

RIEMANNIAN SUBMERSIONS FROM COMPLEX PROJECTIVE SPACE

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To my cousins: Daryl, Gina, Robin, Mark and Daniel

The purpose of this paper is to classify Riemannian submersions from complex projective space onto a Riemannian manifold under the assumption that the fibers are connected, complex, totally geodesic submanifolds. In § 1 we review basic facts about Riemannian submersions needed in the rest of the paper. In § 2 we develop local results used in § 3. Included in § 2 is a decomposition of the second fundamental form for a fibered submanifold. In § 3 we handle the question of uniqueness of submersions from complex projective space, which satisfy the above hypothesis. Specifically, it is shown that any such submersion from complex projective space must fall into one of two classes. No assertion is made in this section about whether these classes are non-empty. In § 4 we discuss the problem of equivalence, and show that any two submersions from the same complex projective space in one of the determined classes differ by a fiber preserving isometry. § 5 gives the main result of the paper and concludes with some remarks and questions.

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1. Let M and B be Riemannian manifolds. By a *Riemannian submersion* we mean a C^∞ mapping $\pi: M \rightarrow B$ such that π is of maximal rank and π_* preserves the lengths of horizontal vectors, i.e., vectors orthogonal to the fiber $\pi^{-1}(x)$ for some $x \in B$. Throughout this paper, g will denote the Riemannian metric on M , and g^* the Riemannian metric on B . For a Riemannian submersion $\pi: M \rightarrow B$, the implicit function theorem tells us that $\pi^{-1}(x)$ is a closed submanifold of M for each $x \in B$. Given a Riemannian submersion π from M onto B , we denote by \mathcal{V} the vector subbundle of the tangent bundle TM of M consisting of the tangent spaces of the fibers of π . \mathcal{V} is called the vertical distribution of π . \mathcal{H} will denote the complementary "horizontal" distribution of \mathcal{V} in TM determined by the metric of M .

If $q \in M$, where M is any Riemannian manifold equipped with connection ∇ , T_qM denotes the tangent space to M at q . If M admits a Riemannian submersion $\pi: M \rightarrow B$, then π determines in a natural way two tensors T and A defined

on M as follows. For vector fields E and F of TM ,

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F,$$

where $\mathcal{V}E$, $\mathcal{H}E$, etc. denote the vertical and horizontal projections of the vector field E . O'Neill [16] has described the following three properties of the tensor T .

(1) T_E is a skew-symmetric operator on M reversing the horizontal and vertical subspaces.

(2) $T_E = T_{\mathcal{H}E}$.

(3) For vertical vector fields V and W , T is symmetric, i.e., $T_V W = T_W V$. In fact, along a fiber T is the second fundamental form provided we restrict ourselves to vertical vector fields.

O'Neill also defined the tensor A which we will call the *integrability tensor* associated with π . For arbitrary vector fields E and F ,

$$A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F.$$

The tensor A enjoys the following properties:

(1') At each point A_E is a skew-symmetric operator on M reversing the horizontal and vertical subspaces.

(2') $A_E = A_{\mathcal{H}E}$.

(3') For any horizontal vector fields X and Y , A is alternating, i.e., $A_X Y = -A_Y X$. Note $A_X Y$ is vertical.

Definition. A *basic vector field* on M is a horizontal vector field X which is π -related to a vector field X_* on B , i.e., $\pi_* X_u = X_{*\pi(u)}$ for all $u \in M$. For basic vector fields we recall the following facts.

Lemma 1.1. *If X and Y are basic vector fields on M which are π -related to X_* and Y_* on B , then each of the following holds:*

(a) $g(X, Y) = g(X_*, Y_*)$.

(b) $\mathcal{H}[X, Y]$ is basic, and is π -related to $[X_*, Y_*]$.

(c) $\mathcal{H}\nabla_X Y$ is basic, and is π -related to $\nabla_{X_*}^* Y_*$ where ∇^* is the Riemannian connection on B .

(d) Suppose $\{Z_{1*}, Z_{2*}, \dots, Z_{n*}\}$ forms a basis for B , and $\{Z_1, Z_2, \dots, Z_n\}$ are the corresponding π -related basic vector fields on M . If $g_p(Y, Z_i) = g_{p'}(Y, Z_i)$ for all $p, p' \in \pi^{-1}(b)$ where $b \in B$, then $\pi_* Y$ is a well defined basic vector field on B . In particular, Y is basic.

Proofs of these results are given in O'Neill [16] and in [3]. For a given Riemannian submersion, we have the following decomposition results which we will need in the sequel.

Lemma 1.2. *Let X and Y be horizontal vector fields, and V a vertical vector field. Then each of the following holds:*

(a) $\nabla_V X = \mathcal{H}\nabla_V X + T_V X$.

(b) If X is basic, then $\mathcal{H}\nabla_V X = A_X V$, and $\nabla_V X = A_X V + T_V X$.

(c) $\nabla_X V = A_X V + \mathcal{V}\nabla_X V$.

(d) $\nabla_x Y = \mathcal{H}\nabla_x Y + A_x Y$.

For a discussion of these properties the reader is referred to [16] and [4].

Denote by R the curvature tensor of M , and by $K(P_E F)$ the sectional curvature of the plane $P_E F$ spanned by linearly independent vectors E and F . In like manner let R^* and K_* denote, respectively, the curvature tensor and sectional curvature of B . Since there is no danger of confusion, we denote the pullback of R^* and K_* to M by π by R^* and K_* , respectively. We recall the following curvature identities which will be needed in the sequel.

Lemma 1.3. *For a Riemannian submersion $\pi: M \rightarrow B$ with totally geodesic fibers, let X, Y, Z and H be horizontal vectors and V a vertical vector. Then*

(a) $g(R^*_{XY}Z, H) = g(R_{XY}Z, H) + 2g(A_X Y, A_Z H) - g(A_Y Z, A_X H) - g(A_Z X, A_Y H)$.

If X, Y and V are of unit length and $g(E, E)$ is denoted by $\|E\|^2$, then the following identities hold:

(b) $K(P_X V) = \|A_X V\|^2$.

