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COMPLEX PARALLELISABLE MANIFOLDS AND THEIR SMALL DEFORMATIONS

IKU NAKAMURA

Introduction

By a complex parallelisable manifold we mean a compact complex manifold with the trivial holomorphic tangent bundle. Wang [8] showed that a complex parallelisable manifold is the quotient space of a simply connected, connected complex Lie group by one of its discrete subgroups.

It is known that if the Lie group corresponding to a parallelisable manifold is semi-simple and does not contain $SL(2\boldsymbol{C})$ as a component, then the first Betti number vanishes and its small deformation is rigid, [2], [5], [6].

In this paper we consider the similar problems in the case where the corresponding Lie group is solvable, and obtain quite different results. We note that a simply connected, connected solvable complex Lie group is biholomorphically equivalent to $\boldsymbol{C}^n$ as a complex manifold where $n = \dim_C G$. If a complex parallelisable manifold has a solvable Lie group as the universal covering, it is called a complex solvable manifold.

In § 1 we summarize some known results and give three lemmas. In § 2 by numerical invariants we classify three-dimensional complex solvable manifolds into four classes III-(1), III-(2), III-(3a), III-(3b), and construct some examples in all cases.

In § 3 we construct Kuranishi families of deformations of three-dimensional complex solvable manifolds constructed in § 2. The base spaces of these Kuranishi families which are reduced complex spaces are irreducible in the cases of III-(2) and III-(3a) but reducible in for case of III-(3b), about which we shall give explicit descriptions.

For a compact complex manifold X we denote by $\mathcal{O}$ and Ω^p the sheaves of germs over X of holomorphic functions and p-forms respectively. Recall $h^{p,q} = \dim_C H^q(X, \Omega^p)$ and $P_m(X) = \dim H^0(X, (\Omega^n)^{\otimes m})$ where $n = \dim_C X$. Also we denote by r, κ and b_i respectively the number of linearly independent closed holomorphic 1-forms, Kodaira dimension of X and the i-th Betti number.

S. Iitaka proposed a problem whether all P_m and κ are deformation invariants [1]. However computing the numerical characters of small deformations obtained in the above examples we have

Theorem 2. *$h^{p,q}$ for $(p, q) \neq (0, 0)$, r, P_m and κ are not necessarily invari-*

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ant under small deformations.

On the other hand we note that small deformations of a complex parallelisable manifold are not necessarily parallelisable.

In § 4 and § 5 we prove the following theorems.

Theorem 3 (*Kodaira*). *Let X be parallelisable such that the corresponding Lie group is nilpotent. Then $h^{0,1} = r$.*

Theorem 4. *For a complex solvable manifold whose Lie algebra has the Chevalley decomposition* (§ 2) *we have $b_1 = 2r$.*

We remark that a complex solvable manifold has C^n as its universal covering.

Theorem 5. *If an n-dimensional complex solvable manifold satisfies the equality $h^{0,1} = r$, then any small deformation has C^n as its universal covering.*

In Theorem 5 we cannot remove the assumption that $h^{0,1} = r$. In fact, in the case of III-(3b) where we have $h^{0,1} > r$, there exist small deformations whose universal covering are not analytically homeomorphic to C^3.

In § 6, following the algorithm shown in § 1 we classify complex solvable manifolds of four and five dimensions.

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1. Preliminaries

Let X be a compact complex manifold of dimension n.

Definition 1.1. X is parallelisable if the holomorphic tangent bundle of X is complex analytically trivial.

This condition is written in the following ways:

(1) $\Theta \cong \mathcal{O}^n$, where Θ is the sheaf of germes of holomorphic vector fields, and $\mathcal{O}$ is the structure sheaf of X.

(2) There exist n holomorphic vector fields $\theta_1, \cdots, \theta_n$ on X which are linearly independent at every point on X.

(3) $\Omega^1 \cong \mathcal{O}^n$, where Ω^1 is the sheaf of germs of holomorphic 1-forms.

(4) There exist n holomorphic 1-forms $\varphi_1, \cdots, \varphi_n$ on X which are linearly independent at every point on X.

It is obvious that $\Omega^p \cong \mathcal{O}^{\binom{n}{p}}$. Hence $H^0(X, \Omega^p)$ is spanned by $\{\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p}$, $1 \leq i_1 < \cdots < i_p \leq n\}$ and $h^{p,0} = \binom{n}{p}$.

Theorem (*Wang* [6]). *Let X be parallelisable. Then there exist a simply connected, connected complex Lie group G and a discrete subgroup Γ of G such that $X \cong G/\Gamma$.*

In particular, $H^0(X, \Theta) \cong \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of G.

Definition 1.2. A complex parallelisable manifold X is solvable (respectively nilpotent) if the corresponding Lie group G is solvable (respectively nilpotent).

Let G be a connected complex Lie group, and Γ one of its discrete subgroups.

Definition 1.3. Γ is uniform in G if G/Γ is compact.

Theorem (*Mostow* [4]). *Let G be a connected solvable complex Lie group, and Γ a uniform subgroup of G. Let N be the connected, maximal nilpotent normal subgroup of G. Then $\Gamma \cap N$ and $\Gamma N/N$ are uniform in N and G/N respectively.*

The original form of this theorem is not restricted to the complex case. This theorem means that for any solvable manifold $X = G/\Gamma$, there is the decomposition $\pi : X \to B$, where $B = (G/N)/(\Gamma N/N)$, and (X, π, B) is a holomorphic fiber bundle with a typical fiber $F \cong N/\Gamma \cap N$. We shall call this decomposition the Mostow decomposition of X. If G is solvable, the commutator group $G' = [G, G]$ is nilpotent. G' is contained in the maximal nilpotent normal subgroup N, so that G/N is abelian. Therefore the base space B is a complex torus.

In an obvious way, we define the pairing

$$H^0(X, \Omega^p) \times H^0(X^p \wedge \Theta) \to C\,, \qquad \varphi \times \theta \rightsquigarrow (\varphi, \theta)\,.$$

The exterior differentiation $d : H^0(X, \Omega^{p-1}) \to H^0(X, \Omega^p)$ induces an adjoint map ${}^t d : H^0(X^p \wedge \Theta) \to H^0(X^{p-1} \wedge \Theta)$. Then we obtain

Lemma 1.1. (1) $({}^t d)(\theta \wedge \theta') = -[\theta, \theta']$, $\theta, \theta' \in H^0(X, \Theta)$.

(2) $({}^t d)(\theta \wedge \theta' \wedge \theta'') = -\theta \wedge ({}^t d)(\theta' \wedge \theta'') - \theta' \wedge ({}^t d)(\theta'' \wedge \theta)$
$- \theta'' \wedge ({}^t d)(\theta \wedge \theta')$, $\theta, \theta', \theta'' \in H^0(X, \Theta)$.

We omit the proof.

(1) of Lemma 1.1 shows

$$(d\varphi, \theta \wedge \theta') = -(\varphi, [\theta, \theta']) \tag{1.1}$$

for $\theta, \theta' \in H^0(X, \Theta)$, and $\varphi \in H^0(X, \Omega^1)$. (1) and (2) show that $d^2 = 0$ is equivalent to the Jacobi's identity.

Let $\mathfrak{g}$ be a solvable Lie algebra defined over C. Then by virtue of Lie's theorem we have a C-basis of $\mathfrak{g} : \varphi_1, \cdots, \varphi_n$ ($n = \dim_C \mathfrak{g}$) such that

$$d\varphi_\nu = \xi_\nu \wedge \varphi_\nu + \eta_\nu, \qquad \nu = 1, \cdots, n\,, \tag{1.2}$$

where ξ_ν, η_ν are represented by $\varphi_1, \cdots, \varphi_{\nu-1}$. Since $d^2\varphi_\nu = 0$, we have $d\xi_\nu = 0$, i.e., ξ_ν is a closed holomorphic 1 form. There it follows from (1.3) that $\xi_\nu = \sum_{\mu=1}^{s} a_{\nu\mu}\varphi_\mu$ for some constants $a_{\nu\mu}$.

Lemma 1.2. *Let X be a compact complex manifold of dimension n, and φ a holomorphic $(n-1)$-form on X. Then $d\varphi = 0$.*

Proof. If $d\varphi \neq 0$, then $i^{-n^2} \int_X d\varphi \wedge d\bar{\varphi} > 0$. On the other hand,

$i^{-n^2}\int_X d\varphi \wedge d\bar{\varphi} = i^{-n^2}\int_X d(\varphi \wedge d\bar{\varphi}) = 0$, a contradiction.

From Lemma 1.2 we infer readily

Lemma 1.3. *Let* $\{\varphi_\lambda\}$ *be a basis of* $H^0(X, \Omega^1)$ *which satisfies* (1.2). *Then*

$$\sum_{\nu=r+1}^{n} \xi_\nu = 0 . \tag{1.3}$$

Proposition 1.4. *Let G be a simply connected, connected solvable complex Lie group. Then G is biholomorphically equivalent to* $\boldsymbol{C}^n$, *where* $n = \dim_{\boldsymbol{C}} G$.

Proof. When $\dim G = 1$, we can prove the proposition easily. By induction on $\dim G$ we shall prove the proposition. When $\dim G \geq 2$, there exists a connected normal Lie subgroup N of dim 1. $(G, \pi, G/N)$ is a holomorphic fiber bundle with fiber N. Calculating homotopy exact sequences of this fiber bundle, we infer readily that N and G/N are simply connected, connected and obviously solvable. By the hypothesis of the induction, G/N and N are biholomorphically equivalent to $\boldsymbol{C}^{n-1}$ and $\boldsymbol{C}$ respectively. From Oka's principle it follows that G is biholomorphically equivalent to $\boldsymbol{C}^n$.

2. Classification of three-dimensional complex solvable manifolds and construction of examples

In this section we shall classify three-dimensional complex solvable manifolds, and use an algorithm to classify higher-dimensional complex solvable manifolds. Let $X = G/\Gamma$ be a three-dimensional solvable manifold, and φ_1, φ_2 φ_3 be a basis of $H^0(X, \Omega^1)$, which satisfies (1.2).

By an elementary calculation together with Lemma 1.4, solvable manifolds X are classified into the following three classes:

III-(1): $d\varphi_\lambda = 0 , \quad \lambda = 1, 2, 3 ,$

III-(2): $d\varphi_1 = 0 , \quad d\varphi_2 = 0 , \quad d\varphi_3 = -\varphi_1 \wedge \varphi_2 ,$

III-(3): $d\varphi_1 = 0 , \quad d\varphi_2 = \varphi_1 \wedge \varphi_2 , \quad d\varphi_3 = -\varphi_1 \wedge \varphi_3 .$

Dualizing III (1)-(3) by virtue of (1.1) we can determine the structures of the Lie algebra $\mathfrak{g}$ of G.

III-(1)′: $[\theta_\lambda, \theta_\nu] = 0 , \quad \lambda, \nu = 1, 2, 3 ,$

III-(2)′: $[\theta_1, \theta_2] = -[\theta_2, \theta_1] = \theta_3 , \quad [\theta_\lambda, \theta_\nu] = 0$ otherwise ,

III-(3)′: $[\theta_1, \theta_2] = -[\theta_2, \theta_1] = -\theta_2 ,$

$[\theta_1, \theta_3] = -[\theta_3, \theta_1] = \theta_3 , \quad [\theta_2, \theta_3] = 0 .$

Case III-(1). It is well known that X is a complex torus.

Case III-(2). In view of Proposition 1.4, $\boldsymbol{C}^3$ is the universal covering of X.

Let 0 be the origin of C^3. We set $\Phi_\nu(z) = \int_0^z \varphi_\nu$, $\nu = 1, 2$. Then Φ_ν is a single valued holomorphic function on C^3, and $\varphi_\nu = d\Phi_\nu$, $\nu = 1, 2$. Thus $d\varphi_3 = -d\Phi_1 \wedge d\Phi_2$, i.e., $d(\varphi_3 + \Phi_1 d\Phi_2) = 0$. We set $\Phi_3(z) = \int_0^z \varphi_3 + \Phi_1 d\Phi_2$. Φ_3 is a single valued holomorphic function on C^3, and $\varphi_3 = d\Phi_3 - \Phi_1 d\Phi_2$. For $g \in \Gamma$, we set $z' = z \cdot g$. Since φ_λ is Γ-invariant, $d\Phi_\nu(z') = d\Phi_\nu(z)$ ($\nu = 1, 2$). Thus we have $\Phi_\nu(z') = \Phi_\nu(z) + \omega_\nu(g)$, where $\omega_\nu(g)$ is a constant depending only on g. Since

$$\begin{aligned}\varphi_3(z') &= d\Phi_3(z') - \Phi_1(z')d\Phi_2(z') \\ &= d\Phi_3(z') - (\Phi_1(z) + \omega_1(g))d\Phi_2(z) ,\end{aligned}$$

we obtain

$$\Phi_3(z') = \Phi_3(z) + \omega_1(g)\Phi_2(z) + \omega_3(g) ,$$

for some constant $\omega_3(g)$ depending only on g. Define a multiplication $*$ of C^3 by

$$(z_1, z_2, z_3) * (y_1, y_2, y_3) = (z_1 + y_1, z_2 + y_2, z_3 + y_1 z_2 + y_3) .$$

This multiplication $*$ makes C^3 a nilpotent complex Lie group with the Lie algebra of type III-(2)$'$. Hence G is isomorphic to $(C^3, *)$ as a complex Lie group.

