

INTEGRAL FORMULAS FOR CLOSED SUBMANIFOLDS OF A RIEMANNIAN MANIFOLD

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Dedicated to Professor Buchin Su on his 57th birthday

1. Introduction

In 1903, H. Minkowski [11] obtained the following two integral formulas for a closed convex surface S in a Euclidean 3-space E^3 :

$$(1.1) \quad \int_S (1 + pH)dV = 0, \quad \int_S (H + pK)dV = 0,$$

where H and K are respectively the mean curvature and the Gaussian curvature of S at a point P whose position vector with respect to the origin O of E^3 is x , dV is the area element of S at P , and p is the scalar product $\langle x, e \rangle$ of x and the unit normal vector e of S at P . In 1954 C. C. Hsiung [5] extended formulas (1.1) to a closed oriented hypersurface M^m in a Euclidean $(m + 1)$ -space E^{m+1} ($m \geq 2$) and obtained characterizations of hyperspheres in E^{m+1} . In 1956 C. C. Hsiung [6] and in 1959 G. F. Feeman and C. C. Hsiung [3] extended Hsiung's integral formulas to the case in which E^{m+1} is a Riemannian space N^{m+1} of constant sectional curvature, and obtained characterizations of umbilical hypersurfaces in N^{m+1} . In 1962, Y. Katsurada [7] extended the afore-said results to a closed oriented hypersurface in N^{m+1} by introducing an infinitesimal conformal vector field ξ to replace the position vector field x . In 1968 and 1969, Y. Katsurada, H. Kôjyô and T. Nagai [8], [9], [10] obtained integral formulas for a closed oriented submanifold M^m of dimension $m (\geq 2)$ in a Riemannian n -manifold N^n ($n > m$) of constant sectional curvature with respect to an infinitesimal conformal vector field ξ and a special unit normal vector field e of M^m , and conditions for M^m to be umbilical with respect to e . In 1971 B. Y. Chen and K. Yano [1] studied the case in which the field e is more general but N^n is Euclidean and ξ is the position vector field x . The purpose of the present paper is to extend the results of Chen and Yano to the general case in which N^n is Riemannian and ξ is an infinitesimal conformal vector field so that all known results are special cases of ours.

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In § 2 we first define the vector product of two tangent vectors of a Riemannian n -manifold N^n at a point P , and then discuss orthonormal frames $Pe_{i_1}e_{i_2}\cdots e_{i_n}$ on N^n at P .

§ 3 contains the fundamental definitions and formulas for a submanifold M^m of dimension $m (\geq 2)$ immersed in $N^n (n > m)$. In particular, some formulas are reduced to simpler forms when N^n is of constant sectional curvature.

Suppose that N^n admits a continuous infinitesimal conformal vector field ξ , and let e be a unit normal vector field over M^m parallel in the normal bundle of M^m . In § 4 we derive integral formulas for a closed oriented M^m in N^n with respect to ξ and e , and in § 5 we obtain various conditions for M^m to be umbilical with respect to e .

We wish to thank Y. Katsurada for her discussion with one of us about some computation involving the infinitesimal conformal vector field ξ .

2. Vector product and orthonormal frames

Throughout this paper unless stated otherwise the ranges of indices are given as follows:

$$(2.1) \quad \begin{aligned} 1 \leq i, j, k, \dots \leq m, \\ 1 \leq \alpha, \beta, \gamma, \dots \leq n, \\ m + 1 \leq A, B, C, \dots \leq n, \quad (m < n). \end{aligned}$$

We shall also follow the usual tensor convention that when a letter appears in any term as a subscript and a superscript, it is understood that this letter is summed over its range.

Let N^n be a Riemannian manifold of dimension $n (\geq 3)$ and class C^3 , (x^1, \dots, x^n) local coordinates of a point P in N^n , and $a_{\alpha\beta}dx^\alpha dx^\beta$ a Riemannian metric of N^n , where $a_{\alpha\beta} = a_{\beta\alpha}$ and the matrix $(a_{\alpha\beta})$ is positive definite so that the determinant $|a_{\alpha\beta}| = a$ is positive.

Let A_1, \dots, A_{n-1} be $n - 1$ tangent vectors of the manifold N^n at the point P , and A_i^α the contravariant components of A_i in the local coordinate system x^1, \dots, x^n . Let $A_1 \times \dots \times A_{n-1}$ denote the vector product of the $n - 1$ vectors A_1, \dots, A_{n-1} , which is defined to be the tangent vector of the manifold M^n at P whose β -th contravariant component is (see, for instance, Feeman and Hsiung [3])

$$(2.2) \quad (A_1 \times \dots \times A_{n-1})^\beta = (-1)^{n-1} a^{-1/2} \begin{vmatrix} \delta_1^\beta & \delta_2^\beta & \dots & \delta_n^\beta \\ a_{\alpha 1} A_1^\alpha & a_{\alpha 2} A_2^\alpha & \dots & a_{\alpha n} A_n^\alpha \\ \dots & \dots & \dots & \dots \\ a_{\alpha 1} A_{n-1}^\alpha & a_{\alpha 2} A_{n-1}^\alpha & \dots & a_{\alpha n} A_{n-1}^\alpha \end{vmatrix},$$

where δ_a^β are the Kronecker deltas. Let T be a tangent vector of the manifold N^n at the point P with contravariant components T^α in x^1, \dots, x^n . From the

definition of the scalar product of any two vectors A_i and A_j , namely,

$$(2.3) \quad \langle A_i, A_j \rangle = a_{\alpha\beta} A_i^\alpha A_j^\beta,$$

it follows that the scalar product of the two vectors T and $A_1 \times \cdots \times A_{n-1}$ is given by

$$(2.4) \quad \langle T, A_1 \times \cdots \times A_{n-1} \rangle = (-1)^{n-1} a^{1/2} |T, A_1, \dots, A_{n-1}|,$$

where $|T, A_1, \dots, A_{n-1}|$ is a determinant, the elements of each of whose columns are the contravariant components of the vector indicated. Thus by (2.4) it is readily seen that the vector $A_1 \times \cdots \times A_{n-1}$ is orthogonal to each of the $n - 1$ vectors A_1, \dots, A_{n-1} .

Now consider an orthonormal frame $Pe_1 \cdots e_n$ on N^n at P , where e_1, \dots, e_n form an ordered set of n mutually orthogonal unit tangent vectors of the manifold N^n at P so that

$$(2.5) \quad \langle e_\alpha, e_\beta \rangle = a_{\gamma\delta} e_\alpha^\gamma e_\beta^\delta = \delta_{\alpha\beta},$$

where $\delta_{\alpha\beta}$ are the Kronecker deltas. The position vector x of the point P is defined to be the tangent vector of the manifold N^n at the point P whose contravariant components are the local coordinates x^1, \dots, x^n of the point P .