(c) $K(P_X Y) = K_*(P_X Y) - 3\|A_X Y\|^2$.

As before, these results are proven in [16].

We have the following structure theorem due to Nagano [15] and Hermann [9]. A nearer related result is found in Muto [14].

Theorem 1.4. *Let $\pi: M \rightarrow B$ be a Riemannian submersion, and assume M to be connected. If M is complete, so is B , and π is a locally trivial fiber space. If, in addition, the fibers are totally geodesic (i.e., $T \equiv 0$), then π is a fiber bundle with structure group the Lie group of isometries of the fiber.*

2. Submanifolds and their lifts

Let M be a connected Riemannian manifold of dimension $n + p$, and let π be a Riemannian submersion from M onto a Riemannian manifold B of dimension n . If P is a closed submanifold of B of dimension r , and Y_* is a vector normal to P in B , then $C^*_{Y_*}$ will denote the second fundamental form of P in B . K^* will denote the covariant derivative of the normal bundle of P in B . Suppose E is tangent to $\pi^{-1}(P)$, and Y is normal to $\pi^{-1}(P)$ in M . Let S_Y denote the second fundamental form of $\pi^{-1}(P)$ in the direction of the horizontal Y , and set $C_Y E = \mathcal{H}S_Y \mathcal{H}E$, where, as in § 1, \mathcal{H} denotes the projection onto the horizontal distribution. In this section K will denote the covariant derivative of the normal bundle of $\pi^{-1}(P)$ in M .

The first result gives a decomposition of the second fundamental form S_Y of $\pi^{-1}(P)$. The local results of this section were part of [2].

Proposition 2.1. (a) *For any horizontal X tangent to $\pi^{-1}(P)$,*

$$S_Y X = C_Y X + A_Y X .$$

*If X and Y are basic vector fields, then $C_Y X$ is π -related to $C^*_{Y_*} X_*$, where X and Y are π -related to X_* and Y_* .*

(b) For any vertical V tangent to $\pi^{-1}(P)$,

$$S_Y V = K_Y Y - \mathcal{H}\nabla_V Y - T_Y Y .$$

In fact, if Y is basic,

$$S_Y V = K_Y Y - A_Y V - T_Y Y .$$

Proof. On B we have

$$(1) \quad \nabla_{X_*}^* Y_* = -C_{Y_*} X_* + K_{X_*}^* Y_* ,$$

where X_* is tangent to P , and ∇^* is the covariant derivative of B . In a similar way on M ,

$$(2) \quad \nabla_X Y = -S_Y X + K_X Y .$$

Using (2), the O'Neill decomposition given in Lemma 1.2, and the alternating property of the tensor of A we get

$$(3) \quad S_Y X = K_X Y - \mathcal{H}\nabla_X Y + A_Y X .$$

Now $A_Y X$ is vertical and hence tangent to $\pi^{-1}(P)$. Since K is the covariant derivative of the normal bundle of $\pi^{-1}(P)$, $K_X Y$ is horizontal. It follows $K_X Y - \mathcal{H}\nabla_X Y$ is exactly $C_Y X$. This gives formula (a). The remaining part of (a) is a straightforward argument depending, in part, on Lemma 1.4 and is omitted.

To show (b), note

$$(4) \quad \nabla_V Y = -S_Y V + K_V Y , \quad \nabla_V Y = \mathcal{H}\nabla_V Y + T_V Y .$$

The second result is from Lemma 1.2. Thus

$$(5) \quad S_Y V = K_V Y - \mathcal{H}\nabla_V Y - T_V Y .$$

In particular, if Y is basic, then $\mathcal{H}\nabla_V Y = A_Y V$, and so $S_Y V = K_V Y - A_Y V - T_V Y$, as asserted.

Our next result concerns minimal submanifolds. A similar result was obtained by Lawson [12] when the fibers were totally geodesic.

Theorem 2.3. *Let $\pi: M \rightarrow B$ be a Riemannian submersion. If the fibers $\pi^{-1}(x)$ are minimal submanifolds of M , then an r -dimensional submanifold P of B is minimal in B if and only if $\pi^{-1}(P)$ is minimal in M .*

Proof. Let $T_q(\pi^{-1}(P))$ denote the tangent space to $\pi^{-1}(P)$ at q . If Y is a normal vector to $\pi^{-1}(P)$, then the second fundamental form of $\pi^{-1}(P)$ in the direction of Y may be viewed as a linear endomorphism $S_Y: T_q(\pi^{-1}(P)) \rightarrow T_q(\pi^{-1}(P))$. With respect to a suitable orthonormal basis $\{X_1, X_2, \dots, X_r, V_1, \dots, V_p\}$ where the X_i are horizontal vectors and V_i are vertical vectors, S_Y

may be interpreted as a real symmetric matrix

$$\left[\begin{array}{c|c} C_Y \circ & K \circ Y - \mathcal{H}V \circ Y \\ \hline A_Y \circ & -T \circ Y \end{array} \right],$$

where $-T \circ Y$ denotes the second fundamental form of the fiber, and $C_Y \circ$ corresponds to $\mathcal{H}S_Y\mathcal{H} \circ$. In fact, if $E = \mathcal{H}E + \mathcal{V}E = X + V$ is a tangent vector to $\pi^{-1}(P)$, then

$$(6) \quad S_Y E = \left[\begin{array}{c|c} C_Y \circ & K \circ Y - \mathcal{H}V \circ Y \\ \hline A_Y \circ & -T \circ Y \end{array} \right] \begin{bmatrix} X \\ V \end{bmatrix}.$$

Now

$$(7) \quad \text{Tr } S_Y \circ = \text{Tr } C_Y \circ - \text{Tr } T \circ Y,$$

where $\text{Tr } L$ denotes the trace of a matrix L . Since the fibers are minimal, $\text{Tr } (-T \circ Y) = 0$. By (7), $\text{Tr } S_Y \circ = 0$ if and only if $\text{Tr } C_Y \circ = 0$. But by Proposition (2.1), $\text{Tr } C_Y \circ = 0$ if and only if $\text{Tr } C^*_{Y_*} \circ = 0$ where $Y_* = \pi_* Y$. Thus $\pi^{-1}(P)$ is minimal in M if and only if P is minimal in B .