Case III-(3). Set

$$\Phi_1(z) = \int_0^z \varphi_1 , \qquad \Phi_2(z) = \int_0^z e^{-\Phi_1}\varphi_2 , \qquad \Phi_3(z) = \int_0^z e^{\Phi_1}\varphi_3 .$$

Since $d\varphi_1 = d(e^{-\Phi_1}\varphi_2) = d(e^{\Phi_1}\varphi_3) = 0$, Φ_λ are single valued holomorphic functions on C^3 and we have $\varphi_1 = d\Phi_1$, $\varphi_2 = e^{\Phi_1}d\Phi_2$, $\varphi_3 = d^{-\Phi_1}d\Phi_3$. By the same argument as in the case of III-(2), we have

$$\begin{aligned}\Phi_1(z') &= \Phi_1(z) + \omega_1(g) , \\ \Phi_2(z') &= e^{-\omega_1(g)}\Phi_2(z) + \omega_2(g) , \\ \Phi_3(z') &= e^{\omega_1(g)}\Phi_3(z) + \omega_3(g) ,\end{aligned}$$

where $z' = z \cdot g$ for $g \in \Gamma$, and $\omega_\nu(g)$'s are constants depending only on g. Define a multiplication $*$ of C_3 by

$$(z_1, z_2, z_3) * (y_1, y_2, y_3) = (z_1 + y_1, e^{-y_1}z_2 + y_2, e^{y_1}z_3 + y_3) .$$

The multiplication $*$ makes C^3 a solvable complex Lie group with the Lie algebra of type III-(3)$'$, so that G is isomorphic to $(C^3, *)$.

Examples. Case III-(2). Set

$$G = \left\{ \begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 1 & 1 \end{pmatrix} ; z_i \in \boldsymbol{C} \right\} \cong \boldsymbol{C}^3 ,$$

$$\Gamma = \left\{ \begin{pmatrix} 1 & \omega_2 & \omega_3 \\ 0 & 1 & \omega_1 \\ 0 & 0 & 1 \end{pmatrix} ; \omega_i \in \boldsymbol{Z} + \boldsymbol{Z}\sqrt{-1} \right\} .$$

The multiplication is defined by

$$\begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \omega_2 & \omega_3 \\ 0 & 1 & \omega_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_2 + \omega_2 & z_3 + \omega_1 z_2 + \omega_3 \\ 0 & 1 & z_1 + \omega_1 \\ 0 & 0 & 1 \end{pmatrix} .$$

$X = G/\Gamma$ is called Iwasawa manifold.

Case III-(3a). We take an algebraic integer α satisfying the equation $\alpha^2 + 5\alpha + 7 = 0$. Let E be an elliptic curve with fundamental periods $\{1, \alpha\}$. Let H be a group of analytic automorphisms of $\boldsymbol{C} \times E \times E$ generated by two automorphisms:

$$\sigma_1 : (z_1, z_2, z_3) \mapsto (z_1 + 2\pi i, z_2, z_3) ,$$
$$\sigma_2 : (z_1, z_2, z_3) \mapsto (z_1 + \beta, (-\alpha - 2)z_2, (\alpha + 3)z_3) ,$$

where $\beta = \log \alpha$, and (z_1, z_2, z_3) are global coordinates of $\boldsymbol{C} \times E \times E$. H acts on $\boldsymbol{C} \times E \times E$ properly discontinuously, and its action has no fixed points. The quotient manifold $X = \boldsymbol{C} \times E \times E/H$ is a parallelisable manifold of type III-(3) with $h^{0,1} = \dim_{\boldsymbol{C}} H^1(X, \mathcal{O}) = 1$.

Case III-(3b). We take a unimodular matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with trace $A \geq 3$. Let α be an eigenvalue of A, and $\beta = \log \alpha > 0$. Let E be an elliptlc curve with fundamental periods $\{1, \tau\}$. Let H be a group of analytic automorphisms of $\boldsymbol{C} \times E \times E$ generated by two automorphisms:

$$\sigma_1 : (z_1, z_2, z_3) \mapsto (z_1 + 2\pi i, z_2, z_3) ,$$
$$\sigma_2 : (z_1, z_2, z_3) \mapsto (z_1 + \beta, az_2 + bz_3, cz_2 + dz_3) ,$$

where (z_1, z_2, z_3) denotes the system of global coordinates of $\boldsymbol{C} \times E \times E$. H operates on $\boldsymbol{C} \times E \times E$ properly discontinuously, and its action has no fixed points. The quotient manifold $X = \boldsymbol{C} \times E \times E/H$ is a parallelisable manifold of type III-(3) with $h^{0,1} = 3$.

By virtue of Theorems 3 and 4 and the proof of Theorem 4, it can be checked that $h^{0,1} \neq 2$ for a solvable manifold of type III-(3). Thus we obtain

Theorem 1. *Three-dimensional solvable manifolds are classified into the following four classes:*

Lie group		b_1	r	$h^{0,1}$	*Structure (Albanese mapping)*
(1)	*abelian*	6	3	3	*complex torus*
(2)	*nilpotent*	4	2	2	T^1*-bundle over* T^2
(3a)	*solvable*	2	1	1	T^2*-bundle over* T^1
(3b)	*solvable*	2	1	3	T^2*-bundle over* T^1

where T^1 *and* T^2 *denote complex tori of dimensions* 1 *and* 2 *respectively.*

In this section, we have shown how to determine the structures of C^3 as solvable Lie groups. Proposition 2.2 and the statement below show that this algorithm is valid for higher dimensional cases.

Let G be a simply connected, connected solvable complex Lie group of dim n.

Definition. A solvable Lie algebra $\mathfrak{g}$ has the Chevalley decomposition if there exist a commutative subalgebra $\mathfrak{a}$ and the maximal nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ (direct sum as vector spaces).

Assume $\mathfrak{g}$ to have the Chevalley decomposition. Then by definition we can choose a basis $\{\theta_\lambda\}$ of $\mathfrak{g}$ such that

$$(2.1)\qquad \begin{aligned} &[\theta_\lambda, \theta_\nu] = \sum_{\mu \geq \max(\lambda,\nu)} c'_{\mu\lambda\nu}\theta_\mu \ , \\ &[\theta_\lambda, \theta_\nu] = 0 \qquad (1 \leq \lambda, \nu \leq s) \ , \\ &[\theta_\lambda, \theta_\nu] = \sum_{\mu > \max(\lambda,\nu)} c'_{\mu\lambda\nu}\theta_\mu \qquad (s + 1 \leq \lambda, \nu \leq n) \ , \end{aligned}$$

where $c'_{\mu\lambda\nu} = -c'_{\mu\nu\lambda}$.

Dualizing (2.1) by (1.1) we conclude that there exists a basis $\{\varphi_\lambda\}$ of right (or left) invariant 1-forms on G such that

$$(2.2)\qquad d\varphi_\mu = \sum c_{\mu\lambda\nu}\varphi_\lambda \wedge \varphi_\nu \ ,$$

where $c_{\mu\lambda\nu} = -c_{\mu\nu\lambda}$. $c_{\mu\lambda\nu} = 0$ if "$1 \leq \lambda, \nu \leq s$" or "$s + 1 \leq \lambda, \nu$ and $\mu \leq \max(\lambda, \nu)$" or "$\mu < \max(\lambda, \nu)$".

Furthermore we can arrange $d\varphi_\mu$ in the following order:

$$Q_1\colon\ d\varphi_1 = 0, \cdots, d\varphi_r = 0, \text{ where } r = \dim H^0(X, d\mathcal{O}),$$

$d\mathcal{O}$ denoting the sheaf of germs of closed holomorphic 1-forms on X;

$$(2.3)\qquad Q_l\colon\ d\varphi_\lambda = \text{sum of } \varphi_\nu \wedge \varphi_\mu\text{'s for } (\nu, \mu) \in \bigcup_{\rho=1}^{l-1} Q_{l-\rho} \times Q_\rho \qquad (l = 2, 3, \cdots);$$

$$Q = \bigcup Q_k \text{ and } Q_\infty = \{1, \cdots, n\} - Q = \{m+1, \cdots, n\} ;$$

any nontrivial linear combination of $d\varphi_{m+1}, \cdots, d\varphi_n$ cannot be represented by a linear combination of $d\varphi_1, \cdots, d\varphi_m$.

Proposition 2.1. *Assume* $\mathfrak{g}$ *to have the Chevalley decomposition, and let* $\{\varphi_\lambda\}$ *be a basis of right invariant holomorphic* 1*-forms on* G *which satisfy* (2.2). *Then there exist holomorphic functions* $\Phi_1, \cdots, \Phi_n$ *on* G *such that*

$$(2.4) \qquad \begin{aligned} \varphi_\lambda &= d\Phi_\lambda \qquad (1 \leq \lambda \leq r) , \\ \varphi_\lambda &= \sum_{\nu=s+1}^{\lambda} F_{\lambda\nu}(\Phi) d\Phi_\nu \qquad (r+1 \leq \lambda \leq n) , \end{aligned}$$

where $F_{\lambda\nu}(\Phi) = \sum_a F_{\lambda\nu a}(\Phi) \exp(a_1\Phi_1 + \cdots + a_s\Phi_s)$, $F_{\lambda\nu a}(\Phi)$ *is a polynomial in* $\Phi_1, \cdots, \Phi_{\lambda-1}$, *and* $F_{\lambda\lambda}(\Phi) = \exp(a_1^\lambda\Phi_1 + \cdots + a_s^\lambda\Phi_s)$.

Proof. By induction on $n = \dim G$ we shall prove the proposition, which is obvious for $n = 1$. Assume (2.1) to be valid for $\nu \leq n-1$. Since $\xi_n = \sum_{\rho=1}^{s} a_\rho^n \varphi_\rho$,

$$\begin{aligned} &d\left(\exp\left(-\sum_{\rho=1}^{s} a_\rho^n \Phi_\rho\right)\varphi_n\right) \\ &= -\exp(-\sum a_\rho^n\Phi_\rho)\xi_n \wedge \varphi_n + \exp(-\sum a_\rho^n\Phi_\rho)(\xi_n \wedge \varphi_n + \eta_n) \\ &= \sum_{1\leq\lambda<\nu\leq n-1} F_{\lambda\nu}^*(\Phi) d\Phi_\lambda \wedge d\Phi_\nu . \end{aligned}$$

By the hypothesis of the induction together with (2.1), (2.2) and (2.3) we have

$$\begin{aligned} F_{\lambda\nu}^*(\Phi) &= 0 \qquad (1 \leq \lambda, \nu \leq s) , \\ F_{\lambda\nu}^*(\Phi) &= \sum_a F_{\lambda\nu a}^* \exp(a_1\Phi_1 + \cdots + a_s\Phi_s) , \end{aligned}$$

wheae $F_{\lambda\nu a}^*(\Phi)$ is a polynomial in $\Phi_1, \cdots, \Phi_{n-2}$. Take G_ν such that

$$\partial G_\nu / \partial \Phi_{n-1} = F_{\nu n-1}^* \quad (\nu > s) , \qquad \partial G_\nu / \partial \Phi_\nu = F_{\nu n-1}^* \quad (\nu \leq s) ,$$

$$\begin{aligned} &d\left(\exp(-\sum a_\rho^n\Phi_\rho)\varphi_n + \sum_{\nu>s} G_\nu d\Phi_\nu - \sum_{\nu\leq s} G_\nu d\Phi_{n-1}\right) \\ &= \sum_{\lambda<\nu} F_{\lambda\nu}^* d\Phi_\lambda \wedge d\Phi_\nu - \sum_{\nu>s} F_{\nu n-1}^* d\Phi_\nu \wedge d\Phi_{n-1} - \sum_{\nu\leq s} F_{\nu n-1}^* d\Phi_\nu \wedge d\Phi_{n-1} \\ &\quad + (\text{terms of } d\Phi_\lambda \wedge d\Phi_\nu,\ \lambda, \nu \leq n-2) \\ &= \sum_{\lambda<\nu\leq n-2} F_{\lambda\nu}^{**} d\Phi_\lambda \wedge d\Phi_\nu, \qquad \text{where } F_{\lambda\nu}^{**} = F_{\lambda\nu}^{**}(\Phi_1, \cdots, \Phi_{n-1}) . \end{aligned}$$

Since $0 = d\left(\sum_{\lambda<\nu\leq n-2} F_{\lambda\nu}^{**} d\Phi_\lambda \wedge d\Phi_\nu\right) = \dfrac{\partial F_{\lambda\nu}^{**}}{\partial \Phi_{n-1}} d\Phi_{n-1} \wedge d\Phi_\lambda \wedge d\Phi_\nu + \cdots$, we have $\partial F^{**}_{\lambda\nu}/\partial \Phi_{n-1} = 0$. Hence $F^{**}_{\lambda\nu} = E^{**}_{\lambda\nu}(\Phi_1, \cdots, \Phi_{n-2})$. Obviously $F^{**}_{\lambda\nu} = 0$ $(1 \leq \lambda, \nu \leq s)$, etc. Thus we obtain the proposition by induction. q.e.d.

By the same way as stated above we can contruct a multiplication $*$ of $\boldsymbol{C}^n$. In order to show that this multiplication defines a Lie group structure of $\boldsymbol{C}^n$ we have only to check the associative law. We can easily do this by using the fact that φ_λ is a right invariant 1-form on G and the multiplication is written explicitly (see below). Hence the multiplication $*$ makes $\boldsymbol{C}^n$ a complex Lie group, and $(\boldsymbol{C}^n, *)$ is isomorphic to the Lie group G.

