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THE LENGTH SPECTRA OF SOME COMPACT MANIFOLDS OF NEGATIVE CURVATURE

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1. Introduction

Let R be a compact Riemannian manifold. In each free homotopy class $\tilde{\gamma}$ of closed paths on R, there exists a geodesic whose length is minimal among the paths in $\tilde{\gamma}$; let $l(\tilde{\gamma})$ be its length. The distinct members of the set of lengths $l(\tilde{\gamma})$ as $\tilde{\gamma}$ varies over all such classes can be arranged in increasing order $0 < l_1 < l_2 < \cdots$. The sequence $\{l_i\}_{i\geq 1}$, finite or infinite, is by definition the length spectrum of R. It may happen that $l(\tilde{\gamma}) = l(\tilde{\gamma}')$ for two distinct classes. Let, for each $i \geq 1$, m_i be the number of free homotopy classes $\tilde{\gamma}$ such that $l(\tilde{\gamma}) = l_i$. The sequence $\{(l_i, m_i)\}_{i\geq 1}$ may be called the length spectrum with multiplicity.

Let Δ be the Laplace-Beltrami operator of R. Then the space $L_2(R)$ (with respect to the Riemannian measure) decomposes as the Hilbert space direct sum of finite dimensional eigenspaces for Δ. Let $\{\lambda_i\}_{i\geq 1}$ be the distinct eigenvalues, and n_i the multiplicity of λ_i. The sequence $\{(\lambda_i, n_i)\}_{i\geq 1}$ is the spectrum of Δ. We may assume the λ_i to be arranged so that $0 \geq \lambda_1 > \lambda_2 > \cdots$.

In this paper, we shall study the length spectrum and its relation to the spectrum of Δ for a very special type of compact manifold of negative sectional curvature. Specifically, we shall consider a compact manifold R whose simply connected Riemannian covering manifold H is a symmetric space of noncompact type and of rank 1. As is well-known, H can then be represented as G/K, where G is a noncompact connected simple Lie group of R-rank one, with finite center, and K is a maximal compact subgroup of G. As a consequence R can be represented as $\Gamma\backslash G/K$, where Γ is a discrete subgroup of G, acting freely on G/K, such that $\Gamma\backslash G$ is compact. Γ can be identified with the fundamental group of R. The metric on R is fixed to be the one obtained from the canonical G-invariant metric on G/K. Cf. [11], [27].

For such a manifold R, let $\{(l_i, m_i)\}_{i\geq 1}$ be the length spectrum with multiplicity, and for any $l \geq 0$, define $Q_1(l) = \sum_{\{i; l_i \leq l\}} m_i$. Thus $Q_1(l)$ is the number of free homotopy classes $\tilde{\gamma}$ such that $l(\tilde{\gamma}) \leq l$. It can be seen easily that $Q_1(l)$ is finite for each finite l. We shall show that the asymptotic behaviour of $Q_1(l)$ as $l \to \infty$ can be described precisely in terms of the covering space G/K. In fact, we find that $Q_1(l) \sim (2|\rho| l)^{-1} \exp 2|\rho| l$ as $l \to \infty$, where ρ is the half

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sum of the positive roots of the symmetric space G/K, and $|\cdot|$ is the usual Cartan-Killing norm. This is the main result of the present paper. In particular the asymptotic behaviour of $Q_1(l)$ depends only on the covering manifold, and is independent of the subgroup Γ, a somewhat unexpected result.

In the course of proving this result, we shall also see that the length spectrum $\{l_i\}_{i\geq 1}$ is determined by the spectrum of the Laplacian Δ. This has been known for certain kinds of manifolds, [1], [19], and the question has been raised whether it is true in general for an arbitrary compact manifold.[1]

A result similar to our main result has been announced by Margulis [13]. See also Sinai [20]. Margulis works in the context of an arbitrary compact manifold of negative curvature; his result is that $Q_1(l) \sim Cl^{-1} \exp dl$ where C, d are positive constants. Bounds for d can be obtained. In our special context, the precise value of d can be obtained in terms of the structure of G/K. Margulis' proof has not appeared as far as the author knows. In any case, his proof is based on ergodic theory and is totally different. Cf. [13].

The free homotopy classes of closed paths on R can be easily seen to be in a natural one-to-one correspondence with the set C_Γ of conjugacy classes of elements of Γ. Thus our main result gives us some information about the distribution of these conjugacy classes. Actually we get somewhat more. An element $\gamma \in \Gamma$, $\gamma \neq 1$, is said to be *primitive* if it cannot be expressed as a positive power of any other element of Γ. Let Pr_Γ be the subset of C_Γ consisting of conjugacy classes of primitive elements of Γ. The corresponding free homotopy classes will be said to be primitive. Let $Q_0(l)$ be the number of primitive classes $\tilde{\gamma}$ such that $l(\tilde{\gamma}) \leq l$. Then we shall see that $Q_0(l)$ has the same asymptotic behaviour as $Q_1(l)$ as $l \to \infty$.

A particular case of our main results was proved by H. Hüber [12], who considered the case of compact Riemann surfaces of genus ≥ 2. Thus $G = SL(2, R)$. Hüber's method is slightly different; it was followed by Berard-Bergery in [1], where the case $G = SO_0(d, 1)$ was considered.

Our method is to apply the Selberg trace formula to the fundamental solution of the heat equation on M, and analyse the resulting theta relation closely. That this is useful for other problems in the context of $\Gamma\backslash G$ is indicated by [4], Eaton [3] or Wallach [22]. In [14] McKean considered $G = SL(2, R)$ and by applying the trace formula to the heat kernel, gave an independent proof of Hüber's result. Our method in proving the main result is a generalization of McKean's method.

Hüber utilizes methods involving the Green's function of the upper half

[1] After this work was completed, the author came to know that recently J. J. Duistermaat and V. W. Guillemin [*The spectrum of positive elliptic operators and periodic bicharacteristics*, Invent. Math. **29** (1975) 39–79] have proved the general result that the length spectrum of any generic compact Riemannian manifold is determined by the spectrum of the Laplacian. The author understands that their method uses the wave equation on M. The method of the present paper uses the heat equation, as will be apparent below.

plane to prove a remarkable formula [12, p. 26], cf. (4.32) below, which is his main tool. We shall indicate below how Hüber's formula can be generalized to our setting by an application of Selberg's trace formula. By using this, one can get some more geometric information. Specifically, for each $x, y \in G$, and $r \geq 0$, let $Q(x, y, r)$ be the number of elements $\gamma \in \Gamma$ such that the Riemannian distance between γxK and yK is less than r. Then the asymptotic behaviour of $Q(x, y, r)$ can be determined. Cf. § 4 below. This may be regarded as a 'local' version of the main result.

2. Preliminaries

Let G be a connected noncompact simple Lie group with finite center, and K a maximal compact subgroup of G. Let $\mathfrak{g}, \mathfrak{k}$ be the respective Lie algebras of G and K, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition, with respect to the involution Θ determined by $\mathfrak{k}$. Denote by $\langle \cdot, \cdot \rangle$ the Cartan Killing form; for any $X \in \mathfrak{g}$, we put $|X|^2 = -\langle X, \Theta X \rangle$. Then $|\cdot|$ is a norm on $\mathfrak{g}$. Let $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$. Throughout this paper, we assume that $\dim \mathfrak{a}_\mathfrak{p} = 1$. Extend $\mathfrak{a}_\mathfrak{p}$ to a maximal abelian Θ-stable subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, so that $\mathfrak{a} = \mathfrak{a}_\mathfrak{k} + \mathfrak{a}_\mathfrak{p}$, where $\mathfrak{a}_\mathfrak{k} = \mathfrak{a} \cap \mathfrak{k}$, $\mathfrak{a}_\mathfrak{p} = \mathfrak{a} \cap \mathfrak{p}$. Then $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Denote by $\mathfrak{g}^C, \mathfrak{a}^C$ etc. the complexifications of $\mathfrak{g}, \mathfrak{a}$, etc, and let $\Phi(\mathfrak{g}^C, \mathfrak{a}^C)$ be the set of roots of $(\mathfrak{g}^C, \mathfrak{a}^C)$. Order the dual spaces of $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{p} + i\mathfrak{a}_\mathfrak{k}$ compactibly as usual (Cf. [11]), and let Φ^+ be the set of positive roots under this order. Let $P_+ = \{\alpha \in \Phi^+ ; \alpha \not\equiv 0 \text{ on } \mathfrak{a}_\mathfrak{p}\}$ and $P_- = \{\alpha \in \Phi^+ ; \alpha \equiv 0 \text{ on } \mathfrak{a}_\mathfrak{p}\}$, and let $\rho = \frac{1}{2} \sum_{\alpha \in P_+} \alpha$. Let X_α be a root vector belonging to $\alpha \in \Phi$, and let $\mathfrak{n}^C = \sum_{\alpha \in P_+} CX_\alpha$. Then, if $\mathfrak{n} = \mathfrak{n}^C \cap \mathfrak{g}$, we have the Iwasawa decompositions $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_\mathfrak{p} + \mathfrak{n}$, $G = KA_\mathfrak{p}N$ where $A_\mathfrak{p} = \exp \mathfrak{a}_\mathfrak{p}$, $N = \exp \mathfrak{n}$. $\mathfrak{n}$ is equal to $\sum_{\alpha \in P_+} RX_\alpha$. Let M be the centralizer of $A_\mathfrak{p}$ in K, M' the normalizer of $A_\mathfrak{p}$ in K, and $W = W(G, A_\mathfrak{p})$ the Weyl group M'/M. W operates naturally on $A_\mathfrak{p}$, $\mathfrak{a}_\mathfrak{p}$, $(\mathfrak{a}_\mathfrak{p})^*$, $(\mathfrak{a}_\mathfrak{p}^C)^*$, etc.

Let Λ be the real dual of $\mathfrak{a}_\mathfrak{p}$, and Λ^C its complexification. For $\lambda \in \Lambda^C$, we put $\lambda = \operatorname{Re} \lambda + i \operatorname{Im} \lambda$ with $\operatorname{Re} \lambda$, $\operatorname{Im} \lambda$ in Λ. We extend the form $\langle \cdot, \cdot \rangle$ to $\mathfrak{a}^C$, Λ^C, in the obvious way. W preserves $\langle \cdot, \cdot \rangle$.

We let dk be the normalized Haar measure on K. Let da, dn be the Haar measures on $A_\mathfrak{p}$, N given by the Euclidean structure on $A_\mathfrak{p}$, n furnished by the inner product $-\langle X, \Theta Y \rangle$, and the exponential map. Then the Haar measure dx on G can be so normalized that for any $f \in C_c(G)$, we have

$$\int_G f(x)dx = \int_K \int_{A_\mathfrak{p}} \int_N f(kan) \exp 2\rho(\log a) dk \, da \, dn \; .$$

These narmalizations will be fixed from now on.

