J. DIFFERENTIAL GEOMETRY
14 (1979) 187–203

A RIGIDITY THEOREM FOR THREE-DIMENSIONAL SUBMANIFOLDS IN EUCLIDEAN SIX-SPACE

KETI TENENBLAT

1. Introduction

In the classical theory of a surface in Euclidean three-space, the asymptotic curves play a special role in the rigidity problems. In this paper, we prove a rigidity theorem for a three-dimensional manifold in the Euclidean six-space E^6, using the asymptotic hyperplanes.

Let M^n be an n-dimensional C^∞ submanifold of E^N, $N = \frac{1}{2}n(n+1)$, with the induced metric and such that the inclusion i: $M^n \to E^N$ is nondegenerate. Let $p \in M^n$, and denote the second fundamental form by s. A q-dimensional, $0 < q < n$, linear subspace L of the tangent space T_pM is called *asymptotic* if there exists a vector ξ normal to T_pM such that $\langle s(X, Y), \xi \rangle = 0 \ \forall X, Y \in L$, where $\langle\ ,\ \rangle$ denotes the Euclidean metric. If L is of codimension one, we have an *asymptotic hyperplane* at p. A q-dimensional submanifold U^q of M^n, $q < n$, is said to be *asymptotic* at $p \in U$ if T_pU is asymptotic, and *asymptotic* if this is true for each $p \in U$. It is not difficult to see that U is an asymptotic hypersurface of M if and only if there exists a normal to the osculating space of U, which is also normal to M. The notion of asymptotic surface in a more general context can be found in [4].

We characterize the set $\mathcal{C}_p$ of all asymptotic hyperplanes at a point $p \in M$ as follows: choose an orthonormal frame $e_1, \cdots, e_N$ defined on a neighborhood of p such that $e_1, \cdots, e_n$ are tangent to M and $e_{n+1}, \cdots, e_N$ are normal to M. Let $\omega^1, \cdots, \omega^N$ be the dual frame. We adopt the following indices convention

$$i, j, k = 1, \cdots, n;\ \mu, \lambda, \alpha, \beta = n+1, \cdots, N;\ A, B, C = 1, \cdots, N$$

and the summation convention with regard to repeated indices. Denote by H^λ the second fundamental forms with respect to this frame, i.e., $s(X, Y) = H^\lambda(X, Y)e_\lambda$. From the theory of a submanifold of a Euclidean space, it follows that $\omega^\lambda = 0$ on M and $H^\lambda = h^\lambda_{ij}\omega^i \otimes \omega^j$, $h^\lambda_{ij} = h^\lambda_{ji}$. An $(n-1)$-dimensional linear subspace of T_pM given by $u_i\omega^i = 0$ is asymptotic if there

Communicated by S. S. Chern, March 14, 1977. Partially supported by CNP_q.

exists $a_\lambda \in \mathbf{R}$, $\lambda = n+1, \cdots, N$ not all zero, such that $a_\lambda H^\lambda = 0$ when restricted to the hyperplane $u_i\omega^i = 0$. This is equivalent to saying that there exist a_λ, $b_i \in \mathbf{R}$ not all zero, such that

$$a_\lambda H^\lambda \equiv u_i\omega^i \otimes b_j\omega^j.$$

This reduces to an $N \times N$ determinant equal to zero, which gives us a homogeneous equation in u_i of degree n.

From now on we restrict ourselves to the case $M^3 \subset E^6$. The set $\mathcal{C}_p$ of all asymptotic hyperplanes at $p \in M$ is given by the planes

$$u_1\omega^1 + u_2\omega^2 + u_3\omega^3 = 0$$

such that

$$(*) \qquad \begin{vmatrix} u_1 & 0 & 0 & u_2 & u_3 & 0 \\ 0 & u_2 & 0 & u_1 & 0 & u_3 \\ 0 & 0 & u_3 & 0 & u_1 & u_2 \\ h^4_{11} & h^4_{22} & h^4_{33} & 2h^4_{12} & 2h^4_{13} & 2h^4_{23} \\ h^5_{11} & h^5_{22} & h^5_{33} & 2h^5_{12} & 2h^5_{13} & 2h^5_{23} \\ h^6_{11} & h^6_{22} & h^6_{33} & 2h^6_{12} & 2h^6_{13} & 2h^6_{23} \end{vmatrix} = 0.$$

Hence $\mathcal{C}_p$ is a homogeneous cubic surface in the cotangent space $T^*_p M$, which can be considered as a cubic curve in a projective plane. We remark that the above characterization can be similarly done for a nonorthonormal frame, keeping e_i and e_λ respectively tangent and normal to M. Moreover, we observe that $(*)$ does not depend on the choice of the normal frame e_λ. In fact, if we choose another normal frame $\bar{e}_\lambda$, keeping e_i fixed, the second fundamental forms $\overline{H}^\lambda = a_{\lambda\beta}H^\beta$, where the determinant $|a_{\lambda\beta}| \neq 0$, and we get $(*)$ multiplied by $|a_{\lambda\beta}|$.

Let M^3 and $\overline{M}^3$ be two C^∞ submanifolds in E^6, and ϕ: $M \to \overline{M}$ an isometry which is the restriction of a rigid motion in E^6. Then a necessary condition is that $\phi^*(\overline{\mathcal{C}}) = \mathcal{C}$. In our theorem we prove that locally this condition is also sufficient; when we restrict to the case in which $\mathcal{C}$ is a nonsingular cubic. More precisely, we have

Theorem 1. *Let M^3 and $\overline{M}^3$ be submanifolds in E^6, and ϕ: $M \to \overline{M}$ an isometry. Suppose that at $p_0 \in M$ there is a neighborhood V in M such that $\forall p \in V$, $\mathcal{C}_p$ and $\overline{\mathcal{C}}_{\phi(p)}$ are nonsingular cubics and $\phi^*(\overline{\mathcal{C}}_{\phi(p)}) = \mathcal{C}_p$. Then there is a Euclidean motion which restricted to V coincides with ϕ.*

The main difficulty in proving this theorem is to deal with $(*)$ which determines the cubic curve. Although there are several ways of obtaining standard equations for a nonsingular cubic, these are not of too much help in our case, where we want to simultaneously reduce the second fundamental forms to canonical types. This is done in the following.

Basic Lemma. *Let M^3 be a submanifold of E^6 such that at $p_0 \in M$, $\mathcal{C}_{p_0}$ is nonsingular. Then there is a frame at p_0 and e_A, $A = 1, \cdots, 6$, (not necessarily orthonormal), where e_i are tangent and e_λ are normal to M, and a real number $\sigma \neq 2, 3, -6$, such that with respect to this frame the second fundamental forms are*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & \sigma \end{bmatrix},$$

and $\mathcal{C}_{p_0}$ is given by

$$u_1^3 + u_2^3 + u_3^3 - u_1^2u_2 - u_1^2u_3 - u_2^2u_1 - u_2^2u_3 - u_3^2u_1 - u_3^2u_2 + \sigma u_1u_2u_3 = 0.$$

The proof of Theorem 1 is based on the choice of a moving frame obtained by the above lemma and the following.