The multiplication $*$ is defined by

$$\begin{aligned}(z_1, \cdots, z_n) * (y_1, \cdots, y_n) = (z_1 + y_1, \cdots, z_r + y_r, \cdots, \\ \exp(-a_1^\nu y_1 - \cdots - a_s^\nu y_s) z_\nu + y_\nu + F_\nu(z, y), \cdots),\end{aligned}$$

where $F_\nu(z, y)$ is expressed in $z_1, \cdots, z_{\nu-1}, y_1, \cdots, y_{\nu-1}$. Therefore we obtain

Proposition 2.2. *Assume $\mathfrak{g}$ to have the Chevalley decomposition. Let $\{\varphi_\lambda\}$ be a basis of right invariant 1-forms on G which satisfy (2.3). By an appropriate choice of a system of coordinates $(z_1, \cdots, z_n)$ of $\boldsymbol{C}^n$, $\{\varphi_\lambda\}$ are represented as follows:*

$$\varphi_\lambda = \begin{cases} dz_\lambda\,, & 1 \leq \lambda \leq r\,, \\ \sum_{\nu=s+1}^{\lambda} F_{\lambda\nu}(z) dz_\nu\,, & r < \lambda\,, \end{cases} \tag{2.5}$$

where $F_{\lambda\nu}(z) = \sum_a F_{\lambda\nu a}(z) \exp\left(\sum_{\rho=1}^{s} a_\rho z_\rho\right)$, $F_{\lambda\nu a}$ is a polynomial in $z_1, \cdots, z_{\lambda-1}$, and $F_{\lambda\lambda} = \exp \sum_{\rho=1}^{s} a_\rho^\lambda z_\rho$.

Dualizing Proposition 2.3 by means of (1.1), we obtain

Proposition 2.3. *Let $\{\theta_\lambda\}$ be a dual basis of right invariant vector fields of $\{\varphi_\lambda\}$. Then by the same system of coordinates of $\boldsymbol{C}^n$ as in Proposition 2.2, $\{\theta_\lambda\}$ are represented as follows:*

$$\theta_\lambda = \begin{cases} \partial/\partial z_\lambda\,, & 1 \leq \lambda \leq r\,, \\ \sum_{\nu=\lambda}^{n} G_{\lambda\nu}(z) \partial/\partial z_\lambda\,, & r < \lambda\,, \end{cases} \tag{2.6}$$

where $G_{\lambda\nu}(z) = \sum_a G_{\lambda\nu a}(z) \exp\left(\sum_{\rho=1}^{s} a_\rho z_\rho\right)$ and $G_{\lambda\nu a}$ is a polynomial in $z_1, \cdots, z_{\nu-1}$.

3. Construction of Kuranishi families of deformations of three-dimensional complex solvable manifolds

In this section we shall calculate small deformations of three-dimensional complex solvable manifolds constructed in §2. In these cases we see that several numerical characters, such as $h^{p,q}$ $(p, q) \neq (0, 0)$, r, P_m $(m \geq 1)$ are not necessarily invariant under small deformations. Moreover, in the case of III-(3b) there are small deformations whose universal covering are not biholomorphically equivalent to $\boldsymbol{C}^3$.

Kodaira first calculated small deformations of Iwasawa manifold. In the first part of this section we shall introduce his result.

Case III-(2) Let $X = \boldsymbol{C}^3/\Gamma$ be Iwasawa manifold. $g \in \Gamma$ operates on $\boldsymbol{C}^3$ as follows:

$$z_1' = z_1 + \omega_1 \ , \quad z_2' = z_2 + \omega_2 \ , \quad z_3' = z_3 + \omega_1 z_2 + \omega_3 \ ,$$

where $g = (\omega_1, \omega_2, \omega_3)$ and $z' = z \cdot g$. There exist holomorhpic 1-forms $\varphi_1, \varphi_2, \varphi_3$ which are linearly independent at every point on X and are given by

$$\varphi_1 = dz_1 \ , \quad \varphi_2 = dz_2 \ , \quad \varphi_3 = dz_3 - z_1 dz_2 \ ,$$

so that

$$d\varphi_1 = d\varphi_2 = 0 \ , \qquad d\varphi_3 = -\varphi_1 \wedge \varphi_2 \ .$$

On the other hand we have holomorphic vector fields $\theta_1, \theta_2, \theta_3$ on X given by

$$\theta_1 = \partial_1 \ , \quad \theta_2 = \partial_2 + z_1 \partial_3 \ , \quad \theta_3 = \partial_3 \ ,$$

where ∂_λ denotes $\partial/\partial z_\lambda$. It is easily seen that

$$[\theta_1, \theta_2] = -[\theta_2, \theta_1] = \theta_3 \ , \qquad [\theta_2, \theta_3] = [\theta_1, \theta_3] = 0 \ .$$

In view of Theorem 2 (§ 4), $H^{0,1}_{\bar\partial}(X)$ is spanned by $\bar\varphi_1, \bar\varphi_2$. Since Θ is isomorphic to $\mathcal{O}^3$, $H^{0,1}_{\bar\partial}(X, \Theta)$ is spanned by $\theta_i \bar\varphi_\lambda$, $i = 1, 2, 3$, $\lambda = 1, 2$.

For vector (0, 1)-forms Ψ, τ, we define

$$[\psi, \tau] = \sum_{\alpha,\beta} (\psi^\alpha \wedge \partial_\alpha \tau^\beta + \tau^\alpha \wedge \partial^\alpha \psi^\beta) \partial_\beta \ ,$$

where $\psi = \sum \psi^\alpha \partial_\alpha$ and $\tau = \sum \tau^\beta \partial_\beta$. (Cf. [3].) We have

$$[\theta_i \bar\varphi_\lambda, \theta_k \bar\varphi_\nu] = [\theta_i, \theta_k] \bar\varphi_\nu \wedge \bar\varphi_\nu \ .$$

We shall construct a vector (0, 1)-form ψ such that

$$\bar\partial \psi - \tfrac{1}{2}[\psi, \psi] = 0 \ . \tag{3.1}$$

Set $\psi = \sum_{\alpha=1}^{\infty} \psi_\alpha(t)$, where $\psi_1(t) = \sum_{i=1}^{3} \sum_{\lambda=1}^{2} t_{i\lambda} \theta_i \bar\varphi_\lambda$, and $\psi_\alpha(t)$ is the homogeneous term of total degree α in $t_{i\lambda}$. Then

$$\bar{\partial}\psi_2(t) = \tfrac{1}{2}[\psi_1(t), \psi_1(t)] = (t_{11}t_{22} - t_{21}t_{12})\theta_3\bar{\varphi}_1 \wedge \bar{\varphi}_2 \ .$$

Set $\psi_2(t) = -(t_{11}t_{22} - t_{21}t_{12})\theta_3\bar{\varphi}_3$. Thus we obtain a solution of (3.1) given by

$$\psi(t) = \sum_{i=1}^{3}\sum_{\lambda=1}^{2} t_{i\lambda}\theta_i\bar{\varphi}_\lambda - (t_{11}t_{22} - t_{21}t_{12})\theta_3\bar{\varphi}_3 \ .$$

This proves the existence of a locally complete complex analytic family of deformations X_t of X depending on 6 effective parameters $t_{i\lambda}$, [3].

Next, by solving the system of differential equations

$$\bar{\partial}\zeta_\nu - \psi(t)\zeta_\nu = 0 \ , \qquad \nu = 1, 2, 3 \ , \tag{3.2}$$

under the initial condition $\zeta_\nu(0) = z_\nu$, we have the solutions:

$$\zeta_1 = z_1 + \sum_{\lambda=1}^{2} t_{1\lambda}\bar{z}_\lambda \ , \qquad \zeta_2 = z_2 + \sum_{\lambda=1}^{2} t_{2\lambda}\bar{z}_\lambda \ ,$$

$$\zeta_3 = z_3 + \sum_{\lambda=1}^{2} (t_{3\lambda} + t_{2\lambda}z_1)\bar{z}_\lambda + A(\bar{z}) - D(t)\bar{z}_3 \ ,$$

where

$$D(t) = t_{11}t_{22} - t_{21}t_{12} \ , \qquad A(\bar{z}) = \tfrac{1}{2}(t_{11}t_{21}\bar{z}_1^2 + 2t_{11}t_{22}\bar{z}_1\bar{z}_2 + t_{12}t_{22}\bar{z}_2^2) \ .$$

Since

$$\begin{aligned} &d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_3 \wedge d\bar{\zeta}_1 \wedge d\bar{\zeta}_2 \wedge d\bar{\zeta}_3 \\ &\qquad = c(t) dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \ , \end{aligned}$$

where $c(t)$ is a differentiable function in $t_{i\lambda}$ with $c(0) = 1$, it follows that $\Phi\colon (z_1, z_2, z_3) \mapsto (\zeta_1, \zeta_2, \zeta_3)$ is a diffeomorphism of C^3 if $\sum_{i,\lambda} |t_{i\lambda}| < \varepsilon$ for sufficiently small positive number ε.

$$\begin{tikzcd}[column sep=huge, row sep=huge] C^3 \arrow[r, "\Phi", "\underset{\text{diffeomorphic}}{\approx}"'] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & C^3 \arrow[d, "\pi_t"] \\ X \arrow[r, "\varphi", "\underset{\text{diffeomorphic}}{\approx}"'] & X_t \end{tikzcd}$$

Since π is a covering map, $\pi_t = \varphi \circ \pi \circ \Phi^{-1}$ is also a covering map from C^3 to X_t. Therefore C^3 is the universal covering of X_t, that is, $X_t = C^3/\Gamma_t$ for a group Γ_t of analytic automorphisms of C^3. The group Γ_t is defined by

$$\zeta_1' = \zeta_1 + \tilde{\omega}_1(t) \ , \qquad \zeta_2' = \zeta_2 + \tilde{\omega}_2(t) \ ,$$

$$\zeta_3' = \zeta_3 + \tilde{\omega}_3(t) + \omega_1\zeta_2 + \left(\sum_{\lambda=1}^{2} t_{2\lambda}\bar{\omega}_\lambda\right)\zeta_1 + A(\bar{\omega}) - D(t)\bar{\omega}_3 ,$$

where $\tilde{\omega}_i(t) = \omega_i + t_{i1}\bar{\omega}_1 + t_{i2}\bar{\omega}_2$ for $(\omega_1\ \omega_2\ \omega_3) \in \Gamma$.

Now we summarize the numerical characters of deformations. *The deformations are divided into the following three classes*:

i) $t_{11} = t_{12} = t_{21} = t_{22} = 0$, X_t is a parallelisable manifold of type III-(2).

ii) $D(t) = 0$ and $(t_{11}\ t_{12}\ t_{21}\ t_{22}) \neq (0, 0, 0, 0)$, X_t is not parallelisable.

iii) $D(t) \neq 0$, X_t is not parallelisable.

	r	$h^{1,0}$	$h^{0,1}$	$h^{2,0}$	$h^{1,1}$	$h^{0,2}$	$h^{3,0}$	$h^{2,1}$	$h^{1,2}$	$h^{0,3}$	$P_m\ (m \geq 1)$
i)	2	3	2	3	6	2	1	6	6	1	1
ii)	2	2	2	2	5	2	1	5	5	1	1
iii)	2	2	2	1	5	2	1	4	4	1	1

$h^{3-p,3-q} = h^{p,q}$

Next we shall calculate small deformations of a solvable manifold of type III-(3) constructed in § 2. As stated before, $h^{0,1}(X) = 1$ or 3 (see the proof of Theorem 3).

First we shall consider the case where $h^{0,1} = 3$. Let $X = C^3/\Gamma$ be a solvable manifold constructed in Example III-(3b). By an appropriate linear transformation of z_2 and z_3, $g \in \Gamma$ operates on C^3 as follows:

$$z_1' = z_1 + \omega_1 , \quad z_2' = e^{-\omega_1}z_2 + \omega_2 , \quad z_3' = e^{\omega_1}z_3 + \omega_3 .$$

There exist holomorphic 1-forms $\varphi_1, \varphi_2, \varphi_3$ on X given by

$$\varphi_1 = dz_1 , \quad \varphi_2 = e^{z_1}dz_2 , \quad \varphi_3 = e^{-z_1}dz_3 ,$$

so that

$$d\varphi_1 = 0 , \quad d\varphi_2 = \varphi_1 \wedge \varphi_2 , \quad d\varphi_3 = -\varphi_1 \wedge \varphi_3 .$$

On the other hand, there exist holomorphic vector fields $\theta_1, \theta_2, \theta_3$ given by

$$\theta_1 = \partial_1 , \quad \theta_2 = e^{-z_1}\partial_2 , \quad \theta_3 = e^{z_1}\partial_3 ,$$

such that

$$[\theta_1, \theta_2] = -[\theta_2, \theta_1] = -\theta_2 , \qquad [\theta_1, \theta_3] = -[\theta_3, \theta_1] = \theta_3 , \qquad [\theta_2, \theta_3] = 0 .$$

$H^{0,1}_{\bar\partial}(X)$ is generated by $\varphi_1^* = d\bar{z}_1$, $\varphi_2^* = e^{z_1}d\bar{z}_2$ and $\varphi_3^* = e^{-z_1}d\bar{z}_3$ (see the proof of Theorem 3). Since $\varphi_2^* = e^{z_1-\bar{z}_1}\bar{\varphi}_2$ and $\varphi_3^* = e^{-z_1+\bar{z}_1}\bar{\varphi}_3$, $H^{0,1}_{\bar\partial}(X, \Theta)$ is spanned by $\theta_i\varphi_\lambda^*$, $i = 1, 2, 3$, $\lambda = 1, 2, 3$. We shall construct a vector (0, 1)-forms ψ satisfying (3.1).