Let $\alpha_1, \dots, \alpha_n$ be distinct and suppose that $1 \leq \alpha_1, \dots, \alpha_n \leq n$. Then we can write

$$(2.6) \quad e_{\alpha_1} \times \cdots \times e_{\alpha_{n-1}} = c e_{\alpha_n},$$

where c is a function of the x 's. In order to find an expression for c , we consider the two matrices

$$(2.7) \quad \phi = (\phi_\beta^i), \quad \psi = (\psi_i^\beta), \quad (i = 1, \dots, n - 1),$$

where

$$(2.8) \quad \phi_\beta^i = a_{\alpha\beta} e_i^\alpha, \quad \psi_i^\beta = e_i^\beta,$$

the superscript of the element ϕ_β^i or ψ_i^β indicating the row to which the element belongs, and the subscript indicating the column. From (2.2) and (2.6) it is easily seen that

$$(2.9) \quad c e_{\alpha_n}^\gamma = (-1)^{n+\gamma} B^\gamma a^{-1/2}, \quad (\gamma = 1, \dots, n),$$

where B^γ is the determinant of the matrix of $(n - 1)$ th order obtained by deleting the γ -th column from the matrix ϕ . Substitution of (2.9) in (2.5) for $\alpha = \beta = \alpha_n$ gives

$$(2.10) \quad c^2 = B,$$

where

$$(2.11) \quad B = \begin{vmatrix} B^1 & -B^2 & \cdots & (-1)^{n-1}B^n \\ e_{\alpha_1}^1 & e_{\alpha_1}^2 & \cdots & e_{\alpha_1}^n \\ \cdot & \cdot & \cdot & \cdot \\ e_{\alpha_{n-1}}^1 & e_{\alpha_{n-1}}^2 & \cdots & e_{\alpha_{n-1}}^n \end{vmatrix},$$

which is equal to the sum of the products of the corresponding determinants of the $(n - 1)$ th order of the two matrices (2.7). By an elementary theorem on determinants (see, for instance, [2, p. 102]), from (2.5) it follows immediately that

$$(2.12) \quad B = |\phi_{\beta}^i \psi_{\beta}^j| = 1,$$

which, together with (2.10), implies that

$$(2.13) \quad c = \pm 1.$$

If the orientations of e_1, \dots, e_n are so chosen that

$$(2.14) \quad |e_1, \dots, e_n| > 0,$$

then by taking the scalar product of the vector e_{α_n} with each side of (2.6) and using (2.4), (2.13), we can easily obtain

$$(2.15) \quad |e_1, \dots, e_n| = a^{-1/2},$$

and therefore

$$(2.16) \quad e_{\alpha_1} \times \cdots \times e_{\alpha_{n-1}} = \delta_{\alpha_1 \dots \alpha_n} e_{\alpha_n},$$

where $\delta_{\alpha_1 \dots \alpha_n} = +1$ or -1 according as the permutation of $\alpha_1, \dots, \alpha_n$ into $1, \dots, n$ is even or odd.

3. Immersed submanifolds

Let $x: M^m \rightarrow N^n$ be an m -dimensional ($2 \leq m < n$) submanifold of class C^3 immersed in a Riemannian n -manifold N^n defined in § 2. For simplicity we shall write $x(M^m)$ as M^m . Let (u^1, \dots, u^m) be local coordinates of a point P on M^m . Then

$$(3.1) \quad x^\alpha = x^\alpha(u^1, \dots, u^m), \quad (\alpha = 1, \dots, n),$$

are of class C^3 , and the first fundamental form of M^m at P is defined to be

$$(3.2) \quad I = \langle dx, dx \rangle = g_{ij} du^i du^j,$$

where d denotes the exterior differentiation, and the matrix (g_{ij}) is positive definite so that the determinant $|g_{ij}| = g > 0$. Let $x^{\alpha}_{,i}$ denotes the covariant derivative of x^α with respect to g_{ij} . Then it is known that

$$(3.3) \quad x^{\alpha}_{,i} = \partial x^\alpha / \partial u^i,$$

$$(3.4) \quad g_{ij} = a_{\alpha\beta} x^{\alpha}_{,i} x^{\beta}_{,j}.$$

The element of volume of M^m at P is given by

$$(3.5) \quad dV = \sqrt{g} du^1 \wedge \dots \wedge du^m,$$

where \wedge denotes the exterior multiplication.

Now we are in a position to introduce the generalized covariant differentiation, which is useful for studying submanifolds of Riemannian manifolds. Let $A^{\alpha}_{\beta i}$ be a mixed tensor of the second order in the x 's and a covariant vector in the u 's, as indicated by the Greek and Latin indices. Then following A. W. Tucker [13], the generalized covariant derivative of $A^{\alpha}_{\beta i}$ with respect to the u 's is defined by

$$(3.6) \quad \nabla_j A^{\alpha}_{\beta i} = \partial A^{\alpha}_{\beta i} / \partial u^j + \Gamma^{\alpha}_{\gamma\delta} A^{\gamma}_{\beta i} x^{\delta}_{,j} - \Gamma^{\gamma}_{\beta\delta} A^{\alpha}_{\gamma i} x^{\delta}_{,j} - \Gamma^k_{ij} A^{\alpha}_{\beta k},$$

where the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ with Greek indices are formed with respect to the $a_{\alpha\beta}$ and the x 's as follows:

$$(3.7) \quad \Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} a^{\alpha\delta} \left(\frac{\partial a_{\beta\delta}}{\partial x^\gamma} + \frac{\partial a_{\gamma\delta}}{\partial x^\beta} - \frac{\partial a_{\beta\gamma}}{\partial x^\delta} \right),$$

$(a^{\alpha\beta})$ being the inverse matrix of $(a_{\alpha\beta})$, and those Γ^i_{jk} with Latin indices are formed with respect to the g_{ij} and the u 's in a similar way. It should be noted that this definition of generalized covariant differentiation can be applied to any tensor in the u 's and the x 's, and that the generalized covariant differentiation of sums and products of tensors obeys the ordinary rules. If a tensor is one with respect to the u 's only, so that only Latin indices appear, its generalized covariant derivative is the same as its covariant derivative with respect to the u 's. Furthermore, in generalized covariant differentiation, the fundamental tensors $a_{\alpha\beta}$ and g_{ij} can be treated as constants. Since x^α is an invariant for the transformation of u 's, its generalized covariant derivative is the same as its covariant derivative with respect to the u 's, so that

$$(3.8) \quad \nabla_i x^\alpha = x^{\alpha}_{,i} = \partial x^\alpha / \partial u^i.$$