Our next result concerns submanifolds of constant mean curvature and their lifts.

Theorem 2.4. *Let $\pi: M \rightarrow B$ be a Riemannian submersion with minimal fibers. Then a closed hypersurface P of B has constant mean curvature in B if and only if $\pi^{-1}(P)$ has constant mean curvature in M .*

We omit the proof, since it is a straightforward application of a result given in [13] and has no direct bearing on the main theorem of this paper.

Under some special restrictions the lift $\pi^{-1}(P)$ of a totally geodesic submanifold P of B is totally geodesic in M . Sufficient conditions are given in the next result.

Theorem 2.5. *Let $\pi: M \rightarrow B$ be a Riemannian submersion with totally geodesic fibers. Assume P is a totally geodesic submanifold of B . Then $\pi^{-1}(P)$ is totally geodesic provided $A_Y X = 0$ whenever X is horizontal and tangent to $\pi^{-1}(P)$ and Y is normal to $\pi^{-1}(P)$.*

Proof. If Y is normal to $\pi^{-1}(P)$, then Y is horizontal. We will show $S_Y X = 0$ and $S_Y V = 0$, where X and V are horizontal and vertical tangent vectors of $\pi^{-1}(P)$. By Proposition 2.1, $S_Y X = C_Y X + A_Y X$. By assumption, $A_Y X = 0$. Since $C_Y X$ is horizontal and π -related to $C^*_{\pi_* X} \pi_* Y$, and since P is totally geodesic, $C_Y X = 0$. Thus $S_Y X = 0$.

Again, by Proposition 2.1, $S_Y V = K_Y Y - \nabla_Y Y - T_Y Y$. Now $T_Y Y = 0$, since the fibers are totally geodesic. Note that $K_Y Y - \nabla_Y Y$ has no vertical component. Let X be a horizontal vector tangent to $\pi^{-1}(P)$, and assume without loss of generality that Y is basic. Then

$$\begin{aligned} g(S_Y V, X) &= g(K_V Y - \mathcal{H}V_V Y, X) = g(-\mathcal{H}V_V Y, X) \\ &= g(-A_Y V, X) = g(V, A_Y X) . \end{aligned}$$

The last two equalities follow from Lemma 1.2 and the fact that A_Y is skew-symmetric. Since $A_Y X = 0$, we conclude $S_Y V = 0$.

Corollary 2.6. *Let $S^{2n+1} \xrightarrow{\pi} CP(n)$ be the standard submersion from a sphere of radius one, and $CP(m)$ a complex projective Kähler submanifold of $CP(n)$. Then $\pi^{-1}(CP(m))$ is totally geodesic in S^{2n+1} . In fact, $\pi^{-1}(CP(m)) = S^{2m+1}$.*

Proof. We refer to our description in [4, § 2] for the standard submersion from $S^{2n+1} \xrightarrow{\pi} CP(n)$ and the work of O'Neill [16].

(a) If X is basic, then $A_X JN$ is also basic where JN is the vector field whose integral curves are the fibers of the submersion. In fact, $A_X JN = JX$, where J is the usual almost complex structure on $CP(n)$.

(b) Since $CP(m)$ is a Kähler submanifold of $CP(n)$, $A_X JN = JX$ is tangent to $CP(m)$ when X is. In fact, let Y be orthogonal to $CP(m)$ and let $V = JN$. Then $g(A_Y X, V) = -g(A_X Y, V) = g(Y, A_X V) = g(Y, JX)$.

(c) Apply the previous theorem. Then the submanifold $\pi^{-1}(CP(m))$ is totally geodesic and complete, since $\pi^{-1}(CP(m))$ is compact in S^{2n+1} . Hence $\pi^{-1}(CP(m))$ is a sphere since the only complete connected, totally geodesic submanifolds of spheres are spheres. One should remark that $\pi^{-1}(CP(m))$ is connected, since $\pi^{-1}(CP(m))$ is a fiber bundle over $CP(m)$ with connected fiber S^1 .

3. The uniqueness question

Let $\rho: CP(r) \rightarrow B$ be a Riemannian submersion from complex projective r -space $CP(r)$ onto a Riemannian manifold B . We equip $CP(r)$ with the standard Fubini-Study metric, normalized so that $1 \leq K \leq 4$ where K denotes the sectional curvature of $CP(r)$. We assume that the fibers of ρ are connected complex totally geodesic subspaces of $CP(r)$. In addition, we make the following restriction on the (real) fiber dimension: for any $b \in B$, $2 \leq \dim \rho^{-1}(b) \leq 2r - 2$.

Following a suggestion of A. Duane Randall, we consider the composite submersion $S^{2r+1} \xrightarrow{\pi} CP(r) \xrightarrow{\rho} B$, where π is the natural Riemannian submersion defined by O'Neill [16] from the unit sphere S^{2r+1} . Then one sees easily that $\rho \circ \pi: S^{2r+1} \rightarrow B$ is a Riemannian submersion.

Now the fibers of $\rho \circ \pi$ are totally geodesic. To see this, note that if $b \in B$, then the fiber $\rho^{-1}(b)$ is a connected totally geodesic complex subspace of $CP(r)$, and hence is isometric to $CP(m)$ with the induced metric. By Corollary 2.6, if $CP(m)$ is totally geodesic in $CP(r)$, then $\pi^{-1}(CP(m))$ is totally geodesic in S^{2r+1} . Thus the fibers of $\rho \circ \pi$ are totally geodesic in S^{2r+1} . Moreover, since $\rho^{-1}(b)$ and the fibers of π are both connected, it follows that $(\rho \circ \pi)^{-1}(b) = \pi^{-1}\rho^{-1}(b)$ is connected.