Set $\psi(t) = \sum_{\alpha=1}^{\infty} \psi_\alpha(t)$, where $\psi_1(t) = \sum_{i=1}^{3}\sum_{\lambda=1}^{3} t_{i\lambda}\theta_i\varphi_\lambda^*$ and $\psi_\alpha(t)$ is the homogeneous term of total degree α in $t_{i\lambda}$. Then we have

$$\begin{aligned}
[\theta_\lambda\varphi_1^*, \theta_\mu\varphi_2^*] &= (\delta_{1\lambda}\theta_\mu + [\theta_\lambda, \theta_\mu])\varphi_1^* \wedge \varphi_2^* , \\
[\theta_\lambda\varphi_1^*, \theta_\mu\varphi_3^*] &= (-\delta_{1\lambda}\theta_\mu + [\theta_\lambda, \theta_\mu])\varphi_1^* \wedge \varphi_3^* , \\
[\theta_\lambda\varphi_2^*, \theta_\mu\varphi_3^*] &= (-\delta_{1\lambda}\theta_\mu - \delta_{1\mu}\theta_\lambda + [\theta_\lambda, \theta_\mu])\varphi_2^* \wedge \varphi_3^* , \\
\tfrac{1}{2}[\psi_1, \psi_1] &= \eta_1\theta_1 + \eta_2\theta_2 + \eta_3\theta_3 ,
\end{aligned}$$

where

$$\begin{aligned}
\eta_1 &= -t_{11}t_{13}\varphi_1^* \wedge \varphi_3^* + t_{11}t_{12}\varphi_1^* \wedge \varphi_2^* - 2t_{12}t_{13}\varphi_2^* \wedge \varphi_3^* , \\
\eta_2 &= (t_{21}t_{13} - 2t_{11}t_{23})\varphi_1^* \wedge \varphi_3^* + t_{12}t_{21}\varphi_1^* \wedge \varphi_2^* - 2t_{12}t_{23}\varphi_2^* \wedge \varphi_3^* , \\
\eta_3 &= -t_{13}t_{31}\varphi_1^* \wedge \varphi_3^* + (2t_{11}t_{32} - t_{31}t_{12})\varphi_1^* \wedge \varphi_2^* - 2t_{13}t_{32}\varphi_2^* \wedge \varphi_3^* .
\end{aligned}$$

Since $\bar\partial\psi_2 = \frac{1}{2}[\psi_1, \psi_1]$, it follows that η_ν is cohomologous to zero in $H_{\bar\partial}^{0,2}(X)$.

Lemma 3.1. *Set* $\eta = A\varphi_1^* \wedge \varphi_3^* + B\varphi_1^* \wedge \varphi_2^* + C\varphi_2^* \wedge \varphi_3^*$, *and assume that* η *is cohomologous to zero in* $H_{\bar\partial}^{0,2}(X)$. *Then* $A = B = C = 0$.

Proof. $\varphi_1^* \wedge \varphi_3^* = e^{\bar z_1 - z_1}\bar\varphi_1 \wedge \bar\varphi_3$, $\varphi_2^* \wedge \varphi_3^* = \bar\varphi_2 \wedge \bar\varphi_3$, $\varphi_1^* \wedge \varphi_2^* = e^{z_1 - \bar z_1}\bar\varphi_1 \wedge \bar\varphi_2$, $\bar\partial(\varphi_\lambda^* \wedge \varphi_\nu^*) = 0$. If f_1, f_2, f_3 are functions in $z_1, \bar z_1$, then

$$\vartheta(f_1\bar\varphi_2 \wedge \bar\varphi_3 + f_2\bar\varphi_1 \wedge \bar\varphi_3 + f_3\bar\varphi_1 \wedge \bar\varphi_2) = -(\partial_1 f_2 + f_2)\bar\varphi_3 - (\partial_1 f_3 - f_3)\bar\varphi_2 ,$$

where ϑ is the adjoint operator of $\bar\partial$ (see the proofs of Theorems 2 and 3). Thus $\vartheta(\varphi_1^* \wedge \varphi_3^*) = \vartheta(\varphi_1^* \wedge \varphi_3^*) = \vartheta(\varphi_1^* \wedge \varphi_2^*) = 0$, and $\varphi_\lambda^* \wedge \varphi_\nu^*$ is harmonic. Hence $\eta = 0$, $A = B = C = 0$. q.e.d.

It follows from Lemma 3.1 that

$$\begin{array}{llll}
 & t_{11}t_{13} = 0 , & t_{11}t_{12} = 0 , & t_{12}t_{13} = 0 , \\
(3.3) & t_{21}t_{13} - 2t_{11}t_{23} = 0 , & t_{12}t_{21} = 0 , & t_{12}t_{23} = 0 , \\
 & t_{31}t_{13} = 0 , & 2t_{11}t_{32} - t_{31}t_{12} = 0 , & t_{13}t_{32} = 0 .
\end{array}$$

Consequently $\psi = \psi_1$.

By solving (3.2) we have the solutions:

$$\begin{aligned}
\eta_1 &= z_1 + t_{11}\bar z_1 - \log(1 - t_{12}e^{z_1}\bar z_2) + \log(1 + t_{13}e^{-z_1}\bar z_3) , \\
\eta_2 &= z_2 + t_{22}\bar z_2 - t_{21}t_{13}e^{-2z_1}\bar z_1\bar z_3 - \frac{t_{21}}{t_{11}}e^{z_1}(e^{-t_{11}\bar z_1} - 1) + \frac{t_{23}e^{-2z_1}\bar z_3}{1 - t_{13}e^{-z_1}\bar z_3} , \\
\zeta_3 &= z_3 + t_{33}\bar z_3 + t_{12}t_{31}e^{2z_1}\bar z_1\bar z_2 + \frac{t_{31}}{t_{11}}e^{z_1}(e^{t_{11}\bar z_1} - 1) + \frac{t_{32}e^{2z_1}\bar z_2}{1 - t_{12}e^{z_1}\bar z_2} .
\end{aligned}$$

Four cases may occur. If $t_{12} = t_{13} = 0$, we infer that C^3 is the universal covring of X_t by the same argument as in the case of III-(2).

Case 1: $t_{11} \neq 0$, $t_{12} = t_{13} = t_{23} = t_{32} = 0$, $\zeta_1 = z_1 + t_{11}\bar{z}_1$,

$$\zeta_2 = z_2 + t_{22}\bar{z}_2 - \frac{t_{21}}{t_{11}} e^{-z_1}(e^{-t_{11}\bar{z}_1} - 1) ,$$

$$\zeta_3 = z_3 + t_{33}\bar{z}_3 + \frac{t_{31}}{t_{11}} e^{z_1}(e^{t_{11}\bar{z}_1} - 1) .$$

$\boldsymbol{C}^3$ is the universal covering of X_t, i.e., $X_t = \boldsymbol{C}^3/\Gamma_t$ for a group Γ_t of analytic automorphisms of $\boldsymbol{C}^3$; the group Γ_t is defined by

$$\zeta_1' = \zeta_1 + \tilde{\omega}_1 ,$$

$$\zeta_2' = e^{-\omega_1}\zeta_2 + \tilde{\omega}_2 + \frac{t_{21}}{t_{11}} e^{-\zeta_1-\omega_1}(1 - e^{-t_{11}\bar{\omega}_1}) ,$$

$$\zeta_3' = e^{\omega_1}\zeta_3 + \tilde{\omega}_3 - \frac{t_{31}}{t_{11}} e^{\zeta_1+\omega_1}(1 - e^{t_{11}\bar{\omega}_1}) ,$$

where $\tilde{\omega}_i = \omega_i + t_{ii}\bar{\omega}_i$ for $(\omega_1, \omega_2, \omega_3) \in \Gamma$.

Case 2: $t_{11} = t_{12} = t_{13} = 0$, $\zeta_1 = z_1$,

$$\zeta_2 = z_2 + t_{22}\bar{z}_2 + t_{21}e^{-z_1}\bar{z}_1 + t_{23}e^{-2z_1}\bar{z}_3 ,$$

$$\zeta_3 = z_3 + t_{33}\bar{z}_3 + t_{31}e^{z_1}\bar{z}_1 + t_{32}e^{2z_1}\bar{z}_2 .$$

$\boldsymbol{C}^3$ is also the universal covering of X_t, i.e., $X_t = \boldsymbol{C}^3/\Gamma_t$; Γ_t is defined by

$$\zeta_1' = \zeta_1 + \omega_1 ,$$

$$\zeta_2' = e^{-\omega_1}\zeta_2 + \tilde{\omega}_2 + t_{21}\bar{\omega}_1 e^{-\zeta_1-\omega_1} + t_{23}\bar{\omega}_3 e^{-2\zeta_1-2\omega_1} ,$$

$$\zeta_3' = e^{\omega_1}\zeta_3 + \tilde{\omega}_3 + t_{31}\bar{\omega}_1 e^{\zeta_1+\omega_1} + t_{32}\bar{\omega}_1 e^{2\zeta_1+2\omega_1} ,$$

where $\tilde{\omega}_i = \omega_i + t_{ii}\bar{\omega}_i$, $i = 2, 3$ for $(\omega_1, \omega_2, \omega_3) \in \Gamma$.

Case 3 (Kodaira): $t_{12} \neq 0$, $t_{11} = t_{13} = t_{21} = t_{23} = t_{31} = 0$,

$$\zeta_1 = z_1 - \log(1 - t_{12}e^{z_1}\bar{z}_2) , \quad \zeta_2 = z_2 + t_{22}\bar{z}_2 ,$$

$$\zeta_3 = z_3 + t_{33}\bar{z}_3 + \frac{t_{32}e^{2z_1}\bar{z}_2}{1 - t_{12}e^{z_1}\bar{z}_2} .$$

Set

$$w = e^{-z_1} , \quad \eta_1 = w - t_{12}\bar{z}_2 , \quad \eta_2 = z_2 + t_{22}\bar{z}_2 ,$$

$$\eta_3 = z_3 + t_{33}\bar{z}_3 - \frac{t_{32}}{t_{12}} \frac{1}{w} .$$

Any $g \in \Gamma$ induces a transformation g_t of W_t as follows:

$$\eta_1' = e^{-\omega_1}(\eta_1 - t_{12}\bar{\omega}_2)\ , \qquad \eta_2' = e^{-\omega_1}(\eta_2 + \tilde{\omega}_2)\ , \qquad \eta_3' = e^{\omega_1}(\eta_3 + \tilde{\omega}_3)\ ,$$

where $W_t = \{(\eta_1, \eta_2, \eta_3) \in C^3\,;\,(1 - |t_{22}|^2)\eta_1 + t_{12}(\bar{\eta}_2 - \overline{t_{22}\eta_2}) \neq 0\}$ and $\tilde{\omega}_i = \omega_i + t_{ii}\bar{\omega}_i$, $i = 2, 3$ for $(\omega_1, \omega_2, \omega_3) \in \Gamma$. Set $\Delta_t = \{g_t\,;\,g \in \Gamma\}$. Then we have $X_t = W_t/\Delta_t$ for $\sum_{i,\lambda} |t_{i\lambda}| < 1$ and $X = X_0 = W_0/\Delta_0$. For $t_{12} \neq 0$, W_t is not a domain of holomorphy. In fact, by virtue of the edge of the wedge theorem [6] any multivalued holomorphic function on W_t extends to C^3. In particular the universal covering manifold $\tilde{W}_t$ of W_t cannot be imbedded into C^n for any n.

Case 4: $\quad t_{12} \neq 0\ , \qquad t_{11} = t_{12} = t_{21} = t_{31} = t_{32} = 0\ .$

By the transformation: $(z_1, z_2, z_3) \mapsto (-z_1, -z_2, -z_3)$, $(\zeta_1, \zeta_2, \zeta_3) \mapsto (-\zeta_1, -\zeta_2, -\zeta_3)$, we can reduce Case 4 to Case 3.

Now we summarize the numerical characters of small deformations in Case 3.

	r	h^{10}	h^{01}	h^{02}	h^{30}	h^{03}	h^{31}	h^{32}	$P_m(m\geq 1)$	κ
i) $t_{\lambda i}=0$ $i,\lambda=1,2,3$	1	3	3	3	1	1	3	3	1	0
ii) $t_{12}\neq 0$	0	0	2	1	0	0	2	1	0	$-\infty$

Thus we obtain

Theorem 2. $h^{p,q}$ $(p, q) \neq (0, 0)$, r, P_m $(m \geq 1)$ *and* κ *are not necessarily invariant under small deformations.*

Secondly we shall consider the case where X is of type III-(3) with $h^{0,1} = 1$. Holomorphic 1-forms and vector fields on X are given as follows:

$$\varphi_1 = dz_1\ , \quad \varphi_2 = e^{z_1}dz_2\ , \quad \varphi_3 = e^{-z_1}dz_3\ ,$$
$$\theta_1 = \partial_1\ , \quad \theta_2 = e^{-z_1}\partial_2\ , \quad \theta_3 = e^{z_1}\partial_3\ .$$

$H^{0,1}(\Theta)$ is spanned by $\theta_1\bar{\varphi}_1$, $\theta_2\bar{\varphi}_1$, $\theta_3\bar{\varphi}_1$, and the vector (0, 1)-form ψ satisfying (3.1) is given by $\psi(t) = \sum_{i=1}^{3} t_i\theta_i\bar{\varphi}_1$. We can construct a locally complete complex analytic family of deformations of X depending on 3 effective parameters t_i.