At a point P on M^m we can choose e_{m+1}, \dots, e_n of the orthonormal frame $Pe_1 \cdots e_n$ on N^n defined in § 2 to be unit normal vectors of M^m . Then we can have (see, for instance, [16, Chapter X])

$$(3.9) \quad \nabla_i x_{,j} = \sum_A \Omega_{A|ij} e_A,$$

$$(3.10) \quad \Omega_{A|ij} = \langle \nabla_j x_{,i}, e_A \rangle,$$

$$(3.11) \quad \nabla_i e_A = -\Omega_{A|ik} g^{kj} x_{,j} + \sum_B \vartheta_{BA|i} e_B,$$

where (g^{ij}) is the inverse matrix of (g_{ij}) , and

$$(3.12) \quad \Omega_{A|ij} = \Omega_{A|ji},$$

$$(3.13) \quad \vartheta_{AB|i} + \vartheta_{BA|i} = 0,$$

so that $\vartheta_{AA|i} = 0$. Thus being defined to be $-\langle dx, de_A \rangle$ the second fundamental form II_A of M^m with respect to e_A is given by

$$(3.14) \quad II_A = \Omega_{A|ij} du^i du^j.$$

The equations of Gauss and Mainardi-Codazzi of M^m in N^n are (see, for instance, [2, p. 162])

$$(3.15) \quad R_{hijk} = \sum_A (\Omega_{A|h k} \Omega_{A|ij} - \Omega_{A|h j} \Omega_{A|ik}) + \bar{R}_{\alpha\beta\gamma\delta} x_{,h}^{\alpha} x_{,i}^{\beta} x_{,j}^{\gamma} x_{,k}^{\delta},$$

$$(3.16) \quad \Omega_{C|ij,k} - \Omega_{C|ik,j} = \sum_B (\vartheta_{BC|k} \Omega_{B|ij} - \vartheta_{BC|j} \Omega_{B|ik}) \\ + \bar{R}_{\alpha\beta\gamma\delta} e_C^{\alpha} x_{,i}^{\beta} x_{,j}^{\gamma} x_{,k}^{\delta},$$

where the Riemann symbols $R_{hijk} = g_{hl} R^l{}_{ijk}$ for M^m formed with respect to the g_{ij} and the u 's are defined by

$$(3.17) \quad R^h{}_{ijk} = \frac{\partial \Gamma^h{}_{ij}}{\partial u^k} - \frac{\partial \Gamma^h{}_{ik}}{\partial u^j} + \Gamma^l{}_{ij} \Gamma^h{}_{lk} - \Gamma^l{}_{ik} \Gamma^h{}_{lj},$$

and the Riemann symbols $\bar{R}_{\alpha\beta\gamma\delta}$ for N^n formed with respect to the $a_{\alpha\beta}$ and the x 's can be similarly defined.

In particular, if the manifold N^n is of constant sectional curvature C , from the definition it follows that

$$(3.18) \quad \bar{R}_{\alpha\beta\gamma\delta} = C(a_{\alpha\delta} a_{\beta\gamma} - a_{\alpha\gamma} a_{\beta\delta}),$$

and therefore (3.15), (3.16) are reduced, in consequence of (3.4), to

$$(3.19) \quad R_{hijk} = \sum_A (\Omega_{A|h k} \Omega_{A|ij} - \Omega_{A|h j} \Omega_{A|ik}) + C(g_{hk} g_{ij} - g_{hj} g_{ik}),$$

$$(3.20) \quad \Omega_{C|ij,k} - \Omega_{C|ik,j} = \sum_B (\vartheta_{BC|k} \Omega_{B|ij} - \vartheta_{BC|j} \Omega_{B|ik}).$$

Moreover, by using (3.11), (3.9), (3.20) we can easily obtain

$$(3.21) \quad \begin{aligned} d^2 e_A &= d(\nabla_i e_A du^i) \\ &= \sum_B \left(\nabla_i \mathcal{D}_{BA|i} - \Omega_{A|ik} \Omega_{B|lj} g^{kj} + \sum_C \mathcal{D}_{CA|i} \mathcal{D}_{BC|l} \right) e_B du^l \wedge du^i . \end{aligned}$$

The principal curvature of M^m at P with respect to a normal vector e_A ($m + 1 \leq A \leq n$) are the eigenvalues $k_1(e_A), \dots, k_m(e_A)$ of the matrix $(\Omega_{A|ij})$ relative to the matrix (g_{ij}) , i.e., the roots of the determinant equation

$$(3.22) \quad \det(\Omega_{A|ij} - \lambda g_{ij}) = 0$$

in λ , and the r th mean curvature of M^m at P with respect to e_A is defined to be the r th elementary symmetric function of $k_1(e_A), \dots, k_m(e_A)$ divided by the number of terms, i.e.,

$$(3.23) \quad \binom{m}{r} K_r(e_A) = \sum_{i_1 < \dots < i_r} k_{i_1}(e_A) \cdots k_{i_r}(e_A), \quad (1 \leq r \leq m, m + 1 \leq A \leq n),$$

where $\binom{m}{r} = m! / (r!(m - r)!)$. For convenience, we assume that $K_0(e_A) = 1$.

$P \in M^m$ is called an *umbilical point* of M^m with respect to e_A if $k_1(e_A) = \dots = k_m(e_A)$ at P , and M^m is called an *umbilical submanifold* of N^n with respect to a vector field e_A if every point of M^m is an umbilical point with respect to e_A at that point. It is well known that a closed oriented hypersurface in a Euclidean space E^{m+1} consisting entirely of umbilical points with respect to the unique normal vector field is a hypersphere.

If k_a is a real simple root of (3.22), then

$$(3.24) \quad (\Omega_{A|ij} - k_a g_{ij}) p_a|^i = 0, \quad (j = 1, \dots, m),$$

define, to within a factor, m quantities $p_a|^i, i = 1, \dots, m$, which are the contravariant components of a real vector in the tangent space of M^m at P , called a *principal vector* of M^m at P corresponding to the principal curvature k_a , as is seen by changing the coordinates and making use of the tensor properties of $\Omega_{A|ij}$ and g_{ij} . If k_b is another real simple root of (3.22), we have a second vector $p_b|^i$ defined by

$$(3.25) \quad (\Omega_{A|ij} - k_b g_{ij}) p_b|^i = 0, \quad (j = 1, \dots, m).$$

Multiplying (3.24) by $p_b|^j$ and (3.25) by $p_a|^j$, summing for j in each case and subtracting, we have, since $k_a \neq k_b$ by hypothesis,

$$(3.26) \quad g_{ij} p_a|^i p_b|^j = 0,$$

that is, the two vectors $p_a|^\xi$ and $p_b|^\xi$ are orthogonal. Hence, as is well known, the m principal vectors $p_1|^\xi, \dots, p_m|^\xi$ corresponding to the m principal curvatures k_1, \dots, k_m with respect to the unit normal vector e_A of M^m at P are mutually orthogonal.