Using the classification 1.1 of [3] (see [4] for its complete proof), we conclude

that the only possible Riemannian submersions from unit spheres S^m with $1 \leq \dim \text{fiber} \leq m - 1$ are the following:

- (i) $S^{2n+1} \xrightarrow{\eta} CP(n)$, where $CP(n)$ is complex projective n -space,
- (ii) $S^{4n+3} \xrightarrow{\eta} QP(n)$, where $QP(n)$ is quaternionic projective n -space,
- (iii) $S^{15} \xrightarrow{\eta} S^8(\frac{1}{2})$, where $S^8(\frac{1}{2})$ is the unit eight-sphere of radius $\frac{1}{2}$.

In (i) and (ii), $1 \leq K_* \leq 4$, where K_* is the curvature of the base space.

Now if $\rho \circ \pi: S^{2n+1} \rightarrow CP(n)$, then $r = n$, so $\rho \circ \pi$ becomes $S^{2n+1} \xrightarrow{\pi} CP(n) \xrightarrow{\rho} CP(n)$. Thus ρ is an isometry. This case is excluded by our assumption on the fibers of ρ . If $\rho \circ \pi: S^{4n+3} \rightarrow QP(n)$, then $4n + 3 = 2(2n + 1) + 1$, so $r = 2n + 1$ and $\rho \circ \pi$ becomes $S^{4n+3} \xrightarrow{\pi} CP(2n + 1) \xrightarrow{\rho} QP(n)$. Finally, if $\rho \circ \pi: S^{15} \rightarrow S^8(\frac{1}{2})$, then we have $15 = 2r + 1$, so $r = 7$ and $\rho \circ \pi$ becomes $S^{15} \xrightarrow{\rho} CP(7) \xrightarrow{\pi} S^8(\frac{1}{2})$. Summarizing, we have the following uniqueness result. No assertion is yet made about whether the classes are nonempty.

Proposition 3.1. *Let $\rho: CP(r) \rightarrow B$ be any Riemannian submersion with connected complete complex and totally geodesic fibers. Assume $2 \leq \dim \text{of fiber} \leq 2r - 2$. Then ρ must have one of the following forms:*

- (i) $\rho: CP(2n + 1) \rightarrow QP(n)$,
- (ii) $\rho: CP(7) \rightarrow S^8(\frac{1}{2})$.

In case (i), $1 \leq K_* \leq 4$ where K_* is the curvature of $QP(n)$ and in case (ii), K_* is the curvature of $S^8(\frac{1}{2})$.

4. The equivalence problem

We begin with a quick review of some elementary facts which will be needed in this section. Suppose that a real vector space W of dimension $4n$ has a positive definite inner product g and a complex structure I with respect to which g is hermitian. We will show the existence of complex structures J and K on W which satisfy the following three properties:

- (i) $IJ = -JI = K$.
- (ii) The metric g on W is hermitian with respect to I, J and K .
- (iii) Suppose S and S' are distinct linear isomorphisms of W belonging to $\{1, I, J, K\}$ where 1 denotes the identity mapping. Then $g(SZ, S'Z) = 0$ for $Z \in W$.

Consider an orthonormal basis of W given by $\{Z_1, IZ_1, Z_2, IZ_2, \dots, Z_{2n}, IZ_{2n}\}$. Define J on these basis elements as follows:

$$\begin{aligned} JZ_1 &= Z_2, & IZ_3 &= Z_4, \dots, JZ_{2n-1} = Z_{2n}, \\ JIZ_1 &= -IZ_2, & JIZ_3 &= -IZ_4, \dots, JIZ_{2n-1} = -IZ_{2n}, \\ JZ_2 &= -Z_1, & JZ_4 &= -Z_3, \dots, JZ_{2n} = -Z_{2n-1}, \\ JIZ_2 &= IZ_1, & JIZ_4 &= IZ_3, \dots, JIZ_{2n} = IZ_{2n-1}. \end{aligned}$$

Then we may extend J linearly to a complex structure of the $4n$ -dimensional

vector space W . Since J maps basis vectors to basis vectors, it is clear that the metric on W is hermitian with respect to J . Now set $K = IJ$. Then one checks easily that $K = IJ = -JI$ and that the metric on W is hermitian with respect to K . With respect to the structures I, J and K we may rewrite the orthonormal basis $\{Z_1, IZ_1, Z_2, IZ_2, \dots, Z_{2n-1}, IZ_{2n-1}, Z_{2n}, IZ_{2n}\}$ as $\{Z_1, IZ_1, JZ_1, KZ_1, \dots, Z_{2n-1}, IZ_{2n-1}, JZ_{2n-1}, KZ_{2n-1}\}$. We have the following lemma.

Lemma 4.1. *Let W be a $4n$ -dimensional real vector space with fixed complex structure I and metric g . If I is hermitian with respect to g , then there exist complex structures J and K on W so that properties (i), (ii), and (iii) hold.*

Proof. We need only to check that (iii) holds and this is checked easily on the basis elements.

Definition. Let W be a $4n$ -dimensional real vector space with a positive definite inner product g endowed with three complex structures I, J and K such that

- (1) $IJ = -JI = K$,
- (2) g is invariant with respect to each such complex structure,
- (3) $g(IX, JX) = g(IX, KX) = g(JX, KX) = 0$ for all $X \in W$.

Then $QX = IX \wedge JX \wedge KX$ is called a *quaternionic structure* or a *quaternionic hermitian structure* on W , where Q is linearized so that it is a tensor of type $(3, 3)$. Note that up to sign QX is independent of I, J and K .

Remark. This definition is the vector space analogue of Gray's *quaternionic hermitian* (see [7, 4.6]).

Definition. Let M be a Riemannian manifold of dimension $4n$. M is said to be *quaternionic hermitian* provided that for each $x \in M$, there exists a neighborhood U of x together with local almost complex structures I, J and K defined on U so that I, J and K give rise to a quaternionic structure on $T_z M$ for each $z \in U$. If $\nabla_E Q = 0$ for all x and for all $E \in T_x M$, then M is *quaternionic*. (Gray calls this a *quaternionic Kählerian manifold*.)

Let $QP(n)$ denote the quaternionic projective space normalized so that $1 \leq K^* \leq 4$, where K^* is the sectional curvature of $QP(n)$. Let R^* denote its curvature tensor and let g^* be its metric. We will need the following lemma later.