Case 1: $\quad t_1 \neq 0\ , \qquad \zeta_1 = z_1 + t_1\bar{z}_1\ ,$

$$\zeta_2 = z_2 - \frac{t_2}{t_1}e^{-z_1}(e^{-t_1\bar{z}_1} - 1)\ , \qquad \zeta_3 = z_3 + \frac{t_3}{t_1}e^{z_1}(e^{t_1\bar{z}_1} - 1)\ .$$

$X^t = C^3/\Gamma_t$, and the group Γ_t is defined by

$$\zeta_1' = \zeta_1 + \omega_1 + t_1\bar{\omega}_1\ , \qquad \zeta_2' = e^{-\omega_1}\zeta_2 + \frac{t_2}{t_1}e^{-\zeta_1-\omega_1}(1 - e^{-t_1\bar{\omega}_1})\ ,$$

$$\zeta_3' = e^{\omega_1}\zeta_3 - \frac{t_3}{t_1}e^{\zeta_1+\omega_1}(1 - e^{t_1\bar{\omega}_1}) \qquad \text{for } (\omega_1, \omega_2, \omega_3) \in \Gamma \ .$$

Case 2: $t_1 = 0$, $\zeta_1 = z_1$, $\zeta_2 = z_2 + t_2e^{-z_1}\bar{z}_1$, $\zeta_3 = z_3 + t_3e^{z_1}\bar{z}_1$.

$x^t = C^3/\Gamma_t$, and the group Γ_t is defined by

$$\zeta_1' = \zeta_1 + \omega_1 \ , \qquad \zeta_2' = e^{-\omega_1}\zeta_2 + \omega_2 + t_2\bar{\omega}_1e^{-\zeta_1-\omega_1} \ ,$$
$$\zeta_3' = e^{\omega_1}\zeta_3 + \omega_3 + t_3\bar{\omega}_1e^{\zeta_1+\omega_1} \qquad \text{for } (\omega_1, \omega_2, \omega_3) \in \Gamma \ .$$

4. Proofs of Theorems 2 and 3

The following theorem is due to Kodaira.

Theorem 3. *If X is nilpotent, then $h^{01} = r$.*

Proof. We shall calculate the dimension of harmonic (0, 1)-forms by the Dolbeault isomorphism $H^1(X, \mathcal{O}) \cong H^{01}_{\bar\partial}(X)$. Let $\{\varphi_\lambda\}$ and $\{\theta_\lambda\}$ be a basis of $H^0(X, \Omega^1)$ and $H^0(X, \Theta)$ dual to each other with respect to (1.1), which satisfy (2.1) and (2.2) respectively. Let φ be a differentiable (0.1)-form on X. Then $\varphi = \sum_{\lambda=1}^{n} f_\lambda\bar{\varphi}_\lambda$, where f_λ's are differentiable function on X, so that

$$\bar\partial\varphi = \sum_{\lambda,\nu} (\bar\theta_\nu f_\nu)\bar\varphi_\nu \wedge \bar\varphi_\lambda + \sum_{\lambda=1}^{n} f_\lambda d\bar\varphi_\lambda = \sum_{\nu<\lambda} (\bar\theta_\lambda f_\nu - \bar\theta_\lambda f_\nu + 2\sum_{\mu=1}^{n} \overline{c_{\mu\nu\lambda}} f_\mu)\bar\varphi_\nu \wedge \bar\varphi_\lambda .$$

For a differentiable (0, 1)-form $\gamma = \sum_{\lambda=1}^{n} g_\lambda\bar\varphi_\lambda$ we define

$$(\varphi, \lambda) = \int_X \sum_{\lambda=1}^{n} f_\lambda\bar{g}_\lambda \, dx \ ,$$

where $dX = i^{-n^2}\varphi_1 \wedge \cdots \wedge \varphi_n \wedge \bar\varphi_1 \cdots \wedge \bar\varphi_n$.

Let ϑ be the adjoint operator or $\bar\partial$ with respect to the inner (,). For a differentiable function g we have

$$(\vartheta\varphi, g) = (\varphi, \bar\partial g) = \int_X \sum_{\lambda=1}^{n} f_\lambda\theta_\lambda\bar{g} \, dX = -\int_X (\sum \theta_\lambda f_\lambda)\bar{g} \, dX \ .$$

Hence $\vartheta\varphi = -\sum_{\lambda=1}^{n} \theta_\lambda f_\lambda$. Assume that φ is harmonic so that $\bar\partial\varphi = 0$, $\vartheta\varphi = 0$. Consequently

$$\bar\theta_\nu f_\lambda - \bar\theta_\lambda f_\nu + 2\sum_{\mu=1}^{n} \overline{c_{\mu\nu\lambda}} f_\mu = 0, \qquad \sum_{\lambda=1}^{n} \theta_\lambda f_\lambda = 0 \ .$$

Define the Laplacian $\square$ by

$$\square = \vartheta\bar\partial + \bar\partial\vartheta \ .$$

Then $\Box f = 0$ implies $\bar{\partial} f = 0$ for a function f. Hence f is holomorphic on X, and is constant.

$$\Box f_\nu = -2 \sum_{\lambda,\mu=1}^{n} \overline{c_{\mu\nu\lambda}}\, \theta_\lambda f_\mu \ .$$

Since X is nilpotent, we have $c_{\mu\nu\lambda} = 0$ ($\nu \geq \mu$ or $\lambda \geq \mu$). Thus $\Box f_n = 0$, which implies that f_n is constant. From this it follows that

$$\Box f_{n-1} = -2 \sum_{\lambda=1}^{n} \overline{c_{n\, n-1\lambda}}\, \theta_\lambda f_n = 0 \ .$$

Thus f_{n-1} is constant. Inductively we conclude that any f_ν is constant. Since $\bar{\partial} = 0$, $f_\lambda = 0$ $(r < \lambda)$. Hence $\varphi = \sum_{\lambda=1}^{n} f_\lambda \bar{\varphi}_\lambda$ where f_λ's are constant, i.e., $h^{0,1} = r$.

Theorem 4. *If X is solvable and its Lie algebra has the Chevalley decomposition, then we have $b_1 = 2r$.*

Proof. First we assume X to be nilpotent. Consider the following exact sequence:

$$0 \longrightarrow \boldsymbol{C} \longrightarrow \mathcal{O} \xrightarrow{d} d\mathcal{O} \longrightarrow 0 \ .$$

Then we hove

$$0 \to H^0(X, d\mathcal{O}) \to H^1(X, \boldsymbol{C}) \to H^1(X, \mathcal{O}) \to \cdots \ .$$

From theorem 3 it follows that $b_1 \leq 2r$, while in general $b_1 \geq 2 \dim_{\boldsymbol{C}} H^0(X, d\mathcal{O})$. Hence we complete the proof in case that X is nilpotent.

Now we assume X not to be nilpotent. Then the Mostow decomposition (X, π, B) is nontrivial. Set $\dim B = s$ (≥ 1) and $\dim X = n$. Then we can take a system of coordinates $(z_1, \cdots, z_n)$ of the universal covering $\boldsymbol{C}^n$ of X satisfying the following two conditions:

(1) π is the projection to the first s factors, and (X, π, B) is a holomorphic fiber bundle with nilpotent F as fiber:

$$\begin{array}{ccc} X = \boldsymbol{C}^n/\Gamma & \ni & (z_1, \cdots, z_n) \\ \pi \big\downarrow & & \big\downarrow \\ B = \boldsymbol{C}^s/\Gamma & \ni & (z_1, \cdots, z_s) \ . \end{array}$$

(2) θ_λ, φ_ν are represented in the forms (2.5) and (2.6), and $g_1 \in \Gamma_1$ induces an analytic automorphism of F and Alb F; hence g_1 operates on $(z_{s+1}, \cdots, z_{s+r(F)})$ as an affine transformation.

Denoting the ν-th coordinate of $z \cdot g_1$ by $(z \cdot g_1)_\nu = z_\nu'$, we have $z_\nu' = \sum_{\mu=s+1}^{\nu} a_{\nu\mu} z_\nu + c_\nu$, $s + 1 \leq \nu \leq s + r(F)$, where $a_{\nu\mu}$ is constant and $c_\nu = c_\nu(z_1, \cdots, z_s, g_1)$. By induction on ν we can check that any c_ν is constant depending only on g_1

in view of the representation (2.5) of φ_ν. Consider the following spectral sequence:

$$E_2^{p,q} = H^p(B, R^q\pi_*\mathcal{O}_X) \Rightarrow H^{p+q}(X, \mathcal{O}) .$$

Then we have the exact sequence:

$$\begin{array}{c} 0 \to H^1(B, O_B) \to H^1(X, \mathcal{O}_X) \to H^0(B, R^1\pi_*\mathcal{O}_x) \to \cdots \\ \wr\Vert \\ H_{\bar\partial}^{0,1}(X) \end{array}$$

Since F is nilpotent, $H_{\bar\partial}^{0,1}(F, O_F)$ is generated by $d\bar z_{s+1}, \cdots, d\bar z_{s+r(F)}$, and therefore any element ψ of $H^0(B, R^1\pi_*\mathcal{O}_X)$ can be written in the form $\psi = \sum_{\lambda=s+1}^{s+r(F)} f_\lambda(z)d\bar z_\lambda$ where $f_\lambda(z)$ is holomorphic in $z_1, \cdots, z_s$. By the above arguments, ψ can be viewed as a (0, 1)-form on X, and can therefore be written as $\psi = \sum_{\lambda=s+1}^{n} g_\lambda\bar\varphi_\lambda$ where $g_\lambda = g_\lambda(z_1, \cdots, z_s, \bar z_1, \cdots, \bar z_n)$ is antiholomorphic in $z_{s+1}, \cdots, z_n$.

By Proposition 2.3 we see readily that $\bar\partial\psi = 0$, $\vartheta\psi = -\sum_{\lambda=s+1}^{n} \theta_\lambda g_\lambda = 0$, and consequently that ψ itself can be viewed as an element of $H^1(X, \mathcal{O}) \cong H_{\bar\partial}^{0,1}(X)$. Hence $H^1(X, \mathcal{O}_X) = H^1(B, \mathcal{O}_B) \oplus H^0(B, R^1\pi_*\mathcal{O}_X)$, that is, any element ψ of $H_{\bar\partial}^{0,1}(X)$ can be represented in the form

$$\psi = \sum_{\lambda=1}^{s} c_\lambda d\bar z_\lambda + \sum_{\lambda=s+1}^{s+r(F)} f_\lambda(z)d\bar z_\lambda = \sum_{\lambda=1}^{s} c_\lambda d\bar z_\lambda + \sum_{\lambda=s+1}^{n} g_\lambda\bar\varphi_\lambda ,$$

where c_λ is constant, and f_λ, g_λ are the same as above. We shall calculate the dimension of real harmonic 1-forms on X. Let φ be a real differentiable 1-form given by

$$\varphi = \sum_{\lambda=1}^{n} \bar g_\lambda\varphi_\lambda + \sum_{\lambda=1}^{n} g_\lambda\bar\varphi_\lambda ,$$

where g_λ is a differentiable function. Set $\psi = \sum_{\lambda=1}^{n} g_\lambda\bar\varphi_\lambda$. Then $\varphi = \bar\psi + \psi$. Define d, δ by

$$d\varphi = (\partial + \bar\partial)(\psi + \bar\psi) , \qquad \delta\varphi = (\vartheta + \bar\vartheta)(\psi + \bar\psi) .$$

Assume φ is harmonic. Then $d\varphi = 0$, $\delta\varphi = 0$, and therefore $\bar\partial\psi = 0$, $\partial\bar\psi + \partial\psi = 0$. Since $\partial\bar\psi + \partial\psi = \sum (\theta_\nu g_\lambda - \bar\theta_\lambda\bar g_\nu)\varphi_\lambda \wedge \bar\varphi_\nu$, we have

$$\theta_\nu g_\lambda = \overline{\theta_\lambda g_\nu} . \tag{4.1}$$

On the other hand,

$$0 = \delta\varphi = -\sum_{\lambda=1}^{n} \overline{\theta_\lambda g_\lambda} - \sum_{\lambda=1}^{n} \theta_\lambda g_\lambda = -2\sum_{\lambda=1}^{n} \theta_\lambda g_\lambda .$$

From this it follows that $\vartheta\psi = -\sum_{\lambda=1}^{n} \theta_\lambda g_\lambda = 0$. Since $\bar\partial\psi = 0$ and $\vartheta\psi = 0$, ψ is an element of $H_{\bar\partial}^{0,1}(X)$. Thus $\psi = \sum_{\lambda=1}^{s} c_\lambda d\bar z_\lambda + \sum_{\lambda=s+1}^{n} g_\lambda \bar\varphi_\lambda$ for some constant $c_\lambda's$. From (4.1) we conclude that any g_λ is constant. Since $d\varphi = 0$, we have $g_\lambda = 0$ $(\lambda > r)$. Accordingly it follows that $\varphi = \sum_{\lambda=1}^{r} c_\lambda \varphi_\lambda + \sum_{\lambda=1}^{r} \overline{c_\lambda \varphi_\lambda}$ where $c_\lambda's$ are complex numbers. This implies that $\dim_{\boldsymbol{R}} H^1(X, \boldsymbol{R}) = 2r$, i.e., $b_1 = 2r$.