Lemma 3.1. *By a suitable choice of the local coordinates u^1, \dots, u^m of M^m at a point P we have*

$$(3.27) \quad \nabla_i e_A = -k_i x_{,i} + \sum_B \mathcal{D}_{BA|} e_B, \quad (i = 1, \dots, m, \text{ not summed}),$$

where k_1, \dots, k_m are the principal curvatures of M^m at P with respect to e_A .

Proof. Choose the local coordinates u^1, \dots, u^m of M^m at P such that $x_{,1}, \dots, x_{,m}$ to be the m principal vectors $p_1|^\xi, \dots, p_m|^\xi$ of M^m at P corresponding to k_1, \dots, k_m , so that $g_{ij} = 0$ for $i \neq j$ at P . The contravariant components $x_{,a}^\alpha$ and $p_a|^\xi$ of the principal vector $p_a|$ in the x 's and the u 's respectively are connected by the relation

$$(3.28) \quad x_{,a}^\alpha = x_{,i}^\alpha p_a|^\xi.$$

Multiplying (3.28) by $a_{\alpha\beta} x_{,b}^\beta$ and summing for α we obtain $g_{ab} = g_{bi} p_a|^\xi$ from which it follows that

$$(3.29) \quad p_a|^\xi = \delta_a^i.$$

Substituting (3.29) in (3.24) gives

$$(3.30) \quad \mathcal{Q}_{A|i} = k_i g_{ij}, \quad (i = 1, \dots, m, \text{ not summed}).$$

From (3.30) and (3.11) follows immediately (3.27). q.e.d.

Let ξ be an infinitesimal conformal vector field on the manifold N^n , and L_ξ the Lie derivative with respect to ξ . Then on N^n we have

$$(3.31) \quad L_\xi a_{\alpha\beta} = \xi_{\alpha,\beta} + \xi_{\beta,\alpha} = 2\rho a_{\alpha\beta},$$

where ρ is a function of x^1, \dots, x^n . The field ξ is said to be homothetic or isometric according as ρ is constant or zero.

Lemma 3.2. *If the local coordinates x^1, \dots, x^n on N^n are so chosen that the Kronecker vector δ_1 , whose contravariant components are the Kronecker deltas $\delta_1^1, \dots, \delta_1^n$, generate an infinitesimal conformal vector field on N^n , then on N^n*

$$(3.32) \quad L_{\delta_1} a_{\alpha\beta} = 2\rho a_{\alpha\beta} = \partial a_{\alpha\beta} / \partial x^1.$$

Proof. From the definition of covariant differentiation with respect to the x 's it follows that

$$(3.33) \quad \delta_{1\alpha,\beta} = a_{\alpha\gamma} \delta_{1,\beta}^\gamma = a_{\alpha\gamma} \Gamma_{1\beta}^\gamma,$$

and similarly $\delta_{1\beta,\alpha} = a_{\beta r} \Gamma_{1\alpha}^r$. By means of (3.7) we readily have $a_{\alpha r} \Gamma_{1\beta}^r + a_{\beta r} \Gamma_{1\alpha}^r = \partial a_{\alpha\beta} / \partial x^1$, which together with (3.31) gives (3.32). q.e.d.

If the vector δ_1 generates an infinitesimal conformal vector field on N^n , then using (3.33) we immediately obtain that on N^n

$$(3.34) \quad d(\delta_1^\alpha) = \delta_{1,r}^\alpha dx^r = \delta_{1,r}^\alpha x_{,i}^r du^i = \Gamma_{1r,i}^\alpha x_{,i}^r du^i,$$

which together with (3.7) implies

$$(3.35) \quad a_{\alpha\beta} d\delta_1^\alpha = \frac{1}{2} \left(\frac{\partial a_{1\beta}}{\partial x^r} + \frac{\partial a_{r\beta}}{\partial x^1} - \frac{\partial a_{1r}}{\partial x^\beta} \right) x_{,i}^r du^i.$$

4. Integral formulas

Let $x: M^m \rightarrow N^n$ be an m -dimensional ($2 \leq m < n$) submanifold of class C^3 immersed in a Riemannian n -manifold N^n , which is of constant sectional curvature and admits a continuous infinitesimal conformal vector field ξ , so that §§ 2 and 3 can be applied. In this section we shall derive some integral formulas for closed oriented M^m with respect to a fixed unit normal vector field, e_{m+1} say, on M^m . For this purpose we choose the orientation of the orthonormal frame $Pe_1 \cdots e_n$ of N^n at a point P defined in § 3 such that (2.14) and therefore (2.15) hold, and we also choose the local coordinates x^1, \dots, x^m and u^1, \dots, u^m of N^n and M^m at P respectively such that the Kronecker vector δ_1 be the infinitesimal conformal vector ξ , and that $x_{,1}, \dots, x_{,m}$ be the m principal vectors $p_1|^\xi, \dots, p_m|^\xi$ of M^m at P with respect to e_{m+1} , so that at P

$$(4.1) \quad g_{ij} = 0, \quad (i \neq j),$$

$$(4.2) \quad e_i = x_{,i} / [i], \quad (i = 1, \dots, m),$$

where

$$(4.3) \quad [i] = \sqrt{g_{ii}}.$$

Now we are in a position to evaluate the following exact differential m -form for $1 \leq i \leq m$:

$$(4.4) \quad \begin{aligned} & d(\sqrt{a} | \delta_1, \underbrace{dx, \dots, dx}_{m-i}, \underbrace{de_{m+1}, \dots, de_{m+1}}_{i-1}, e_{m+1}, \dots, e_n) \\ &= (I) + (-1)^{m-i}(i-1)(II) + (-1)^{m-1}(III) \\ & \quad + (-1)^{m-1}(IV) + (-1)^{m-1} \sum_{\alpha=3}^{n-m} (V)_\alpha, \end{aligned}$$

