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RIEMANNIAN S-MANIFOLDS

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1. Let M be an n-dimensional connected Riemannian manifold, and $I(M)$ the group of isometries of M. If there is a map $s: M \to I(M)$ such that for every $x \in M$ the image $s(x) = s_x$ is an isometry of M having x as an isolated fixed point, then the isometry s_x is called Riemannian symmetry at x or simply symmetry at x. The Riemannian manifold M with this property is called Riemannian s-manifold. If there is a positive integer k such that $s_x^k = \text{id.}, \forall x \in M$, then M is called a Riemannian s-manifold of order k or simply k-symmetric Riemannian space. The usual Riemannian symmetric spaces are Riemannian s-manifolds of order 2.

The aim of the present paper is to prove that every Riemannian s-manifold M can carry another s'-structure $\{s'_x : x \in M\}$ such that M with $\{s'_x : x \in M\}$ becomes a k-symmetric Riemannian space. The decomposition of a simply connected Riemannian s-manifold into simply connected irreducible Riemannian s-manifolds is also studied. Finally, the problem of Riemannian s-manifolds is reduced to the study of special Lie algebras.

2. We do not assume that the map $s: M \to I(M)$ is continuous. The point $x \in M$ for this symmetry s_x is an isolated fixed point if and only if the orthogonal transformation $(s_x)_{*x}$ on the tangent space $T_x(M)$ of M at x does not have eigenvalue 1.

The following Theorem in known [6, p. 451].

Theorem 2.1. *The group of all isometries $I(M)$ on a Riemannian s-manifold M acts transitively on it.*

From this theorem we conclude that the Riemannian s-manfold M is a homogeneous space, which is $M = I(M)/H$, where H is the isotropy subgroup of $I(M)$ at any arbitrary point of M.

It can be easily proved, applying the same method as in [2], that the subgroup G of $I(M)$ generated by the symmetries of M acts transitively on M. Therefore we can state the theorem.

Theorem 2.2. *Let M be a Riemannian s-manifold. Then $M = G/H$, where G is the closed subgroup of all symmetries of M, and H is the isotropy subgroup of G at any point of M.*

Let s_x be the symmetry at the point $x \in M$. We can consider $(ds_x)_x$ as an element of the orthogonal group $O(n)$. Let f be a real-valued function on $O(n)$ defined by

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$$f: O(n) \to \boldsymbol{R}\,, \qquad f: (ds_x)_x = A \to f(A) = |A - I| \in \boldsymbol{R}\,,$$

where I is the identity matrix. The function f is continuous. Since $|(ds_x)_{x-1}| \neq 0$, we conclude that there exists a neighborhood of $(ds_x)_x$ whose all elements do not have eigenvalue 1 and hence a neighborhood of s_x containing only symmetries of M.

From the above we have the theorem.

Theorem 2.3. *Let $M = G/H$ be a Riemannian s-manifold. If $s \in G$, then there is a neighborhood of s consisting only of symmetries of M.*

Now we prove

Theorem 2.4. *Let M be a connected Riemannian s-manifold. There exists another s'-structure $\{s'_x : x \in M\}$ on M such that M with $\{s'_x : x \in M\}$ becomes a k-symmetric Riemannian space.*

It is known that $M = G/H$, where G is the group of isometries. H is called the origin of M and is denoted by 0. Let s_O be the symmetry of M at 0. The following relation holds: $(ds_O)_O = \mathrm{ad}\,(s_O)$, where ad is the adjoint representation.

We assume that s_O does not have finite order. It is possible to choose another symmetry s'_O of finite order.