Lemma 4.2. *Suppose $x \in QP(n)$ and let $\bar{I}, \bar{J}, \bar{K}$ be complex structures on $T_x QP(n)$ which give rise to the quaternionic structure Q on $T_x QP(n)$. Assume \bar{X} and \bar{Y} are unit vectors in $T_x QP(n)$ with \bar{Y} in the complementary subspace of the space spanned by $\{\bar{X}, \bar{I}\bar{X}, \bar{J}\bar{X}, \bar{K}\bar{X}\}$. Then $g^*(R^*_{x, \bar{S}\bar{X}} \bar{Y}, \bar{S}\bar{Y}) = 2$, for any $\bar{S} \in \{\bar{I}, \bar{J}, \bar{K}\}$.*

Proof. Suppose $\bar{S} = \bar{J}$. Consider the Riemannian submersion $\pi: S^{4n+3} \rightarrow QP(n)$, where S^{4n+3} is the unit sphere and π is the natural submersion described in [4, pp. 266–267]. For these complex structures, we showed in Step C of [4, Lemma 2.5], that $g(R^*_{x, JX} Y, JY) = 2$, where g is the metric on S^{4n+3} . X and Y were unit horizontal vectors, and Y was in the complementary horizontal subspace of the space spanned by $\{X, IX, JX, KX\}$. Thus $g_p^*(R^*_{x, JX} Y, JY) = 2$ for some $p \in QP(n)$, since the R^* on S^{4n+3} is the pullback by π of R^* on $QP(n)$. Define a linear isometry L from $T_x QP(n)$ to $T_p QP(n)$ which maps \bar{I} to I, \bar{J} to

J , and \bar{K} to K . Then, up to sign, L preserves the quaternionic structure Q . Thus L preserves the sectional curvatures [7], and L induces an isometry f of $QP(n)$, so $f_{*p} = L$. Suppose \bar{X}, \bar{Y} satisfy the conditions in the lemma, and set $f_*\bar{X} = X, f_*\bar{Y} = Y$. Then $Y \in \{X, IX, JX, KX\}^\perp \subset T_pQP(n)$. By [4], $2 = g_p^*(R^*_{XJX}Y, JY) = g_x^*(R^*_{X,JX}\bar{Y}, J\bar{Y})$, since an isometry preserves both the curvature tensor and the metric. If $\bar{S} = \bar{I}$ or \bar{K} , a similar argument gives the desired result.

Before proceeding to the main result on equivalence, we recall this concept which was introduced in [3]. Let π_1 and π_2 be Riemannian submersions from some connected complete M onto B . Assume the fibers of π_1 and π_2 are connected and totally geodesic. π_1 and π_2 are said to be *equivalent* provided there exists an isometry f of M which induces an isometry \underline{f} of B , so that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ B & \xrightarrow{\underline{f}} & B \end{array}$$

In [3] we announced the following result which is crucial to Proposition 4.4.

Theorem 4.3. *Let π_1 and π_2 be Riemannian submersions from M onto B satisfying the above hypotheses. Suppose f is an isometry of M satisfying the following two conditions alone:*

- (i) $f_{*p}: \mathcal{H}_{1_p} \rightarrow \mathcal{H}_{2_{f(p)}}$ is an isometry from the horizontal distribution \mathcal{H}_{1_p} of π_1 at p onto the horizontal distribution $\mathcal{H}_{2_{f(p)}}$ of π_2 at $f(p)$.
- (ii) For $E, F \in T_p(M)$, the tangent space to M at p , $f_*(A_{1E}F) = A_{2_{f*E}}f_*F$, where A_i are the integrability tensors of π_i .

Then f induces an isometry \underline{f} of B so that π_1 and π_2 are equivalent.

The next proposition is the main result of this section. K will denote the curvature of $CP(2n + 1)$.

Proposition 4.4. *Let ρ_1 and ρ_2 be two Riemannian submersions from $CP(2n + 1) \rightarrow QP(n)$. Assume the fibers are complex totally geodesic subspaces, $n \geq 2$ and $1 \leq K \leq 4$. Then there exists an isometry f of $CP(2n + 1)$ which induces an isometry \underline{f} of $QP(n)$ so that the following diagram commutes:*

$$\begin{array}{ccc} CP(2n + 1) & \xrightarrow{f} & CP(2n + 1) \\ \rho_1 \downarrow & & \downarrow \rho_2 \\ QP(n) & \xrightarrow{\underline{f}} & QP(n) \end{array}$$

Proof. By Proposition 3.1, $1 \leq K_* \leq 4$, where K_* denotes the curvature of $QP(n)$. Since the ρ_i are fiber bundles by Theorem 1.4, a simple homotopy argument guarantees that the fibers are connected. The idea of the proof is to con-

construct a linear isometry L which corresponds to f_{*q} of the last theorem. The construction of such an L with the desired properties is given in steps A through E .

Let I be the natural complex structure on $CP(2n + 1)$; let θ and A be the integrability tensors of ρ_1 and ρ_2 respectively. Suppose q and q' are in $CP(2n + 1)$ with $\rho_1(q) = p$ and $\rho_2(q') = p'$. We denote the horizontal distribution of ρ_1 at q by \mathcal{H}_{1q} , and the horizontal distribution of ρ_2 at q' by $\mathcal{H}_{2q'}$. By Lemma 4.1, we may define three complex structures I, J and K on \mathcal{H}_{1q} so that $I = I_q$. In a similar way, we may define complex structure \bar{I}, \bar{J} and \bar{K} on $\mathcal{H}_{2q'}$, enjoying the same three properties with $\bar{I} = I_{q'}$. Choose an orthonormal family $\{X_1, X_2, X_3, \dots, X_n\}$ of \mathcal{H}_{1q} . For $j \neq i$, assume $X_j \in \{X_i, IX_i, JX_i, KX_i\}^\perp \subset \mathcal{H}_{1q}$, where $\{X_i, IX_i, JX_i, KX_i\}^\perp$ denotes the orthogonal complement in \mathcal{H}_{1q} of the subspace spanned by these vectors. Then $\{X_1, IX_1, JX_1, KX_1, X_2, IX_2, JX_2, KX_2, \dots, X_n, IX_n, JX_n, KX_n\}$ is an orthonormal basis of \mathcal{H}_{1q} . In a similar way, we may choose an orthonormal family of vectors $\{\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n\}$ of $\mathcal{H}_{2q'}$ so that $\{\bar{X}_1, \bar{I}\bar{X}_1, \bar{J}\bar{X}_1, \bar{K}\bar{X}_1, \dots, \bar{X}_n, \bar{I}\bar{X}_n, \bar{J}\bar{X}_n, \bar{K}\bar{X}_n\}$ is an orthonormal basis of $\mathcal{H}_{2q'}$. This notation is used in Step A through Step D .

