J. DIFFERENTIAL GEOMETRY
12 (1977) 629-633

A NOTE ON A THEOREM OF NIRENBERG

TAKAAKI NISHIDA

The abstract forms of the nonlinear Cauchy-Kowalewski theorem are investigated in [1] and [2] in a little different formulations. We note here that the Nirenberg's formulation and proof in [1] can be simplified to give an improved abstract nonlinear Cauchy-Kowalewski theorem in a scale of Banach spaces, which contains both theorems in [1] and [2]. The proof follows that of Nirenberg exactly except one point.

Definition. Let $S = \{B_\rho\}_{\rho>0}$ be a scale of Banach spaces, and let all B_ρ for $\rho > 0$ be linear subspaces of B_0. It is assumed that $B_\rho \subset B_{\rho'}$, $\|\cdot\|_{\rho'} \leq \|\cdot\|_\rho$ for any $\rho' \leq \rho$, where $\|\cdot\|_\rho$ denotes the norm in B_ρ.

Consider in S the initial value problem of the form

$$\frac{du}{dt} = F(u(t), t) , \qquad |t| < \delta , \tag{1}$$

$$u(0) = 0 . \tag{2}$$

Assume the following conditions on F:

(i) For some numbers $R > 0$, $\eta > 0$, $\rho_0 > 0$ and every pair of numbers ρ, ρ' such that $0 \leq \rho' < \rho < \rho_0$, $(u, t) \to F(u, t)$ is a continuous mapping of

$$\{u \in B_\rho ; \|u\|_\rho < R\} \times \{t ; |t| < \eta\} \text{ into } B_{\rho'} . \tag{3}$$

(ii) For any $\rho' < \rho < \rho_0$ and all $u, v \in B_\rho$ with $\|u\|_\rho < R$, $\|v\|_\rho < R$, and for any t, $|t| < \eta$, F satisfies

$$\|F(u, t) - F(v, t)\|_{\rho'} \leq C \|u - v\|_\rho/(\rho - \rho') , \tag{4}$$

where C is a constant independent of t, u, v, ρ or ρ'.

(iii) $F(0, t)$ is a continuous function of t, $|t| < \eta$ with values in B_ρ for every $\rho < \rho_0$ and satisfies, with a fixed constant K,

$$\|F(0, t)\|_\rho \leq K/(\rho_0 - \rho) , \qquad 0 \leq \rho < \rho_0 . \tag{5}$$

Theorem. *Under the preceding hypotheses there is a positive constant a such that there exists a unique function $u(t)$ which, for every positive $\rho < \rho_0$ and $|t| < a(\rho_0 - \rho)$, is a continuously differentiable function of t with values in B_ρ, $\|u(t)\|_\rho < R$, and satisfies* (1), (2).

Communicated by L. Nirenberg, February 12, 1976.

Remark. The assumption (ii) on F is simpler than that in [1] or [2].

Proof. Let B be the Banach space of functions $u(t)$ which, for every non-negative $\rho < \rho_0$ and $|t| < a(\rho_0 - \rho)$, are continuous functions of t with values in B_ρ, and have the norm

$$(6) \qquad M[u] = \sup_{\substack{0 \leq \rho < \rho_0 \\ |t| < a(\rho_0 - \rho)}} \|u(t)\|_\rho \left(\frac{a(\rho_0 - \rho)}{|t|} - 1 \right) < +\infty .$$