Remark 4.1. In the proof of Theorem 3 we have given an explicit description of elements of $H_{\bar\partial}^{0,1}(X)$. Since Θ is isomorphic to $\mathcal{O}^n$, $H_{\bar\partial}^{0,1}(X, \Theta)$ is spanned by $\theta_i\varphi$ $(i = 1, \cdots, n)$ for elements φ of $H_{\bar\partial}^{0,0}(X)$.

Remark 4.2. If X is not nilpotent, then $X = \boldsymbol{C}^r \times F/\Gamma_1$ for a nilpotent manifold F and a group Γ_1 of analytic automorphisms of $\boldsymbol{C}^s \times F$. Any element g of Γ_1 induces an automorphism g^* of $\boldsymbol{C}^s \times \operatorname{Alb} F$. Set $\Gamma_1^* = \{g^* ; g \in \Gamma_1\}$. Since Γ_1 operates on $\boldsymbol{C}^s \times F$ properly discontinuously without fixed points, Γ_1^* operates on $\boldsymbol{C}^s \times \operatorname{Alb} F$ in the same way. Thus $X^* = \boldsymbol{C}^s \times \operatorname{Alb} F/\Gamma_1^*$ is a compact complex manifold, and is therefore parallelisable and solvable. Using this fact we infer that *a parallelisable manifold with the following basis $\{\varphi_\lambda\}$ of $H^0(X, \Omega^1)$ does not exist*:

$$d\varphi_1 = 0 , \quad d\varphi_2 = \varphi_1 \wedge \varphi_2 , \quad d\varphi_3 = -2(\mu + 1)\varphi_1 \wedge \varphi_3 ,$$
$$d\varphi_4 = \mu\varphi_1 \wedge \varphi_4 , \quad d\varphi_5 = (\mu + 1)\varphi_1 \wedge \varphi_5 + \varphi_2 \wedge \varphi_4 ,$$

where μ is constant, and $\mu(\mu + 1) \neq 0$.

Proof. If a parallelisable manifold X of this type exists, X^* is a parallelisable manifold with a basis $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ of $H^0(X^*, \Omega^1)$ such that $d\psi_1 = 0$, $d\psi_2 = \psi_1 \wedge \psi_2$, $d\psi_3 = -2(\mu + 1)\psi_1 \wedge \psi_3$, $d\psi_4 = \mu\psi_1 \wedge \psi_4$. This contradicts Lemma 1.4.

5. Proof of Theorem 5

First for brevity we assume $\mathfrak{g}$ to have the Chevalley decomposition. Let $\{\varphi_\lambda\}$ and $\{\theta_i\}$ be dual bases of $H^0(X, \Omega^1)$ and $H^0(X, \Theta)$ which satisfy (2.1) and (2.2) respectively. The assumption means that $H_{\bar\partial}^{0,1}(X, \Theta)$ is generated by $\theta_i\bar\varphi_\lambda$, $\lambda = 1, \cdots, n$, and $i = 1, \cdots, r$. Define a $(n - r,\ n - r)$ matrix $A = (A_{ij})$ by

$$A_{ij} = 2v_0 \sum_{\lambda<\nu} c_{i+r\lambda\nu}\overline{c_{j+r\lambda\nu}} , \qquad i, j = 1, \cdots, n - r ,$$

where $v_0 = \int_X dX = i^{-n^2}\int_X \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \bar\varphi_1 \wedge \cdots \wedge \bar\varphi_n$.

Lemma 5.1. $\det(A_{ij}) \neq 0$.

Proof. $(\partial\varphi_{i+r}, \partial\varphi_{j+r}) = (d\varphi_{i+r}, d\varphi_{j+r}) = v_0 \sum_{\lambda,\nu} c_{i+r\lambda\nu}\overline{c_{j+r\lambda\nu}} = 2A_{ij}$. Thus, in order to prove Lemma 5.1, it suffices to show the following:

If for a 1-form $\psi = \sum_{\lambda=r+1}^{n} c_\lambda \varphi_\lambda$, $(\partial\psi, \partial\varphi_\nu) = 0$, $\nu = r+1, \cdots, n$, then we have $\psi = 0$, where c_λ's are constant. However this is obvious. q.e.d.

It follows from Lemma 5.1. that there exists $(n-r, n-r)$ matrix (A^{lj}) such that $\sum_{l=1}^{n-r} A_{kl}A^{lj} = \delta_{kj}$.

Lemma 5.2. *For a* $(0,2)$*-form* $\varphi = \sum_{\lambda<\nu} a_{\lambda\nu}\bar{\varphi}_\lambda \wedge \bar{\varphi}_\nu$ *with some constants* $a_{\lambda\nu}$, φ *is cohomologous to zero in* $H^{02}_{\bar\partial}(X)$ *if and only if* $\varphi = \sum_{\lambda=r+1}^{n} a_\lambda d\bar{\varphi}_\lambda$, *where* a_λ'*s are constants.*

Proof. For a $(0,2)$-form $\varphi = \sum_{\lambda<\nu} a_{\lambda\nu}\bar{\varphi}_\lambda \wedge \bar{\varphi}_\nu$ the adjoint operator ϑ of $\bar\partial$ is defined by

$$\vartheta\varphi = 2\sum_{\substack{i\\ \lambda<\nu}} c_{i\lambda\nu}a_{\lambda\nu}\bar{\varphi}_i = 2\sum_{\substack{i\\ \lambda<\nu}} c_{i+r\lambda\nu}a_{\lambda\nu}\bar{\varphi}_{i+r} .$$

Set

$$H\varphi = \varphi - v_0 \sum_{\substack{i,j\\ \lambda<\nu}} A^{ji}c_{i+r\lambda\nu}a_{\lambda\nu}d\bar{\varphi}_{j+r} .$$

If $\bar\partial\varphi = 0$, then $H\varphi$ is harmonic, i.e., $\bar\partial(H\varphi) = 0$, $\vartheta(H\varphi) = 0$. In fact, $\bar\partial(H\varphi) = \bar\partial\varphi = 0$. Moreover,

$$\begin{aligned}\vartheta(H\varphi) &= 2\sum_{\substack{i\\ \lambda<\nu}} c_{i\lambda\nu}a_{\lambda\nu}\bar{\varphi}_i - 4v_0 \sum_{\substack{ijk\\ \lambda<\nu,\alpha<\beta}} c_{i+r\lambda\nu}a_{\lambda\nu}A^{ji}c_{k+r\alpha\beta}\overline{c_{j+r\alpha\beta}}\bar{\varphi}_{k+r} \\ &= 2\sum c_{i\lambda\nu}a_{\lambda\nu}\bar{\varphi}_i - 2\sum c_{i+r\lambda\nu}a_{\lambda\nu}A^{ji}A_{kj}\bar{\varphi}_{k+r} = 0 .\end{aligned}$$

Since H is nothing but the projection of the harmonic part, we have $H\varphi = 0$ if φ is cohomologous to zero. "If" part of the lemma is obvious.

Lemma 5.3. *Under some algebraic relations between* $t_{i\lambda}$ $(i = 1, \cdots, n$, *and* $\lambda = 1, \cdots, r)$, *there exists a vector* $(0,1)$*-form* $\psi = \sum_{\alpha=1}^{n_1} \psi_\alpha(t)$ *for some* $n_1 \le n$ *such that*

$$\bar\partial\psi - \tfrac{1}{2}[\psi, \psi] = 0 , \tag{5.1}$$

where $\psi_1 = \sum_{i=1}^{n}\sum_{\lambda=1}^{r} t_{i\lambda}\theta_i\bar{\varphi}_\lambda$, *and* ψ_α *is the homogeneous term of total degree* α *in* $t_{i\lambda}$.

Proof. Set $\psi = \sum_{\alpha=1}^{\infty} \psi_\alpha(t)$ and $\psi_1 = \sum_{i=0}^{n}\sum_{\lambda=1}^{r} t_{i\lambda}\theta_i\bar{\varphi}_\lambda$, where ψ_α is the homogeneous term of total degree α in $t_{i\lambda}$. Since $[\theta_i\bar{\varphi}_\lambda, \theta_k\bar{\varphi}_\nu] = [\theta_i, \theta_k]\bar{\varphi}_\lambda \wedge \bar{\varphi}_\lambda$, we have[1]

$$\bar{\partial}\psi_2 = \tfrac{1}{2}[\psi_1, \psi_1] - \sum_{\substack{i=1 \\ \lambda,\nu\in Q_1}}^{n} \sum_{\lambda<\nu} a^1_{\lambda\nu i}(t)\theta_i\bar{\varphi}_\lambda \wedge \bar{\varphi}_\nu \ .$$

$\eta^1_i = \sum_{\lambda<\nu} a^1_{\lambda\nu i}\bar{\varphi}_\lambda \wedge \bar{\varphi}_\lambda$ is cohomologous to zero in $H^{0,2}_{\bar{\partial}}(X)$. Hence from Lemma 5.2 it follows[2] that $\eta^1_i = \sum_{\mu\in Q_2} b^1_{\mu i}(t)d\bar{\varphi}_\mu$ for some $b^1_{\mu i}(t)$ and

$$(5.2)_1 \qquad a^1_{\lambda\nu i}(t) = 2\sum_{\mu} b^1_{\mu i}(t)c_{\mu\lambda\nu} \ .$$

(In general $(5.2)_1$ is nontrivial; see (3.3).) Then we have

$$\bar{\partial}\psi_3 = [\psi_1, \psi_2] = \sum_{i=0} \sum_{\substack{\lambda\in Q_1 \\ \nu\in Q_2}} a^2_{\lambda\nu i}\theta_i\bar{\varphi}_\lambda \wedge \bar{\varphi}_\nu \ .$$

Again from Lemma 5.2 it follows that

$$\eta^2_i = \sum_{\substack{\lambda\in Q_1 \\ \nu\in Q_2}} a^2_{\lambda\nu i}\bar{\varphi}_\lambda \wedge \bar{\varphi}_\lambda = \sum_{\beta\in Q_3} b^2_{\mu i}d\bar{\varphi}_\mu \qquad \text{for some } b^2_{\mu i} \ ,$$

$$(5.2)_2 \qquad a^2_{\lambda\nu i}(t) = 2\sum_{\mu} b^2_{\mu i}(t)c_{\mu\lambda\nu} \ .$$

Inductively we define ψ_α and $b^\alpha_{\mu i}(t)$ under additional relations $(5.2)_2, \cdots, (5.2)_{n_1-1}$. Since $\bigcup_\mu Q_\mu$ is bounded, we obtain the desired ψ after finite steps of processes. q.e.d.

$(5.2)_1, \cdots, (5.2)_{n_1-1}$ define an algebraic set A in C^{nr}. Set $A_\varepsilon = \{(t_{i\lambda}) \in A\,;\ \sum_{i=0} \sum_{\lambda=1} |t_{i\lambda}| < \varepsilon\}$ for a sufficiently small positive number ε. Lemma 5.3 implies that there exists a maximal complex analytic family of deformations of X depending on nr parameters $t_{i\lambda}$. A_ε is the Kuranishi space of deformations of X, [3].

Lemma 5.4. *The system of differential equations*

$$(5.3) \qquad \bar{\partial}\zeta_\alpha - \psi(t)\zeta_\alpha = 0 \ , \qquad \alpha = 1, \cdots, n$$

can be solved in $C^n \times A_\varepsilon$ under the initial condition $\zeta_\alpha(0) = z_\alpha$, where $0 \in A_\alpha$ denotes the origin of C^{nr}.

Proof. Since (5.3) is the integrability condition of the system of differential equations (5.3), we can formally solve it by the interation method. To this end we must show that the formal solutions converge in $C^n \times A_\varepsilon$ for a sufficient small positive number ε.

In view of Propositions 2.3 and 2.4 together with Lemma 5.3, ψ is represented by

[1,2] See (1.4) as for Q_1, Q_2, etc.

$$\psi = \sum_{i=1}^{n} \sum_{\lambda=1}^{r} t_{i\lambda}\theta_i\bar{\varphi}_\lambda + \sum_{k=r+1}^{n} \sum_{\mu=r+1}^{m} \sum_{\beta=2}^{n_1} a^\beta_{k\mu}(t)\theta_k\bar{\varphi}_\mu ,$$

where $m = \sharp Q$ (see (1.2)), and $a^\beta_{k\mu}$ denotes the homogeneous term of total degree β in $t_{i\lambda}$. Then we have $\sum_{\mu=r+1}^{m} a^\beta_{k\mu}\bar{\varphi}_\mu = \sum_{\mu=r+1}^{m} b^\beta_{k\mu}(t, \bar{z})d\bar{z}_\mu$ for a polynomial $b^\beta_{k\mu}(t, \bar{z})$ in $\bar{z}_1, \cdots, \bar{z}_{m-1}, t_{i\lambda}$, which is of degree β in $t_{i\lambda}$. Therefore the system of differential equations (5.3) is equivalent to the system of equations:

$$\begin{aligned}
&\bar{\partial}_1\zeta = \sum_{i=1}^{n} t_{i1}\theta_i\zeta + \sum_{\beta=2}^{n_1} \sum_{i=r+1}^{n} b^\beta_{i1}(t, \bar{z})\theta_i\zeta ,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
(5.4)\qquad &\bar{\partial}_r\zeta = \sum_{i=1}^{n} t_{ir}\theta_i\zeta + \sum_{\beta=2}^{n_1} \sum_{i=r+1}^{n} b^\beta_{ir}(t, \bar{z})\theta_i\zeta ,\\
&\bar{\partial}_\mu\zeta = \sum_{\beta=l}^{n_1} \sum_{i=r+1}^{n} b^\beta_{i\mu}\theta_i\zeta \qquad \text{for } \mu \in Q_l (l \geq 2) ,\\
&\phantom{\bar{\partial}_\mu\zeta} = 0 \qquad\qquad\qquad\quad \text{for } \mu \in Q_\infty .
\end{aligned}$$

Set $\zeta = \sum_{\beta=0}^{\infty} \zeta_\beta$ where ζ_β denotes the homogeneous term of total degree β in $t_{i\lambda}$.