Denote by $C_c^\infty(K \backslash G/K)$ the subspace of $C^\infty(G)$ consisting of those $f \in C_c^\infty(G)$ such that $f(k_1 x k_2) = f(x)$, $x \in G$, $k_1, k_2 \in K$. Such functions are said to be spherical. The spaces $L_1(K \backslash G/K)$, $L_2(K \backslash G/K)$ etc. are defined analogously. For any $x \in G$, let $H(x) \in \mathfrak{a}_\mathfrak{p}$ be the unique element of $\mathfrak{a}_\mathfrak{p}$ such that $x \in K \exp H(x)N$. Then for any $\lambda \in \Lambda^C$, the function $\phi_\lambda(x) = \int_K \exp(i\lambda - \rho)(H(xk))dk$ is the *elementary* spherical function corresponding to λ. For $f \in L_1(K\backslash G/K)$ and $\lambda \in \Lambda^C$, define the spherical Fourier transform

$$\hat{f}(\lambda) = \int_G f(x)\phi_\lambda(x)dx \,, \tag{2.1}$$

where dx is the Haar measure on G.

Let $f \in L_1(K\backslash G/K)$, and define

$$F_f(a) = \exp \rho(\log a) \int_N f(an)dn \,. \tag{2.2}$$

Then $F_f \in L_1(A_\mathfrak{p})$. F_f is the so-called Abel transform of f, and it is known that

$$\hat{f}(\lambda) = \int_{A_\mathfrak{p}} F_f(a) \exp i\lambda(\log a)da = F_f^*(\lambda) \,, \tag{2.3}$$

where $F_f^*(\lambda)$ is the Euclidean Fourier transform of F_f.

For $x \in G$, we have $x = k \exp X$, $k \in K$, $X \in \mathfrak{p}$. Put $\sigma(X) = |X|$. $\sigma(x)$ is spherical, smooth and will play a role below. Let $\Xi(x)$ be the elementary spherical function $\phi_0(x) = \int_K \exp -\rho(H(xk))dk$. The Harish-Chandra-Schwartz space $\mathscr{C}(G)$ is then defined as in [10]. For each left or right invariant differential operator D on G, and an integer $r \geq 0$, define $\tau_{D,r}(f) = \mathrm{Sup}_{x \in G}\, \Xi(x)^{-1}(1 + \sigma(x))^r\, |Df(X)|$, for $f \in C^\infty(G)$. $\mathscr{C}(G)$ then consists of those $f \in C^\infty(G)$ for which $\tau_{D,r}(f) < \infty$ for all D, r. $\mathscr{C}(G)$ is a Frechét space under these seminorms. Similarly we define seminorms $\nu_{D,r}(f) = \mathrm{Sup}_{x \in G}\, \Xi(x)^{-2}(1 + \sigma(x))^r\, |Df(x)|$, and put $\mathscr{C}_1(G) = \{f \in C^\infty(G)\,;\, \nu_{D,r}(f) < \infty \text{ for all } D, r\}$. Then $\mathscr{C}_1(G) \subset \mathscr{C}(G) \subset L_2(G)$. $\mathscr{C}_1(G) \subset L_1(G)$. The space $\mathscr{C}_1(G)$ was introduced and studied by Trombi-Varadarajan [21].

The spaces of spherical functions in $\mathscr{C}(G)$, $\mathscr{C}_1(G)$ will be denoted by $\mathscr{C}(K\backslash G/K)$, $\mathscr{C}_1(K\backslash G/K)$ respectively.

Let Σ be the set of restrictions to $\mathfrak{a}_\mathfrak{p}$ of elements of P_+. Then one knows, since rank $(G/K) = 1$, that we can select $\beta \in \Sigma$ such that 2β is the only other possible element of Σ. Let $\mathfrak{p}$ be the number of roots in P_+ whose restriction to $\mathfrak{a}_\mathfrak{p}$ equals β, and let q be the number of remaining elements. Let H_0 be the element of $\mathfrak{a}_\mathfrak{p}$ such that $\beta(H_0) = 1$, and H_β the element such that $\langle H, H_\beta\rangle = \beta(H)$, $H \in \mathfrak{a}_\mathfrak{p}$. Then it is known that $\langle H_0, H_0\rangle = 2p + 8q$, $\rho(H_0) = \frac{1}{2}(p + 2q)$ and $H_\beta = (2p + 8q)^{-1}H_0$. It follows that $\langle \rho, \rho\rangle = \frac{1}{4}(p + 2q)^2(2p + 8q)^{-1}$, which will be used below.

3. The trace formula

Let Γ be a discrete subgroup of G such that $\Gamma\backslash G$ is compact. Fix a G-invariant measure dx on $\Gamma\backslash G$ by requiring that for each $f \in C_c(G)$, we have $\int_G f(x)dx = \int_{\Gamma\backslash G} (\sum_{\gamma\in\Gamma} f(\gamma x)d\dot{x}$. Let T be an irreducible unitary representation of Γ on a finite dimensional vector space V, and denote by U the representation of G induced by T. Thus U acts on the Hilbert space H consisting of functions $f: G \to V$ which satisfy (i) $f(\gamma x) = T(\gamma)f(x)$ and (ii) $\int_{\Gamma\backslash G} (f(x), f(x))d\dot{x} < \infty$ where $(\cdot, \cdot)$ is the inner product on V. The action of G on H is by right translation. Thus $(U(x)f)(y) = f(yx), x, y \in G, f \in H$. U is a unitary representation of G. Under our assumption of compactness for $\Gamma\backslash G$, it is well known that U is a discrete direct sum of irreducible unitary representations of G, each occurring with finite multiplicity. Denoting by $\mathscr{E}(G)$ the set of equivalence classes of irreducible unitary representations of G, we let $n_\Gamma(\omega, T)$ be the number of summands of U which lie in the class ω. Then we can write $U \cong \sum_{\omega\in\mathscr{E}(G)} n_\Gamma(\omega, T)\omega$, and $n_\Gamma(\omega, T) < \infty$ for each ω.

For $f \in L_1(G)$ let $U(f) = \int_G f(x)U(x)dx$. $U(f)$ is a bounded operator on H. As in [18], [7], we say that f is admissible if (i) the series $\sum_\gamma f(y^{-1}\gamma x)T(\gamma)$ converges absolutely, uniformly on compacts of $G \times G$, to a continuous End (V)-valued function $F(x, y, T)$ and (ii) the operator $U(f)$ is of trace class. When f is admissible, we have the trace formula

$$\sum_{\omega\in\mathscr{E}(G)} n_\Gamma(\omega, T) \operatorname{Trace} U_\omega(f) = \int_{\Gamma\backslash G} \operatorname{Trace} F(x, x, T)d\dot{x} , \tag{3.1}$$

where U_ω is a representation of class $\omega \in \mathscr{E}(G)$. Of course, $U_\omega(f)$ has a trace because $U(f)$ does.

As in [18], one rewrites the right side of (3.1) to get the Selberg trace formula

$$\sum_{\omega\in\mathscr{E}(G)} n_\Gamma(\omega, T) \operatorname{Trace} U_\omega(f) = \sum_{\gamma\in C_\Gamma} \operatorname{Trace} T(\gamma) \operatorname{Vol} (\Gamma_\gamma\backslash G_\gamma)I_\gamma(f) , \tag{3.2}$$

where C_Γ is a complete set of representatives in Γ of the conjugacy classes of elements of Γ, and G_γ is the centralizer of γ in $G, \Gamma_\gamma = \Gamma \cap G_\gamma$. Since $\Gamma\backslash G$ is compact, every element of Γ is semisimple, and G_γ is reductive, and $\Gamma_\gamma\backslash G_\gamma$ is compact. We fix a Haar measure dx_γ on G_γ in a manner analogous to the manner in which the Haar measure on G was fixed, following the Iwasawa decomposition of G_γ, and put $d\dot{x}_\gamma$ for the invariant measure on $\Gamma_\gamma\backslash G_\gamma$. The volume Vol $(\Gamma_\gamma\backslash G_\gamma)$ is computed with respect to this measure. Finally, $I_\gamma(f) = \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx_\gamma^*$, where dx_γ^* is the G-invariant measure on $G_\gamma\backslash G$ normalized so that $dx = dx_\gamma dx_\gamma^*$.

The use of (3.2) depends on having a stock of admissible functions. The following proposition was proved in [7].

Proposition 3.1. *Let* $f \in \mathscr{C}_1(K\backslash G/K)$. *then* f *is admissible.*

A similar assertion holds if $f \in \mathscr{C}_1(G)$ and is left and right K-finite. We shall only need this special case.

Let $f \in \mathscr{C}_1(K\backslash G/K)$. Then $U_\omega(f) = 0$ unless is of class one with respect to K, i.e., unless the restriction of U_ω to K contains the trivial representation of K. When U_ω is of class one, there is associated with it a unique positive definite elementary spherical function ϕ_{λ_ω} say, $\lambda_\omega \in \Lambda_C$. Then Trace $U_\omega(f) = \hat{f}(\lambda_\omega)$, where $\hat{f}$ is as in (2.1). (Cf. [6]). Thus, when $f \in \mathscr{C}_1(K\backslash G/K)$, we get

$$\sum_{\omega \in \mathscr{E}(G,1)} n_\Gamma(\omega, T)\cdot \hat{f}(\lambda_\omega) = \sum_{\gamma \in C_\Gamma} \operatorname{Trace} T(\gamma).\ \operatorname{Vol}(\Gamma_\gamma \backslash G_\gamma) I_\gamma(f)\ , \tag{3.3}$$

where $\mathscr{E}(G, 1)$ stands for these elements in $\mathscr{E}(G)$ which are of class one.

We shall now compute the integrals $I_\gamma(f)$ for $f \in \mathscr{C}_1(K\backslash G/K)$, in a form suitable for use in § 4.

An element $x \in G$ is said to be *elliptic*, if it is conjugate to some element of K and is then automatically semisimple. $x \in G$ is said to be *hyperbolic*, if it is semisimple but not elliptic. In all other cases x is said to be *parabolic*. When G/Γ is compact, Γ does not contain parabolic elements.

It is well-known that $\gamma \in \Gamma$ is elliptic if and only if it is of finite order. Both these properties are equivalent to the property that γ has a fixed point on G/K. We assume throughout that Γ contains no nontrivial elliptic elements. Thus each $\gamma \in \Gamma$, $\gamma \neq 1$, is hyperbolic.

The integrals $I_\gamma(f)$ can be computed for hyperbolic γ quite simply, and can be expressed in terms of the Abel transform F_f of (2.2) when f is spherical.

Let J be a Cartan subgroup of G with Lie algebra $\mathfrak{j}$, Φ^+ a set of positive roots for $\Phi = \Phi(\mathfrak{g}^C, \mathfrak{j}^C)$. For any $\alpha \in \Phi^+$ let ξ_α be the corresponding character of J. Put $\rho_J = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$, and $\xi_{\rho_J} = \exp \rho(\log h)$. We may assume that ξ_{ρ_J} is a well-defined character of J. Put $\Delta_J(h) = \xi_\rho(h) \prod_{\alpha \in \Phi^+} (1 - \xi_\alpha(h)^{-1})$, $h \in J$, and let Φ_f^J be the invariant integral of f relative to J (cf. [9]). Thus

$$\Phi_f^J(h) = \varepsilon_R^J(h)\Delta_J(h) \int_{J\backslash G} f(x^{-1}\gamma x)dx^*\ . \tag{3.4}$$

Here $\varepsilon_R^J(h) = \operatorname{sign} \prod_{\alpha \in \Phi_R^+} (1 - \xi_\alpha(h)^{-1})$, the product being over the set Φ_R^+ of real roots in Φ^+, i.e., those which are real on $\mathfrak{j}$, the Lie algebra of J. The Haar measure dh on J is normalized as mentioned in § 2 above, and dx^* is the G-invariant measure on $J\backslash G$ such that $dx = dh\, dx^*$. Φ_f^J is defined and smooth on $J' = J \cap G'$ = the regular points in J.

