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MANIFOLDS WITH REFLECTING BOUNDARY

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1. Introduction

Let M be a compact oriented C^∞ manifold of dimension $4k$ with boundary B of dimension $4k - 1$, and let g be a Riemannian metric on M. Being given a power series with real coefficients $P \in R[[x_{4i}]]$ in variables $x_{4i}, i = 1, 2, \cdots$ one may replace x_{4i} by the real $4i$-form P_i which gives the i-th Pontrjagin class $\mathscr{P}_i$ of M, expressed in the standard way in terms of the curvature 2-forms Ω_{jk} of the Riemannian metric g, and integrate the component of dimension $4k$ over M to obtain a real number

$$P(M, g) = \int_M P(P_1, P_2, \cdots) .$$

If M is closed, *i.e.*, B is empty, then $P(M, g)$ is independent of the Riemannian metric g, and many be obtained by replacing x_{4i} by the i-th Pontrjagin class $\mathscr{P}_i$ of M, and evaluating the 4k-dimensional component of the resulting cohomology class on the fundamental homology class of M; *i.e.*,

$$P(M, g) = \langle P(\mathscr{P}_1, \mathscr{P}_2, \cdots), [M] \rangle .$$

In [5], C. C. Hsiung has introduced another class of manifolds for which these numbers are well behaved, which he calls manifolds with reflecting boundary. Specifically, one considers a manifold M together with an orientation reversing involution $\pi: B \to B$. For such a pair (M, π) one considers a "nice" Riemannian metric g on M, which satisfies the conditions that π is an isometry of the manifold B with induced Riemannian metric g/B and that, on a tubular neighborhood $B \times [0, 1)$ of $B = B \times 0$ in M, g is given by a product metric. Such metrics always exist.

Proposition 1. *If (M, π) is a manifold with reflecting boundary with nice Riemannian metric g, then $P(M, g)$ is independent of the nice metric. Further,*

a) *if $P \in Z[[x_{4i}]]$ is a power series with integral coefficients, then $P(M, g)$ belongs to $\frac{1}{2}Z$,*

b) *if P is a power series of the form $Q \cdot L$ where Q, L are the rational power series given by considering x_{4i} as the i-th elementary symmetric function in variables y_j (of dimension 2), with Q any symmetric polynomial over the integers in the variables $e^{y_j} + e^{-y_j} - 2$ and with L the product of the classes*

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$y_j/\tanh y_j$, *then* $P(M, g)$ *belongs to* $Z[\frac{1}{2}]$, *and*

c) *if P is the power series L, then* $P(M, g)$ *is an integer.*

Proof. Let N be the closed 4k-manifold obtained from $M \times \{0, 1\}$ by identifying $(b, 0) \in B \times 0$ with $(\pi(b), 1) \in B \times 1$, using the differentiable structure arising from the tubular neighborhood $B \times [0, 1)$ of B in M, with the Riemannian metric obtained from g on each copy of M (note that π is an isometry and that the metric is compatible with the product structure near B) and with the orientation obtained from the given orientation on each copy of M. Then the induced metric h on N is symmetric in the sense of Hsiung, and one has

$$\begin{aligned} 2P(M, g) &= \left(\int_{M \times 0} + \int_{M \times 1}\right) P(P_1, P_2, \cdots) \\ &= \int_N P(P_1, P_2, \cdots) = P(N, h) \\ &= \langle P(\mathscr{P}_1, \mathscr{P}_2, \cdots), [N] \rangle , \end{aligned}$$

which is independent of the metric. Part a) follows from integrality of the Pontrjagin classes of N, while part b) is the condition $P(N, h) \in Z[\frac{1}{2}]$ coming from the Riemann-Roch theorem (see [7, p. 204]), and finally, part c) is immediate from Hsiung's result that $P(M, g)$ is the signature sign (M, B) of the manifold M with boundary B. q.e.d.

While part b) of the proposition may seem unwieldy and dull, it has one intriguing consequence.

Corollary. *Suppose that* (M, π) *is a manifold with reflecting boundary with nice Riemannian metric g. Then there is a closed manifold* M' *with metric* g' *for which* $P(M, g) = P(M', g')$ *for all P if and only if* $P(M, g)$ *is integral for all power series with integral coefficients.*

Proof. As noted in [7, p. 207] all relations among the Pontrjagin numbers of closed manifolds follow from the Riemann-Roch theorem and integrality of the Pontrjagin classes. q.e.d.

Another way to phrase this is to say that there is a closed manifold M' with metric g' so that $P(M, g) = P(M', g')$ for all P if and only if all Pontrjagin numbers $\langle \mathscr{P}_{i_1} \cdots \mathscr{P}_{i_r}, [N] \rangle$, $i_1 + \cdots + i_r = k$, are even.