Case 1. Assume $\zeta(0) = z_\alpha$ or requivalently $\zeta_0 = z_\alpha$ $(\alpha = 1, \cdots, r)$. Then we have

$$\bar{\partial}_\mu\zeta_1 = t_{\alpha\mu} , \quad \mu \leq r ; \qquad \bar{\partial}_\mu\zeta_1 = 0 , \quad \mu > r .$$

Hence setting $\zeta_1 = \sum_{\mu=1}^{r} t_{\alpha\mu}\bar{z}_\mu$, we obtain the solution $\zeta = z_\alpha + \sum_{\mu=1}^{r} t_{\alpha\mu}\bar{z}_\mu$.

Case 2. Assume $\zeta_0 = z_\alpha$ $(\alpha \in Q)$. Denote by $D_\lambda(f)$ the degree of a polynominal f with respect to z_λ. Since $\bar{\partial}_\mu\zeta_1 = t_{\alpha\mu}$ $(\mu \leq r)$, $\bar{\partial}_\mu\zeta_1 = 0$ $(\mu > r)$, we have $\zeta_1 = \sum_{\mu=1}^{r} t_{\alpha\mu}\bar{z}_\mu$. From Proposition 2.4 it follows that for μ $(\leq m)$

$$\theta_1 z = 0 \ (i > \mu \text{ or } i \leq s) , \quad \theta_\mu z_\mu = 1 , \quad \theta_i z_\mu = G_{i\mu} \ (\mu > i > s) ,$$

where $G_{i\mu}$ is a polynominal in $z_1, \cdots, z_{i-1}$. Hence we have $(D_\alpha(\zeta_2) = 0$, $D_\gamma(\zeta_2) = 0$ $(\gamma > \alpha)$. If $D_{\alpha-1}(\zeta_2) = N$, then $D_{\alpha-1}(\zeta_3) = N - 1$, $D_\gamma(\zeta_3) = 0$ $(\gamma \geq \alpha)$. Inductively we obtain $D_{\alpha-1}(\zeta_{N+2}) = 0$ $D_\gamma(\zeta_{N+2}) = 0$ $(\gamma \geq \alpha)$. For a sufficiently large integer N_1 we have

$$D_\gamma(\zeta_{N_1+\delta}) = 0 , \quad \gamma = 1, \cdots, n , \quad \delta = 1, \cdots, n_1 ,$$

so that we may set $\zeta_\beta = 0$ for any $\beta > N_1 + \delta$. Hence $\zeta = \sum_{\beta=0}^{N_1+\delta} \zeta_\beta$ is the desired solution.

Case 3. Assume $\zeta_0 = z_\alpha$ ($\alpha \in Q_\infty$, i.e., $\alpha \geq m+1$). Similarly, as in Case 2 we have $D_\gamma(\zeta_{N+\delta}) = 0$, $\gamma = s+1, \cdots, n$, $\delta = 1, \cdots, n_1$. Therefore the problem is reduced to the case where

$$r = s\ , \quad \text{and} \quad \zeta_0 \text{ is a polynomial in } e^{z_1}, z_1, \cdots, z_r\ ,$$

and it suffices to prove the covergence of the series $\sum_{\beta=0}^{\infty} \zeta_\beta$ only for $\zeta_0 = e^{z_1} z_1^{e_1} \cdots z_r^{e_r}$. Moreover, the system of differential equations (5.4) takes the form:

$$\bar{\partial}_\mu \zeta = \sum_{\lambda=1}^{r} t_{\mu\lambda} \partial_\lambda \zeta\ .$$

Define the norms $\|\ \ \|$ by

$$\|z\| = \sum_{\lambda=1}^{r} |z_\lambda|\ , \qquad \|t\| = \sum_{i=1}^{n} \sum_{\lambda=1}^{r} |t_{i\lambda}|\ , \qquad \|f\| = \sum_{\beta_1, \cdots, \beta_r} |a_{\beta_1} \cdots \beta_r|\, |z_1^{\beta_1} \cdots z_r^{\beta_r}|$$

for a polynomial $f = \sum a_{\beta_1 \cdots \beta_r} z_1^{\beta_1} \cdots z_r^{\beta_r}$. Since $\bar{\partial}_\mu \zeta_1 = \sum t_{\mu\lambda} \partial_\lambda \zeta_0$, we have

$$\zeta_1 = \sum t_{\mu\lambda} (\partial_\lambda \zeta_0) \overline{z_\mu}\ ,$$

and therefore $|\zeta_1| \leq \|t\|\, \|z\|\, \|\zeta_0'\|$, $\bar{\partial}_\mu \zeta_2 = \sum t_{\mu\lambda} \partial_\lambda \zeta_1$. Thus $|\zeta_2| \leq \dfrac{\|t\|^2 \|z\|^2}{2!} \|\zeta_0''\|$. Idductively we have $|\zeta_k| \leq \dfrac{\|t\|^k \|z\|^k}{k!} \|\zeta_0^{(k)}\|$, and

$$\zeta_0^{(k)} = \sum_{k_0 + \cdots + k_r = k} \frac{k!}{k_0! k_1! \cdots k_r!}\, e^{z_1} (z_1^{e_1})^{(k_1)} \cdots (z_r^{e_r})^{(k_r)}\ .$$

Therefore $\quad \|\zeta_0^{(k)}\| \leq \dfrac{k!}{(k-e_0)!} M \cdot G(z) \quad$ where $\quad e_0 = \sum_{\nu=1}^{r} e_\nu, \quad G(z) = |e^{z_1}| \sum_{0 \leq j_\nu \leq e_\nu} |z_1^{j_1}|\, |z_2^{j_2}| \cdots |z_r^{j_r}|$, and M is a sufficiently large positive number independent of k. Hence

$$\begin{aligned}
\left|\sum_{k=0}^{\infty} \zeta_k\right| &\leq \left|\sum_{k=0}^{e_0-1} \zeta_k\right| + \left|\sum_{k=e_0}^{\infty} \zeta_k\right| \\
&\leq \left|\sum_{k=0}^{e_0-1} \zeta_k\right| + M \cdot G(z) (\|t\|\, \|z\|)^{e_0} \sum_{k=0}^{\infty} \frac{1}{k!} \|t\|^k \|z\|^k \\
&= \left|\sum_{k=0}^{e_0-1} \zeta_k\right| + M \cdot G(z) (\|t\|\, \|z\|)^{e_0} \exp(\|t\|\, \|z\|)\ . \quad \text{q.e.d.}
\end{aligned}$$

In view of the proof of Lemma 5.4 we have

$$d\zeta_1 \wedge \cdots \wedge d\zeta_n \wedge d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_n$$

$$= c(t)dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n ,$$

where a differentiable function $c(t)$ of $t_{i\lambda}$ is independent of $z_1, \cdots, z_n$ and satisfies $c(0) = 1$. Hence by the same argument as in §3 we conclude that any small deformation X_t ($t \in A_\varepsilon$) has $\boldsymbol{C}^n$ as the universal covering for a sufficiently small positive number ε.

In the general case we can apply the following lemma which is a weaker form of Propositions 2.2 and 2.3.

Lemma. *Let G be a connected solvable complex Lie group. Then we can choose a global coordinate $(z_1, \cdots, z_n)$ of G ($\cong \boldsymbol{C}^n$) and a basis $\{\varphi_\lambda\}$, $\{\theta_\lambda\}$ of right invariant 1-forms and vector fields respectively such that*

$$\varphi_\lambda = \sum_{\nu=1}^{\lambda} F_{\lambda\nu}(z)dz_\nu , \qquad \theta_\lambda = \sum_{\nu=\lambda}^{n} G_{\lambda\nu}(z)\partial/\partial z_\nu ,$$

where $F_{\lambda\nu} = F_{\lambda\nu}(z_1, \cdots, z_{\nu-1})$, $G_{\lambda\nu} = G_{\lambda\nu}(z_1, \cdots, z_{\nu-1})$ and $(\theta_\lambda, \varphi_\nu) = \delta_{\lambda\nu}$.

By quite similar arguments we can also prove Theorem 5.

6. Classification of four- and five-dimensional complex solvable manifolds

By an elementary calculation together with Lemma 1.4 and Remark 4.2 we classify four and five-dimensional complex solvable Lie groups which may have uniform subgroups as follows:

Type IV:

1. $d\varphi_\lambda = 0 , \quad 1 \le \lambda \le 4 .$
2. $d\varphi_\lambda = 0 , \quad 1 \le \lambda \le 3 , \quad d\varphi = -\varphi_2 \wedge \varphi_3 .$
3. $d\varphi_1 = 0 , \quad d\varphi_2 = 0 , \quad d\varphi_3 = -\varphi_1 \wedge \varphi_2 , \quad d\varphi_4 = -2\varphi_1 \wedge \varphi_3 .$
4. $d\varphi_1 = 0 , \quad d\varphi_2 = 0 , \quad d\varphi_3 = \varphi_2 \wedge \varphi_3 , \quad d\varphi_4 = -\varphi_2 \wedge \varphi_4 .$
5. $d\varphi_1 = 0 , \quad d\varphi_2 = \varphi_1 \wedge \varphi_2 , \quad d\varphi_3 = \alpha\varphi_1 \wedge \varphi_3 ,$
   $d\varphi_4 = -(1 + \alpha)\varphi_1 \wedge \varphi_4 , \quad \alpha(1 + \alpha) \neq 0 .$
6. $d\varphi_1 = 0 , \quad d\varphi_2 = \varphi_1 \wedge \varphi_2 , \quad d\varphi_3 = -\varphi_1 \wedge \varphi_3 , \quad d\varphi_4 = -\varphi_2 \wedge \varphi_3 .$
7. $d\varphi_1 = 0 , \quad d\varphi_2 = \varphi_1 \wedge \varphi_2 , \quad d\varphi_3 = -2\varphi_1 \wedge \varphi_3 ,$
   $d\varphi_4 = d\varphi_1 \wedge \varphi_4 - \varphi_1 \wedge \varphi_2 .$

Type V:

1. $d\varphi_\lambda = 0 , \quad 1 \le \lambda \le 5 .$
2. $d\varphi_\lambda = 0 , \quad 1 \le \lambda \le 4 , \quad d\varphi_5 = -\varphi_3 \wedge \varphi_4 .$
3. $d\varphi_\lambda = 0 , \quad 1 \le \lambda \le 4 , \quad d\varphi_5 = -\varphi_1 \wedge \varphi_3 - \varphi_2 \wedge \varphi_4 .$
4. $d\varphi_\lambda = 0 , \quad 1 \le \lambda \le 3 , \quad d\varphi_4 = -\varphi_1 \wedge \varphi_2 , \quad d\varphi_5 = -\varphi_1 \wedge \varphi_3 .$
5. $d\varphi_\lambda = 0 , \quad 1 \le \lambda \le 3 , \quad d\varphi_4 = -\varphi_2 \wedge \varphi_3 , \quad d\varphi_5 = -2\varphi_2 \wedge \varphi_4 .$