For $\gamma \in \Gamma$, let G_γ be its centralizer with Lie algebra $\mathfrak{g}_\gamma$, and let $\mathfrak{j}_\gamma$ be a Θ-stable Cartan subalgebra of $\mathfrak{g}_\gamma$ which is fundamental. Then one knowns t hat $I_\gamma(f)$ and $\Phi_f^{J_\gamma}$ are related to each other, thanks to a theorem of Harish-Cha ndra

[10, p. 33]. If we let Φ_γ^+ be the set of positive roots of $(\mathfrak{g}_\gamma^C, \mathfrak{j}_\gamma^C)$, and put $\Pi_\gamma = \prod_{\alpha\in\Phi_\gamma^+} H_\gamma$, then we know that

$$I_\gamma(f) = C_\gamma \Phi_f^{J_\gamma}(\gamma\,;\Pi_\gamma)\,; \qquad C_\gamma \neq 0\,, \tag{3.5}$$

where $\Phi_f^{J_\gamma}(\gamma\,;\Pi_\gamma)$ is the result of applying the differential operator Π_γ to the function $\Phi_f^{J_\gamma}$, and evaluating the result at γ. All this is well-known and can be found, e.g., in [23].

The value of C_γ will be useful for us. It can be computed by using [10, Lemma 23], and [23, II, Chap. 8]. One should bear in mind that our normalizations of Haar measure differ from those used in [23, II, Chap. 8]. The value of C_γ is found to be

$$\begin{aligned} C_\gamma &= (-1)^{m_\gamma}[W_{K_\gamma}] \prod_{\alpha\in\Phi_{\gamma,K}^+} \langle\alpha, \rho_{K_\gamma}\rangle^{-1}\cdot(2)^{-m_\gamma}\cdot 2^{-n_\gamma} \\ &\quad\times\left\{\xi_\rho(\gamma)\prod_{\alpha\in\Phi_{\mathfrak{g}/\mathfrak{g}_\gamma}^+}(1-\xi_\alpha(\gamma)^{-1})^{-1}\right\}\varepsilon_R^J(\gamma)\,. \end{aligned} \tag{3.6}$$

Here $m_\gamma = \frac{1}{2}(\dim G_\gamma - \operatorname{rank} G_\gamma - \dim K_\gamma + \operatorname{rank} K_\gamma)$, $n_\gamma = \frac{1}{2}(\dim(G_\gamma/K_\gamma) - \operatorname{rank}(G_\gamma/K_\gamma))$, W_{K_γ} is the Weyl group of K_γ and $[W_{K_\gamma}]$ is its cardinality, $\Phi_{\gamma,K}^+$ stands for the compact roots in Φ_γ^+, ρ_{K_γ} is the half sum of these roots, and $\Phi_{\mathfrak{g}/\mathfrak{g}_\gamma}^+$ is the complement of Φ_γ^+ in Φ^+.

Recall that we have assumed that rank $(G/K) = 1$. In this case there can be at most two nonconjugate Cartan subgroups. One of these is always noncompact, namely $A = A_\mathfrak{t} A_\mathfrak{p}$, and $\dim A_\mathfrak{p} = 1$. When another nonconjugate Cartan subgroup exists, it is compact, and we may call it B. Thus there are two invariant integrals Φ_f^A and Φ_f^B.

We shall compute Φ_f^A for $f \in \mathscr{C}_1(K\backslash G/K)$ and relate it to F_f. Let a be a regular element of A, and let $a = a_\mathfrak{t} a_\mathfrak{p}$, $a_\mathfrak{t} \in A_\mathfrak{t}$, $a_\mathfrak{p} \in A_\mathfrak{p}$. Then

$$F_f(a_\mathfrak{p}) = \xi_\rho(a_\mathfrak{p})\int_N f(a_\mathfrak{p} n)dn = \xi_\rho(a_\mathfrak{p})\int_N f(an)dn\,, \tag{3.7}$$

Since $f(an) = f(a_\mathfrak{t} a_\mathfrak{p} n) = f(a_\mathfrak{p} n)$.

For regular a, the map $n \to a^{-1}n^{-1}an$ is a diffeomorphism of N onto N whose Jacobian is computable. (See e.g. [11, Chapter X]). Thus

$$\begin{aligned} F_f(a_\mathfrak{p}) &= \xi_\rho(a_\mathfrak{p})\left|\prod_{\alpha\in P_+}(1-\xi_\alpha(a)^{-1})\right|\int_N f(n^{-1}an)dn \\ &= \xi_\rho(a_\mathfrak{p})\left|\prod_{\alpha\in P_+}(1-\xi_\alpha(a)^{-1})\right|\int_K\int_N f(k^{-1}n^{-1}ank)dn\,dk\,. \end{aligned} \tag{3.8}$$

since f is spherical.

The last integral can be transformed as in [10]. It equals $\int_{A_\mathfrak{p}\backslash G} f(x^{-1}ax)dx_\mathfrak{p}^*$, where $dx = da_\mathfrak{p}\, dx_\mathfrak{p}^*$. Since $A_\mathfrak{t}$ is compact and carries normalized Haar measure, this last integral equals $\int_{A\backslash G} f(x^{-1}ax)dx^*$. Also, if $\alpha \in P_+$, so does $\bar{\alpha}$. Hence the product $\prod_{\alpha\in P_+}(1-\xi_\alpha(a)^{-1})$ is real and has the same sign as $\prod_{\substack{\alpha\in P_+\\ \alpha \text{ real}}}(1-\xi_\alpha(a)^{-1})$, which of course is precisely $\varepsilon_R^A(a)$. Using all this, we get

$$
(3.9)\qquad \begin{aligned} F_f(a_\mathfrak{p}) &= \xi_\rho(a_\mathfrak{p})\varepsilon_R^A(a)\prod_{\alpha\in P_+}(1-\xi_\alpha(a)^{-1})\cdot\int_{A\backslash G} f(x^{-1}ax)dx^* \\ &= \xi_\rho(a_\mathfrak{t})^{-1}\prod_{\alpha\in P_-}(1-\xi_\alpha(a_\mathfrak{t})^{-1})^{-1}\Phi_f^A(a)\ , \end{aligned}
$$

where we have used the fact that for $\alpha\in P_-$, $\xi_\alpha(a_\mathfrak{p}) = 1$ so that $\xi_\alpha(a) = \xi_\alpha(a_\mathfrak{t})$. Thus finally, we have

$$
(3.10)\qquad \Phi_f^A(a) = \xi_\rho(a_\mathfrak{t})\prod_{\alpha\in P_-}(1-\xi_\alpha(a_\mathfrak{t})^{-1})\cdot F_f(a_\mathfrak{p})\ ,\qquad a\in A'\ .
$$

Now suppose that $\gamma\in\Gamma$, $\gamma\neq 1$, so that γ is hyperbolic. Let $h = h(\gamma)$ be an element of A to which γ is conjugate. Then $I_\gamma(f) = I_h(f)$. Let $h = h_\mathfrak{p}h_\mathfrak{t}$; then $h_\mathfrak{p}\neq 1$, since γ is hyperbolic. Clearly, $\mathfrak{a}^C$ is a Cartan subalgebra of $\mathfrak{g}_h^C$. If $\alpha\in\Phi^+(\mathfrak{g}_h^C, \mathfrak{a}^C)$, then $\xi_\alpha(h) = 1$, so $\xi_\alpha(h_\mathfrak{p})\xi_\alpha(h_\mathfrak{t}) = 1$. Since α is real on $\mathfrak{a}_\mathfrak{p}$, and purely imaginary on $\mathfrak{a}_\mathfrak{t}$, it follows that $\xi_\alpha(h_\mathfrak{p}) = 1$. Since $\dim a_\mathfrak{p} = 1$, and ξ_α is real on $A_\mathfrak{p}$, we conclude that $\xi_\alpha\equiv 1$ on $A_\mathfrak{p}$, and so α vanishes on $\mathfrak{a}_\mathfrak{p}$. Thus $\alpha\in P_-$. Therefore $\Phi^+(\mathfrak{g}_h^C, \mathfrak{a}^C)\subset P_-$. It follows that $G_h\subset MA_\mathfrak{p}$, and $A_\mathfrak{p}$ is in the center of G_h. Hence A is fundamental in G_h. The operator Π_h equals $\prod_{\{\alpha\in P_-;\ \xi_\alpha(h)=1\}} H_\alpha$. In particular, each H_α occurring here is in $\mathfrak{a}_\mathfrak{t}$. Thus, in applying Π_h to (3.10), we need only worry about the factor $\xi_\rho(a_\mathfrak{t})\prod_{\alpha\in P_-}(1 - \xi_\alpha(a_\mathfrak{t})^{-1})$, since Π_h will not act on $F_f(a_\mathfrak{p})$ at all. The result of applying Π_h to this function and evaluating the result at h is seen to be equal to

$$
[W_{K_h}]\prod_{\{\alpha\in P_-;\ \xi_\alpha(h)=1\}}\langle\alpha, \rho_{K_h}\rangle\times\xi_\rho(h_\mathfrak{t})\times\prod_{\{\alpha\in P_-;\ \xi_\alpha(h)\neq 1\}}(1-\xi_\alpha(h)^{-1})\ .
$$

Cf. [10, Lemma 24] for a similar computation. Using (3.5), (3.6), we have the following proposition.

Proposition 3.2. *Let γ be a hyperbolic element of G, and let $h = h(\gamma)$ be an element of A to which it is conjugate. Let $h = h_\mathfrak{t}h_\mathfrak{p}$, $h_\mathfrak{t}\in A_\mathfrak{t}$, $h_\mathfrak{p}\in A_\mathfrak{p}$. Then*

$$
(3.11)\qquad I_\gamma(f) = I_h(f) = C(h)\cdot F_f(h_\mathfrak{p})\ ,\qquad f\in\mathscr{C}(K\backslash G/K)\ ,
$$

where $C(h) = \varepsilon_R^A(h)(\xi_\rho(h_\mathfrak{p})\prod_{\alpha\in P_+}(1-\xi_\alpha(h)^{-1}))^{-1}$.

One should note that $C(h)$ is actually positive. For later use, we shall examine $C(h)$ a little more carefully. Since $\xi_\rho(h_\mathfrak{p}) = \exp\rho(\log h_\mathfrak{p}) = \exp\frac{1}{2}\prod_{\alpha\in P_+}\alpha(\log h_\mathfrak{p})$, we see that $C(h)$ equals

$$
\varepsilon_R^A(h)\prod_{\alpha\in P_+}(\exp\tfrac{1}{2}\alpha(\log h_\mathfrak{p}) - \xi_\alpha(h_\mathfrak{t})^{-1}\exp - \tfrac{1}{2}\alpha(\log h_\mathfrak{p}))^{-1}\ .
$$

Since any α is purely imaginary on $\mathfrak{a}_{\mathfrak{k}}$, we must have $\xi_\alpha(h_{\mathfrak{k}})^{-1} = \overline{\xi_\alpha(h_{\mathfrak{k}})}$; if α is a real root, then of course $\xi_\alpha(h_{\mathfrak{k}}) = 1$. Now it is well-known in our case that there is at most one real root in P_+. Denote this root by α_0 when it exists. Then the factor corresponding to it is $\exp \alpha_0(\log h_{\mathfrak{p}})/2 - \exp - \alpha_0(\log h_{\mathfrak{p}})/2$. The remaining roots in P_+ will be denoted by P_+^0. These are all complex, and occur in conjugate pairs $\alpha, \bar{\alpha}$. Thus we can find a subset Q_+^0 of P_+^0 so that $P_+^0 = Q_+^0 \cup \overline{Q_+^0}$.