Define a linear isometry from \mathcal{H}_{1q} onto $\mathcal{H}_{2q'}$ as follows:

$$\begin{aligned} L: X_i &\rightarrow \bar{X}_i, \\ L: IX_i &\rightarrow \bar{I}\bar{X}_i, \\ L: JX_i &\rightarrow \bar{J}\bar{X}_i, \\ L: KX_i &\rightarrow \bar{K}\bar{X}_i. \end{aligned}$$

One checks easily that $L \circ I = \bar{I} \circ L$, $L \circ J = \bar{J} \circ L$, and $L \circ K = \bar{K} \circ L$.

Step A.

- (a) $\|\theta_X IX\| = \|A_X \bar{I}\bar{X}\| = 0$,
- (b) $\|\theta_X JX\| = \|A_X \bar{J}\bar{X}\| = 1$,
- (c) $\|\theta_X KX\| = \|A_X \bar{K}\bar{X}\| = 1$,

for any unit vector $X \in \mathcal{H}_{1q}$ and $\bar{X} = L(X) \in \mathcal{H}_{2q'}$.

To see (a) note that by Lemma 1.3 the following formula holds:

$$(8) \quad K_*(P_X Y) = K(P_X Y) + 3\|\theta_X Y\|^2,$$

where K_* denotes the sectional curvature of $QP(n)$ lifted to q . By assumption, $1 \leq K \leq 4$. Using [10, formula 7.2, p. 167] (normalizing the metric and recalling $I = I_q$ is induced from the natural complex structure on $CP(2n + 1)$), we have $K(P_X IX) = 4$. On the other hand, from Proposition 3.1, we have $1 \leq K_* \leq 4$. By a formula in Kraines [11], (suitably normalized), we have $K_*(P_X IX) = 4$. Setting $Y = IX$ in (8), we conclude $\|\theta_X IX\| = 0$. Exactly the same argument shows $\|A_X \bar{I}\bar{X}\| = 0$.

To prove (b), we again use [10, formula 7.2, p. 167] (Note their J is our I and we multiply their formula by 4.) On the other hand, by the formula of Kraines [11], $K_*(P_X JX) = 4$. Thus letting $Y = JX$ in (8) we get $\|\theta_X JX\| = 1$. The same argument shows $\|A_X \bar{J}\bar{X}\| = 1$. The proof of (c) is similar to that of (b).

Step B. Let $Y \in \{X, IX, JX, KX\}^\perp \subset \mathcal{H}_{1_q}$ where, as before, \perp denotes the orthogonal complement of the indicated subspace of \mathcal{H}_{1_q} . Suppose $\bar{Y} \in \{\bar{X}, \bar{I}\bar{X}, \bar{J}\bar{X}, \bar{K}\bar{X}\}^\perp \subset \mathcal{H}_{2_{q'}}$. Assume both Y and \bar{Y} have unit length. We claim

- (a) $\theta_X Y = 0$,
- (b) $A_X X = 0$.

By suitably normalizing [10, formula 7.2, p. 167], it follows that $K(P_X Y) = 1$. On the other hand, by the formula in Kraines [11] (suitably normalized), $K_*(P_X Y) = 1$. Direct substitution into formula (8) of Step A shows $\theta_X Y = 0$. In a similar way, $A_X \bar{Y} = 0$.

Step C.

- (a) $\theta_X S X = \theta_Y S Y$ for all X, Y in \mathcal{H}_{1_q} of unit length and $S \in \{I, J, K\}$.
- (b) $A_X S \bar{X} = A_X S \bar{Y}$ for all \bar{X}, \bar{Y} in $\mathcal{H}_{2_{q'}}$ of unit length and $S \in \{\bar{I}, \bar{J}, \bar{K}\}$.

To prove (a), assume $Y \in \{X, IX, JX, KX\}^\perp \subset \mathcal{H}_{1_q}$. From step A, $\theta_X I X = \theta_Y I Y = 0$. We will show $\theta_X J X = \theta_Y J Y$. Using Lemma 1.3 we have

$$(9) \quad \begin{aligned} g(R^*_{XJX} Y, JY) &= g(R_{XJX} Y, JY) + 2g(\theta_X J X, \theta_Y J Y) \\ &\quad - g(\theta_{JX} Y, \theta_X J Y) - g(\theta_Y X, \theta_{JX} J Y), \end{aligned}$$

where R^* is the curvature tensor of $QP(n)$ lifted to q while R is the curvature tensor of $CP(2n + 1)$. Note that $\theta_Y X = 0$ and $\theta_{JX} Y = -\theta_Y J Y = 0$ follow from Step B and our assumptions on X and Y . Recall that I , and not J , is the complex structure on \mathcal{H}_{1_q} induced by the coordinates of $CP(2n + 1)$. Keeping this fact in mind, we use a formula of [10, p. 166] to conclude $g(R_{XJX} Y, JY) = 0$.

Caution. The curvature tensor $R_{XY} Z$ used in this paper differs by a sign from that used in [10].

On the other hand, by Lemma 4.2, $g(R^*_{XJX} Y, JY) = 2$. Again by Lemma 4.2, this equality is independent of the structure J which together with I and K gives the quaternionic structure on \mathcal{H}_{1_q} . Now for our J , $\|\theta_X J X\| = \|\theta_Y J Y\| = 1$. This follows from Step A. Substituting in (9), we have $2 = 2g(\theta_X J X, \theta_Y J Y)$, so $\theta_X J X = \theta_Y J Y$.