Let H^0 be the identity component of H. If $s_O \in H^0$, then there is a maximal torus T in H^0, passing through s_O. Therefore ad (s_O) can be written as a matrix in the form

$$\mathrm{ad}\,(s_O) = \begin{pmatrix} \cos 2\pi\vartheta_1(t) & \sin 2\pi\vartheta_1(t) & & & \\ -\sin 2\pi\vartheta_1(t) & \cos 2\pi\vartheta_1(t) & & & \\ & & \ddots & & \\ & & & \cos 2\pi\vartheta_l(t) & \sin 2\pi\vartheta_l(t) \\ & & & -\sin 2\pi\vartheta_l(t) & \cos 2\pi\vartheta_l(t) \end{pmatrix},$$

where $\vartheta_1, \cdots, \vartheta_l$ are homomorphisms of T into $S^1 = \boldsymbol{R}/Z$ which induce real linear forms $a_i : L(T) \to \boldsymbol{R}$, where a_i, $i = 1, \cdots, l$, are called the roots of H with respect to the torus T. From the above we obtain the following commutative diagram

$$\begin{array}{ccc} L(T) & \xrightarrow{\quad a_i \quad} & \boldsymbol{R} \\ (x_1, \cdots, x_m) & & a_i(x_1, \cdots, x_m) = b_{i1}x_1 + \cdots + b_{im}x_m \\ \downarrow & & \downarrow \\ T & \xrightarrow{\quad \vartheta_i \quad} & S^1 = \boldsymbol{R}/T \\ (x_1, \cdots, x_m), \bmod Z^m & & \vartheta_i(x_1, \cdots, x_m), \bmod Z^m \\ & & = b_{i1}x_1 + \cdots + b_{im}x_m, \bmod Z \end{array}$$

where m is the rank of H, and $b_{j_1 j_2} \in Z$, $1 \leq j_1 \leq l$, $1 \leq j_2 \leq m$.

We assume that the symmetry s_O has infinite order, which means that at least one of the values ϑ_i, $1 \leq i \leq l$, is an irrational number. From this we conclude that at least one of x_j, $1 \leq j \leq m$, is irrational. Therefore some, or all of the m-tuple numbers $(x_1, \cdots, x_m)$, to which the symmetry s_O corresponds, are irrational. We substitute these irrational numbers by rational ones as close to them as we wish. Hence we obtain another symmetry s'_O, which has finite order.

Now we assume that $s_O \notin H^0$. Therefore there exists an integer λ such that $s_O^\lambda \in H^0$. Since s_O has infinite order, equally so does s_O^λ. Let T_1 be the maximal torus in H^0 passing through s_O^λ.

The symmetry s_O can be considered as an orthogonal matrix. Therefore another orthogonal matrix β exists such that

$$\beta s_O \beta^{-1} = \begin{pmatrix} \cos 2\pi\tau_1 & \sin 2\pi\tau_1 & & & \\ -\sin 2\pi\tau_1 & \cos 2\pi\tau_1 & & & \\ & & \ddots & & \\ & & & \cos 2\pi\tau_m & \sin 2\pi\tau_m \\ & & & -\sin 2\pi\tau_m & \cos 2\pi\tau_m \end{pmatrix},$$

where at least one of the numbers $\tau_1, \cdots, \tau_m$ is irrational. From the above we obtain

$$\beta s_O^\lambda \beta^{-1} = \begin{pmatrix} \cos 2\pi\lambda\tau_1 & \sin 2\pi\lambda\tau_1 & & & \\ -\sin 2\pi\lambda\tau_1 & \cos 2\pi\lambda\tau_1 & & & \\ & & \ddots & & \\ & & & \cos 2\pi\lambda\tau_m & \sin 2\pi\lambda\tau_m \\ & & & -\sin 2\pi\lambda\tau_m & \cos 2\pi\lambda\tau_m \end{pmatrix}.$$

Since $s_O^\lambda \in T_1$, there is another base such that s_O^λ can be written

$$s_O^\lambda = \begin{pmatrix} \cos 2\pi\lambda\tau'_1 & \sin 2\pi\lambda\tau'_1 & & & \\ -\sin 2\pi\lambda\tau'_1 & \mathrm{con}\, 2\pi\lambda\tau'_1 & & & \\ & & \ddots & & \\ & & & \cos 2\pi\lambda\tau'_m & \sin 2\pi\lambda\tau'_m \\ & & & -\sin 2\pi\lambda\tau'_m & \cos 2\pi\lambda\tau'_m \end{pmatrix},$$

where at least two of the numbers $(1, \lambda\tau'_1, \cdots, \lambda\tau'_m)$ are linearly independent of the field of rational numbers. Therefore s_O^λ generates at least one-dimensional torus $T'_1 \subseteq T_1$ and closure $\{s_O^{\lambda m}, m \geq n_0\} = T'_1$ and the elements of T'_1 commute with s_O.