Proposition 1. *Consider the nonsingular cubics*

$$\Gamma\colon x^3 + y^3 + z^3 - x^2y - x^2z - y^2x - y^2z - z^2x - z^2y + \sigma xyz = 0,$$

$$\bar{\Gamma}\colon \bar{x}^3 + \bar{y}^3 + \bar{z}^3 - \bar{x}^2\bar{y} - \bar{x}^2\bar{z} - \bar{y}^2\bar{x} - \bar{y}^2\bar{z} - \bar{z}^2\bar{x} - \bar{z}^2\bar{y} + \bar{\sigma}\bar{x}\bar{y}\bar{z} = 0.$$

If Γ and $\bar{\Gamma}$ are projectively equivalent, then the only real linear transformations, which will take Γ into $\bar{\Gamma}$, are permutations of x, y, z multiplied by a nonzero constant. Therefore $\sigma = \bar{\sigma}$.

In §2, assuming the Basic Lemma and the proposition above, we prove Theorem 1, using the theory of a nonorthonormal moving frame and the fundamental theorem for submanifolds of Euclidean space.

In §3, we consider a family $\mathcal{F} = \{\sum_{i=1}^3 \lambda_i\varphi_i, \lambda_i \in \mathbf{R}\}$ generated by three linearly independent quadratic forms φ_i in three variables with real coefficients, such that any pair of quadratic forms among φ_i may not be expressed in less than three linearly independent variables (this last condition is suggested by Proposition 3 in §4). Under nonsingular linear transformations on the variables and also such transformations of the λ_i, we obtain a complete classification of canonical generators for the family $\mathcal{F}$.

In §4, we first prove Proposition 3, which asserts that if the cubic $\mathcal{C}_p$ is nonsingular, then the second fundamental forms H^λ at p generate a family $\mathcal{F}$, as described in the preceding paragraph. The Basic Lemma is proved using the classification obtained in §3. Another result obtained in this section asserts that if the sectional curvatures at $p \in M$ are zero, then $\mathcal{C}_p$ is reducible to the product of three lines. This is a first step in the classification of $\mathcal{C}$ according to the sectional curvature. Finally we prove Proposition 1.

The author would like to acknowledge the helpful suggestions and comments of Professor S. S. Chern, during the elaboration of this paper.

2. Proof of main result

In this section we prove Theorem 1, assuming the Basic Lemma and Proposition 1 stated in the introduction. The proof is based on the choice of a special moving frame, obtained from the Basic Lemma. Since this frame is not necessarily orthonormal, we first recall the theory of a nonorthonormal moving frame, defined locally on a submanifold of a Euclidean space.

Let U be an open subset of E^N, and $e_1, \cdots, e_N$ differentiable vector fields on U. We consider the differential forms $\omega^1, \cdots, \omega^N$ such that $\omega^A(e_B) = \delta^A_B$, $A, B = 1, \cdots, N$, and the connection forms $\omega_A{}^B$ defined by

$$de_A = \omega_A{}^B e_B.$$

We get the structure equations in E^N

$$d\omega^A = \omega^B \wedge \omega_B{}^A,\ d\omega_A{}^B = \omega_A{}^C \wedge \omega_C{}^B, \quad A, B, C = 1, \cdots, N.$$

We remark that the matrix $\omega_A{}^B$ is not necessarily skew-symmetric, since we are not assuming $e_1, \cdots, e_N$ to be orthonormal.

Now we consider an n-dimensional C^∞ submanifold M^n in E^N, $N = n + p$. We choose $e_1, \cdots, e_n, e_{n+1}, \cdots, e_N$ locally defined in such a way that $e_1, \cdots, e_n$ are tangent to M and $e_{n+1}, \cdots, e_N$ are normal to M. With the following indices convention,

$$i, j, k = 1, \cdots, n;\ \mu, \lambda, \alpha, \beta = n + 1, \cdots, N;\ A, B, C = 1, \cdots, N,$$

when we restrict to M, we get

$$\omega^\lambda = 0,\ d\omega^i = \omega^j \wedge \omega_j{}^i,\ d\omega^\lambda = \omega^i \wedge \omega_i{}^\lambda = 0.$$

Hence there exist h^λ_{ij} such that

$$\omega_i{}^\lambda = h^\lambda_{ij}\omega^j,\ h^\lambda_{ij} = h^\lambda_{ji}. \tag{1}$$

The second structure equation restricted to M is decomposed as follows:

$$d\omega_i{}^j = \omega_i{}^k \wedge \omega_k{}^j + \Omega_i{}^j, \tag{2}$$

$$d\omega_i{}^\lambda = \omega_i{}^k \wedge \omega_k{}^\lambda + \omega_i{}^\alpha \wedge \omega_\alpha{}^\lambda, \tag{3}$$

$$d\omega_\lambda{}^i = \omega_\lambda{}^k \wedge \omega_k{}^i + \omega_\lambda{}^\alpha \wedge \omega_\alpha{}^i, \tag{4}$$

$$d\omega_\lambda{}^\mu = \omega_\lambda{}^k \wedge \omega_k{}^\mu + \omega_\lambda{}^\alpha \wedge \omega_\alpha{}^\mu,$$

where

$$\Omega_i{}^j = \omega_i{}^\lambda \wedge \omega_\lambda{}^j. \tag{5}$$

Next we will relate $\omega_A{}^B$ with $\omega_B{}^A$. We denote the inner product $g_{AB} = \langle e_A, e_B \rangle$ and remark that $g_{i\lambda} = g_{\lambda i} = 0$. Taking exterior derivative we get

$$\omega_A{}^C g_{CB} + g_{AC}\omega_B{}^C = dg_{AB}.$$

In particular we have

$$\omega_i{}^k g_{kj} + g_{ik}\omega_j{}^k = dg_{ij},$$

$$\omega_i{}^\mu g_{\mu\lambda} + g_{ij}\omega_\lambda{}^j = 0, \tag{6}$$

$$\omega_\mu{}^\alpha g_{\alpha\lambda} + g_{\mu\alpha}\omega_\lambda{}^\alpha = dg_{\mu\lambda}. \tag{7}$$

It follows from (6) and (1) that

$$\omega_\lambda{}^i = -h^\mu_{kl} g_{\mu\lambda} g^{li}\omega^k, \tag{8}$$

where g^{ij} is the inverse matrix of g_{ij}. Hence using (5), (1) and (8) we have

$$\Omega_i{}^j = h^\lambda_{ik} h^\mu_{rl} g^{lj} g_{\mu\lambda}\omega^r \wedge \omega^k. \tag{9}$$