We seek a solution of

$$(7) \qquad u(t) = \int_0^t F(u(\tau), \tau) d\tau$$

with finite norm $M[u]$ with a suitably small. Our solution will be obtained as the limit of a sequence u_k defined recursively by

$$(8) \qquad u_0 = 0 , \qquad u_{k+1} = u_k + v_k ,$$

where

$$(9) \qquad \|u_k(t)\|_\rho < R \qquad \text{for } |t| < a_k(\rho_0 - \rho) ,$$

and v_k is defined by

$$(10) \qquad v_k(t) = \int_0^t F(u_k(\tau), \tau) d\tau - u_k(t) ,$$

i.e.,

$$u_{k+1}(t) = \int_0^t F(u_k(\tau), \tau) d\tau .$$

Here, for every $\rho < \rho_0$ and $|t| < a_k(\rho_0 - \rho)$, $u_k(t)$ and $v_k(t)$ are continuous functions of t with values in B_ρ for which $M_k[v_k]$ are finite, where

$$(11) \qquad M_k[v] = \sup_{\substack{0 \leq \rho < \rho_0 \\ |t| < a_k(\rho_0 - \rho)}} \|v(t)\|_\rho \left(\frac{a_k(\rho_0 - \rho)}{|t|} - 1 \right) ,$$

the numbers a_k being defined by

$$(12) \qquad a_{k+1} = a_k(1 - (k+2)^{-2}) , \qquad k = 0, 1, 2, \cdots ,$$

so that

$$(13) \qquad a = a_0 \prod_0^{+\infty} (1 - (k+2)^{-2}) > 0 ,$$

and a_0 will be chosen suitably small later.

Let us imagine that u_i are determined for $i \leq k$ with $M_i[u_i] < +\infty$ and $\|u_i(t)\|_\rho < \frac{1}{2}R$ for $|t| < a_i(\rho_0 - \rho)$. By the assumption (i), $v_k(t)$ is well defined. Set

$$\lambda_k = M_k[v_k] < +\infty \ . \tag{14}$$

Then

$$\|v_k(t)\|_\rho \leq \frac{\lambda_k}{a_k/a_{k+1} - 1} \qquad \text{for } |t| < a_{k+1}(\rho_0 - \rho) \ ,$$

and it follows that for $|t| < a_{k+1}(\rho_0 - \rho)$

$$\|u_{k+1}(t)\|_\rho \leq \frac{\lambda_k}{a_k/a_{k+1} - 1} + \|u_k(t)\|_\rho \ ,$$

and so, by recursion,

$$\|u_{k+1}(t)\|_\rho \leq \sum_0^k \lambda_j/(a_j/a_{j+1} - 1) \ . \tag{15}$$

We will require that

$$\sum_0^k \lambda_j/(a_j/a_{j+1} - 1) < \tfrac{1}{2}R \ . \tag{16}$$

Then for $|t| < a_{k+1}(\rho_0 - \rho)$ we have $\|u_{k+1}(t)\|_\rho < \frac{1}{2}R$ and so $F(u_{k+1}(t), t)$ is defined.

Our aim is to estimate λ_k so that $\lambda_k \to 0$ as $k \to +\infty$, and (16) holds for any $k \geq 0$. By (8) and (10) we have

$$v_{k+1}(t) = \int_0^t [F(u_{k+1}(\tau), \tau) - F(u_k(\tau), \tau)]d\tau \ .$$

Thus for $|t| < a_{k+1}(\rho_0 - \rho)$, we see from the assumption (ii) that

$$\|v_{k+1}(t)\|_\rho \leq C \left| \int_0^t \frac{\|v_k(\tau)\|_{\rho(\tau)}}{\rho(\tau) - \rho} d\tau \right|$$

for some choice of $\rho(\tau) < \rho_0 - |\tau|/a_k$.

We may set $\rho(\tau) = \frac{1}{2}(\rho_0 - |t|/a_{k+1} + \rho)$. Then we find by virtue of (14) (assuming, say, $t > 0$)

$$\begin{aligned} \|v_{k+1}(t)\|_\rho &\leq 4Ca_{k+1}\lambda_k \int_0^t \tau(a_{k+1}(\rho_0 - \rho) - \tau)^{-2} d\tau \\ &\leq 4Ca_{k+1}\lambda_k t \int_0^t (a_{k+1}(\rho_0 - \rho) - \tau)^{-2} d\tau \end{aligned}$$

$$= 4Ca_{k+1}\lambda_k \frac{t}{a_{k+1}(\rho_0-\rho)} \Big/ \left(\frac{a_{k+1}(\rho_0-\rho)}{t} - 1\right) .$$