From the above we conclude that there exists an element $\alpha \in T_1$ which can be written

$$\alpha = \begin{pmatrix} \cos 2\pi(p'_1 - \tau'_1) & \sin 2\pi(p'_1 - \tau'_1) & & & \\ -\sin 2\pi(p'_1 - \tau'_1) & \cos 2\pi(p'_1 - \tau'_1) & & & \\ & & \ddots & & \\ & & & \cos 2\pi(p'_m - \tau'_m) & \sin 2\pi(p'_m - \tau'_m) \\ & & & -\sin 2\pi(p'_m - \tau'_m) & \cos 2\pi(p'_m - \tau'_m) \end{pmatrix},$$

where $p'_1, \cdots, p'_m$ are rational numbers close to $\tau'_1, \cdots, \tau'_m$, as we wish, respectively, and $p'_i = \tau'_i$, if τ'_i is rational.

The same element α with respect to the old base can be written

$$\beta\alpha\beta^{-1} = \begin{pmatrix} \cos 2\pi(p_1 - \tau_1) & \sin 2\pi(p_1 - \tau_1) & & & \\ -\sin 2\pi(p_1 - \tau_1) & \cos 2\pi(p_1 - \tau_1) & & & \\ & & \ddots & & \\ & & & \cos 2\pi(p_m - \tau_m) & \sin 2\pi(p_m - \tau_m) \\ & & & -\sin 2\pi(p_m - \tau_m) & \cos 2\pi(p_m - \tau_m) \end{pmatrix}.$$

Since α and s_O commute, we obtain

$$\beta\alpha s_O\beta^{-1} = \beta\alpha\beta^{-1}\beta s_O\beta^{-1} = \begin{pmatrix} \cos 2\pi p_1 & \sin 2\pi p_1 & & & \\ -\sin 2\pi p_1 & \cos 2\pi p_1 & & & \\ & & \ddots & & \\ & & & \cos 2\pi p_m & \sin 2\pi p_m \\ & & & -\sin 2\pi p_m & \cos 2\pi p_m \end{pmatrix},$$

where p_i, $i = 1, \cdots, m$, have the same meaning as p'_i.

Therefore the symmetry αs_O belongs to the same component of H as the given symmetry s_O, having finite order.

Proposition 2.5. *Let $M = G/H$ be a compact Riemannian s-manifold. The symmetry s_O belongs to the identity component H^0 of H if and only if rank G = rank H.*

We assume that the symmetry s_O belongs to H^0. From s_O we obtain an automorphism A on G:

$$A : G \to G , \qquad A : v \to A(v) = s_O v s_O^{-1} ,$$

and an automorphism α on the Lie algebra g of G:

$$\alpha : g = h + m \to g = h + m , \quad \alpha : X \to \alpha(X) \in h , \qquad \forall X \in h .$$

Let T_1, T_2 be the maximal tori of H and G, respectively, through the element s_O. Since $T_1 \subseteq T_2$ and all the elements of T_2 commute with s_O, so do the elements of T_1. Since the vectors belonging to the tangent space of T_2 at the identity element are invariant by α, we conclude that $T_2 \subseteq H$ and therefore rank G = rank H.

The inverse is an immediate consequence of the assumption $\operatorname{rank} G = \operatorname{rank} H$; then we have that $s_O \in H^0$.

Corollary 2.6. *Let $M = G/H$ be a Riemannian homogeneous space such that H is the largest isotropy subgroup of G at one point of M. If H is connected and* $\dim H$ *is odd, then M can never be a Riemannian s-manifold.*

If we assume that M is a Riemannian s-manifold, then $s_O \in H$ and there is always a maximal torus T in H through s_O. However since $\dim M$ is odd we obtain $\operatorname{ad}(s_O)$ having an eigenvalue 1. So we reach to a contradiction because $\operatorname{ad}(s_O)$ never has an eigenvalue 1. Therefore M can not be a Riemannian s-manifold.