The main result of this paper is a converse to Proposition 1, namely,

Proposition 2. *If* $\phi\colon (R[x_{4i}])_{4k} \to R$ *is a homomorphism of real vector spaces defined on the homogeneous polynomials of degree* $4k$ *in variables* x_{4i}, *and satisfying*

a) $\phi(P)$ *is in* $\frac{1}{2}Z$ *if P has integral coefficients,*

b) *if P is the* $4k$*-dimensional component of a power series* $Q \cdot L$ *as in Proposition* 1 b), *then* $\phi(P) \in Z[\frac{1}{2}]$, *and*

c) *if P is the* $4k$*-dimensional component of L, then* $\phi(P)$ *is integral,*

then there is a manifold with reflecting boundary (M, π) *of dimension* $4k$ *with nice metric g, so that* $\phi(P) = P(M, g)$ *for all P.*

This is, of course, a completely dishonest statement. Since 2ϕ is given by $2\phi(P) = P(N', h')$ for some closed manifold N', by [7, p. 207], with N' having even index by c), one is really showing:

Proposition 2′. *Every closed 4k-dimensional oriented manifold with even index has the same Pontrjagin numbers as the "double in the sense of Hsiung" of some manifold with reflecting boundary.*

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2. Cobordism groups

Consider pairs (M, π) where M is a compact oriented C^∞ manifold of dimension n with boundary B of dimension $n - 1$, and $\pi\colon B \to B$ is an orientation reversing involution on B. Two pairs (M, π) and (M', π') will be said to be equivalent or cobordant if there is a compact oriented manifold V with orientation reversing involution ρ with the boundary of V being the disjoint union of B and $-B'$, i.e., B' with opposite orientation, so that $\rho | B = \pi, \rho | B' = \pi'$, and if the closed oriented manifold W obtained from $M \cup (-V) \cup (-M')$ by identifying the two copies of B and the two copies of B' is the boundary of some compact oriented manifold X. The set of equivalence classes of manifolds with reflecting boundary becomes an abelian group using the operation induced by taking the disjoint union, with inverse given by reversing orientation. This group will be denoted $\Omega_n(Z_2 - \partial)$.

One has a boundary homomorphism $\partial\colon \Omega_n(Z_2 - \partial) \to \Omega_{n-1}(Z_2 -)$ into the cobordism group of oriented manifolds with orientation reversing involution (denoted $\mathcal{O}_{n-1}^-$ in [6, p. 206]), assigning to (M, π) the class of (B, π). One has a forgetful homomorphism, augmentation, $\varepsilon\colon \Omega_n(Z_2 -) \to \Omega_n$ into the ordinary oriented cobordism group assigning to (B, π) the class of B, and a forgetful homomorphism $i\colon \Omega_n \to \Omega_n(Z_2 - \partial)$ assigning to a closed manifold M the pair (M, ϕ) where ϕ is the "empty" involution on the boundary of M, which is empty. The sequence

$$\begin{array}{ccc} \Omega_* & \xrightarrow{\ i\ } & \Omega_*(Z_2 - \partial) \\ {\scriptstyle\varepsilon}\nwarrow & & \swarrow{\scriptstyle\partial} \\ & \Omega_*(Z_2 -) & \end{array}$$

is then exact; i.e., $\Omega_*(Z_2 - \partial)$ is the relative cobordism theory associated to oriented manifolds and oriented manifolds with orientation reversing involution in the sense of [7, p. 9].

Being given a manifold with reflecting boundary (M, π), one may as above form the closed oriented manifold N from $M \times \{0, 1\}$ by identifying $(b, 0) \in$

$B \times 0$ with $(\pi(b), 1) \in B \times 1$. If (M, π) and (M', π') are equivalent, then in the notation used for equivalence, one has a compact manifold with boundary T obtained from $X \times \{0, 1\}$ by identifying $(v, 0) \in V \times 0$ with $(\rho(v), 1) \in V \times 1$ and the boundary of T is the disjoint union of N and $-N'$. Thus the doubling construction defines a homomorphism

$$D\colon \Omega_n(Z_2 - \partial) \to \Omega_n \,.$$

The composite $D \circ i\colon \Omega_n \to \Omega_n$ is clearly multiplication by 2.

One now recalls that the graded group Ω_* is actually a commutative graded ring with the product $[M]\cdot[N] = [M \times N]$. Then $\Omega_*(Z_2 -)$ and $\Omega_*(Z_2, - \partial)$ are modules over Ω_*; given a closed manifold Q and manifold with involution $(M, \pi), Q \times M$ has the involution $1 \times \pi, (1 \times \pi)(q, m) = (q, \pi(m))$, while if π is only an involution on the boundary of $M, 1 \times \pi$ is an involution on the boundary of $Q \times M$. Each of the homomorphisms ∂, ε, i, and D is easily seen to be an Ω_* module homomorphism.