On the other hand we know that

$$\Omega_i{}^j = \tfrac{1}{2} R_{lirk} g^{lj}\omega^r \wedge \omega^k, \tag{10}$$

where R_{lirk} is the Riemannian curvature tensor. Therefore from (9) and (10) we get that $\forall l, i, k \neq r$

$$R_{lirk} g^{lj} = \left(h^\mu_{lr} h^\lambda_{ik} - h^\mu_{lk} h^\lambda_{ir}\right) g_{\mu\lambda} g^{lj},$$

and hence $\forall j, i, k \neq r$

$$R_{jirk} = \left(h^\mu_{jr} h^\lambda_{ik} - h^\mu_{jk} h^\lambda_{ir}\right) g_{\mu\lambda}. \tag{11}$$

The second fundamental forms with respect to this frame will be denoted by $H^\lambda = h^\lambda_{ij}\omega^i\omega^j$.

After these preliminaries we restrict ourselves to 3-dimensional submanifolds in E^6 such that the inclusions are nondegenerate. Let M^3 and $\overline{M}^3$ be C^∞ submanifolds in E^6 and let $\phi\colon M \to \overline{M}$ be an isometry. We choose frames e_A and $\bar{e}_A$ locally defined such that e_i is tangent to M, $\bar{e}_i = \phi_*(e_i)$, e_λ and $\bar{e}_\lambda$ are respectively normal to M and $\overline{M}$. We prove the following.

Proposition 2. *Let M^3 and $\overline{M}^3$ be submanifolds in E^6 and let $\phi\colon M \to \overline{M}$ be an isometry. Suppose there is a neighborhood V of $p_0 \in M$ such that for each $p \in V$, the cubic $\mathcal{C}_p$ is nonsingular and $\phi^*\overline{H}^\lambda$ is a linear combination of H^μ. Then there is a Euclidean motion which coincides with ϕ in V.*

Proof. We want to define a normal bundle isomorphism $\tilde{\phi}\colon V^\perp \to \overline{V}^\perp$ covering $\phi\colon V \to \overline{V}$ such that $\tilde{\phi}$ preserves inner products, second fundamental forms and normal connections. Then it will follow from the fundamental theorem for submanifolds of Euclidean space, that there is a Euclidean motion which, restricted to V, coincides with ϕ.

For each $p \in V$, $\mathcal{C}_p$ is nonsingular, therefore we can apply the Basic Lemma. Using the fact that there are only a finite number of linear transformations which reduce the quadratic forms as in the lemma and the smoothness of H^λ, we get a differentiable frame e_A and a differentiable real function σ on V such that the quadratic forms H^λ, with respect to this frame, are respectively

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & \sigma \end{bmatrix}. \tag{12}$$

We remark also that σ does not assume the real values 2, 3, –6. We consider $\bar{e}_i = \phi_*(e_i)$, and since $\phi^*\overline{H}^\lambda$ is a linear combination of H^μ, we can obtain $\bar{e}_\lambda$ normal to $\overline{V} = \phi(V)$ such that the quadratic forms $\overline{H}^\lambda$ with respect to the frame $\bar{e}_A$ are also as in (12).

Now we define a bundle isomorphism $\tilde{\phi}$: $V^\perp \to \overline{V}^\perp$ covering ϕ by taking $\tilde{\phi}(e_\lambda) = \bar{e}_\lambda$ and extending linearly.

Let ω^A be the dual frame of e_A, $\omega_A{}^B$ the connection forms and similarly, $\overline{\omega}^A$, $\overline{\omega}_A{}^B$ for $\bar{e}_A$. Then identifying $\bar{e}_i$ with e_i and using the fact that ϕ is an isometry we have

$$\omega^\lambda = \overline{\omega}^\lambda = 0, \quad \omega^i = \overline{\omega}^i,$$
$$g_{ij} = \langle e_i, e_j \rangle = \langle \bar{e}_i, \bar{e}_j \rangle = \bar{g}_{ij},$$
$$\omega_i^{\ j} = \overline{\omega}_i^{\ j}, \quad \omega_i^{\ \lambda} = h_{ij}^\lambda \omega^j = \overline{h_{ij}^\lambda} \overline{\omega}^j = \overline{\omega}_i^{\ \lambda},$$

where $h_{ij}^\lambda = \overline{h_{ij}^\lambda}$ is now the ij-element of H^λ in (12).

We want to prove that $g_{\mu\lambda} = \bar{g}_{\mu\lambda}$. Since M and $\overline{M}$ are isometric, we have that

$$R_{jirk} = \overline{R}_{jirk}.$$

Therefore from (11) we get

$$\left(h_{jr}^\mu h_{ik}^\lambda - h_{jk}^\mu h_{ir}^\lambda\right)\left(g_{\mu\lambda} - \bar{g}_{\mu\lambda}\right) = 0, \quad \forall i, j, k \neq r.$$

Using the notation $Z_{\mu\lambda} = g_{\mu\lambda} - \bar{g}_{\mu\lambda}$ and substituting h_{ij}^λ for its value in (12), we obtain

$$\begin{aligned}
Z_{44} \quad + Z_{45} - \sigma Z_{46} \qquad\qquad + \qquad\quad Z_{66} &= 0,\\
Z_{55} + Z_{45} \qquad\qquad - \sigma Z_{56} + \qquad\quad Z_{66} &= 0,\\
Z_{45} \qquad\qquad\qquad\qquad - \qquad\quad Z_{66} &= 0,\\
Z_{46} + Z_{56} + (1 - \sigma)\, Z_{66} &= 0,\\
Z_{56} - \qquad\quad Z_{66} &= 0,\\
Z_{46} \qquad\quad - \qquad\quad Z_{66} &= 0.
\end{aligned}$$

The determinant of the coefficients is equal to $\sigma - 3 \neq 0$. Hence $Z_{\mu\lambda} = g_{\mu\lambda} - \bar{g}_{\mu\lambda} = 0$, $\forall \mu, \lambda$. As a consequence, we get from (8) that

$$\omega_\lambda{}^i = \bar{\omega}_\lambda{}^i \quad \forall i, \lambda.$$

It remains to prove that $\omega_\mu{}^\lambda = \bar{\omega}_\mu{}^\lambda$. In order to do so, we use (3) and (7). Since $\omega_i{}^\lambda = \bar{\omega}_i{}^\lambda$ and $\omega_i{}^j = \bar{\omega}_i{}^j$ it follows from (3) that

$$\omega_i{}^\mu \wedge (\omega_\mu{}^\lambda - \bar{\omega}_\mu{}^\lambda) = 0,$$

i.e.,

$$h_{ij}^\mu \omega^j \wedge (\omega_\mu{}^\lambda - \bar{\omega}_\mu{}^\lambda) = 0, \quad \forall i, \lambda.$$