Remark 2.7. From the above we conclude that all Riemannian s-manifolds form a proper subset of all Riemannian homogeneous spaces.

3. Let $M = G/H$ be a simply connected homogeneous space. It is known that M is isometric to the direct product $M_0 \times M_1 \times \cdots \times M_r$ and that the identity component $I^0(M)$ of the group of isometries $I(M)$ is naturally isomorphic to the group $I^0(M_0) \times I^0(M_1) \times \cdots \times I^0(M_r)$.

We shall prove that each of the homogeneous spaces $M_0, M_1, \cdots, M_r$ is a Riemannian s-manifold. To this aim we distinguish two cases.

(i) If $s \in I^0(M)$, then we have

$$\begin{aligned} s &: M = M_0 \times M_1 \times \cdots \times M_r \to M = M_0 \times M_1 \times \cdots \times M_r , \\ s &: 0 = (0_0, 0_1, \cdots, 0_r) \to 0 = (0_0, 0_1, \cdots, 0_r) , \\ s &: x = (x_0, x_1, \cdots, x_r) \to s(x) = (y_0, y_1, \cdots, y_r) , \end{aligned}$$

where $y_i = s_i(x_i) = p_i(s(x))$, p_i is the natural projection of M into M_i, and s_i is an isometry of M_i [4, p. 241].

By considering the de Rham decomposition theorem for the tangent space of M at 0, we have

$$T_0(M) = T_0^{(0)}(M) \oplus T_0^{(1)}(M) \oplus \cdots \oplus T_0^{(r)}(M) . \tag{3.1}$$

Since $s \in I^0(M)$, we have $\operatorname{ad}(s)(T_0^{(i)}(M)) = T_0^{(i)}(M)$, where $i = 0, 1, \cdots, r$ or $\operatorname{ad}(s_i)(T_0^{(i)}(M)) = T_0^{(i)}(M) = \operatorname{ad}(s)(T_0^i(M))$, [4, p. 240]. We also have $s_i : M_i \to M_i$, $s_i : 0_i \to 0_i$ and hence s_i is symmetry at 0_i for the manifold M_i. Therefore M_i, $i = 0, 1, \cdots, r$, is a Riemannian s-manifold. The order of s is the least common multiple of the integers $\{k_0, k_1, \cdots, k_r\}$ where k_i, $i = 0, 1, \cdots, r$, is the order of s_i.

(ii) If $s \notin I^0(M)$, then we obtain an orbit $(M_i^1, M_i^2, \cdots, M_i^r)$ of the permutation group defined by s, and consider the product

$$M_{(i)} = M_i^1 \times M_i^2 \times \cdots \times M_i^{ri} .$$

If $r_1 > 1$, then we can order $M_i^1, M_i^2, \cdots, M_i^{ri}$ such that s maps M_i^j isometrically onto $M_i^{\lambda+1}$, where $1 \leq \lambda \leq r_i - 1$, and $M_i^{r_i}$ isometrically onto M_i^1. This can always be done after some identifications. Therefore M be written

$$M = M_0 \times M_{(1)} \times \cdots \times M_{(\mu)} ,$$

where M_0 is the Euclidean part of M and $M_{(i)}$, $i = 1, \cdots, \mu$, have the above meaning.

With the same technique, as in case (i), we can prove that s can be written $s = (\psi_0, \psi_1, \cdots, \psi_\mu)$, where ψ_i, $i = 1, \cdots, \mu$, is a symmetry on the manifold $M_{(i)}$ having also the following properties

$$\psi_i : M_i^1 \times M_i^2 \times \cdots \times M_i^{r_i} \to M_i^1 \times M_i^2 \times \cdots \times M_i^{r_i} ,$$

$$\psi_i : (0_1, 0_2, \cdots, 0_{r_i}) \to (0_1, 0_2, \cdots, 0_{r_i}) , \tag{3.2}$$

$$\psi_i : (M_i^1 \times 0_2 \times \cdots \times 0_{r_i}) \to (0_1 \times M_i^2 \times \cdots \times 0_{r_i}) , \tag{3.3}$$

$$\cdots\cdots\cdots\cdots$$

$$\begin{aligned} \psi_i : (0_1 \times 0_2 \times \cdots \times 0_{r_i-2} \times M^{r_i-1} \times 0_{r_i}) \\ \to (0_1 \times 0_2 \times \cdots \times 0_{r_i-1} \times M_i^{r_i}) , \end{aligned} \tag{3.4}$$

$$\psi_i : (0_1 \times 0_2 \times \cdots \times 0_{r_i-1} \times M_i^{r_i}) \to (M_i^1 \times 0_2 \times \cdots \times 0_{r_i}) . \tag{3.5}$$