Now let $\alpha \in Q_+^0$, and consider the factors corresponding to α and $\bar{\alpha}$ in the above product. We have $\xi_\alpha(h_{\mathfrak{k}})^{-1} = \overline{\xi_\alpha(h_{\mathfrak{k}})} = \xi_{\bar\alpha}(h_{\mathfrak{k}})$. Let $\Theta_\alpha(h_{\mathfrak{k}})$ be the argument of $\xi_\alpha(h_{\mathfrak{k}})$. Thus $\xi_\alpha(h_{\mathfrak{k}}) = \exp i\Theta_\alpha(h_{\mathfrak{k}})$. Then these two factors have the product $\exp \alpha(\log h_{\mathfrak{p}}) + \exp - \alpha(\log h_{\mathfrak{p}}) - 2 \cos \Theta_\alpha(h_{\mathfrak{k}})$. Now all the numbers $\alpha(\log h_{\mathfrak{p}})$ are of the same sign, depending on which Weyl chamber $h_{\mathfrak{p}}$ lies in. Using this remark one quickly finds that

$$\begin{aligned} C(h) = \exp - |\rho(\log h_{\mathfrak{p}})| &\times (1 - \exp - |\alpha_0(\log h_{\mathfrak{p}})|)^{-1} \\ &\times \prod_{\alpha \in Q_+^0} (1 - 2\cos \Theta_\alpha(h_{\mathfrak{k}}) \exp - |\alpha(\log h_{\mathfrak{p}})| \\ &\qquad + \exp - 2|\alpha(\log h_{\mathfrak{p}})|)^{-1} ; \end{aligned} \tag{3.12}$$

when P_+ contains no real root, the factor corresponding to α_0 is, of course, absent.

4. The length spectrum

As we have said in § 1, our results follow from applying the trace formula to suitable admissible functions, mainly to the fundamental solution of the heat equation on G/K.

Let Ω be the Casimir operator of G, and for $t > 0$ let $g_t(x)$ be the fundamental solution of the heat equation $\Omega u = \partial u/\partial t$ on G/K, with u assumed spherical. The properties of g_t are discussed in [4]. Let us briefly recall them. As a function on G, g_t is spherical, nonnegative real valued, and $g_{t+s} = g_t^* g_s$, for $t, s > 0$. g_t is the fundamental solution in the sense that for any $f \in C_c^\infty(K\backslash G/K)$, for example, the function $U(x, t) = (g_t^* f)(x)$ is the unique spherical solution of $\Omega u = \partial u/\partial t$ such that $u(x, t) - f(x) \to 0$ uniformly on compact sets as $t \to 0$. The function g_t is in $L_1(K\backslash G/K)$ for each $t > 0$, and $\hat{g}_t$ can be computed. Indeed, $\hat{g}_t(\lambda) = \exp - (\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle)t$. Since g_t is integrable, $\hat{g}_t$ is defined for all λ such that φ_λ is bounded, thus in the tube $\Lambda + iC_\rho$, and the above formula for $\hat{g}_t$ holds there. It follows, for example by using [21], that $g_t \in \mathscr{C}_1(K\backslash G/K)$. In particular, g_t is admissible.

Since $\hat{g}_t(\lambda)$ is known, it is possible to compute the Abel transform F_{g_t} by using the Fourier inversion formula. We get, remembering $\dim A_{\mathfrak{p}} = 1$,

$$F_{g_t}(a_{\mathfrak{p}}) = (4\pi t)^{-1/2} \exp - (t\langle \rho, \rho \rangle + |\log a_{\mathfrak{p}}|^2/(4t)) . \tag{4.1}$$

Of course, a similar formula would hold when the dimension of $A_{\mathfrak{p}} > 1$, but we would not be using it.

Now applying (3.3) to g_t, using (3.5) and (3.6) we find

$$\sum_{\omega \in \mathcal{E}(G,1)} n_\Gamma(\omega, T) \exp - (\langle \lambda_\omega, \lambda_\omega \rangle + \langle \rho, \rho \rangle)t \tag{4.2}$$
$$= \sum_{\gamma \in C_\Gamma} \text{Trace } T(\gamma) \text{ Vol } (\Gamma_\gamma \backslash G_\gamma) \cdot I_\gamma(g_t) \ .$$

On the right side we get from the term corresponding to $\gamma = 1$, the contribution $g_t(1)$ (degree T). Vol $(\Gamma \backslash G)$. The remaining elements C_Γ are all hyperbolic since Γ is assumed torsion-free. Call the sum of these remaining terms $J_H(t)$.

It can be shown (cf. Eaton [3], or [4]) that $\lim_{t\to 0} J_H(t) = 0$. This is actually done in Eaton [3] under the additional hypothesis that T is the trivial representation. But the expression for $J_H(t)$ when T is nontrivial is clearly dominated in absolute value by a multiple of the correspondiug expression when T is trivial, since $g_t \geq 0$. Hence $J_H(t) \to 0$ in our case also. If $L(t)$ denotes the left side of (4.2), it follows that

$$\lim_{t\to 0} t^{n/2} L(t) = \left(\lim_{t \to 0} t^{n/2} g_t(1)\right)(\text{Vol. } \Gamma \backslash G) \text{ (degree } T) \ .$$

Here $n = \dim (G/K)$.

It is shown in [4] that $\lim_{t\to 0} t^{n/2} g_t(1)$ exists and equals C'_G, a constant which depends only on G. Thus $\lim_{t \to 0} t^{n/2} L(t) = C'_G$ Vol. $(\Gamma \backslash G)$ degree (T). Now introduce, for $r > 0$, the function

$$N(r, T) = \sum_{\substack{\omega \in \mathcal{E}(G,1) \\ |\Omega_\omega| \leq r}} n_\Gamma(\omega, T) \ , \tag{4.3}$$

where Ω_ω is the scalar with which the Casimir element acts in any representation of class ω. When ω is of class one, one can compute Ω_ω and find that $\Omega_\omega = -\langle \lambda_\omega, \lambda_\omega \rangle - \langle \rho, \rho \rangle$. Thus we conclude that $L(t) = \int_0^\infty e^{-tr} dN(r, T)$, showing that $L(t)$ is the Laplace transform of $N(r, T)$. Of course, the admissibility of g_t shows that $N(r, T)$ is finite for each r, and $L(t)$ exists.

Arguing as in [4], we now find by Karamata's theorem that, as $r \to \infty$,

$$r^{-n/2} N(r, T) \sim C'_G \Gamma\left(\frac{n}{2} + 1\right)^{-1} \text{Vol. } (\Gamma \backslash G). \text{ degree } (T) \ , \tag{4.4}$$

which is analogous to a classical result of H. Weyl [24]. When T is trivial this result is implied by that of Minakshisundaram and Pleijel [15]. Of course (4.4) is just a step away from Eaton's result.[2]

When T is trivial, $N(r, T)$ is just the Weyl function of the manifold $\Gamma\backslash G/K$. More precisely, if $\{(\lambda_i, n_i)\}_{i\geq 1}$ is the spectrum of the Laplacian on $\Gamma\backslash G/K$, is easily seen that $N(r, 1) = \sum_{\{i;|\lambda_i|\leq r\}} n_i$. We shall write $N(r)$ for $N(r, 1)$. Clearly the knowledge of $N(r)$ is equivalent to that of the spectrum of the Laplacian. In particular, the spectrum of the Laplacian on $\Gamma\backslash G/K$ determines Vol $(\Gamma\backslash G)$. Cf. [4], [15]. This will be needed below.

We now turn to the consideration of the length spectrum of $R = \Gamma\backslash G/K$. For this purpose, we have to compute the terms in (4.2) explicitly, with $T = 1$; this will be done next, resulting in (4.7) below.

Cearly G/K is the simply connected covering manifold of R, and we can identify Γ with the fundamental group $\pi_1(R)$. It is well-known that the free homotopy classes of closed paths on R are in a natural one-to-one correspondence with the set of conjugacy classes of Γ, and hence with the set C_Γ. For any $\gamma \in C_\Gamma$, the corresponding free homotopy class always contains a periodic geodesic g_γ say, which has minimum length among all the paths in that class [2]. Let $l(\gamma)$ be the length of g_γ. Any closed path in this homotopy class can be lifted to a path of equal length on G/K which joins some point $m \in G/K$ to the point γm. It follows that the length $l(\gamma)$ of g_γ is the minimum of the lengths of paths joining some point $m \in G/K$ to its image γm under γ. In fact, $l(\gamma) = \inf_{m\in G/K} d(m, \gamma m)$ where $d(\cdot, \cdot)$ is the Riemannian distance on G/K. Now, if $m = xK$ with $x \in G$, we have $d(m, \gamma m) = d(xK, \gamma xK) = d(K, x^{-1}\gamma xK) = \sigma(x^{-1}\gamma x)$, where σ is the function introduced in § 1. It follows that $l(\gamma) = \inf_{x\in G} \sigma(x^{-1}\gamma x)$. Notice that $l(\gamma)$ depends only on the conjugacy class of γ, as it should. Moreover, for the computation of $l(\gamma)$, we can replace γ by *any* element h of G conjugate to γ, even if h does not lie in Γ at all. This remark enables one to compute $l(\gamma)$ more explicitly. Recall that γ is conjugate to an element $h = h(\gamma) \in A$. Let $h = h_\mathfrak{p} h_\mathfrak{t}$; h acts as an isometry on G/K, with no fixed points. Since G/K is of negative curvature, it follows from [2], [16] that there is exactly one geodesic of G/K which is stabilized by h. This geodesic is characterized by the property that a point $p \in G/K$ is on the geodesic if and

[2] Actually, one does not need to assume that Γ is torsion free. In that case the right side of (4.2) splits into three terms, namely $J_C(t), J_E(t), J_H(t)$, coming respectively from central, elliptic and hyperbolic elements in C_Γ. Cf. [4]. One sees that $J_C(t) = g_t(1) \cdot$Vol. $(\Gamma\backslash G) \sum_{\gamma\in Z\cap C_\Gamma}$ Trace $T(\gamma)$, where $Z =$ center (G), so that

$$\lim_{t\to 0} t^{n/2} J_C(t) = C'_G \text{ Vol. } (\Gamma\backslash G) \sum_{\gamma\in Z\cap C_\Gamma} \text{Trace } T(\gamma) .$$

One can show as in [3] that $\lim_{t\to 0} t^{n/2} J_E(t) = 0$, so that one gets $r^{-n/2}N(r,T) \to C'_G\Gamma(n/2 + 1)^{-1}$ Vol. $(\Gamma\backslash G) \sum_{\gamma\in Z\cap C_\Gamma}$ Trace $T(\gamma)$, which implies that $\sum_{\gamma\in Z\cap C_\Gamma}$ Trace $T(\gamma)$ must be nonnegative. If now T is irreducible, then $T(\gamma)$ is a scalar for $\gamma \in Z \cap \Gamma$ by Schur's lemma, and $T(\gamma) = \chi(\gamma)$. Identity, where χ is a character of the finite abelian group $Z \cap \Gamma$. If χ is nontrivial character, it follows that $\sum_{\gamma\in Z\cap C_\Gamma} T(\gamma) = 0$, so that $r^{-n2/}N(r, T) \to 0$ if $T|_{Z\cap C_\Gamma}$ is a nontrivial irreducible representation. When T is not irreducible, $\sum_{\gamma\in Z\cap C_\Gamma} \chi(\gamma) = \sum \deg T_i$, where T_i runs over those irreducible summands of T which restrict to the trivial character of $Z \cap \Gamma$.

only if $d(p, hp) = \inf_{m \in G/K} d(m, hm)$. Now it is easy to see that the geodesic $\operatorname{Exp} A_{\mathfrak{p}}$ (where Exp is the exponential map of G/K from $\mathfrak{p}$ to G/K) is stabilized by h, (recall here that $\dim A_{\mathfrak{p}} = 1$). Moreover, if $p \in \operatorname{Exp} \mathfrak{a}_{\mathfrak{p}}$, then $d(p, hp) = \sigma(h)$. This shows that $\inf_{m \in G/K} d(m, hm) = \sigma(h)$, so that $l(\gamma) = \sigma(h(\gamma))$. Of course, $\sigma(h(\gamma)) = |\log h_{\mathfrak{p}}(\gamma)|$.