Adopting the notation $D_\mu{}^\lambda = \omega_\mu{}^\lambda - \bar{\omega}_\mu{}^\lambda$ and applying the above equality to the pair of vectors (e_1, e_2), (e_1, e_3) (e_2, e_3) we have

$$h_{il}^\mu D_\mu{}^\lambda(e_2) - h_{i2}^\mu D_\mu{}^\lambda(e_1) = 0,$$
$$h_{il}^\mu D_\mu{}^\lambda(e_3) - h_{i3}^\mu D_\mu{}^\lambda(e_1) = 0,$$
$$h_{i2}^\mu D_\mu{}^\lambda(e_3) - h_{i3}^\mu D_\mu{}^\lambda(e_2) = 0, \quad \forall i, \lambda.$$

We substitute the values of h_{ij}^μ from (12), and we get

$$\begin{aligned} D_4{}^\lambda(e_2) &= D_4{}^\lambda(e_3) = D_5{}^\lambda(e_1) = D_5{}^\lambda(e_3) = D_6{}^\lambda(e_1) \\ &= D_6{}^\lambda(e_2) = D_6{}^\lambda(e_3) = a^\lambda, \\ D_4{}^\lambda(e_1) &= (\sigma - 2)a^\lambda, \quad D_5{}^\lambda(e_2) = (\sigma - 2)a^\lambda. \end{aligned}$$

Now we prove that $a^\lambda = 0$ $\forall\lambda$. In fact, since $g_{\mu\lambda} = \bar{g}_{\mu\lambda}$ it follows from (7) that

$$D_\mu{}^\alpha g_{\alpha\lambda} + g_{\mu\alpha} D_\lambda{}^\alpha = 0, \quad \forall \mu, \lambda.$$

In particular, for $\mu = \lambda$ we get

$$D_\mu{}^\alpha g_{\alpha\mu} = 0, \quad \forall \mu,$$

which applied to e_3 gives the system of equations

$$a^\alpha g_{\alpha\mu} = 0, \quad \mu = 4, 5, 6.$$

Since $\det(g_{\alpha\mu}) \neq 0$, we conclude that $a^\alpha = 0$, $\forall \alpha$ and hence $D_\mu{}^\lambda(e_i) = 0$, $\forall i, \mu, \lambda$. Therefore $\omega_\mu{}^\lambda = \bar{\omega}_\mu{}^\lambda$, and this concludes the proof of Proposition 2.

We can now prove our main result as a consequence of Propositions 1 and 2.

Proof of Theorem 1. For each $p \in V$, $\mathcal{C}_p$ is nonsingular, hence applying Basic Lemma we get a differentiable frame e_A, $A = 1, \cdots, 6$ and a differentiable real function σ on V such that with respect to this frame the quadratic forms H^λ are as in (12), and the cubic $\mathcal{C}$ is given by

$$u_1^3 + u_2^3 + u_3^3 - u_1^2u_2 - u_1^2u_3 - u_2^2u_1 - u_2^2u_3 - u_3^2u_1 - u_3^2u_2 + \sigma u_1u_2u_3 = 0.$$

Similarly for $\phi(V)$ we get a frame $\bar{e}_A, \bar{\sigma}$ such that with respect to this frame the quadratic forms $\overline{H}^\lambda$ are as in (12), and the cubic $\overline{\mathcal{C}}$ is given by

$$\bar{u}_1^3 + \bar{u}_2^3 + \bar{u}_3^3 - \bar{u}_1^2\bar{u}_2 - \bar{u}_1^2\bar{u}_3 - \bar{u}_2^2\bar{u}_1 - \bar{u}_2^2\bar{u}_3 - \bar{u}_3^2\bar{u}_1 - \bar{u}_3^2\bar{u}_2 + \bar{\sigma}\bar{u}_2\bar{u}_3 = 0.$$

Since $\phi^*(\overline{\mathcal{C}}) = \mathcal{C}$, it follows from Proposition 1 that the tangent frame $\bar{e}_i$ is a permutation of the frame $\phi_* e_i$ multiplied by real valued function $h \neq 0$. Moreover $\phi(\sigma) = \bar{\sigma}$. Now we consider a new frame on $\phi(V)$ defined as follows:

$$\bar{\bar{e}}_i = \phi_*(e_i), \bar{\bar{e}}_\lambda = \bar{e}_\lambda.$$

With respect to this frame the quadratic forms $\overline{\overline{H}}^\lambda$ will be linear combinations of $h^2\overline{H}^\mu$, and $\phi^*\overline{\overline{H}}^\lambda$ are linear combinations of H^μ since $\phi(\sigma) = \bar{\sigma}$. We conclude the proof using Proposition 2.

3. Reduction of a family of three quadratic forms in three variables

Consider a family

$$\mathcal{F} = \left\{ \sum_{i=1}^{3} \lambda_i \varphi_i, \lambda_i \in \mathbf{R} \right\}$$

where φ_i are linearly independent quadratic forms in three variables x_1, x_2, x_3 with real coefficients. Moreover, we suppose that any pair of quadratic forms among φ_i may not be expressed in less than three variables under a linear transformation on the variables. In this section, we want to find canonical generators for $\mathcal{F}$ by linear transformations with real coefficients on the variables and also by such transformations on the λ_i. Each linear transformation on the λ_i amounts to consider new generators for $\mathcal{F}$, which are linear combinations of the initial quadratic forms. Where we say a linear transformation we mean a nonsingular linear transformation with real coefficients.

3.1 We start by considering two quadratic forms φ_1 and φ_2 which may not be expressed in less than three variables. If we consider only linear transformations on the variables, we have a complete classification of types, to which φ_1 and φ_2 can be reduced (see [2]). According as the determinant $|\lambda\varphi_1 + \mu\varphi_2|$ is or is not identically zero, the case is said to be singular or nonsingular.