We can identify the manifold M_i^1 with $M_i^2, \cdots, M_i^{r_i}$ by virtue of the following mappings

$$f_v : M_i^1 \to M_i^v , \qquad v = 2, \cdots, r_i ,$$

where $f_2 = p_i^{(2)} \circ \psi_i$, $f_3 = f_2 \circ p_i^{(3)} \circ \psi_i, \cdots, f_{r_i} = f_{r_i-1} \circ \cdots \circ f_2 \circ p_i^{(r_i)} \circ \psi_i$, and $p_i^{(2)}, \cdots, p_i^{(r_i)}$ are the natural projections of $M_{(i)}$ into $M_i^2, \cdots, M_i^{r_i}$, respectively.

The mapping, defined by (3.5), can be considered as an isometry of $M_i^{r_i}$ onto M_i^1 after the following identification

$$f_1 : M_i^1 \to M_i^1 , \qquad f_1 = f_{r_i} \circ f_{r_i-1} \circ \cdots \circ f_2 \circ p_i^{(1)} \circ \psi_i ,$$

where $p_i^{(1)}$ is the natural projection of $M_{(i)}$ into M_i^1. From the construction of f_1, we conclude that f_1 has 0_1 as a fixed point,

Let $T_{0'}(M_{(i)})$ be the tangent space of $M_{(i)}$ at the point $0' = (0_1, 0_2, \cdots, 0_{r_i})$. Then we have

$$T_{0'}(M_{(i)}) = T_{0'}^{(1)}(M_{(i)}) \oplus T_{0'}^{(2)}(M_{(i)}) \oplus \cdots \oplus T_{0'}^{(r_i)}(M_{(i)}) ,$$

and ad (ψ_i) has the properties:

$$\operatorname{ad}(\psi_i): T_{0'}^{\lambda}(M_{(i)}) \to T_{0'}^{\lambda+1}(M_{(i)}) , \qquad \lambda = 1, \cdots, r_i - 1 ,$$
$$\operatorname{ad}(\psi_i): T_{0'}^{r_i}(M_{(i)}) \to T_{0'}^{(1)}(M_{(i)}) ,$$

from which we obtain $\operatorname{ad}(\psi_i) = A_1 \times A_2 \times \cdots \times A_{r_i}$, where $A_j, j = 1, \cdots, r_i$, are defined as follows

$$A_\mu : T_{0'}^{\mu}(M_{(i)}) \to T_{0'}^{\mu+1}(M_{(i)}) , \qquad \mu = 1, \cdots, r_i - 1 ,$$
$$A_{r_i} : T_{0'}^{r_i}(M_{(i)}) \to T_{0'}^{1}(M_{(i)}) .$$

We assume that the mapping f_1 is not a symmetry for the point 0_1 of M_i^1. Therefore there is a vector $u_1 \in T_{0'}^1(M_{(i)}) = T_{0_1}(M_{(i)}^1)$ which is invariant under $d(f_1)_{0_1} = \operatorname{ad}(f_1)$. From this vector we obtain the following sequence of vectors: $u_2 = \operatorname{ad}(f_2)(u_1) \in T_{0'}^2(M_{(i)}), \cdots, u_{r_i-1} = \operatorname{ad}(f_{r_i-1})u_{r_i-2}) \in T_{0'}^{r_i-1}(M_{(i)})$, $u_{r_i} = \operatorname{ad}(f_{r_i})(u_{r_i-1}) \in T^{r_i}(M_{(i)})$, $\operatorname{ad}(f_1)(u_{r_i}) = u_1 \in T_{0'}^1(M_{(i)})$. Hence $\operatorname{ad}(\psi_i)$, by the form of a matrix, can be written

$$B = \begin{bmatrix} 0 & A_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_2 & \cdots & 0 & 0 \\ \cdot & \cdot & & \cdots & & \cdot \\ 0 & 0 & 0 & \cdots & A_{r_i-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & A_{r_i-1} \\ A_{r_i} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} .$$