where we have used $d^2x = 0$ and put

$$(4.5) \quad (I) = \sqrt{a} |d\delta_1, \underbrace{dx, \dots, dx}_{m-i}, \underbrace{de_{m+1}, \dots, de_{m+1}}_{i-1}, e_{m+1}, \dots, e_n|,$$

$$(4.6) \quad (II) = \sqrt{a} |\delta_1, \underbrace{dx, \dots, dx}_{m-i}, d^2e_{m+1}, \underbrace{de_{m+1}, \dots, de_{m+1}}_{i-2}, e_{m+1}, \dots, e_n|,$$

$$(4.7) \quad (III) = \sqrt{a} |\delta_1, \underbrace{dx, \dots, dx}_{m-i}, \underbrace{de_{m+1}, \dots, de_{m+1}}_i, e_{m+2}, \dots, e_n|,$$

$$(4.8) \quad (IV) = \sqrt{a} |\delta_1, \underbrace{dx, \dots, dx}_{m-i}, \underbrace{de_{m+1}, \dots, de_{m+1}}_{i-1}, e_{m+1}, de_{m+2}, \\ e_{m+3}, \dots, e_n|,$$

$$(4.9) \quad (V)_\alpha = \sqrt{a} |\delta_1, \underbrace{dx, \dots, dx}_{m-i}, \underbrace{de_{m+1}, \dots, de_{m+1}}_{i-1}, \\ e_{m+1}, e_{m+2}, \dots, de_{m+\alpha}, \dots, e_n|, \quad (3 \leq \alpha \leq n - m).$$

By means of (3.27) for $A = m + 1$, (4.2), (4.3), (2.4), (2.3), (2.16), (3.35), (3.32), (3.4), (3.5), (4.1) we obtain

$$(4.10) \quad \begin{aligned} (I) &= (-1)^{i-1} \sqrt{a} |d\delta_1, [j_1]e_{j_1}du^{j_1}, \dots, [j_{m-i}]e_{j_{m-i}}du^{j_{m-i}}, \\ &\quad k_{j_{m-i+1}}[j_{m-i+1}]e_{j_{m-i+1}}du^{j_{m-i+1}}, \\ &\quad \dots, k_{j_{m-1}}[j_{m-1}]e_{j_{m-1}}du^{j_{m-1}}, e_{m+1}, \dots, e_n| \\ &= (-1)^{m+i}(m-i)!(i-1)! a_{\alpha\beta} d\delta_1^\alpha \wedge \frac{x_{j_1}^\beta}{[j_m]} [j_1] \dots [j_{m-1}] \\ &\quad \cdot \delta_{j_1 \dots j_m} k_{j_{m-i+1}} \dots k_{j_{m-1}} du^{j_1} \wedge \dots \wedge du^{j_{m-1}} \\ &= (-1)^{n+i}(m-i)!(i-1)! \langle d\delta_1, ([j_1] \dots [j_{m-1}]e_{j_1} \\ &\quad \times \dots \times e_{j_{m-1}} \times e_{j_{m+1}} \times \dots \times e_n) \\ &\quad \cdot k_{j_{m-i+1}} \dots k_{j_{m-1}} du^{j_1} \wedge \dots \wedge du^{j_{m-1}} \rangle \\ &= (-1)^{m+i} \frac{(m-i)!(i-1)!}{2[j_m]} [j_1] \dots [j_{m-1}] \frac{\partial a_{\alpha\beta}}{\partial x^1} x_{j_m}^\alpha x_{j_m}^\beta \\ &\quad \cdot \delta_{j_1 \dots j_m} k_{j_{m-i+1}} \dots k_{j_{m-1}} du^{j_m} \wedge du^{j_1} \wedge \dots \wedge du^{j_{m-1}} \\ &= (-1)^{i-1} \rho(m-i)!(i-1)!(m-i+1) \\ &\quad \sum_{j_{m-i+1}, \dots, j_{m-1}} k_{j_{m-i+1}} \dots k_{j_{m-1}} dV. \end{aligned}$$

It should be remarked that in the summation on j_1, \dots, j_m in (4.10) for fixed $j_{m-i+1}, \dots, j_{m-1}$, $m-i$ other j 's are together and their order is immaterial, and the remaining j can take any one of the other $(m-i+1)$ j 's, namely, j_1, \dots, j_{m-i}, j_m , so that we get the factor $m-i+1$. From (4.10) and (3.23) follows immediately

$$(4.11) \quad (I) = (-1)^{t-1} m! \rho K_{i-1}(e_{m+1}) dV .$$

Substituting (3.21) for $A = m + 1$ in (4.6) gives readily

$$(4.12) \quad (II) = 0 .$$

Using the same method as above we can easily obtain

$$(4.13) \quad (III) = (-1)^{m+i} m! \langle \delta_1, e_{m+1} \rangle K_i(e_{m+1}) dV ,$$

$$(4.14) \quad (IV) = (-1)^{m+i} (m-i)! (i-1)! \langle \delta_1, e_{m+2} \rangle \cdot \sum_{j_{m-i+1}, \dots, j_m} k_{j_{m-i+1}} \cdots k_{j_{m-1}} \Omega_{m+2|j_k} g^{k j_m} dV .$$

The vector field δ_1 can be decomposed into two parts:

$$(4.15) \quad \delta_1 = \delta_{1|t} + \delta_{1|n} ,$$

where $\delta_{1|t}$ is tangent to M^m , and $\delta_{1|n}$ normal to M^m . Let e and \bar{e} be two unit normal vector fields over M^m coplanar with $\delta_{1|n}$. Then

$$(4.16) \quad \delta_{1|n} = \langle \delta_{1|n}, e \rangle e + \langle \delta_{1|n}, \bar{e} \rangle \bar{e} .$$

Now suppose that the unit normal vector field e_{m+1} is parallel in the normal bundle of M^m , i.e., by the definition, de_{m+1} is tangent to M^m everywhere. Then by choosing $e_{m+1} = e$ and $e_{m+2} = \bar{e}$ everywhere on M^m and using (4.15) and (4.16) we obtain

$$(4.17) \quad \langle \delta_1, e_{m+\alpha} \rangle = 0 , \quad (\alpha = 3, \dots, n-m) ,$$

and therefore

$$(4.18) \quad (V)_\alpha = 0 , \quad (3 \leq \alpha \leq n-m) .$$

Combination of (4.4), (4.11), \dots , (4.14), (4.18) gives

$$(4.19) \quad \begin{aligned} & d(\sqrt{a} |\delta_1, \underbrace{dx, \dots, dx}_{m-i}, \underbrace{de_{m+1}, \dots, de_{m+1}}_{i-1}, e_{m+1}, \dots, e_n) \\ & = (-1)^{t-1} m! [\rho K_{i-1}(e_{m+1}) + \langle \delta_1, e_{m+1} \rangle K_i(e_{m+1}) + F_i(e_{m+1})] dV , \\ & \quad (i = 1, \dots, m) . \end{aligned}$$

where

$$(4.20) \quad F_i(e_{m+1}) = \frac{(m-i)! (i-1)!}{m!} \langle \delta_1, e_{m+2} \rangle \sum_{j_{m-i+1}, \dots, j_{m-1}} k_{j_{m-i+1}} \cdots k_{j_{m-1}} \Omega_{m+2|j_k} g^{k j_m} .$$