Note that $l(\gamma) = l(\gamma^{-1})$, (indeed the geodesics in the homotopy class γ^{-1} are just reverse to those in γ), and $l(\gamma^j) = jl(\gamma)$ for any integer $j \geq 1$.

Lemma 4.1. *Let $\gamma \in \Gamma$, $\gamma \neq 1$. Then Γ_γ is isomorphic to $\boldsymbol{Z}$.*

Proof. γ is hyperbolic, and by conjugation, we may assume $\gamma \in A$, $\gamma_{\mathfrak{p}} \neq 1$. Let $\gamma', \gamma'' \in \Gamma_\gamma$ and suppose $\gamma' = \gamma'_{\mathfrak{p}}\gamma'_{\mathfrak{k}}$, $\gamma'' = \gamma'_{\mathfrak{p}}\gamma'_{\mathfrak{k}}$. Since $G_\gamma \subset MA_{\mathfrak{p}}$ as we have seen above, and $\gamma'_{\mathfrak{k}}$ commutes with γ, we have $\gamma'_{\mathfrak{k}} \in MA_{\mathfrak{p}}$. Thus $\gamma'(\gamma'')^{-1} = \gamma'_{\mathfrak{p}}(\gamma'_{\mathfrak{p}})^{-1}\gamma'_{\mathfrak{k}}(\gamma'_{\mathfrak{k}})^{-1}$. It follows that the set of elements $\{\gamma'_{\mathfrak{p}}, \gamma' \in \Gamma_\gamma\}$ is a subgroup of $A_{\mathfrak{p}}$. Clearly this is a discrete subgroup, hence it is isomorphic to $\boldsymbol{Z}$. Let $\delta_{\mathfrak{p}}$ be a generator for it, and let $\delta \in \Gamma_\gamma$ be such that $\delta = \delta_{\mathfrak{p}}\delta_{\mathfrak{k}}$. We claim that δ generates Γ_γ freely. In fact let $\gamma' \in \Gamma_\gamma$. Then $\gamma'_{\mathfrak{p}} = \delta_{\mathfrak{p}}^j$ for some $j \in \boldsymbol{Z}$. We claim that $\gamma' = \delta^j$. Indeed, $\gamma'\delta^{-j} = \gamma'_{\mathfrak{p}}\delta_{\mathfrak{p}}^{-j}\gamma'_{\mathfrak{k}}\delta_{\mathfrak{k}}^{-j} = \gamma'_{\mathfrak{k}}\delta_{\mathfrak{k}}^{-j}$. Thus $\gamma'\delta^{-j} \in \Gamma \cap K$, so that $\gamma'\delta^{-j} = 1$ since Γ contains no elliptic elements $\neq 1$. Hence $\gamma' = \delta^j$ and our assertion follows.

Remark. Using the negative curvature of G/K, this result could also have been deduced from the theorem of Preismann [17], which is more general. In our special case, the above proof is more direct.

Definition 4.2. An element $\gamma \in \Gamma$, $\gamma \neq 1$, will be said to be *primitive* if γ is a generator of Γ_γ.

Clearly every $\gamma \in \Gamma$, $\gamma \neq 1$, can be written as δ^j with $j \geq 1$ integral, and δ primitive. The integer j is unique and will be denoted by $j(\gamma)$.

We will next compute Vol. $(\Gamma_\gamma \backslash G_\gamma)$. We may again assume $\gamma \in A$. Then $G_\gamma \subset MA_{\mathfrak{p}}$. In fact $G_\gamma = M_\gamma A_{\mathfrak{p}}$, where $M_\gamma = M \cap G_\gamma$. Let $\gamma = \gamma_{\mathfrak{p}}\gamma_{\mathfrak{k}}$. Each element of M_γ commutes with both γ and $\gamma_{\mathfrak{p}}$, hence with $\gamma_{\mathfrak{k}}$. If follows that $\gamma_{\mathfrak{k}}$ commutes with G_γ, so $\gamma_{\mathfrak{k}}$ acts trivially on G_γ/K_γ. Thus the action of γ on G_γ/K_γ is the same as the action of $\gamma_{\mathfrak{p}}$. Now it is clear that $K_\gamma = K \cap G_\gamma = M_\gamma$, and since $G_\gamma = M_\gamma A_{\mathfrak{p}}$ we conclude that the action of Γ_γ on G_γ/K_γ is the same as the action of $\{\delta_{\mathfrak{p}}^j, j \in \boldsymbol{Z}\}$ on $A_{\mathfrak{p}}$, acting by left translation. Here we identify $A_{\mathfrak{p}} \cong G_\gamma/K_\gamma$. We thus get (recalling that the measures have been so normalized that K_γ carries normalized Haar measure),

$$\text{Vol.}\,(\Gamma_\gamma \backslash G_\gamma) = \text{Vol.}\,(\Gamma_\gamma \backslash G_\gamma/K_\gamma) = \text{Vol.}\,(A_{\mathfrak{p}}/\{\delta_{\mathfrak{p}}^j, j \in \boldsymbol{Z}\})\ . \tag{4.5}$$

The last term is clearly equal to $|\log \delta_{\mathfrak{p}}| = l(\delta)$. Moreover, since $\gamma = \delta^{j(\gamma)}$, we have $l(\gamma) = j(\gamma)l(\delta)$. Thus

$$\text{Vol.}\,(\Gamma_\gamma \backslash G_\gamma) = l(\gamma)j(\gamma)^{-1}\ . \tag{4.6}$$

Using all this in the trace formula, (3.3) with $T = 1$

$$
\begin{aligned}
L(t) &= \sum_{\omega \in \mathcal{E}(G,1)} n_\Gamma(\omega, 1) \exp - (\langle \lambda_\omega, \lambda_\omega \rangle + \langle \rho, \rho \rangle) t \\
&= g_t(1) \operatorname{Vol.} (\Gamma \backslash G) + \sum_{\gamma \in C_\Gamma - \{1\}} l(\gamma) j(\gamma)^{-1} I_\gamma(g_t) \ .
\end{aligned} \tag{4.7}
$$

Moreover, if γ is conjugate to $h = h(\gamma) \in A$, we also know that

$$
\begin{aligned}
I_\gamma(g_t) &= I_h(g_t) \\
&= (4\pi t)^{-1/2} C(h(\gamma)) \exp - (\langle \rho, \rho \rangle t + \tfrac{1}{4} |\log h_{\mathfrak{p}}(\gamma)|^2 / t) \\
&= (4\pi t)^{-1/2} C(h(\gamma)) \exp - (\langle \rho, \rho \rangle t + \tfrac{1}{4} l(\gamma)^2 / t) \ ,
\end{aligned} \tag{4.8}
$$

because, as we have seen above, $l(\gamma) = |\log h_p(\gamma)|$.

It follows that for each $t > 0$, the series $\sum_{\gamma \in C_\Gamma - \{1\}} l(\gamma) j(\gamma)^{-1} \exp - \frac{1}{4} l(\gamma)^2 / t$ is convergent; one sees from this that the numbers $\{l(\gamma), \gamma \in C_\Gamma - \{1\}\}$ have no finite point of accumulation. In particular, one may indeed order them $0 < l_1 < l_2 \cdots$, and the multiplicity m_i of each l_i is finite. (This can also be inferred on general grounds of course.)

One immediate consequence of (4.7) is that the length spectrum $\{l_i\}_{i \geq 1}$ of R is determined by the spectrum of the Laplacian, or what is the same, by the function $L(t)$. For, as we saw before, $L(t)$ determines the volume Vol. $(\Gamma \backslash G)$, and hence the first term on the right side of (4.7). Then the smallest of the numbers $\{l(\gamma) ; \gamma \in C_\Gamma - \{1\}\}$, which is of course l_1, is seen to be equal to the supremum of the set

$$
\left\{ \varepsilon > 0 ; \lim_{t \to 0} ((4\pi t)^{1/2} \exp (\langle \rho, \rho \rangle t + \tfrac{1}{4} \varepsilon^2 / t)(L(t) - g_s(1) \operatorname{Vol.} (\Gamma \backslash G))) = 0 \right\} .
$$

This means that l_1 is determined by $L(t)$. Moreover, it is seen that

$$
\begin{aligned}
&\lim_{t \to 0} (4\pi t)^{1/2} \exp (\langle \rho, \rho \rangle t + \tfrac{1}{4} l_1^2 / t)(L(t) - g_t(1) \operatorname{Vol.} (\Gamma \backslash G)) \\
&\qquad = \sum_{\{\gamma ; l(\gamma) = l_1\}} l(\gamma) j(\gamma)^{-1} C(h(\gamma)) = l_1 \sum_{\{\gamma ; l(\gamma) = l_1\}} j(\gamma)^{-1} C(h(\gamma)) \ ,
\end{aligned}
$$

which is positive. Call this number ε_1. One can now subtract off the contribution to $L(t)$ from $\{\gamma ; l(\gamma) = l_1\}$, and putting

$$
L_2(t) = L(t) - g_t(1) \operatorname{Vol.} (\Gamma \backslash G) - \{(4\pi t)^{-1/2} \varepsilon_1 \exp - (\langle \rho, \rho \rangle t + \tfrac{1}{4} l_1^2 / t)\} \ ,
$$

we find l_2 to be the supremum of

$$
\left\{ \varepsilon > 0 ; \lim_{t \to 0} ((4\pi t)^{1/2} \exp (\langle \rho, \rho \rangle t + \tfrac{1}{4} \varepsilon^2 / t) \cdot L_2(t)) = 0 \right\} ,
$$

and that $\lim_{t \to 0} (4\pi t)^{1/2} \exp (\langle \rho, \rho \rangle t + \frac{1}{4} l_2^2 / t) L_2(t)$ is positive and equals $\varepsilon_2 = l_2 \sum_{\{\gamma ; l(\gamma) = l_2\}} j(\gamma)^{-1} C(h(\gamma))$.