Let u be the vector of $T_{0'}(M_{(i)})$ with coordinates $u_1, u_2, \cdots, u_{r_i}$. Then we have

$$(3.6) \quad Bu = \begin{bmatrix} 0 & A_1 & 0 & \cdots & 0 \\ 0 & A_2 & A_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & A_{r_i-1} \\ A_{r_i} & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{r_i} \end{bmatrix} = \begin{bmatrix} A_{r_i}u_1 \\ A_1u_2 \\ \vdots \\ A_{r_i-1}u_{r_i} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{r_i} \end{bmatrix} = u .$$

From (3.6) we conclude that $\operatorname{ad}(\psi_i)$ leaves the vector u fixed, and therefore ψ_i is not a symmetry. But this is not true because ψ_i is a symmetry. Therefore f_1 is a symmetry.

The order of the k-symmetric Riemannian space M is the least common multiple of the orders $k_0, k_1, \cdots, k_\mu$ of the manifolds $M_0, M_{(1)}, \cdots, M_{(\mu)}$, respectively. Each order $k_i, i = 0, 1, \cdots, \mu$, has the form $r_i q$, where q is the least common multiple of (rank $(A_1), \cdots,$ rank (A_{r_i})). Hence we have

Theorem 3.1. *Let M be a simply connected Riemannian s-manifold. This manifold splits into the product manifolds $M_0 \times M_1 \times \cdots \times M_r$ each of which is a simply connected, irreducible Riemannian s-manifold.*

4. Let $M = G/H$ be a k-symmetric Riemannian space, and s_O the symmetry of M at its origin 0. From this symmetry s_O we obtain an automorphism A on G defined by

$$A : G \to G \ , \qquad A : v \to A(v) = s_O v s_O^{-1} \ .$$

Proposition 4.1. *Let $M = G/H$ be a k-symmetric Riemannian space. Then the automorphism A on G has order k and preserves the isotropy subgroup H.*

From the definition of A we have

$$\begin{aligned} &A : G \to G \ , \qquad A : v \to A(v) = s_O v s_O^{-1} \ , \\ &A : s_O v s_O^{-1} \to A(s_O v s_O^{-1}) = s_O s_O v s_O^{-1} s_O^{-1} = s_O^2 v (s_O^{-1})^2 \ , \\ &A : s_O^{k-1} v (s_O^{-1})^{k-1} \to A(s_O^{k-1} v (s_O^{-1})^{k-1}) \to s_O^k v (s_O^{-1})^k = v \ . \end{aligned}$$

Thus we conclude that $A^k = \text{id.}$, that is, A has order k. If $\mu \in H$, then we obtain $A(\mu) = s_O \mu s_O^{-1}$. It is known that $s_O : M \to M$, $\mu : M \to M$, $s_O^{-1} : M \to M$, $s_O : 0 \to s_O(O) = 0$, $\mu : 0 \to \mu(0) = 0$, $s_O^{-1} : O \to s_O^{-1}(O) = 0$, from which we obtain $s_O \mu s_O^{-1} \in H$, that is, A preserves H.

Definition 4.2. The triplet (G, H, A) is called a k-symmetric Lie group, where G is a Lie group, H is a closed subgroup of G, and A is an automorphism on G of order k with the property $A(H) \subseteq H$.