Integrating (4.19) over an oriented M^m and applying Stokes' theorem we hence arrive at

Theorem 4.1. *Let $x: M^m \rightarrow N^n$ be a closed oriented m -dimensional ($2 \leq m < n$) submanifold of class C^3 immersed in a Riemannian n -manifold N^n , which is of constant sectional curvature and admits a continuous infinitesimal conformal vector field ξ . If e_{m+1} and e_{m+2} are unit normal vector fields over M^m such that e_{m+1} is parallel in the normal bundle of M^m , and e_{m+1}, e_{m+2} are coplanar with the normal component of ξ , then*

$$(4.21) \quad \int_{M^m} [\rho K_{i-1}(e_{m+1}) + \langle \delta, e_{m+1} \rangle K_i(e_{m+1})] dV = - \int_{M^m} F_i(e_{m+1}) dV, \\ (i = 1, \dots, m),$$

where ρ is given by (3.32).

Remarks. 1. If $n - m = 1$, then $F_i(e_{m+1}) = 0$, $i = 1, \dots, m$, hold automatically, and formulas (4.21) are due to Hsiung [5] for Euclidean N^n with ξ generated by the position vector x of a general point of M^m with respect to a fixed point 0 in N^n , due to Hsiung [6] and Feeman and Hsiung [3] for a Riemannian N^n and a special ξ , and due to Katsurada [7] for a Riemannian N^n and a general ξ .

2. For Euclidean N^n and general n with the position vector field x as ξ , formulas (4.21) are due to Chen and Yano [1], and due to Yano [14], [15] under some additional conditions.

3. For Euclidean N^n , the condition of the parallelism of e_{m+1} in the normal bundle of M^m can be replaced by the condition that M^m be immersed in a hypersphere of N^n centered at the origin of N^n .

4. For a special e_{m+1} , formulas (4.21) are due to Katsurada and Kôjyô [13], and Katsurada [8].

5. Characterizations of umbilical submanifolds

In this section we use integral formula (4.21) to derive various conditions for a submanifold of a Riemannian manifold to be umbilical with respect to a given normal vector field. For this purpose we first state the following three lemmas which will be needed for the proofs of our main theorems. The proofs of the lemmas are omitted here, but can be found in [4, pp. 52, 104–105].

Lemma 5.1. *Let $K_i(e_A)$, $i = 1, \dots, m$, be given by (3.23). Then*

$$(5.1) \quad K_i(e_A)^2 - K_{i-1}(e_A)K_{i+1}(e_A) \geq 0, \quad (i = 1, \dots, m - 1),$$

where the equality implies that $k_1(e_A) = \dots = k_m(e_A)$.

Lemma 5.2. *If $K_i(e_A), K_{i-1}(e_A), \dots, K_{i-j-1}(e_A) > 0$, $1 \leq j < i \leq m$, then*

$$(5.2) \quad \frac{K_{i-1}(e_A)}{K_i(e_A)} \geq \frac{K_{i-2}(e_A)}{K_{i-1}(e_A)} \geq \dots \geq \frac{K_{i-j-1}(e_A)}{K_{i-j}(e_A)},$$

where the equality at any stage implies that $k_1(e_A) = \dots = k_m(e_A)$.

Lemma 5.3. *If $K_1(e_A), \dots, K_j(e_A) > 0, j \leq m$, then*

$$(5.3) \quad K_1(e_A) \geq K_2(e_A)^{1/2} \geq K_3(e_A)^{1/3} \geq \dots \geq K_j(e_A)^{1/j},$$

where the equality at any stage implies that $k_1(e_A) = \dots = k_m(e_A)$.

In the remainder of this section we shall use the following notation:

N^n : A Riemannian n -manifold ($n > 2$) having constant sectional curvature and admitting a continuous infinitesimal conformal vector field ξ so that $L_\xi a_{\alpha\beta} = 2\rho a_{\alpha\beta}$ where $a_{\alpha\beta}$ is the Riemannian metric tensor of N^n .

M^m : A closed oriented m -dimensional ($n > m \geq 2$) submanifold of class C^3 immersed in N^n .

e : A unit normal vector field on M^m parallel in the normal bundle of M^m .
 k_i, K_i, F_i , and p : $k_i(e), K_i(e), F_i(e)$ for $i = 1, \dots, m$, and $\langle \xi, e \rangle$, respectively.

Theorem 5.1. *M^m is umbilical with respect to e if at all points of M^m for an integer or $i, 1 \leq i \leq m$,*

- (i) $\rho/K_i > 0$,
- (ii) $p \leq -\rho K_{i-1}/K_i$ (or $p \geq -\rho K_{i-1}/K_i$),
- (iii) $F_i = F_{i+1} = 0$ for $1 \leq i < m$, and $F_i = F_{i-1} = 0$ for $i = m$.

For Euclidean N^n with the position vector field x as ξ , Theorem 5.1 is due to Hsiung [5] for $n = m + 1$ and due to Chen and Yano [1] for general n and $2 < i < m$. For Riemannian N^n with a special e and $i = 1$, Theorem 5.1 is due to Katsurada [8].

Proof. By (ii), the integrand of (4.21) for $e_{m+1} = e$ is either nonpositive or nonnegative, and therefore we have

$$(5.4) \quad p = \rho K_{i-1}/K_i.$$

For $i < m$, substituting (5.4) in (4.21), where i is replaced by $i + 1$, gives

$$(5.5) \quad \int_{M^m} \frac{\rho}{K_i} (K_i^2 - K_{i-1}K_{i+1}) dV = 0.$$

Due to (i) and (5.1), the integrand of (5.5) is nonnegative, and therefore (5.5) holds only when, at all points of $M^m, K_i^2 - K_{i-1}K_{i+1} = 0$. From Lemma 5.1 it follows that $k_1 = \dots = k_m$ at all points of M^m , and hence M^m is umbilical with respect to e .