Proceeding in this way, we see that $L(t)$ determines both the numbers $\{l_i\}_{i\geq 1}$ and $\{\varepsilon_i\}_{i\geq 1}$, where $\varepsilon_i = l_i \sum_{\{\gamma \in C_\Gamma; l(\gamma)=l_i\}} j(\gamma)^{-1}C(h(\gamma))$. Conversely, a knowledge of these numbers and of Vol. $(\Gamma\backslash G)$ clearly determines $L(t)$, and hence the spectrum of the Laplacian; indeed

$$L(t) = g_t(1) \text{ Vol. } (\Gamma\backslash G) + \sum_{i\geq 1} (4\pi t)^{-1/2}\varepsilon_i \exp - (\langle\rho, \rho\rangle t + \tfrac{1}{4}l_i^2/t) .$$

When $G = SL(2, R)$, $C(h(\gamma))$ depends on γ only via $l(\gamma)$. In fact $C(h(\gamma)) = 2\cosh(l(\gamma)/2\sqrt{2})$, and so $\varepsilon_i = 2l_i \cosh(l_i/2\sqrt{2}) \sum_{\{\gamma \in C_\Gamma, l(\gamma)=l_i\}} j(\gamma)^{-1}$. Thus in this case, knowledge of the sequence $\{(l_i, \varepsilon_i)\}$ is equivalent to the knowledge of the sequence $\{(l_i, \eta_i)\}$, where $\eta_i = \sum_{\{\gamma \in C_\Gamma, l(\gamma)=l_i\}} j(\gamma)^{-1}$. Since $\{(l_i, \varepsilon_i)\}$ characterizes $L(t)$, we see that in this special case $\{(l_i, \eta_i)\}$ characterizes $L(t)$. This result was originally observed by Hüber [12]. As we have seen in § 3, the expression for $C(h(\gamma))$ is more complicated in the general case, and does not depend merely on $l(\gamma)$.

Returning to the general case, we let Pr_Γ be the set of primitive elements in $C_\Gamma - \{1\}$. Then we can write

$$L(t) = g_t(1) \text{ Vol. } (\Gamma\backslash G) + \sum_{\delta \in \mathrm{Pr}_\Gamma} \sum_{j\geq 1} l(\delta)I_{\delta^j}(g_t) , \tag{4.9}$$

where

$$I_{\delta^j}(g_t) = (4\pi t)^{-1/2}C(h(\delta^j)) \exp - (\langle\rho, \rho\rangle t + \tfrac{1}{4}j^2 l(\delta)^2/t) . \tag{4.10}$$

The set $\{l(\delta); \delta \in \mathrm{Pr}_\Gamma\}$ can be ordered in a sequence $0 < r_1 < r_2 < \cdots$; let p_i be the cardinality of the set $\{\delta \in \mathrm{Pr}_\Gamma; l(\delta) = r_i\}$. We call the sequence $\{r_i\}$ the primitive length spectrum, and the sequence $\{(r_i, p_i)\}$ the primitive length spectrum with multiplicity. One can ask to what extent these are determined by $L(t)$. Obviously, the set $\{r_i\}$ is contained in the set $\{l_i\}$, which is determined by $L(t)$. So one must try and decide from a knowledge of $L(t)$ whether a given number l_j is in the set $\{r_i\}$ or not, i.e., if it is a primitive length or not. Obviously, if l_j is not a multiple of some smaller l_k, it must be a primitive length. However, if l_j is a multiple of some smaller l_k, it could happen that l_j is also the length of some other primitive geodesic as well. The author has not been able to decide this question in general by using the above formula. However, when $G = SL(2, R)$, one can answer this question. Indeed in this case, $L(t)$ is characterized by $\{(l_i, \eta_i)\}$ which we can assume known. Now l_1 is obviously equal to r_1, and η_1 eqnals p_1, since $j(\gamma) = 1$ for all γ such that $l(\gamma) = l_1$. Now consider $2r_1$. It must be one of the numbers $\{l_i\}_{i\geq 2}$. Suppose $2r_1 = l_{i_1}$. Then the numbers $\{l_s; s \leq i_1 - 1\}$ must all be primitive lengths. Thus $r_s = l_s$ and $\eta_s = p_s$ for all $s \leq i_1 - 1$. We can now decide whether l_{i_1} is a primitive length or not. For if $l_{i_1} = r_{i_1}$, then we should have $\eta_{i_1} = \frac{1}{2}p_1 + p_{i_1}$, and $p_{i_1} > 0$. Thus, if $\eta_{i_1} > \frac{1}{2}p_1 = \frac{1}{2}\eta_1$, we can conclude that l_{i_1} is a primitive length, $l_{i_1} = r_{i_1}$ and $p_{i_1} = \eta_{i_1} - \frac{1}{2}\eta_i$. On the other hand if $\eta_{i_1} = \frac{1}{2}p_{i_1}$ then l_{i_1} is not a primitive length. Next, let l_{i_2} be the smallest member of the set $\{l_i\}_{i>i_1}$, which is an integral multiple of some number l_j smaller than it. By the definition of l_{i_2}, it is clear that the numbers $\{l_s; i_1 < s < i_2\}$ are primitive lengths, and so $\eta_s = p_s$ for these. As to l_{i_2} itself, we can decide whether it is a primitive length by comparing η_{i_2} with the sum $\sum_{\{(k,j); jr_k = l_{i_2}, j>1\}} 1/j$. If η_{i_2} is strictly larger, then l_{i_2} is a primitive length, and the difference between η_{i_2} and this sum gives its multiplicity. Proceeding in this way, we see that $L(t)$ determines both the primitive length spectrum and its multiplicity. Finally, let $S_i = \{k \geq 1, jr_k = l_i$ for some $j > 1\}$. Then we have $m_i = \sum_{k \in S_i} p_k$. Hence the length spectrum with multiplicity is also determined by $L(t)$ in this case. When G is not $SL(2, R)$, these questions are not settled by the present method, and a close look at the computations seems to indicate that in general $L(t)$ probably would not determine the primitive length spectrum or the multiplicities.

To return to our main topic, define for any $l \geq 0$,

$$(4.11) \quad Q_0(l) = [\{\delta \in \mathrm{Pr}_\Gamma ; l(\delta) \leq l\}] , \qquad Q_1(l) = [\{\gamma \in C_\Gamma - \{1\}, l(\gamma) \leq l\}] .$$

$[S]$ stands for the cardinality of S.

We shall now determine the asymptotic behaviour of the functions $Q_0(l)$, $Q_1(l)$ as $l \to \infty$. For $h \in A$, with $h_{\mathfrak{p}} \neq 1$ put

$$(4.12) \qquad C_+(h) = \exp - |\rho(\log h_{\mathfrak{p}})| \prod_{\alpha \in P_+} (1 + \exp - |\alpha(\log h_{\mathfrak{p}})|)^{-1} ,$$

$$(4.13) \qquad C_-(h) = \exp - |\rho(\log h_{\mathfrak{p}})| \prod_{\alpha \in P_+} (1 - \exp - |\alpha(\log h_{\mathfrak{p}})|)^{-1} ,$$

$$(4.14) \qquad C_0(h) = \exp - |\rho(\log h_{\mathfrak{p}})| ,$$

and define

$$(4.15) \quad F(t) = (4\pi t)^{-1/2}(\exp - \langle\rho, \rho\rangle t) \sum_{\gamma \in C_\Gamma - \{1\}} l(\gamma) j(\gamma)^{-1} C(h(\gamma)) \exp - \tfrac{1}{4} l(\gamma)^2/t ,$$

and let F_+, F_-, F_0 be defined analogously by replacing $C(h)$ by $C_+(h), C_-(h)$, $C_0(h)$ in (4.15).

Lemma 4.3. *Let $H(t)$ be any of the four functions $F(t), F_+(t), F_-(t), F_0(t)$, and let, for $r > 0$, $\tilde{H}(r) = \int_0^\infty e^{-rt}H(t)dt$. Then $H(t) \to 0$ as $t \to 0$, $H(t) \to 1$ as $t \to \infty$, and $r\tilde{H}(r) \to 1$ as $r \to 0$.*

Proof. We know that for $\gamma \in C_\Gamma - \{1\}$, $l(\gamma) = |\log h_{\mathfrak{p}}(\gamma)|$ is bounded away from zero. Hence, if $\beta = \sup_{\alpha \in P_+, \gamma \in C_\Gamma - \{1\}} \exp - |\alpha(\log h_{\mathfrak{p}}(\gamma))|$, we conclude that $\beta < 1$. Let $D = ((1 + \beta)/(1 - \beta))^{[P_+]}$. Then for each $\gamma \in C_\Gamma - \{1\}$,

$$(4.16) \qquad C_+(h(\gamma)) \leq C(h(\gamma)) \leq C_-(h(\gamma)) \leq D \cdot C_+(h(\gamma)) ,$$

where we used the expression (3.12) for $C(h)$. Therefore

$$F_+(t) \leq F(t) \leq F_-(t) \leq DF_+(t) , \tag{4.17}$$

and similarly

$$F_0(t) \leq F_-(t) . \tag{4.18}$$

Now we know, by the remarks immediately following (4.2), that $F(t)$ (called $J_H(t)$ there) approaches zero as $t \to 0$. From (4.17), (4.18) it follows that $F_+(t), F_-(t)$ and $F_0(t)$ all do the same.

We next claim that $F(t) \to 1$ as $t \to \infty$. In fact by

$$\begin{aligned} F(t) = 1 + \sum_{\substack{\omega \in \mathscr{E}(G,1) \\ \omega \neq 1}} n_\Gamma(\omega, 1) \cdot \exp - (\langle \lambda_\omega, \lambda_\omega \rangle + \langle \rho, \rho \rangle)t \\ - g_t(1) \operatorname{Vol.} (\Gamma \backslash G) . \end{aligned} \tag{4.19}$$

As $t \to \infty$, each term in the sum approaches monotonely to zero, because $\langle \lambda_\omega, \lambda_\omega \rangle + \langle \rho, \rho \rangle \geq 0$; so the whole sum approaches zero. Next, we know [4] that

$$g_t(x) = [W(G, A_\mathfrak{p})]^{-1} \int_A \exp - (\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle)t \cdot \phi_\lambda(x) \, |c(\lambda)|^{-2} \, d\lambda ,$$

where $c(\lambda)$ is the Harish-Chandra c-function. It follows that

$$g_t(1) = [W]^{-1} \int_A \exp - (\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle)t \, |c(\lambda)|^{-2} \, d\lambda$$

again by monotone convergence, we conclude that $g_t(1) \to 0$ as $t \to \infty$. Now (4.19) shows $F(t) \to 1$ as $t \to \infty$.