Consequently

$$\lambda_{k+1} = M_{k+1}[v_{k+1}] \leq 4Ca_{k+1}\lambda_k \sup_{\substack{0\leq\rho<\rho_0 \\ |t|<a_{k+1}(\rho_0-\rho)}} \frac{t}{a_{k+1}(\rho_0-\rho)} \frac{a_{k+1}(\rho_0-\rho)/t-1}{a_{k+1}(\rho_0-\rho)/t-1}$$
$$\leq 4Ca_{k+1}\lambda_k \leq 4Ca_0\lambda_k .$$

Hence for $k = 0, 1, 2, \cdots$

$$\lambda_{k+1} \leq 4Ca_0\lambda_k . \tag{17}$$

Now we can choose a_0. Using the assumption (iii) we know that

$$\lambda_0 = M_0\left[\int_0^t F(0,\tau)d\tau\right] \leq K \sup_{\substack{|t|<a_0(\rho_0-\rho) \\ 0\leq\rho<\rho_0}} \frac{|t|}{\rho_0-\rho}\left(\frac{a_0(\rho_0-\rho)}{|t|} - 1\right) \leq a_0K .$$

We shall require that for $j = 0, 1, 2, \cdots$

$$\lambda_j \leq 2^4a_0K(j+2)^{-4} . \tag{18}$$

Assuming (18) to be true for λ_k we find from (12) and (17)

$$\lambda_{k+1} \leq 4Ca_0 2^4a_0K(k+2)^{-4}$$
$$\leq 2^4a_0K(k+3)^{-4}\left(4Ca_0\left(\frac{k+3}{k+2}\right)^4\right) \leq 2^4a_0K(k+3)^{-4} ,$$

provided $a_0 \leq a'$ independent of k. We have to verify (16). From (12) and (18)

$$\sum_0^k \lambda_j/(a_j/a_{j+1}-1) \leq \sum_0^k \lambda_j/(1-a_{j+1}/a_j) = \sum_0^k \lambda_j(j+2)^2$$
$$\leq 2^4a_0K\sum_0^k (j+2)^{-2} < 2^4a_0K\sum_0^\infty (j+2)^{-2} < \tfrac{1}{2}R ,$$

provided $a_0 \leq a''$. If we choose $a_0 \leq a'$ and $a_0 \leq a''$, we find the functions u_k are defined for all k with

$$\|u_k(t)\|_\rho < \tfrac{1}{2}R , \qquad \text{for } |t| < a_k(\rho_0-\rho) . \tag{19}$$

Furthermore, from (14) we have for $|t| < a(\rho_0-\rho) < a_k(\rho_0-\rho)$

$$\|u_{k+1}(t) - u_k(t)\|_\rho \leq \lambda_k \Big/ \left(\frac{a_k(\rho_0-\rho)}{|t|} - 1\right) < \lambda_k \Big/ \left(\frac{a(\rho_0-\rho)}{|t|} - 1\right) ,$$
$$M[u_{k+1} - u_k] \leq \lambda_k .$$

Since $\sum \lambda_k < +\infty$, it follows that u_k converges to some $u(t)$ in B. From (19) we have $\|u(t)\|_\rho \leq \frac{1}{2}R$ for $|t| < a(\rho_0 - \rho)$. The limit $u(t)$ is the unique solution of (7) and therefore of (1) and (2) because of the same arguments in [1].

Added in proof. The theorem can be generalized for the integral equation of the form

$$u(t) = u_0(t) + \int_0^t F(t - s, s, u(s))ds ,$$

the proof of which will appear in the appendix of the paper by T. Kano and T. Nishida, *Sur les ondes surfaces de l'eau avec une justification mathématique de l'équation de l'eau peu profonde*, J. Math. Kyoto Univ., 1978.

References

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KYOTO UNIVERSITY