For $i = m$, substituting (5.4) in (4.21) where i is replaced by $i - 1$, we obtain

$$(5.6) \quad \int_{M^m} \frac{\rho}{K_m} (K_{m-1}^2 - K_{m-2}K_m) dV = 0.$$

By applying Lemma 5.1 with the same argument as above, we can show that M^m is also umbilical with respect to e .

Theorem 5.2. M^m is umbilical with respect to e if at all points of M^m for an integer i , $1 < i < m$,

- (i) $\rho, K_{i+1}, K_i, K_{i-1} > 0$,
- (ii) $p \geq -\rho K_{i-1}/K_i$,
- (iii) $F_{i+1} = 0$.

For Euclidean N^n with the position vector field x as ξ , Theorem 5.2 is due to Chen and Yano [1]. It should be remarked that we may have a similar theorem by assuming $\rho < 0$ instead of $\rho > 0$.

Proof. By (ii) and Lemma 5.2 we have

$$(5.7) \quad p \geq -\rho K_{i-1}/K_i \geq -\rho K_i/K_{i+1}.$$

(4.21), with i replaced by $i + 1$, and (5.7) imply that the equality holds in (5.7), and hence M^m is umbilical with respect to e by Lemma 5.2.

Theorem 5.3. M^m is umbilical with respect to e if at all points of M^m for an integer s , $1 \leq s \leq m$,

- (i) p is of the same sign,
- (ii) $K_i > 0$, $i = 1, \dots, s$,
- (iii) K_s is constant,
- (iv) ρ is of the same sign,
- (v) $F_1 = F_{s+1} = 0$ for $1 < s < m$, and $F_1 = F_s = 0$ for $s = m$.

For Euclidean N^n and $n = m + 1$, Theorem 5.3 is due to Hsiung [5].

Proof. Case 1. $s < m$. By (ii) and inequality (5.1) for $i = 1, \dots, s$ we obtain

$$K_1/K_0 \geq K_2/K_1 \geq \dots \geq K_{s+1}/K_s,$$

and, in particular,

$$(5.8) \quad K_1 K_s \geq K_{s+1},$$

where the equality holds only when $k_1 = \dots = k_m$ in view of Lemma 5.1. Here we assume $\rho > 0$. Then from (4.21) for $i = 1$ and assumptions (i), (ii), (v) it follows that p is negative. (For the case $\rho < 0$, the arguments in the proof of our theorem will be exactly the same, except that p would be positive.) Multiplying both sides of inequality (5.8) by p , integrating over M^m , and applying (4.21) for $i = 1$ and $i = s + 1$, we can readily obtain, in consequence of (iii) and (v),

$$-\int_{M^m} \rho K_s dV = \int_{M^m} p K_1 K_s dV \leq \int_{M^m} p K_{s+1} dV = -\int_{M^m} \rho K_s dV,$$

from which it follows that

$$(5.9) \quad \int_{M^m} p(K_1 K_s - K_{s+1}) dV = 0.$$

Since p is negative, from (5.8) we see that the integrand in (5.9) is nonpositive and therefore must be zero. Thus the equality holds in (5.8) so that $k_1 = \dots = k_m$ everywhere by Lemma 5.1. Hence M^m is umbilical with respect to e .

Case 2. $s = m$. From (ii), (iii) and Lemma 5.3 it follows that

$$(5.10) \quad K_1 \geq K_2^{1/2} \geq \dots \geq K_{m-1}^{1/(m-1)} \geq K_m^{1/m} = c,$$

where c is a positive constant. By means of (4.21) for $i = m$, assumption (v) and inequalities (5.10), we obtain

$$(5.11) \quad \int_{M^m} p K_m dV = - \int_{M^m} \rho K_{m-1} dV < -c^{m-1} \int_{M^m} \rho dV.$$

On the other hand, using (4.21) for $i = 1$, (v), (5.10) and the fact that $p < 0$, we have

$$(5.12) \quad \begin{aligned} \int_{M^m} p K_m dV &= \int_{M^m} p c^m dV = c^{m-1} \int_{M^m} p K_m^{1/m} dV \\ &\geq c^{m-1} \int_{M^m} p K_1 dV = -c^{m-1} \int_{M^m} \rho dV. \end{aligned}$$

Combination of (5.11) and (5.12) shows immediately that the equality holds in (5.12) and therefore that

$$(5.13) \quad \int_{M^m} p(K_m^{1/m} - K_1) dV = 0.$$

Since $p < 0$, (5.10) implies that the integrand of (5.13) is nonnegative and therefore that $K_1 = K_m^{1/m}$. Thus by Lemma 5.3, $k_1 = \dots = k_m$ at all points of M^m . Hence the proof of Theorem 5.3 is complete.

Theorem 5.4. M^m is umbilical with respect to e if at all points of M^m for two integers i and s , $i \leq i < s \leq m$,

- (i) $K_i, K_{i+1}, \dots, K_s > 0$,
- (ii) $K_s = \sum_{j=1}^{s-1} c_j K_j$, for some constants $c_j \geq 0$, $i \leq j \leq s - 1$,
- (iii) ρ is of the same sign,
- (iv) $F_j = 0$, $j = 1, \dots, s - 1$.

Proof. We observe

$$(5.14) \quad \frac{K_j}{K_s} - \frac{K_{j-1}}{K_{s-1}} = \frac{K_j}{K_{s-1}} \left(\frac{K_{s-1}}{K_s} - \frac{K_{j-1}}{K_j} \right).$$

In view of Lemma 5.2, the right side of (5.14) is nonnegative for $i \leq j \leq s - 1$. Thus

$$(5.15) \quad K_j/K_s \geq K_{j-1}/K_{s-1},$$

where the equality holds only when $k_1 = \dots = k_m$. By (ii) and (5.15) we obtain

$$1 = \sum_{j=i}^{s-1} c_j K_j/K_s \geq \sum_{j=i}^{s-1} c_j K_{j-1}/K_{s-1},$$

or

$$(5.16) \quad K_{s-1} - \sum_{j=i}^{s-1} c_j K_{j-1} \geq 0,$$

where the equality holds only when $k_1 = \dots = k_m$. Thus by means of (4.21), (iv) and (ii) we obtain

$$(5.17) \quad \int_{M^m} \rho \left(K_{s-1} - \sum_{j=i}^{s-1} c_j K_{j-1} \right) dV = - \int_{M^m} p \left(K_s - \sum_{j=i}^{s-1} c_j K_j \right) dV = 0.$$

(5.16), (5.17), (iii) show immediately that the equality holds in (5.16). Hence M^m is umbilical with respect to e .