If Z is a unit vector in the space spanned by $\{X, IX, JX, KX\}$, then using the above arguments we see $\theta_Z J Z = \theta_Y J Y = \theta_X J X$. Thus, for any unit X and Y in \mathcal{H}_{1_q} , $\theta_X J X = \theta_Y J Y$. It should be noticed that this is the point where we use the assumption $n \geq 2$. In a similar way, we see $\theta_X K X = \theta_Y J Y$ for any unit vectors X and Y of \mathcal{H}_{1_q} . Part (b) follows in the same way.

Step D.

(a) For any horizontal unit X in \mathcal{H}_{1_q} , $\theta_X J X$ is orthogonal to $\theta_X K X$. Moreover, $I\theta_X J X = \theta_X K X$, and $I\theta_X K X = -\theta_X J X$.

(b) For any horizontal unit \bar{X} in $\mathcal{H}_{2_{q'}}$, $A_X \bar{J} \bar{X}$ is orthogonal to $A_X \bar{K} \bar{X}$. Also, $\bar{I} A_X \bar{J} \bar{X} = A_X \bar{K} \bar{X}$, and $\bar{I} A_X \bar{K} \bar{X} = -A_X \bar{J} \bar{X}$.

To obtain (a), set $V_1 = \theta_X J X$ and $V_2 = \theta_X K X$. V_1 and V_2 are vertical vectors in the tangent space to the fiber π_1 at q_1 , since the integrability tensor θ_X reverses horizontal and vertical subspaces as was pointed out in § 1. If g is the met-

ric on $CP(2n + 1)$, then $g(V_1, V_2) = g(\theta_X JX, V_2) = -g(JX, \theta_X V_2)$. For the last equality we use the fact that θ_X is a skew symmetric operation as described in § 1. By Step A above, $1 = g(\theta_Y JX, \theta_X JX)$. It follows that $1 = g(\theta_X JX, V_1) = -g(JX, \theta_X V_1)$. But $g(JX, JX) = 1$. Hence $\theta_X V_1 = -JX$. Similarly, $\theta_X V_2 = -KX$. Thus $g(\theta_X JX, \theta_X KX) = g(\theta_X JX, V_2) = -g(JX, \theta_X V_2) = -g(JX, -KX) = 0$, which means $\theta_X JX$ is orthogonal to $\theta_X KX$.

To see that $I\theta_X JX = \theta_X KX$, note $I \circ \nabla = \nabla \circ I$ where ∇ is the connection on $CP(2n + 1)$. Since \mathcal{H}_{1q} is $I = I_q$ invariant, it follows the vertical distribution of ρ_1 at q is I invariant. By definition, $\theta_X JX = \mathcal{V}\nabla_X JX$ where \mathcal{V} denotes the projection onto the vertical distribution. Hence $I\theta_X JX = \theta_X IJX = \theta_X KX$. The other relations follow in a similar way. To show (b), reproduce the arguments above.

Step E. We define a linear isometry $L: T_q CP(2n + 1) \rightarrow T_{q'} CP(2n + 1)$ as follows. Choose a family $\{X_1, X_2, \dots, X_n\}$ a family of orthonormal vectors of \mathcal{H}_{1q} , so $X_j \in \{X_i, IX_i, JX_i, KX_i\}^\perp \subset \mathcal{H}_{1q}$ for $j \neq i$. In a similar way, select a family of orthonormal vectors $\{\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n\}$ of $\mathcal{H}_{2q'}$, so $X_j \in \{X_i, IX_i, JX_i, KX_i\}^\perp \subset \mathcal{H}_{2q'}$ for $j \neq i$. Set

$$\begin{aligned} L: X_i &\rightarrow \bar{X}_i, \\ L: IX_i &\rightarrow \bar{I}\bar{X}_i, \\ L: JX_i &\rightarrow \bar{J}\bar{X}_i, \\ L: KX_i &\rightarrow \bar{K}\bar{X}_i, \\ L: \theta_{X_1} IX_1 &\rightarrow A_{X_1} \bar{I}\bar{X}_1, \\ L: \theta_{X_1} JX_1 &\rightarrow A_{X_1} \bar{J}\bar{X}_1, \\ L: \theta_{X_1} KX_1 &\rightarrow A_{X_1} \bar{K}\bar{X}_1. \end{aligned}$$

This defines L on a basis of $T_q CP(2n + 1)$. That L is an isometry is obvious from the above steps.

We claim $L\theta_E F = A_{L(E)} L(F)$ for any $E, F \in T_q CP(2n + 1)$. Let us check this on the horizontal distribution. If $i \neq j$, then $L\theta_{X_i} SX_j = A_{LX_i} LSX_j$ for any $S \in \{1, I, J, K\}$. This assertion is a consequence of Step B. If $i = j$, then $L\theta_{X_j} SX_j = L\theta_{X_1} SX_1 = A_{X_1} \bar{S}\bar{X}_1 = A_{X_j} \bar{S}\bar{X}_j = A_{LX_j} LSX_j$, where $S \in \{1, I, J, K\}$ and $\bar{S} \in \{1, \bar{I}, \bar{J}, \bar{K}\}$. The first quality follows from Step C and the fact that $\theta_{X_j} X_j = 0 = A_{X_j} \bar{X}_j$. Thus $L\theta_{X_j} SX_j = A_{LX_j} LSX_j$.

To show $L\theta_{SX_i} S'X_j = A_{LSX_i} LS'X_j$ for all i, j and $S, S' \in \{1, I, J, K\}$, one renumbers the basis elements and proceeds as above. We omit the details. Thus $L\theta_X Y = A_{LX} LY$ for all $X, Y \in \mathcal{H}_{1q}$. An argument given in the proof of [4, Lemma 2.4] shows $L\theta_E F = A_{LE} LF$ for all $E, F \in T_q CP(2n + 1)$.

Note that $L \circ I = \bar{I} \circ L$, which follows from the definition of L on \mathcal{H}_{1q} , the relations of the complex structures, and the work of Step D.