In the nonsingular case, if $|\varphi_1| \neq 0$ we have the following classification according to the different types of elementary divisors of $|\lambda\varphi_1 - \varphi_2|$:

	Type of elementary divisors	$\lvert\varphi_1\rvert \neq 0$, φ_1 and φ_2 are reduced to
I	$(\lambda - c_1), (\lambda - c_2), (\lambda - c_3)$ $c_1 = a + bi,\ c_2 = a - bi,$ $b \neq 0, a, b, c_3 \in \mathbf{R}$	$\varphi_1 = x_1^2 - x_2^2 + k_1 x_3^2$ $\varphi_2 = a(x_1^2 - x_2^2) - 2bx_1x_2 + k_1 c_3 x_3^2,\ k_1 = \pm 1$
II	$(\lambda - c_1), (\lambda - c_2), (\lambda - c_3)$ $c_1 \neq c_2 \neq c_3,\ c_i \in \mathbf{R}$	$\varphi_1 = k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2$ $\varphi_2 = k_1 c_1 x_1^2 + k_2 c_2 x_2^2 + k_3 c_3 x_3^2,\ k_i = \pm 1$
III	$(\lambda - c_1), (\lambda - c_1)\,(\lambda - c_3)$ $c_1 \neq c_3,\ c_i \in \mathbf{R}$	$\varphi_1 = k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2$ $\varphi_2 = c_1(k_1 x_1^2 + k_2 x_2^2) + k_3 c_3 x_3^2,\ k_i = \pm 1$
IV	$(\lambda - c_1), (\lambda - c_1), (\lambda - c_1)$ $c_1 \in \mathbf{R}$	$\varphi_1 = k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2$ $\varphi_2 = c_1(k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2),\ k_i = \pm 1$
V	$(\lambda - c_1)^2, (\lambda - c_3)$ $c_1 \neq c_3,\ c_i \in \mathbf{R}$	$\varphi_1 = k_1 2x_1x_2 + k_2 x_3^2$ $\varphi_2 = k_1(c_1 2x_1x_2 + x_2^2) + k_2 c_3 x_3^2,\ k_i = \pm 1$
VI	$(\lambda - c_1)^2, (\lambda - c_1)$ $c_1 \in \mathbf{R}$	$\varphi_1 = k_1 2x_1x_2 + k_2 x_3^2$ $\varphi_2 = k_1(c_1 2x_1x_2 + x_2^2) + k_2 c_1 x_3^2,\ k_i = \pm 1$
VII	$(\lambda - c_1)^3$ $c_1 \in \mathbf{R}$	$\varphi_1 = k_1(2x_1x_2 + x_3^2)$ $\varphi_2 = k_1\,[c_1(2x_1x_2 + x_3^2) + 2x_1x_3],\ k_1 = \pm 1$

TABLE 1

In the singular case, φ_1 and φ_2 are reduced to

VIII $\varphi_1 = 2x_1x_2,\ \varphi_2 = 2x_2x_3.$

For the reduction of a family of three quadratic forms, we need to remove the restriction $|\varphi_1| \neq 0$, i.e., in the nonsingular case if $|\varphi_1| = |\varphi_2| = 0$, then we can obtain a pair of forms $\bar{\varphi}_1$, $\bar{\varphi}_2$, to which the preceding classifications can be applied. Let g, h, g', h' be fixed real numbers for which $|g\varphi_1 + h\varphi_2| \neq 0$ and $gh' - g'h = 1$. Then

$$\bar{\varphi}_1 = g\varphi_1 + h\varphi_2, \quad \bar{\varphi}_2 = g'\varphi_1 + h'\varphi_2$$

can be reduced to one of the types in Table 1. After this reduction we obtain

$$\varphi_1 = h'\bar{\varphi}_1 - h\bar{\varphi}_2, \quad \varphi_2 = g\bar{\varphi}_2 - g'\bar{\varphi}_1.$$

In the above conditions, it follows from the choice of g, h, g', h' that $\bar{\varphi}_1$ and $\bar{\varphi}_2$ can only be reduced to types II, III and V.

We conclude that if $|\varphi_1| = |\varphi_2| = 0$, then under a linear transformation on the variables, φ_1 and φ_2 are reduced to one of the following types:

$$
\begin{array}{llll}
\text{(II}'\text{)} & \varphi_1 = k_1x_1^2 + k_3(h' - hc_3)x_3^2, & \varphi_2 = k_2x_2^2 + k_3(gc_3 - g')x_3^2, \\
\text{(III}'\text{)} & \varphi_1 = k_3x_3^2, & \varphi_2 = k_1x_1^2 + k_2x_2^2, \\
\text{(V}'\text{)} & \varphi_1 = k_1 2x_1x_2 - k_1hx_2^2, & \varphi_2 = k_1gx_2^2 + k_2x_3^2, \\
\text{(VIII)} & \varphi_1 = 2x_1x_2, & \varphi_2 = 2x_2x_3 \text{ (singular case)},
\end{array}
$$

where k_i is 1 or -1, and g, h, g', h' are fixed as above.

3.2. We now consider the reduction of a family

$$\mathcal{F} = \left\{ \sum_{i=1}^{3} \lambda_i\varphi_i, \lambda_i \in \mathbf{R} \right\}$$

as described at the beginning of this section. To each triple of values for λ_i, it corresponds an element of $\mathcal{F}$, which represents a conic to be called the conic of the point $P = (\lambda_1, \lambda_2, \lambda_3)$; its determinant Δ is homogeneous of order three in $\lambda_1, \lambda_2, \lambda_3$. Hence the points P whose conic is decomposable in two lines is a cubic $\Delta = 0$.

Let P_1 and P_2 be any two points of the projective plane of λ_i, and let ψ_1 and ψ_2 be the corresponding conics. To each point of the line l, determined by P_1, P_2, will correspond the conics of the family $\mathcal{F}'$ generated by ψ_1 and ψ_2. Now the points of intersection of the line l with the cubic Δ will correspond to the forms of $\mathcal{F}'$ whose determinant is zero. This condition is expressed by an homogeneous equation of order three in λ_1, λ_2.

After these preliminaries, we consider as the sides $\lambda_1, \lambda_2, \lambda_3$ for the triangle of reference and specially for λ_3, lines which have an invariant relation with the cubic Δ. For an irreducible cubic curve we have the following classification: since the cubic cannot have more than one double point, it is called a nonsingular cubic if it has no double point, otherwise it is called an acnodal, crunodal or cuspidal cubic according to the type of singularity of the double point (see [3]).

(a) If Δ is irreducible and is a nonsingular or an acnodal cubic, then there are exactly three collinear real inflexion points. We consider this line as λ_3 and for λ_1 and λ_2 we choose the tangents at two of these points of inflexion. Moreover we choose homogeneous coordinates such that the tangent to the third point of inflexion is $\lambda_1 + \lambda_2 + \rho\lambda_3 = 0$ where $\rho = 0$ if the three tangent lines are concurrent.

(b) If Δ is irreducible and is a crunodal or cuspidal cubic, then there is exactly one real point of inflexion. We choose λ_3 to be the line determined by the point of inflexion and the double point.

(b.1) If Δ is a crunodal cubic, we choose λ_1 to be the tangent line at the point of inflexion and λ_2 the polar harmonic of the point of inflexion.

(b.2) If Δ is a cuspidal cubic, we choose λ_1 to be the tangent line at the cusp and λ_2 the tangent at the point of inflexion.