Let $M = G/H$ be a k-symmetric Riemannian space. We consider the Lie algebras g, h of G and H, respectively. Then we have

$$g = h + m \ ,$$

where m can be identified with the tangent space $T_0(M)$ of M at its origin 0. From s_O we can also obtain an automorphism α on g defined as follows:

$$\alpha : g = h + m \to g = h + m \ , \qquad \alpha : X \to \alpha(X) = \operatorname{Ad}(s_O)X \ ,$$

where $\operatorname{Ad}(s_O) = \operatorname{ad}_*(s_O)$. The following is also known:

$$\exp : g \to G \ , \qquad \exp : X \to \exp X \ ,$$

$$\exp\{\operatorname{Ad}(s_O)X\} = s_O \exp X s_O^{-1} \ . \tag{4.1}$$

Proposition 4.3. *Let $M = G/H$ be a k-symmetric Riemannian space, α the automorphism on $g = h + m$ obtained by s_O. Then h is preserved by α, which has order k.*

If $X \in h$, then $\exp X = \lambda \in H$. Since $\lambda \in H$, we have $s_O \lambda s_O^{-1} \in H$, which implies $s_O \exp X s_O^{-1} \in H$. From this and (4.1) we obtain

$$\exp\{\operatorname{Ad}(s_O)(X)\} = s_O \exp X s_O^{-1} \in H \ ,$$

which gives $\mathrm{Ad}\,(s_O)(X) \in h$. Therefore h is preserved by $\alpha = \mathrm{Ad}\,(s_O)$.

From the definition of α and formula (4.1) we have

$$\begin{gathered}
\alpha: g \to g\,, \quad \alpha: X \to \alpha(X) = \alpha(X) = \mathrm{Ad}\,(s_O)(X)\,, \\
\exp\{\mathrm{Ad}\,(s_O)(X)\} = s_O \exp X s_O^{-1}\,, \\
\alpha: \mathrm{Ad}\,(s_O)(X) \to \mathrm{Ad}\,(s_O)\{\mathrm{Ad}\,(s_O)(X)\} = \mathrm{Ad}^2\,(s_O)X\,, \\
\exp\{\mathrm{Ad}^2\,(s_O)(X)\} = s_O\{\exp((\mathrm{Ad}\,(s_O))(X)\}s_O^{-1} = s_O\{s_O \exp X s_O^{-1}\}s_O^{-1} \\
= s_O^2 \exp X (s_O^{-1})^2\,,
\end{gathered}$$

which imply

$$\exp\{\mathrm{Ad}^k\,(s_O)(X)\} = s_O^k \exp X (s_O^{-1})^k\,,$$

showing that $\alpha = \mathrm{Ad}\,(s_O)$ has order k.

Definition 4.4. The triplet (g, h, α) is called a k-symmetric Lie algebra, where g is a Lie algebra, h is a Lie subalgebra of g, and α is an automorphism on g of order k with the property $\alpha(h) \subseteq h$.

Let $M = G/H$ be a k-symmetric Riemannian space. If g and h are the Lie algebras of G and H, respectively, then we have

$$g = h + m\,, \qquad \alpha(h) \subseteq h\,,$$

where α is the automorphism on g of order k, and $m = g/h$. It is known that the Riemannian metric $\bar{g}$ on M is G-invariant, which gives an $\mathrm{Ad}\,(H)$-invariant nondegenerate symmetric bilinear form B on $m = g/h$ defined by

$$B(\bar{X}, \bar{Y}) = \bar{g}(X, Y)\,, \qquad X, Y \in g\,,$$

where $\bar{X}, \bar{Y}$ are the elements of g/h represented by X, Y, respectively.

From the above we conclude that given a k-symmetric Riemannian space we then have a k-symmetric Lie group (G, H, A), a k-symmetric Lie algebra (g, h, α), and an $\mathrm{Ad}\,(H)$-invariant nondegenerate symmetric bilinear form on $m = g/h$.

Definition 4.5. Let $M = G/H$ be a k-symmetric Riemannian space. If the symmetry s_O commutes with all the elements of H, then M is called a regular k-symmetric Riemannian space or regular Riemannian s-manifold of order k.

If a k-symmetric Riemannian manifold $M = G/H$ is regular, then the automorphism A on G preserves the subgroup H as pointwise so that $A(v) = v$, $\forall v \in H$. The same is true of the automorphism α on the Lie algebra g of G which preserves the Lie algebra h of H pointwise so that $\alpha(X) = X$, $\forall X \in h$.

