} D = n$. At a point $x \in M$ the tangent space $T_xM$ has a complement $K_x$ so that

$$T_xM + K_x = \mathbf{C}^n, \tag{12}$$

and (12) is the local splitting in purely real and imaginary coordinates. The direction from $x$ into $D$ is given by a cone $V$ in $K_x$ (the cone defining the tube domain).

**Remark 7.** Realizing $D$ as a tube domain $\{x + iy \mid y \in V\}$ (e.g., the upper half-plane) the splitting (12) is simply in $x$ and $y$, the real and imaginary coordinates.

Suppose $T$: $D \to D$ is biholomorphic, and consider $x \in M$. $M$ is also stable under $T$ and the differential on $M$

$$T_*: T_xM \to T_{T(x)}M.$$

$T_*$ on $D$ is complex-linear and preserves $D$ so that $T_*$ is of the form (relative to the splitting (12) at $x$ and $T(x)$)

$$T_* = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where $A = B$ by complex linearity, and $A$ is the differential of $T$ on $M$. Here $B$ must preserve $V$ (since $T$ preserves $D$), hence $A$ preserves $V$ which proves that $T$ on $M$ is locally causality preserving, where we equip $M$ with the cone-field defined by $V$. In other words, locally we have that $i$ times a timelike direction on $M$ is a direction into $D$. Note also that in the same way one gets that a holomorphic transformation from a domain of tube type to another, on the Shilov boundary is causality-preserving (as for example the Cayley transform).

It would seem that reversing the argument (given a causality-preserving transformation on $M$, extend it to an isomorphism of $D$) requires a little more work.

The interpretation of Huygen's principle in terms of objects on $D$ is apparently not known; but it seems that hyperbolicity and causality on the boundary somehow reflects the geometry of $D$.

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