(c) If Δ is reducible, then we choose λ_3 to be any linear factor of Δ.

Let us examine each case:

(a) By the choice of the sides for the triangle of reference, we get

$$\mathfrak{F} = \left\{ \sum_{i=1}^{3} \lambda_i \psi_i, \lambda_i \in \mathbf{R} \right\}, \quad \Delta = a\lambda_3^3 + \lambda_1\lambda_2(b\lambda_1 + b\lambda_2 + c\lambda_3).$$

The line λ_3 intersects the cubic Δ in three real distinct points, i.e., there exist three distinct pairs of real numbers λ_1, λ_2 such that the determinant

$$|\lambda_1\psi_1 + \lambda_2\psi_2| = 0.$$

Clearly $|\psi_1| = |\psi_2| = 0$ and by a linear transformation on the variables, ψ_1 and ψ_2 can only be reduced to type II′ in §3.1. Hence we have

$$\begin{aligned} \psi_1 &= k_1x_1^2 + k_3(h' - hc_3)x_3^2, \\ \psi_2 &= k_2x_2^2 + k_3(gc_3 - g')x_3^2. \end{aligned}$$

Since the coefficients of $\lambda_1^2\lambda_2$ and $\lambda_2^2\lambda_1$ in Δ are equal, we get

$$h' - hc_3 = gc_3 - g' \neq 0,$$

Hence we can reduce ψ_1 and ψ_2 to

$$\begin{aligned} \psi_1 &= k_1x_1^2 + k_3x_3^2, \\ \psi_2 &= k_2x_2^2 + k_3x_3^2, \quad k_i = 1 \text{ or } -1. \end{aligned}$$

Now let $\psi_3 = \sum_{i,j=1}^3 a_{ij}x_ix_j$, $a_{ij} = a_{ji}$, then

$$\begin{aligned} \Delta = &\begin{vmatrix} \lambda_1k_1 + \lambda_3a_{11} & \lambda_3a_{12} & \lambda_3a_{13} \\ \lambda_3a_{12} & \lambda_2k_2 + \lambda_3a_{22} & \lambda_3a_{23} \\ \lambda_3a_{13} & \lambda_3a_{23} & \lambda_1k_3 + \lambda_2k_3 + \lambda_3a_{33} \end{vmatrix} \\ &\equiv a\lambda_3^3 + \lambda_1\lambda_2(b\lambda_1 + b\lambda_2 + c\lambda_3). \end{aligned}$$

From the fact that the coefficients of $\lambda_1^2\lambda_3$, $\lambda_2^2\lambda_3$, $\lambda_1\lambda_3^2$ and $\lambda_2\lambda_3^2$ are zero, it follows that

$$a_{11} = a_{22} = 0, \quad k_1a_{23}^2 = k_2a_{13}^2 = -k_3a_{12}^2.$$

Since Δ is irreducible, we get

$$a_{12}^2 = a_{13}^2 = a_{23}^2 \neq 0, \quad k_1 = k_2 = -k_3, \tag{13}$$

i.e.,

$$\psi_3 = 2(a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3) + \sigma a_{33}x_3^2,$$

where a_{12}, a_{13}, a_{23} may differ by a sign. But we can reduce them to the same sign by multiplying x_1, x_2, x_3 and ψ_3 by 1 or -1 conveniently, without changing ψ_1 and ψ_2. Now let $d \neq 0$ be the common value of a_{12}, a_{13} and a_{23}. By changing ψ_3 by $\frac{1}{d}\psi_3$, we get

$$\psi_3 = 2(x_1x_2 + x_1x_3 + x_2x_3) + \sigma x_3^2,$$

where $\sigma = a_{33}/d$. Using (13) we conclude that when Δ is a nonsingular or acnodal cubic, the canonical generators are

$$\begin{aligned} &\psi_1 = x_1^2 - x_3^2, \quad \psi_2 = x_2^2 - x_3^2, \\ &\psi_3 = 2(x_1x_2 + x_1x_3 + x_2x_3) + \sigma x_3^2, \end{aligned} \tag{14}$$

where σ is an invariant. We remark that now we have

$$\Delta = (2 - \sigma)\lambda_3^3 + \lambda_1\lambda_2(-\lambda_1 - \lambda_2 + \sigma\lambda_3),$$

and for $\sigma = 2$, Δ is reducible (this case will be examined later), hence we will consider $\sigma \neq 2$.

Some special values for σ are the followng: $\sigma = 0$ corresponds to the case in which Δ is a nonsingular cubic with concurrent inflexional tangents; it Δ is an acnodal cubic, then $\sigma = 3$ or $\sigma = -6$.

A final remark has to be done on the above method of obtaining the generators (14). One would expect to find as many different values for the invariant σ, as there are couples of real points of inflexion on Δ. But it is not difficult to see that σ is unique.

(b.1) Δ is a crunodal cubic, λ_3 is the line determined by the real point of inflexion and the crunode, λ_1 is the tangent line at the point of inflexion and the crunode, λ_1 is the tangent line at the point of inflexion, and λ_2 is the polar harmonic of the point of inflexion. Then $\mathcal{F} = \{\Sigma_{i=1}^3 \lambda_i\psi_i, \lambda_i \in \mathbf{R}\}$ and

$$\Delta = b\lambda_3^3 + c\lambda_1(\lambda_2^2 - \lambda_3^2).$$

The intersection of λ_3 with Δ consists of the point of inflexion and two coincident points at the crunode, i.e., $|\psi_1| = |\psi_2| = 0$, and ψ_1, ψ_2 can only be reduced to III$'$ and V$'$ in §3.1.

(b.1.1) If ψ_1 and ψ_2 are reduced to III′ with a similar argument as in case (a), we get the generators

$$\psi_1 = x_3^2, \qquad \psi_2 = x_1^2 - x_2^2,$$
$$\psi_3 = a_{11}(x_1^2 + x_2^2) + 2(a_{12}x_1x_2 + a_{13}x_1x_3 + k_1a_{13}x_2x_3).$$

(b.1.2) If ψ_1 and ψ_2 are reduced to V′, we get

$$\psi_1 = k_1x_2^2 + k_2x_3^2, \qquad \psi_2 = 2x_1x_2 - x_2^2,$$
$$\psi_3 = a_{22}x_2^2 + 2(a_{13}x_1x_3 + a_{23}x_2x_3).$$

(b.2) Δ is a cuspidal cubic, λ_3 is the line determined by the real point of inflexion and the cusp, λ_2 is the tangent at the point of inflexion, and λ_1 is the tangent line at the cusp. Then

$$\mathcal{F} = \left\{ \sum_{i=1}^{3} \lambda_i\psi_i, \lambda_i \in \mathbf{R} \right\}, \qquad \Delta = b\lambda_3^3 + c\lambda_1\lambda_2^2.$$

The intersection of λ_3 with Δ consists of the point of inflexion and two coincident points at the cusp. Therefore ψ_1 and ψ_2 can only be reduced to III′ and V′ in §3.1.