The triplets (G, H, A) and (g, h, α), which are obtained by a regular k-symmetric Riemannian space, are called a regular k-symmetric Lie group and a regular k-symmetric Lie algebra, respectively.

Theorem 4.6. *Let $M = G/H$ be a regular Riemannian s-manifold. Then M is a reductive homogeneous space.*

Let g and h be the Lie algebras of G and H respectively. Then we have $g = h + m$, where m can be identified with the tangent space of M at its origin.

If $\mathrm{ad}\,(H)m \subseteq m$, then M is a reductive homogenous space. We assume that there exist $X \in m$ and $\beta \in H$ such that $\mathrm{ad}\,(\beta)(X) = Y \in h$. Since $\mathrm{ad}\,(\beta) \circ \mathrm{ad}\,(s_O) = \mathrm{ad}\,(s_O) \circ \mathrm{ad}\,(\beta)$, we have $\mathrm{ad}\,(\beta) \circ \mathrm{ad}\,(s_O)(X) = \mathrm{ad}\,(s_O) \circ \mathrm{ad}\,(\beta)(X)$, which implies $\mathrm{ad}\,(\beta)(Z) = Y$, where $Z = \mathrm{ad}\,(s_O)(X) \in m$. From $\mathrm{ad}^k\,(s_O)(X) = X$ and the fact that $\mathrm{ad}\,(\beta)$ is an automorphism, we conclude that $Z = X$ and hence $X = \mathrm{ad}\,(s_O)X$ which is impossible because s_O is a symmetry. Hence we have reached a contradiction to our assumption. This implies $\mathrm{ad}\,(\beta)(m) \subseteq m$.

Theorem 4.7. *Let (G, H, A) be a regular k-symmetric Lie group. Then there is a Riemannian metric on the homogeneous space $M = G/H$, which makes M a regular k-symmetric Riemannian space.*

First, we shall construct for each point P of $M = G/H$ a diffeomorphism s_P of order k on M, having P as an isolated fixed point. For the origin 0 of M we have the diffeomorphism s_O defined as follows:

$$s_O : M = G/H \to M = G/H \;, \qquad s_O : vH \to s_O(vH) = A(v)H \;.$$

Let $v(O)$ be a fixed point of s_O, where $v \in G$. Then $A(v) \in vH$. By putting $\mu = v^{-1}A(v) \in H$, since $v \in H$ we have $\mu^2 = \mu A(\mu) = v^{-1}A(v)A(v^{-1})A^2(v)$ and therefore $\mu^2 = v^{-1}A^2(v)$. But $\mu^2 \in H$ implies $A(\mu^2) = \mu^2$. Thus $\mu^2 = A(v^{-1})A(v^2)$. Similarly, for $r < k$ we obtain $\mu^r = A(v^{-1})A^{r+1}(v)$ and finally $\mu^k = v^{-1}A(v)A(v^{-1})A^k(v) = \mathrm{id}$ since $A^k = \mathrm{id}$. Thus μ^k is the identity element of H. Now assume that v is sufficiently close to the identity element so that μ is also near the identity element. Then μ itself must be the identity element and therefore $A(v) = v$. Being invariant by A and near the identity element, v lies in the identity component of G_A, where G_A is the setwise of G by A and hence in H. Thus $v(O) = 0$ proving our assertion that O is an isolated fixed point of s_O.

For the point $P = v(0)$ we obtain as a diffeomorphism $s_P = v \circ s_O \circ v^{-1}$. Then s_P has P as an isolated fixed point, and its order is k. This is independent of the choice of v such that $P = v(0)$.

The Lie algebra g of G can be written in the known decomposition

$$g = h + m \;.$$

We consider a special $\mathrm{ad}\,(H)$-invariant nondegenerate symmetric bilinear form B on m. From B we obtain a G-invariant Riemannian metric $\bar{g}$ on $M = G/H$, which is given by the formula $B(X, Y) = \bar{g}_0(X, Y)$ for $X, Y \in m$. It can be easily obtained that s_P is a Riemannian symmetry of order k on M at P. Hence $M = G/H$ is a regular k-symmetric Riemannian space.

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