Step F. Since I and \bar{I} are L -related, and both arise from the complex structure on $CP(2n + 1)$, we see that L preserves the holomorphic sectional curvature. Hence L preserves the sectional curvature, that is, $K_q(P_E F) = K_{q'}(P_{LE} LE)$

for $E, F \in T_q CP(2n + 1)$. Since $CP(2n + 1)$ is simply connected, there exists an isometry f of $CP(2n + 1)$ so that $f_{*q} = L$. Hence, f_{*q} satisfies properties (i) and (ii) of Theorem 4.3. It follows that f induces an isometry \underline{f} of $QP(n)$ so that

$$\begin{array}{ccc} CP(2n + 1) & \xrightarrow{f} & CP(2n + 1) \\ \downarrow \rho_1 & & \downarrow \rho_2 \\ QP(n) & \xrightarrow{\underline{f}} & QP(n) \end{array}$$

commutes. This completes the proof of Proposition 4.4.

It might be observed that since q and q' were arbitrary points of $CP(2n + 1)$, any submersion satisfying the stated hypotheses of Proposition 4.4 is *homogeneous* in the sense of [3] and [4].

5. The existence problem

In § 3 we showed that the only possible Riemannian submersions from $CP(r)$ onto B with complex connected totally geodesic fibers, $2 \leq \dim \text{fiber} \leq 2r - 2$, fell into the following two classes:

- (i) $\rho: CP(2n + 1) \rightarrow QP(n)$,
- (ii) $\rho: CP(7) \rightarrow S^8(\frac{1}{2})$.

We show that submersions exist in class (i). To see this consider the unit sphere $S^{4n+3} \subset R^{4n+4}$. Let I, J, K be the almost complex structures R^{4n+4} , and let N denote the outward unit normal to S^{4n+3} . Then IN, JN, KN generate a foliation of S^{4n+3} . Identifying the leaves we obtain $\eta: S^{4n+3} \rightarrow QP(n)$, where η is a Riemannian submersion with totally geodesic fibers, as in [7]. Consider the action of the one-parameter group generated by IN . This group is a copy of S^1 and gives rise to a Riemannian submersion $\pi: S^{4n+3} \rightarrow CP(2n + 1)$ with connected totally geodesic fibers as in [16]. Now $\eta: S^{4n+3} \rightarrow QP(n)$ is a principal S^3 bundle, and S^1 is a closed subgroup of S^3 . Since the action described above is the restriction of that of S^3 , it follows there exists a mapping $\rho: CP(2n + 1) \rightarrow QP(n)$ so that the following diagram commutes:

$$\begin{array}{ccc} S^{4n+3} & & \\ \eta \downarrow & \searrow \pi & \\ QP(n) & & CP(2n + 1) \\ & \swarrow \rho & \end{array}$$

Now since η and π are Riemannian submersions, ρ is a Riemannian submersion by [1]. On the other hand, $\eta^{-1}(b) = \pi^{-1}\rho^{-1}(b)$ is totally geodesic for $b \in QP(n)$. A local result, Corollary 2.6, requires that $\rho^{-1}(b)$ must be totally geodesic in $CP(2n + 1)$. It remains to show that the fibers of ρ are complex submanifolds of $CP(2n + 1)$. First, note $\eta^{-1}(b)$ is isometric to $S^3 \subset S^{4n+3}$. Now $\{IN, JN, KN\}$ are tangent to S^3 . On the other hand, $\eta^{-1}(b) = S^3 = \pi^{-1}\rho^{-1}(b)$. Consider

the vector space spanned by $\{JN, KN\}$. This space is π horizontal and is I invariant. Now we may choose π basic vector fields X and Y which span $\{JN, KN\}$. In fact, by O'Neill's work, [16], we may choose $Y = IX$. Moreover, π commutes with I (see [16]), and induces the natural almost complex structure on $CP(2n + 1)$. It follows that π_*X and $\pi_*IX = I\pi_*X$ span the tangent space to $\rho^{-1}(b)$. Thus $\rho^{-1}(b)$ is a totally geodesic I invariant submanifold of $CP(2n + 1)$, so $\rho^{-1}(b) = CP(1) = S^2$, since it is obvious that $\rho^{-1}(b)$ is connected. We have the following result.

Proposition 3.1. *There exists a Riemannian submersion $\rho: CP(2n + 1) \rightarrow QP(n)$ with connected complex totally geodesic fibers.*

By the assumption on the fibers in Proposition 1.1 any submersion of type (i) must have fiber isometric with $CP(1)$, and any submersion of type (ii) must have fiber isometric with $CP(3)$. Summarizing the results of this paper we have the following main result. We assume $1 \leq K \leq 4$, where K is the curvature of $CP(r)$.

Theorem 3.2. *Any submersion $\rho: CP(r) \rightarrow B$ with connected complex totally geodesic fibers and with $2 \leq \dim \text{fiber} \leq 2r - 2$ must fall into one of the following two classes:*

$$\begin{array}{ccc}
 \text{(i)} & S^2 \longrightarrow CP(2n + 1) & \text{(ii)} & CP(3) \longrightarrow CP(7) \\
 & \downarrow \rho & & \downarrow \rho \\
 & QP(n) & & S^8(\frac{1}{2})
 \end{array}$$

In fact, $1 \leq K_* \leq 4$, where K_* denotes the curvature of $QP(n)$, and $S^8(\frac{1}{2})$ denotes the sphere of radius $\frac{1}{2}$. Moreover, class (i) is not empty. Finally, if $n \geq 2$, any two submersions in class (i) are equivalent.

Remarks. (1) The author does not know whether or not class (ii) is empty. The existence of such a ρ in class (ii) would be of interest. In particular, it would imply that the estimate ν_{14} of Ferus [5] is a best possible result. There is a related question: Does there exist a fiber bundle $S^4 \rightarrow QP(3)$?

$$\begin{array}{c}
 \downarrow \\
 S^8
 \end{array}$$

Remarks at the end of [6] are of interest, although they deal with fiberings of $CP(2r)$ and $QP(2r)$.

(2) Can one drop the assumption $n \geq 2$ in Proposition 4.4? In a similar vein, can one drop the assumption $n \geq 2$ in [4, Theorem 3.5]?

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