(b.2.1) If ψ_1 and ψ_2 are reduced to III′, we get

$$\psi_1 = x_3^2, \qquad \psi_2 = x_1^2 - x_2^2,$$
$$\psi_3 = a_{11}(x_1^2 + x_2^2) + 2(k_1a_{11}x_1x_2 + a_{13}x_1x_3 + k_2a_{13}x_2x_3).$$

(b.2.2) If ψ_1 and ψ_2 are reduced to V′, we get $\Delta = c\lambda_1\lambda_2^2$, which is reducible and will be examined in the next case.

(c) Δ is reducible, and λ_3 is any fixed linear factor of Δ. Then

$$\mathcal{F} = \left\{ \sum_{i=1}^{3} \lambda_i\psi_i, \lambda_i \in \mathbf{R} \right\},$$
$$\Delta = \lambda_3(A\lambda_3^2 + B\lambda_1^2 + C\lambda_2^2 + D\lambda_1\lambda_3 + E\lambda_2\lambda_3 + F\lambda_1\lambda_2).$$

The intersection of λ_3 with Δ is identically zero $\forall\lambda_1, \lambda_2$, hence $|\lambda_1\psi_1 + \lambda_2\psi_2| \equiv 0$, i.e., ψ_1 and ψ_2 are in the singular case. Therefore

$$\psi_1 = 2x_1x_2, \psi_2 = 2x_2x_3, \psi_3 = \sum_{i,j=1}^{3} a_{ij}x_ix_j, a_{ij} = a_{ji}.$$

We summarize the above discussion in the following.

(I) $\psi_1 = x_1^2 - x_3^2$	$\psi_2 = x_2^2 - x_3^2$	$\psi_3 = 2(x_1x_2 + x_1x_3 + x_2x_3) + \sigma x_3^2,\ \sigma \neq 2$
(II) $\psi_1 = x_3^2$	$\psi_2 = x_1^2 - x_2^2$	$\psi_3 = a_{11}(x_1^2 + x_2^2) + 2(a_{12}x_1x_2 + a_{13}x_1x_3 + k_1a_{13}x_2x_3)$
(III) $\psi_1 = k_1x_2^2 + k_2x_3^2$	$\psi_2 = 2x_1x_2 - x_2^2$	$\psi_3 = a_{22}x_2^2 + 2(a_{13}x_1x_3 + a_{23}x_2x_3)$
(IV) $\psi_1 = x_3^2$	$\psi_2 = x_1^2 - x_2^2$	$\psi_3 = a_{11}(x_1^2 + x_2^2) + 2(k_1a_{11}x_1x_2 + a_{13}x_1x_3 + k_2a_{13}x_2x_3)$
(V) $\psi_1 = 2x_1x_2$	$\psi_2 = 2x_2x_3$	$\psi_3 = \sum_{i,j=1}^{3} a_{ij}x_ix_j,\ a_{ij} = a_{ji}$

where k_i is 1 or -1.

TABLE 2

We conclude observing that the method used in this section to reduce a family of three quadratic forms is essentially due to C. Jordan [5].

4. Proof of Basic Lemma, Proposition 1 and other results

Let M^3 be a submanifold of E^6 such that the inclusion is nondegenerate. In a neighborhood of a point $p_0 \in M$ we choose an orthonormal frame e_A, $A = 1, \cdots, 6$, such that e_i, $i = 1, 2, 3$, are tangent to M and e_λ, $\lambda = 4, 5, 6$, are normal to M. Let ω^A be the dual frame, and $H^\lambda = h_i^\lambda{}_j\omega^i\omega^j$ the second fundamental forms. The following result is needed for the proof of the Basic Lemma.

Proposition 3. *Suppose that at $p_0 \in M$ there are two quadratic forms among H^λ, which may be expressed in less than three linearly independent linear differential forms. Then the cubic $\mathcal{C}$ at p_0 is reducible.*

Proof. We may suppose without loss of generality that H^4 and H^5 may be expressed in less than three linear differential forms, i.e., after a linear transformation on ω^i we get $H^\lambda = h_i^\lambda{}_j\omega^i\omega^j$ where $h_{i3}^4 = h_{i3}^5 = 0$, $\forall i = 1, 2, 3$.

Then the cubic at p_0 is given by

$$\begin{vmatrix} u_1 & 0 & 0 & 0 & u_3 & u_2 \\ 0 & u_2 & 0 & u_3 & 0 & u_1 \\ 0 & 0 & u_3 & u_2 & u_1 & 0 \\ h^4_{11} & h^4_{22} & 0 & 0 & 0 & 2h^4_{12} \\ h^5_{11} & h^5_{22} & 0 & 0 & 0 & 2h^5_{12} \\ h^6_{11} & h^6_{22} & h^6_{33} & 2h^6_{23} & 2h^6_{13} & 2h^6_{12} \end{vmatrix} = 0,$$

which is factorable by u_3.

Remark. If all the second fundamental forms are expressable in less than three linearly independent linear differential forms, by a similar argument we can prove that $\mathcal{C}$ reduces to $u_3^3 = 0$.

The following result gives the projective type of $\mathcal{C}$ at a point where the curvature is zero.

Proposition 4. *Suppose that at $p_0 \in M$ all sectional curvatures are zero, then $\mathcal{C}$ at p_0 is reducible to the product of three lines.*

Proof. It is an immediate consequence of a theorem of Cartan [1] on exteriorly orthogonal quadratic forms. First we observe that from the above remark it follows that it is sufficient to consider the case where the fundamental forms H^λ may not be expressed in less than three linear differential forms. In this case, since all sectional curvatures at p_0 are zero, H^λ are exteriorly orthogonal and it follows from Cartan's theorem [1] that there exist a real orthogonal matrix b_i^{λ} and three linearly independent linear differential forms ω^i such that $H^\lambda = b_i^{\lambda}\omega^i\omega^i$. Therefore, by an appropriate choice of the normal vectors e^λ, the fundamental forms H^λ with respect to the dual basis to ω^i will be

$$H^4 = \omega^1\omega^1, \quad H^5 = \omega^2\omega^2, \quad H^6 = \omega^3\omega^3.$$

Hence the cubic at p_0 reduces to $u_1u_2u_3 = 0$.

Before proving the Basic Lemma, we remark that for n-dimensional submanifolds of E^N, $N = n(n+1)/2$, the generalization of Proposition 3 and the subsequent remark can be proved, using the same arguments when $n > 2$.