Theorem 5.5. M^m is umbilical with respect to e if at all points of M^m for two integers i and s , $0 \leq i < s < m$,

- (i) $K_i, \dots, K_{s+1} > 0$,
- (ii) $K_s = \sum_{j=i}^{s-1} c_j K_j$, for some constants $c_j \geq 0$, $i \leq j \leq s-1$,
- (iii) p is of the same sign,
- (iv) $F_j = 0$, $j = 1, \dots, s-1$.

Proof. By Lemma 5.2 we have

$$(5.18) \quad \frac{K_j}{K_s} - \frac{K_{j+1}}{K_{s+1}} = \frac{K_j}{K_{s+1}} \left(\frac{K_{s+1}}{K_s} - \frac{K_{j+1}}{K_j} \right) \leq 0,$$

where the equality holds only when $k_1 = \dots = k_m$. From (ii), (5.18) it follows that

$$1 = \sum_{j=i}^{s-1} c_j K_j/K_s \leq \sum_{j=i}^{s-1} c_j K_{j+1}/K_{s+1},$$

or

$$(5.19) \quad K_{s+1} - \sum_{j=i}^{s-1} c_j K_{j+1} \leq 0,$$

where the equality holds only when $k_1 = \dots = k_m$. Thus by means of (4.21), (iv) and (ii) we obtain

$$(5.20) \quad \int_{M^m} p\left(K_{s+1} - \sum_{j=i}^{s-1} c_j K_{j+1}\right) dV = - \int_{M^m} \rho\left(K_s - \sum_{j=i}^{s-1} c_j K_j\right) dV = 0.$$

(5.19), (5.20), (iii) show immediately that the equality holds in (5.19). Hence M^m is umbilical with respect to e .

Theorem 5.6. M^m is umbilical with respect to e if at all points of M^m for an integer i , $1 < i \leq m$,

- (i) $K_i > 0$,
- (ii) $K_i = cK_{i-1}$, for some constant c ,
- (iii) ρ is of the same sign,
- (iv) $F_{i-1} = F_i = 0$.

Proof. Due to (i), c cannot be zero and K_{i-1} must be of a fixed sign. Using (ii) and Lemma 5.1 we have

$$K_{i-1}(K_{i-1} - cK_{i-2}) = K_{i-1}^2 - K_i K_{i-2} \geq 0,$$

so that

$$(5.21) \quad K_{i-1} - cK_{i-2} \quad \text{is of fixed sign,}$$

and vanishes identically only when $k_1 = \dots = k_m$. Thus by means of (4.21), (iv) and (ii) we obtain

$$(5.22) \quad \int_{M^m} \rho(K_{i-1} - cK_{i-2}) dV = - \int_{M^m} \rho(K_i - cK_{i-1}) dV = 0.$$

(5.21), (5.22), (iii) imply immediately that $K_{i-1} = cK_{i-2}$. Hence Theorem 5.6 is proved.

Corollary 5.6. M^m is umbilical with respect to e if at all points of M^m

- (i) $K_m > 0$,
- (ii) $\sum_{i=1}^m (1/k_i) = \text{constant}$,
- (iii) ρ is of the same sign,
- (iv) $F_{m-1} = F_m = 0$.

Proof. By (ii) and the definition (3.23) of K_i we obtain

$$mK_{m-1}/K_m = \sum_{j=1}^m (1/k_j) = \text{constant},$$

so that

$$K_m = cK_{m-1}, \quad \text{for some constant } c.$$

Hence Corollary 5.6 is an immediate consequence of Theorem 5.6 for $i = m$.

Theorem 5.7. M^m is umbilical with respect to e if at all points for an integer s , $1 < s \leq m$, and a constant c

- (i) $K_i > 0$ for $i = 1, \dots, s$,

- (ii) $K_{s-1}^{1/(s-1)} \geq c \geq K_s^{1/s}$,
- (iii) p is of the same sign,
- (iv) ρ is of the same sign,
- (v) $F_i = F_2 = F_s = 0$.

Proof. As in the proof of Theorem 5.3 we may assume $\rho > 0$. Then due to (iii), (v) and (i) for $i = 1$, (4.21) for $i = 1$ implies $p < 0$. By (5.3), (ii) we have $K_1 > K_{s-1}^{1/(s-1)} \geq c$, and therefore, in consequence of (ii), (4.21) for $i = s$ and $i = 1$,

$$\begin{aligned}
 (5.23) \quad & - \int_{M^m} c^{s-1} p K_1 dV \geq - \int_{M^m} c^s p dV \geq - \int_{M^m} p K_s dV \\
 & = \int_{M^m} \rho K_{s-1} dV \geq \int_{M^m} \rho c^{s-1} dV \\
 & = - \int_{M^m} c^{s-1} p K_1 dV .
 \end{aligned}$$

Thus the equality holds everywhere in (5.23), so that

$$\int_{M^m} p(K_1 - c) dV = 0 ,$$

which implies that $K_1 = c$. Hence, by Theorem 5.3 for $s = 1$, M^m is umbilical with respect to e .

Theorem 5.8. M^m is umbilical with respect to e if at all points of M^m for an integer s , $1 < s \leq m$, and a constant c

- (i) $K_{s-1}, K_s > 0$,
- (ii) $K_{s-1}/K_s \geq c \geq K_{s-2}/K_{s-1}$,
- (iii) p is of the same sign,
- (iv) ρ is of the same sign,
- (v) $F_{s-1} = F_s = 0$.

Proof. As before we may assume $\rho > 0$. Then due to (i), (iii) and (v), (4.21) implies $p < 0$. By using (ii), (4.21) for $i = s - 1$ and $i = s$ we have

$$\begin{aligned}
 (5.24) \quad & \int_{M^m} \rho K_{s-2} dV = - \int_{M^m} p K_{s-1} dV \geq - \int_{M^m} c p K_s dV \\
 & = \int_{M^m} c \rho K_{s-1} dV \geq \int_{M^m} \rho K_{s-2} dV .
 \end{aligned}$$

Thus the equality holds everywhere in (5.24), so that

$$(5.25) \quad \int_{M^m} p(K_{s-1} - cK_s) dV = 0 .$$

Since $p(K_{s-1} - cK_s) \leq 0$, (5.25) implies that $K_{s-1} = cK_s$ at all points of M^m . Hence, by Theorem 5.6 for $i = s$, M^m is umbilical with respect to e .

Theorems 5.4, 5.5, 5.6 and Corollary 5.6 are due to Chen and Yano [1] for Euclidean N^n with the position vector field x as ξ . Theorems 5.4, \dots , 5.8 are due to Strong [12] for $n = m + 1$ with the position vector field x as ξ .

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