We will now show that $F_+(t) \to 1$ as $t \to \infty$. The other functions F_-, F_0 can be treated similarly. Using (3.12) it is easy to see that $C_+(h(\gamma))/C(h(\gamma)) \to 1$ as $l(\gamma) = |\log h_\mathfrak{p}(\gamma)| \to \infty$. Let $\varepsilon > 0$ be given, and choose and fix N so large that for $l(\gamma) \geq N$, we have

$$(1 - \varepsilon)C(h(\gamma)) \leq C_+(h(\gamma)) \leq (1 + \varepsilon)C(h(\gamma)) . \tag{4.20}$$

Let $F^N(t), F_+^N(t)$ be the tails of the series defining $F(t), F_+(t)$ beyond $l(\gamma) > N$. Then one sees

$$(1 - \varepsilon)F^N(t) \leq F_+^N(t) \leq (1 + \varepsilon)F^N(t) . \tag{4.21}$$

For each fixed N, the sum

$$(4\pi t)^{-1/2} \exp - \langle \rho, \rho \rangle t \sum_{l(\gamma) \leq N} l(\gamma) j(\gamma)^{-1} C(h(\gamma)) \exp - \tfrac{1}{4} l(\gamma)^2 / t$$

is a finite sum and approaches zero as $t \to \infty$. Since $F(t) \to 1$ as $t \to \infty$, it follows that $F^N(t) \to 1$ as $t \to \infty$. Thus from (4.21) we deduce

$$(1-\varepsilon) \leq \varliminf_{t\to\infty} F_+^N(t) \leq \varlimsup_{t\to\infty} F_+^N(t) \leq 1+\varepsilon .$$

Now by examining the sum $F_+(t) - F_+^N(t)$ we can similarly conclude that $\lim_{t\to\infty} (F_+(t) - F_+^N(t)) = 0$. This together with the above shows that

$$(1-\varepsilon) \leq \varliminf F_+(t) \leq \varlimsup F_+(t) \leq 1+\varepsilon . \tag{4.23}$$

Since ε is arbitrary, we conclude $F_+(t) \to 1$ as $t \to \infty$. The first assertion of the lemma is proved by proceeding similarly for F_-, F_0.

Since $F(t)$ is nonnegative and $F(t) \to 1$ as $t \to \infty$, Karamata's theorem [25] shows that

$$r\tilde{F}(r) \to 1 \text{ as } r \to 0, \text{ where } \tilde{F}(r) = \int_0^\infty e^{-rt} dF(t) .$$

Also, the functions $F_+(t) - F(t)$, $F_-(t) - F(t)$ do not change sign, and approach 0 as $t \to \infty$. So by the same theorem, we must have $r(\tilde{F}_+(r) - \tilde{F}(r)) \to 0$, $r(\tilde{F}_-(r) - \tilde{F}(r)) \to 0$ as $r \to 0$. Finally, $F_0(t) - F_-(t)$ does not change sign, and approaches 0 as $t \to \infty$. So we get $r(\tilde{F}_0(r) - \tilde{F}_-(r)) \to 0$ as $r \to 0$. Since $r\tilde{F}(r) \to 1$ as $r \to 0$, the proof is finished.

Theorem 4.4. *Let $Q_0(l)$, $Q_1(l)$ be the functions defined in* (4.11). *Then we have*

$$\begin{aligned} &2|\rho| l \exp - (2|\rho| l) Q_0(l) \to 1 \qquad \textit{as } l \to \infty , \\ &2|\rho| l \exp - (2|\rho| l) Q_1(l) \to 1 \qquad \textit{as } l \to \infty , \end{aligned} \tag{4.25}$$

where $2|\rho| = 2\langle \rho, \rho\rangle^{1/2} = (p+2q)(2p+8q)^{-1/2}$.

Proof. We deal first with $Q_0(l)$. The result for $Q_1(l)$ will be deduced from it. Recall first the notations of § 1.

Let $h(\gamma)$ be in A, and $h(\gamma)$ conjugate to $\gamma \in C_\Gamma - \{1\}$. $\log h_{\mathfrak{p}}(\gamma)$ is a multiple of H_0; say it equals $u_\gamma H_0$. Then $l(\gamma) = |\log h_{\mathfrak{p}}(\gamma)| = |u_\gamma| . \ |H_0|$. Also $|\rho(\log h_{\mathfrak{p}}(\gamma))| = |u_\gamma| |\rho(H_0)|$. Then

$$|\rho(\log h_{\mathfrak{p}}(\gamma))| = l(\gamma) \cdot |\rho(H_0)|/|H_0| .$$

It can be computed easily that $|\rho(H_0)||H_0| = \frac{1}{2}(p+2q)(2p+8q)^{-1/2} = |\rho|$. Hence $2|\rho| = (p+2q)(2p+8q)^{-1/2}$ and $|\rho(\log h_{\mathfrak{p}}(\gamma))| = |\rho| l(\gamma)$. Since each γ equals $\delta^{j(\gamma)}$ with δ primitive, and $l(\gamma) = j(\gamma) l(\delta)$, we have

$$F_0(t) = (4\pi t)^{-1/2} \exp - |\rho|^2 t \sum_{\delta \in \Pr\Gamma} \sum_{j\geq 1} \exp - (j|\rho| l(\delta) + \tfrac{1}{4} j^2 l(\delta)^2/t) . \tag{4.26}$$

Thus

$$\int_0^\infty e^{-rt} F_0(t) dt$$

$$(4.27)\qquad = \sum_{\delta\in\mathrm{Pr}_\Gamma}\sum_{j\geq 1} l(\delta)\exp - j|\rho|\, l(\delta) \cdot\int_0^\infty (4\pi t)^{-1/2}\exp - ((|\rho|^2+r)t + \tfrac{1}{4}j^2 l(\delta)^2/t)dt\,.$$

Use the formula $\int_0^\infty (4\pi t)^{-1/2}\exp(-x^2t - \frac{1}{4}y^2/t)dt = (2x)^{-1}\exp - xy$ to get

$$(4.28)\qquad \begin{aligned}\tilde{F}_0(r) &= \frac{1}{2}(r+|\rho|^2)^{-1/2}\sum_{\delta\in\mathrm{Pr}_\Gamma}\sum_{j\geq 1} l(\delta)\exp - (jl(\delta)(|\rho| + \sqrt{r+|\rho|^2}))\\ &= \frac{1}{2}(r+|\rho|^2)^{-1/2}\sum_{\delta\in\mathrm{Pr}_\Gamma} l(\delta)\,\frac{\exp - l(\delta)(|\rho|+\sqrt{r+|\rho|^2})}{1-\exp - l(\delta)(|\rho|+\sqrt{r+|\rho|^2})}\,.\end{aligned}$$

Let

$$(4.29)\qquad G_0(r) = \tfrac{1}{2}(r+|\rho|^2)^{-1/2}\sum_{\delta\in\mathrm{Pr}_\Gamma} l(\delta)\exp - l(\delta)(|\rho|+\sqrt{r+|\rho|^2})\,,$$

which converges by comparison with $\tilde{F}_0(r)$. The ratio of the corresponding terms in $G_0(r)$ and $\tilde{F}_0(r)$ approaches 1 as $l(\delta)\to\infty$. So an argument similar to that of Lemma 4.3 shows that $rG_0(r)$ and $r\tilde{F}_0(r)$ have the same limit as $r\to 0$. Since we know $r\tilde{F}_0(r)\to 1$ as $r\to 0$, we conclude $rG_0(r)\to 1$ as $r\to 0$. Now

$$(4.30)\qquad \begin{aligned} rG_0(r) &= \tfrac{1}{2}r(r+|\rho|^2)^{-1/2}\sum_{\delta\in\mathrm{Pr}_\Gamma} l(\delta)\cdot\exp - l(\delta)(|\rho|+\sqrt{r+|\rho|^2})\\ &= \tfrac{1}{2}r(r+|\rho|^2)^{-1/2}\int_0^\infty l\exp - |\rho|\, l\cdot\exp - l\sqrt{r+|\rho|^2}\cdot dQ_0(l)\\ &= \frac{r}{2\sqrt{r+|\rho|^2}\,(\sqrt{r+|\rho|^2}-|\rho|)}\times(\sqrt{|\rho|^2+r}-|\rho|)\\ &\quad\cdot\int_0^\infty l\exp - 2|\rho|\, l\cdot\exp - (\sqrt{r+|\rho|^2}-|\rho|)l\cdot dQ_0(l)\,.\end{aligned}$$

Writing $z=\sqrt{r+|\rho|^2}-|\rho|$, we see that $z\to 0$ as $r\to 0$. Letting $r\to 0$ in the above expression we conclude

$$\lim_{z\to 0} z\int_0^\infty \exp - zl\cdot l\exp - 2|\rho|\, l\cdot dQ_0(l) = 1\,.$$

Now Karamata's theorem gives us the first conclusion of the theorem. (See the note added in proof.)

As to $Q_1(l)$, we have

$$(4.31)\qquad \begin{aligned} Q_0(l) &= [\{\delta\,;\,\delta\in\mathrm{Pr}_\Gamma,\, l(\delta)\leq l\}]\\ &\leq Q_1(l) = [\{\gamma\in C_\Gamma - \{1\}\,;\, l(\gamma)\leq l\}]\\ &= [\{(\delta,j)\,;\,\delta\in\mathrm{Pr}_\Gamma,\, j\geq 1,\, jl(\delta)\leq l\}]\\ &\leq \sum_{\{\delta\in\mathrm{Pr}_\Gamma;\, l(\delta)\leq l\}}\frac{l}{l(\delta)} = \int_0^l \frac{l}{y}dQ_0(y) = Q_0(l) + \int_0^l\frac{l}{y^2}Q_0(y)dy\,.\end{aligned}$$

Since we know the asymptotic estimate for $Q_0(l)$, the estimate for $Q_1(l)$ follows easily from this expression. This finishes the proof of the main result.

One notes that the asymptotic behaviour of Q_0 and Q_1 depends only on the metric structure of the covering manifold G/K and not on the particular manifold R (or what is the same, on Γ).

This theorem generalizes a result of H. Hüber [12] who treated the case $G = SL(2, R)$. Hüber's method is slightly different; it was followed by Berard-Bergery [1] to $G = SO_0(d, 1)$, $d \geq 2$; Our method generalizes the method of McKean [14] who works with $G = SL(2, R)$. These authors use a metric on G/K which gives it curvature -1 in their cases. Our metric is somewhat different. This introduces an inessential discrepancy between the values of $|\rho|$ which they get there and we get here. Hüber also proved the remarkable formula [12, p. 26],

$$
\begin{aligned}
&\frac{2\sqrt{\pi}\,\Gamma(s)}{(s-1)\Gamma(s-\frac{1}{2})} \\
&\quad + \frac{2^{s-1}}{\Gamma(s-\frac{1}{2})} \sum_{\substack{\omega \in \mathcal{E}(G,1) \\ \omega \neq 1}} n_\Gamma(\omega, 1)\Gamma\Big(\frac{1}{2}(s - s_-(\lambda_\omega))\Big)\Gamma\Big(\frac{1}{2}(s - s_+(\lambda_\omega))\Big) \\
&= \frac{1}{\sqrt{\pi}} \frac{\Gamma(s)}{\Gamma(s-\frac{1}{2})} \mathrm{Vol.}\,(\Gamma/G) \\
&\quad + \sum_{\gamma \in C_\Gamma - \{1\}} l(\gamma) j(\gamma)^{-1} (\cosh l(\gamma) - 1)^{-1/2} (\cosh l(\gamma))^{-s+1/2}\,,
\end{aligned}
\tag{4.32}
$$

where $s_\pm(\lambda_\omega)$ are the roots of $S^2 - S - \Delta_\omega = 0$, and $G = SL(2, R)$. Δ_ω is the eigenvalue of the Laplacian. One must bear in mind that Hüber used the metric which gives curvature -1 to G/K.