Proof of Basic Lemma. We start with an orthonormal frame as described at the beginning of this section. Since we are assuming that the cubic $\mathcal{C}$ at p_0 is nonsingular, it follows from Proposition 3 that any pair of quadratic forms among H^λ may not be expressed in less than three linearly independent linear differential forms. Therefore, if we consider the family

$$\mathcal{F} = \left\{ \sum_{\lambda=4}^{6} a_\lambda H^\lambda, a_\lambda \in \mathbf{R} \right\},$$

by real linear transformations on the ω^i and also by such transformations on the a_λ, the generators for $\mathcal{F}$ can be reduced to one of the types on Table 2 in §3.2. In what follows, we prove that if $\mathcal{C}$ is nonsingular, then the generators for $\mathcal{F}$ can only be reduced to type I. Let us consider each case.

(I) In this case the cubic $\mathcal{C}$ at p_0 is reduced to

$$\begin{vmatrix} u_1 & 0 & 0 & 0 & u_3 & u_2 \\ 0 & u_2 & 0 & u_3 & 0 & u_1 \\ 0 & 0 & u_3 & u_2 & u_1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & \sigma & 2 & 2 & 2 \end{vmatrix} = 0,$$

where $\sigma \neq 2$, i.e.,

$$u_1^3 + u_2^3 + u_3^3 - u_1^2u_2 - u_1^2u_3 - u_2^2u_1 - u_2^2u_3 - u_3^2u_1 - u_3^2u_2 + \sigma u_1u_2u_3 = 0.$$

Moreover, since $\mathcal{C}$ is nonsingular, $\sigma \neq 3$ and $\sigma \neq -6$. In fact, if $\sigma = 3$, $\mathcal{C}$ has a double point at (1, 1, 1), and if $\sigma = -6$ then $\mathcal{C}$ can be factored by $u_1 + u_2 + u_3$.

An easy computation shows that in case (II) $\mathcal{C}$ has a double at (0, 0, 1); in case (III) $\mathcal{C}$ can be factored by u_1; in (IV) $\mathcal{C}$ can be factored by $u_1 - k_1u_2$; in (V) $\mathcal{C}$ can be factored by u_2.

We conclude that if $\mathcal{C}$ at p_0 is nonsingular, then there is a frame e_A at p_0, where e_i are tangent to M, and e_λ are normal to M, and a real number $\sigma \neq 2$, 3, -6 such that the second fundamental forms at p_0 with respect to this frame are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & \sigma \end{bmatrix}.$$

Moreover $\mathcal{C}$ at p_0 is

$$u_1^3 + u_2^3 + u_3^3 - u_1^2u_2 - u_1^2u_3 - u_2^2u_1 - u_2^2u_3 - u_3^2u_1 - u_3^2u_2 + \sigma u_1u_2u_3 = 0.$$

This completes the proof of the Basic Lemma. q.e.d.

We now prove Proposition 1. First we state some properties of nonsingular cubics, which can be found in [3].

Nonsingular cubics can be classified into two main divisions: *one circuited cubics* having a single odd circuit, and *two-circuited cubics* having an odd and an even circuit. From a point P on the even circuit of a cubic, no real tangent can be drawn other than the tangent at P. From a point P on the odd circuit two real tangents can be drawn to the even circuit (if any). Moreover, two real tangents can be drawn from P to the odd curcuit. The cubic has nine points of inflexion, of which exactly three are real and collinear and lie on the odd circuit of the cubic.

We observe that from the above properties it follows that if L is a real linear transformation which takes a nonsingular cubic Γ into $\bar{\Gamma}$, then it takes the odd circuit of Γ into the odd circuit of $\bar{\Gamma}$, and the real points of inflexion of Γ into those of $\bar{\Gamma}$. Moreover, if P is a point on the odd circuit of Γ, let l_1 be the tangent at P, and l_2 the other real tangent from P to the odd circuit. Then $L(l_1)$ will be the tangent to $\bar{\Gamma}$ at $L(P)$, and $L(l_2)$ will be the other tangent from $L(P)$ to the odd circuit of $\bar{\Gamma}$. Now we can prove Proposition 1.

Proof of Proposition 1. The nonsingular cubics Γ and $\bar{\Gamma}$ are given respectively by

$$x^3 + y^3 + z^3 - x^2y - x^2z - y^2x - y^2z - z^2x - z^2y + \sigma xyz = 0,$$
$$\bar{x}^3 + \bar{y}^3 + \bar{z}^3 - \bar{x}^2\bar{y} - \bar{x}^2\bar{z} - \bar{y}^2\bar{x} - \bar{y}^2\bar{z} - \bar{z}^2\bar{x} - \bar{z}^2\bar{y} + \bar{\sigma}\bar{x}\bar{y}\bar{z} = 0.$$

It is not difficult to see that the real points of inflexion of Γ are $P_1 = (0, 1, -1)$, $P_2 = (1, 0, -1)$, $P_3 = (1, -1, 0)$. Moreover, the real tangents from P_i to the odd circuit of Γ are the tangent at P_i and $x = 0, y = 0, z = 0$ respectively for $i = 1, 2, 3$. Similarly for $\bar{\Gamma}$, the points of inflexion are $\bar{P}_1 = (0, 1 - 1)$, $\bar{P}_2 = (1, 0, -1)$, $\bar{P}_3 = (1, -1, 0)$, and the tangent from P_i to the odd circuit of $\bar{\Gamma}$ which is not tangent at P_i is $\bar{x} = 0, \bar{y} = 0, \bar{z} = 0$ respectively.

If Γ and $\bar{\Gamma}$ are projectively equivalent, it follows from the above considerations that any real linear transformation taking Γ into $\bar{\Gamma}$ is a permutation of x, y, z multiplied by a nonzero constant, and consequently $\sigma = \bar{\sigma}$.

References

[1] E. Cartan, *Sur les variétés de courbure constant d'un espace euclidien ou non-euclidien*, Bull. Soc. Math. France **47** (1919) 125–160, **48** (1920) 132–208.

[2] L. E. Dickson, *Equivalence of pairs of bilinear or quadratic forms under rational transformations*, Trans. Amer. Math. Soc. **10** (1909) 347–360.

[3] H. Hilton, *Plane algebraic curves*, Oxford University Press, Oxford, 1932.

[4] H. Jacobowitz, *Deformations leaving a hypersurface fixed*, Proc. Sympos. Pure Math., Amer. Math. Soc. **23** (1971) 343–351.

[5] C. Jordan, *Réduction d'un réseau de formes quadratique ou bilinéaires*, J. de Mathématiques (6) **2** (1906) 403–438, **3** (1907) 5–51.

UNIVERSITY OF CALIFORNIA, BERKELEY
UNIVERSIDADE DE BRASILIA, BRASILIA