6. $d\varphi_\lambda = 0\,,\quad 1 \leq \lambda \leq 3\,,\quad d\varphi_4 = -\varphi_1 \wedge \varphi_2\,,$
$d\varphi_5 = -2\varphi_1 \wedge \varphi_4 - \varphi_2 \wedge \varphi_3\,.$
7. $d\varphi_\lambda = 0\,,\quad 1 \leq \lambda \leq 3\,,\quad d\varphi_4 = \varphi_3 \wedge \varphi_4\,,\quad d\varphi_5 = -\varphi_3 \wedge \varphi_5\,.$
8. $d\varphi_\lambda = 0\,,\quad \lambda = 1, 2\,,\quad d\varphi_3 = -\varphi_1 \wedge \varphi_2\,,\quad d\varphi_4 = -2\varphi_1 \wedge \varphi_3\,,$
$d\varphi_5 = -2\varphi_2 \wedge \varphi_3\,.$
9. $d\varphi_\lambda = 0\,,\quad \lambda = 1, 2\,,\quad d\varphi_3 = -\varphi_1 \wedge \varphi_2\,,\quad d\varphi_4 = -2\varphi_1 \wedge \varphi_3\,,$
$d\varphi_5 = -3\varphi_1 \wedge \varphi_4\,.$
10. $d\varphi_\lambda = 0\,,\quad \lambda = 1, 2\,,\quad d\varphi_3 = -\varphi_1 \wedge \varphi_2\,,\quad d\varphi_4 = -2\varphi_1 \wedge \varphi_3\,,$
$d\varphi_5 = -3\varphi_1 \wedge \varphi_4 - \varphi_2 \wedge \varphi_3\,.$
11. $d\varphi_\lambda = 0\,,\quad \lambda = 1, 2\,,\quad d\varphi_3 = -\varphi_1 \wedge \varphi_2\,,\quad d\varphi_4 = \varphi_1 \wedge \varphi_4\,,$
$d\varphi_5 = \varphi_1 \wedge \varphi_5\,.$
12. $d\varphi_\lambda = 0\,,\quad \lambda = 1, 2\,,\quad d\varphi_3 = \varphi_1 \wedge \varphi_3\,,\quad d\varphi_4 = \varphi_2 \wedge \varphi_4\,,$
$d\varphi_5 = -(\varphi_1 + \varphi_2) \wedge \varphi_5\,.$
13. $d\varphi_\lambda = 0\,,\quad \lambda = 1, 2\,,\quad d\varphi_3 = \varphi_2 \wedge \varphi_3\,,\quad d\varphi_4 = \alpha\varphi_2 \wedge \varphi_4\,,$
$d\varphi_5 = -(1 + \alpha)\varphi_2 \wedge \varphi_5\,,\qquad \alpha(1 + \alpha) \neq 0\,.$
14. $d\varphi_\lambda = 0\,,\quad \lambda = 1, 2\,,\quad d\varphi_3 = \varphi_1 \wedge \varphi_3\,,\quad d\varphi_4 = -2\varphi_1 \wedge \varphi_4\,,$
$d\varphi_5 = \varphi_1 \wedge \varphi_5 - \varphi_1 \wedge \varphi_3\,.$
15. $d\varphi_\lambda = 0\,,\quad \lambda = 1, 2\,,\quad d\varphi_3 = \varphi_2 \wedge \varphi_3\,,\quad d\varphi_4 = -\varphi_2 \wedge \varphi_4\,,$
$d\varphi_5 = -\varphi_3 \wedge \varphi_4\,.$
16. $d\varphi_\lambda = 0\,,\quad \lambda = 1, 2\,,\quad d\varphi_3 = \varphi_1 \wedge \varphi_3\,,\quad d\varphi_4 = -\varphi_1 \wedge \varphi_4\,,$
$d\varphi_5 = -\varphi_3 \wedge \varphi_4 - \varphi_1 \wedge \varphi_2\,.$
17. $d\varphi_1 = 0\,,\quad d\varphi_2 = \varphi_1 \wedge \varphi_2\,,\quad d\varphi_3 = \alpha\varphi_1 \wedge \varphi_3\,,\quad d\varphi_4 = \beta\varphi_1 \wedge \varphi_4\,,$
$d\varphi_5 = -(1 + \alpha + \beta)\varphi_1 \wedge \varphi_5\,,\qquad \alpha\beta(1 + \alpha + \beta) \neq 0\,.$
18. $d\varphi_1 = 0\,,\quad d\varphi_2 = -3\varphi_1 \wedge \varphi_2\,,\quad d\varphi_3 = \varphi_1 \wedge \varphi_3\,,$
$d\varphi_4 = \varphi_1 \wedge \varphi_4 - \varphi_1 \wedge \varphi_3\,,\qquad d\varphi_5 = \varphi_1 \wedge \varphi_5 - \varphi_1 \wedge \varphi_3\,.$
19. $d\varphi_1 = 0\,,\quad d\varphi_2 = \varphi_1 \wedge \varphi_2\,,\quad d\varphi_3 = -\varphi_1 \wedge \varphi_3\,,$
$d\varphi_4 = \varphi_1 \wedge \varphi_4 - \varphi_1 \wedge \varphi_2\,,\qquad d\varphi_5 = -\varphi_1 \wedge \varphi_5 - \varphi_1 \wedge \varphi_3\,.$
20. $d\varphi_1 = 0\,,\quad d\varphi_2 = \varphi_1 \wedge \varphi_2\,,\quad d\varphi_3 = \varphi_1 \wedge \varphi_3 - \varphi_1 \wedge \varphi_2\,,$
$d\varphi_4 = \alpha\varphi_1 \wedge \varphi_4\,,\quad d\varphi_5 = -(2 + \alpha)\varphi_1 \wedge \varphi_5\,,\quad \alpha(2 + \alpha) \neq 0\,.$

Lemma 6.1. *Let A be a 3×3 matrix which induces an automorphism of a complex torus of dimension 3. Assume that A has eigenvalues α, α and α^{-2}. Then α is a root of unity.*

Proof. Let Φ be the proper polynomial of A. Then

$$\Psi(x) = \Phi(x)\overline{\Phi(x)} \in Z[x]\,.$$

Assume α is not a root of 1. We shall prove Ψ is irreducible in $Z[x]$. In fact, if Ψ is not irreducible, there exist $\Phi_1, \Phi_2 \in \boldsymbol{Z}[x]$ such that $\Psi = \Phi_1\Phi_2$. We may assume $\deg \Phi_1 \geq \deg \Phi_2 \geq 2$. Two cases may occur. First, we assume $\deg \Phi_1 = 4$, $\deg \Phi_2 = 2$. Put $F(x) = (x - \alpha)(x - \bar{\alpha}) \in \boldsymbol{R}[x]$. If $F \mid \Phi_2$ in $\boldsymbol{R}[x]$, then $F = \pm \Phi_2$. Since the constant term of Ψ equals ± 1, that of Φ_2 equals ± 1. Hence $|\alpha| = 1$. Any conjugate of α has absolute value 1. Therefore α is a root of 1. This is a contradiction. If $F \nmid \Phi_2$ in $\boldsymbol{R}[x]$, then $F^2 \mid \Phi_1$ in $\boldsymbol{R}[x]$, hence $F^2 = \pm \Phi_1$. Similarly, we are led to the contradiction. In case $\deg \Phi_1 = \deg \Phi_2 = 3$, we also have a contradiction. Thus Ψ is proved to be irreducible. This contradicts the fact that Ψ has a double root. q.e.d.

Similarly we obtain

Lemma 6.2. *Let A be a 4×4 matrix which induces an automorphism of a 4-dimensional complex torus. Assume A has eigenvalues $\alpha, \alpha, \alpha, \alpha^{-3}$. Then α is a root of 1.*

From these lemmas, we conclude that a parallelisable manifold of type IV-6, V-14 or V-18 does not exist. In fact, in the case of IV-7 we consider the Mostow decomposition $\pi: X \to B$. Then B is an elliptic curve, the fiber F is a complex torus of dimension 3,

$$X = \boldsymbol{C}^4/\Gamma\ , \qquad g = (\omega_1, \omega_2, \omega_3, \omega_4) \in \Gamma\ ,$$

and $g: F \to F$ is given by

$$(z_2, z_3, z_4) \mapsto (e^{-\omega_1}z_2 + \omega_2, e^{2\omega_1}z_3 + \omega_3, e^{-\omega_1}z_4 + e^{-\omega_1}\omega_1 z_2 + \omega_4)\ .$$

Lemma 6.1 shows that e^{ω_1} is a root of 1. This contradicts the fact that $\{\omega_1; (\omega_1, \omega_2, \omega_3, \omega_4) \in \Gamma\}$ are periods of the elliptic curve B.

Similarly it can be proved that a parallelisable manifold of type V-15 or V-18 does not exist.

The author does not know whether there exist parallelisable manifolds of types IV-5, V-11, V-13, V-16, V-19, V-20. In other cases we can construct examples of each type. Now we summarize the results. In the following table we omit $z_i + y_i$ for simplicity. For example, $z*y = (z_3 + y_3 + y_1z_2)$ implies $z*y = (z_1 + y_1, z_2 + y_2, z_3 + y_3 + y_1z_2)$.

Type IV		$z*y$
1	abelian	
2	nilpotent	$z_4+y_4+y_2z_3$
3	nilpotent	$z_3+y_3+y_1z_2,\ z_4+y_4+2y_1z_3+y_1^2z_2$
4	solvable	$e^{-y_2}z_3+y_3,\ e^{y_2}z_4+y_4$
5	solvable	$e^{-y_1}z_2+y_2,\ e^{-\alpha y_1}z_3+y_3.\ e^{(1+\alpha)y_1}z_4+y_4$
6	solvable	$e^{-y_1}z_2+y_2,\ e^{y_1}z_3+y_3,\ z_4+y_4+e^{y_1}y_2z_3$

Type V		$z*y$
1	abelian	
2	nilpotent	$z_5+y_5+y_3z_4$
3	nilpotent	$z_5+y_5+y_1z_3+y_2z_4$
4	nilpotent	$z_4+y_4+y_1z_2,\ z_5+y_5+y_1z_3$
5	nilpotent	$z_4+y_4+y_2z_3,\ z_5+y_5+2y_2z_4+y_2^2z_3$
6	nilpotent	$z_4+y_4+y_1z_2,\ z_5+y_5+2y_1z_4+y_2z_3+y_1^2z_2$
7	solvable	$e^{-y_3}z_4+y_4,\ e^{y_3}z_5+y_5$
8	nilpotent	$z_3+y_3+y_1z_2,\ z_4+y_4+2y_1z_3+y_1^2z_2,$ $z_5+y_5+2y_2z_3+y_1z_2^2+2y_1y_2z_2$
9	nilpotent	$z_3+y_3+y_1z_2,\ z_4+y_4+2y_1z_3+y_1^2z_2,$ $z_5+y_5+3y_1z_4+3y_1^2z_3+y_1^3z_2$
10	nilpotent	$z_3+y_3+y_1z_2,\ z_4+y_4+2y_1z_3+y_1^2z_2,$ $z_5+y_5+3y_1z_4+(3y_1^2+2y_2)z_3+y_1z_2^2+(y_1^3+2y_1y_2)z_2$
11	solvable	$z_3+y_3+y_1z_2,\ e^{-y_1}z_4+y_4,\ e^{y_1}z_5+y_5$
12	solvable	$e^{-y_1}z_3+y_3,\ e^{-y_2}z_4+y_4,\ e^{y_1+y_2}z_5+y_5$
13	solvable	$e^{-y_2}z_3+y_3,\ e^{-\alpha y_2}z_4+y_4,\ e^{(1+\alpha)y_2}z_5+y_5$
15	solvable	$e^{-y_2}z_3+y_3,\ e^{y_2}z_4+y_4,\ z_5+y_5+e^{y_2}y_3z_4$
16	solvable	$e^{-y_1}z_3+y_3,\ e^{y_1}z_4+y_4,\ z_5+y_5+e^{y_1}y_3z_4+y_1z_2$
17	solvable	$e^{-y_1}z_2+y_2,\ e^{-\alpha y_1}z_3+y_3,\ e^{-\beta y_1}z_4+y_4,\ e^{(1+\alpha+\beta)}z_5+y_5$
19	solvable	$e^{-y_1}z_2,\ e^{y_1}z_3+y_3,\ e^{-y_1}z_4+e^{-y_1}y_1z_2,\ e^{y_1}z_5+y_5+e^{y_1}y_1z_3$
20	solvable	$e^{-y_1}z_2+y_2,\ e^{-y_1}z_3+y_3+e^{-y_1}y_1z_2,\ e^{-\alpha y_1}z_4+y_4,$ $e^{-(\alpha+2)y_1}z_5+y_5$

Complex solvable manifolds of dimensions 4, 5 are classified as follows:

		r	$h^{0,1}$	structure (Albanese mapping)
IV:	1	4	4	complex torus
	2	3	3	T^1-bundle over T^3
	3	2	2	T^2-bundle over T^2
	4	2	2,4	T^2-bundle over T^2
	5	1	1,2,4	T^3-bundle over T^1?
	6	1	1,3	(III-2)-bundle over T^1
V:	1	5	5	complex torus
	2	4	4	T^1-bundle over T^4
	3	4	4	T^1-bundle over T^4
	4	3	3	T^2-bundle over T^3
	5	3	3	T^2-bundle over T^3
	6	3	3	T^2-bundle over T^3
	7	3	3,5	T^2-bundle over T^3
	8	2	2	T^3-bundle over T^2

	r	$h^{0,1}$	structure (Albanese mapping)
9	2	2	T^3-bundle over T^2
10	2	2	T^3-bundle over T^2
11	2	2,4	T^3-bundle over T^2?
12	2	2,3,5	T^3-bundle over T^2
13	2	2,3,5	T^3-bundle over T^2?
15	2	2,4	(III-2)-bundle over T^2
16	2	2,4	(III-2)-bundle over T^2?
17	1	1,2,3,5	T^4-bundle over T^1
19	1	1,3	T^4-bundle over T^1?
20	1	1,2,4	T^4-bundle over T^1?

Here $r = \dim H^0(X, d\mathcal{O})$, $h^{0,1} = \dim H^1(X, \mathcal{O})$, $T^n =$ a complex torus of dimension n.

Remark. *A* solvable manifold of dimension 4 or 5 has a Lie algebra with the Chevalley decomposition, and so from Theorem 3 it follows that $b_1 = 2r$.

References

[1] S. Iitaka, *Plurigenera and classification of algebraic varieties,* Sugaku **24** (1972) 14–27 (in Japanese).

[2] S. Kaneyuki & T. Nagano, *On the first Betti numbers of compact quotient spaces of complex semi-simple Lie groups by discrete subgroups,* Sci. Papers College Gen. Ed. Univ. Tokyo **12** (1962) 1–11.

[3] M. Kuranishi, *On the locally complete families of complex analytic structures,* Ann. of Math. **75** (1962) 536–577.

[4] G. D. Mostow, *Factor spaces of solvable groups,* Ann. of Math. **60** (1954) 1–27.

[5] M. S. Raghunathan, *Vanishing theorem for cohomology groups associated to discrete subgroups of semi-simple Lie groups,* Osaka J. Math. **3** (1966) 243–256.

[6] ——, *Discrete subgroups of Lie groups,* Springer, Berlin, 1972.

[7] V. S. Vladimirov, *Methods of the theory of functions of many complex variables,* Massachusetts Institute of Technology Press, 1966 (English Translation).

[8] H. C. Wang, *Complex parallelisable manifolds,* Proc. Amer. Math. Soc. **5** (1954) 771–776.

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