Hüber's proof of (4.32) utilizes methods involving the Green's function of the upper half-plane. Hüber used the above formula together with the theorem of Ikehara to get the analogue of Theorem 4.4 for $G = SL(2, R)$. A generalization of (4.32) for $G = SO_0(d, 1)$ is presented by Berard-Bergery in [1, p. 118], and is used there similarly to obtain Theorem 4.4 for $G = SO_0(d, 1)$.

Both (4.32) and its generalization to $SO_0(d, 1)$ in [1] result from the trace-formula by the choice of a suitable admissible function f_s. One must, of course, compute $\hat{f}_s$ and F_{f_s}. In fact, let $x \in G$, and $x = ka_\mathfrak{p}k'$, $k, k' \in K$, $a_\mathfrak{p} \in A_\mathfrak{p}$, be its polar decomposition. Put $|H_0| = c$ (recall that this equals $\sqrt{2p + 8q}$). Let $\beta \in \Sigma$ be as in § 2, and put $t = t(a_\mathfrak{p}) = \beta(\log a_\mathfrak{p})$. Then t can be regarded as a coordinate on $A_\mathfrak{p}$. Consider, for a complex S, the function $f_s(x) = (\cosh t)^{-s}$ where $t = t(a_\mathfrak{p})$ and $x = ka_\mathfrak{p}k'$. f_s is clearly spherical. If $\mathrm{Re}\, s > p + 2q$, one can show that $f_s \in \mathscr{C}_1(K\backslash G/K)$, so that f_s is admissible. (4.32) and its generalization result from applying the trace formula to this f_s. It is possible to compute the analogue of (4.32) for all the groups of rank $(G/K) = 1$ by computing $\hat{f}_s$, F_{f_s} directly. Since the main application of these formulas was to get

Theorem 4.4 which we have obtained by other means, it does not seem worthwhile to give details of the derivation. We will content ourselves with quoting the result, which may amuse the reader:

$$
\begin{aligned}
&\sum_{\omega\in\mathscr{E}(G,1)} n_\Gamma(\omega,1)\pi^{(p+q+1)/2}\frac{\Gamma(\frac{1}{2}(s-s_-(\lambda_\omega)))\Gamma(\frac{1}{2}(s-s_+(\lambda_\omega)))}{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}(s-q+1))} \\
(4.33)\qquad &= \mathrm{Vol.}\,(\Gamma/G) + \pi^{(p+q+1)/2}\cdot 2^{1-s+(p+2q)/2}\frac{\Gamma(s-\frac{1}{2}(p+2q))}{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}(s-q+1))} \\
&\times \sum_{\gamma\in C_\Gamma-\{1\}} l(\gamma)j(\gamma)^{-1}C(h(\gamma))(\cosh l(\gamma))^{-s+(p+2q)/2},
\end{aligned}
$$

where $s_\pm(\lambda_\omega)$ are the roots of the equation

$$s^2 - s(p+2q) + \tfrac{1}{4}(p+2q)^2 + \lambda_\omega(H_0)^2 = 0 .$$

Thus

$$s_\pm(\lambda_\omega) = \tfrac{1}{2}(p+2q) \pm \sqrt{-\lambda_\omega(H_0)^2} = \rho(H_0) \pm i\lambda_\omega(H_0) .$$

The reader will easily check that when $p = d-1$, $q = 0$ (which is appropriate for $G = SO_0(d,1)$), one gets from this the formula of [1, p. 118]. (4.32) results from $p = 1$, $q = 0$. The difference of metrics must be borne in mind. For the other groups G, the values of p, q are as follows: When $G = SU(d,1)$, $p = 2(d-1)$ and $q = 1$; When $G = Sp(d,1)$, $p = 4(d-1)$ and $q = 3$. When $G = F_{4(-20)}$, $p = 8$ and $q = 7$.

A final application of these methods which may be worth mentioning is the following. Let $x, y \in G$, and let for any $r > 0$, $Q(x,y,r)$ be the number of elements $\gamma \in \Gamma$, such that $\sigma(y^{-1}\gamma x) \le r$. $Q(x,y,r)$ is the number of points k on G/K which lie in a ball of radius r around the point yK.

The computation of $\hat{f}_s$ alluded to above enables us to find the asymptotic behaviour of $Q(x,y,r)$ as $r \to \infty$; (cf. [1]). Briefly, the method is as follows: Since f_s is admissible, $\sum_{\gamma\in\Gamma} f_s(x\gamma y^{-1})$ converges nicely and can be expanded as a series $\sum_{\omega\in\mathscr{E}(G,1)}\sum_{i=1}\hat{f}_s(\lambda_\omega)\cdot\psi^i_{\lambda_\omega}(x)\overline{\psi^i_{\lambda_\omega}(y)}$, where $\psi^i_{\lambda_\omega}$, $1 \le i \le n_\Gamma(\omega,1)$, are eigenfunctions of Ω in $L_2(\Gamma\backslash G/K)$, corresponding to the eigenvalue Ω_ω. Now $\sum_\Gamma f_s(y^{-1}\gamma x) = \sum_\Gamma (\cosh\alpha(y^{-1}\gamma x)/c)^{-s}$, with $c = \sqrt{2p+8q}$ as before, which can be viewed as a Dirichlet series, convergent if Re $s > p+2q$. On the right side, the computation of $\hat{f}_s$ allows one to conclude that this Dirichlet series has a single simple pole at $s = p+2q$ whose residue can be computed. Applying the theorem of Wiener-Ikehara one gets

$$(4.34)\quad Q(x,y,r) \sim \frac{2\cdot\pi^{(p+q+1)/2}}{\Gamma(\frac{1}{2}(p+q+1))\cdot 2|\rho|\,\mathrm{Vol.}\,(\Gamma\backslash G)}\cdot\frac{e^{2|\rho|r}}{2^{p+2q}},$$

$$\text{as } r \to \infty .$$

We leave the details to the reader.

A result analogous to Theorem 4.4 has been announced by Margulis [13]. See also Sinai [20]. These authors use ergodic theory. Margulis' result is the stronger one. His context is that of an arbitrary compact manifold of negative curvature, and he shows that $Q_0(l) \sim Cl^{-1} \exp dl$, for some positive d. In our special situation, we have been able to relate this constant d to the structure of the manifold. Margulis' proofs have not appeared, as far as the author knows.

Added in proof. After this paper went to press, D. Hejhal pointed out to me that the proof of Theorem 4.4, as well as of the analogous theorem in McKean's paper, is based on an incorrect application of Karamata's theorem. However, the conclusion of the theorem is correct. There are several ways of filling the gap. One is to use Hüber's method as indicated above, exploiting (4.33). The other is to use the heat kernel in the trace formula, and to study the behaviour of that formula for complex t in a sector. The third, and the most satisfactory, method is to study the Dirichlet series $\sum l(\delta) \exp -sl(\delta)$, $s \in \boldsymbol{C}$. By using the analytic properties of the Selberg zeta function (See R. Gangolli, Ill. J. Math. **21** (1977) 1–41), one can show that this series is meromorphic in $\operatorname{Re}(s) > 2|\rho| - \varepsilon$ for some $\varepsilon > 0$, and has a single simple pole at $s = 2|\rho|$ with residue $\frac{1}{2}|\rho|$. Now Wiener-Ikehara's theorem yields Theorem 4.4. (This method is described for noncompact G/Γ in a forthcoming paper of G. Warner and the author.) For yet another method, and a better result, see D. DeGeorge, Ann. Sci. École Norm. Sup. **10**(1977) 133–153.

Bibliography

[1] L. Berard-Bergéry, *Laplacien et geodesiques fermeés sur les formes despace hyperbolique compactes*, Séminaire Bourbaki, 24ième Annee, 1971–72, Exp. 406.

[2] R. L. Bishop & B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Soc. **145** (1969) 1–48.

[3] T. Eaton, Thesis, University of Washington, Seattle, 1972.

[4] R. Gangolli, *Asymptotic behaviour of spectra of compact quotients of certain symmetric spaces*, Acta Math. **121** (1968) 151–192.

[5] ——, *Spherical functions on semisimple Lie groups*, Geometry and analysis on symmetric spaces, Marcel Dekker, New York, 1972.

[6] ——, *Spectra of discrete uniform subgroups*, Geometry and analysis on symmetric spaces, Marcel Dekker, New York, 1972.

[7] R. Gangolli & G. Warner, *On Selberg's trace formula*, J. Math. Soc. Japan **27** (1975) 328–343.

[8] Harish-Chandra, *Spherical functions on semisimple Lie groups*. I, II, Amer. J. Math. **80** (1958) 241–310, 533–613.

[9] ——, *Invariant eigendistributions on a semisimple Lie group*, Trans. Amer. Math. Soc. **119** (1965) 457–508.

[10] ——, *Discrete series for semisimple Lie groups*. II, Acta Math. **116** (1966) 1–111.

[11] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

[12] H. Huber, *Zur analytischen theorie hyperbolisher raumformen und bewegungsgruppen*. I, Math. Ann. **138** (1959) 1–26.

[13] G. A. Margulis, *Certain applications of ergodic theory to the investigation of manifolds of negative curvature*, J. Functional Anal. i Priložen **3** (1969) 89–90, (Russian).
[14] H. P. McKean, *Selberg's trace formula as applied to a compact Riemann surface*, Comm. Pure Appl. Math. **25** (1972) 225–246.
[15] S. Minakshisundaram & A. Pleijel, *Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds*, Canad. J. Math. **1** (1949) 242–256.
[16] V. Ozols, *On the critical points of the displacement function of an isometry*, J. Differential Geometry **3** (1969) 411–432.
[17] A. Preismann, *Quelques proprietes globales des espaces de Riemann*, Comment. Math. Helv. **15** (1943) 175–216.
[18] A. Selberg, *Harmonic analysis and discontinuous subgroups in weekly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956) 47–87.
[19] Séminaire Berger, *Variétés à courbures negative*, Université Paris VII, 1970/71.
[20] Y. Sinai, *The asymptotic behaviour of the number of closed geodesics on a compact manifold of negative curvature*, Izv. Akad. Nauk. SSSR Ser. Mat. **30** (1966) 1275–1296.
[21] P. Trombi & V. S. Varadarajan, *Spherical transforms on semisimple Lie groups*, Ann. of Math. **94** (1971) 246–303.
[22] N. Wallach, *An asymptotic formula of Gelfand and Gangolli for the spectrum of* $\Gamma\backslash G$, J. Differential Geometry **11** (1976) 91–101.
[23] G. Warner, *Harmonic analysis on semisimple Lie groups*. I, II, Springer, Berlin, 1972.
[24] H. Weyl, *Das asymptotische Verteilungsgesetz der eigenschwingungen eines beliebig gestalteten elastischen korpers*, Rend. Circ. Mat. Palermo **39** (1915) 1–50.
[25] D. Widder, *The Laplace transform*, Princeton University Press, Princeton, 1941.
[26] N. Wiener, *Tauberian theorems*, Ann. of Math. **33** (1932) 1–100.
[27] J. A. Wolf, *Spaces of constant curvature*, McGraw-Hill, New York, 1967.

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