Similarly, there exist unoriented versions of each of these groups, in which it is only necessary to forget all mention of orientation and consider involutions, without using the adjective "orientation reversing". One then obtains an exact sequence

$$\begin{array}{ccc} \mathcal{N}_* & \xrightarrow{\;i\;} & \mathcal{N}_*(Z_2\partial) \\ & {}^{\varepsilon}\nwarrow \qquad \swarrow{}^{\partial} & \\ & \mathcal{N}_*(Z_2) & \end{array}$$

and a homomorphism

$$D\colon \mathcal{N}_*(Z_2\partial) \to \mathcal{N}_*$$

with $D \circ i$ being doubling, which is zero.

Ignoring the orientation defines a restriction homomorphism, generically denoted ρ, and one has commutative diagrams:

$$\begin{array}{ccccc} \Omega_* & & \xrightarrow{\;i\;} & & \Omega_*(Z_2 - \partial) \\ \Big\downarrow{\scriptstyle\rho} & {}^{\varepsilon}\nwarrow & & \swarrow{}^{\partial} & \Big\downarrow{\scriptstyle\rho} \\ & & \Omega_*(Z_2 -) & & \\ & & \Big\downarrow{\scriptstyle\rho} & & \\ \mathcal{N}^* & & \xrightarrow{\;i\;} & & \mathcal{N}_*(Z_2\partial) \\ & {}^{\varepsilon}\nwarrow & \Big\downarrow & \swarrow{}^{\partial} & \\ & & \mathcal{N}_*(Z_2) & & \end{array} \qquad \begin{array}{ccc} \Omega*(Z_2 - \partial) & \xrightarrow{\;D\;} & \Omega_* \\ \Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle\rho} \\ \mathcal{N}_*(Z_2\partial) & \xrightarrow{\;D\;} & \mathcal{N}_* \end{array}$$

In the unoriented case one may understand all of the homomorphisms quite easily. First, the homomorphism $\varepsilon\colon \mathcal{N}_*(Z_2) \to \mathcal{N}_*$ is epic with a splitting homomorphism $s\colon \mathcal{N}_* \to \mathcal{N}_*(Z_2)$ given by sending the class of M to the class of M with the trivial involution $1\colon 1(m) = m$. Obviously $\varepsilon s =$ identity. Thus one has a short exact sequence

$$0 \longrightarrow \mathcal{N}_{n+1}(Z_2\partial) \xrightarrow{\partial} \mathcal{N}_n(Z_2) \xrightarrow{\varepsilon} \mathcal{N}_n \longrightarrow 0$$

which is split.

One may then define a homomorphism $D'\colon \mathcal{N}_n(Z_2) \to \mathcal{N}_{n+1}$ for which $D = D' \circ \partial$ by $D'(x) = D(\partial^{-1}(x + s\varepsilon(x)))$.

Lemma. *The homomorphism $D'\colon \mathcal{N}_*(Z_2) \to \mathcal{N}_*$ is the homomorphism K_1 defined by Conner and Floyd* [2], *which assigns to the manifold M with involution t the class of the manifold $(M \times S^1)/(t \times a)$ obtained from $M \times S^1$ by identifying $(t(m), -z)$ with (m, z).*

Proof. If $x = [M, t], s\varepsilon(x)$ is the class of $[M, 1]$, and $(M, t) \cup (M, 1)$ is the boundary of $M \times [0, 1]$. Joining boundaries on two copies of $M \times [0, 1]$ by t along $M \times 0$ and by the identity along $M \times 1$ gives $(M \times S^1)(t \times a)$.

Note. D' is zero on the image of s, so D' and D have the same image, which was completely analyzed by Conner and Floyd [2]. The image of D consists of all classes with even Euler characteristic.

Understanding the oriented case will be somewhat more difficult, and one will need to make use of some of the results of Wall [9] on the structure of Ω_*.

First, recall that assigning to a closed $4k$-manifold N the Pontrjagin number $\langle \mathcal{P}_{i_1} \cdots \mathcal{P}_{i_r}, [N]\rangle \in Z, i_1 + \cdots + i_r = k$, defines a homomorphism $\Omega_{4k} \to Z$ with kernel containing the torsion subgroup, and in fact two closed oriented manifolds have the same Pontrjagin numbers if and only if they represent the same element of Ω_*/Tor. According to Wall, Ω_*/Tor is a polynomial ring over Z on generators y_{4i} of dimension $4i$.

Letting $q\colon \Omega_* \to \Omega_*/\text{Tor}$ be the quotient homomorphism, one sees that the image of qD contains $2(\Omega_*/\text{Tor})$, and one is then interested in the image of qD in $(\Omega_*/\text{Tor}) \otimes Z_2$.

According to Wall, $(\Omega_*/\text{Tor}) \otimes Z_2$ is the polynomial ring over Z_2 on the classes y_{4i}, and one can take y_{4i} to be the class of the complex projective space $\mathrm{CP}(2i)$. The class of a manifold in $(\Omega_*/\text{Tor}) \otimes Z_2$ is determined by the Pontrjagin numbers reduced mod 2. Since the mod 2 reduction of the Pontrjagin class $\mathcal{P}_i$ is the Stiefel-Whitney class $w_{2i}{}^2$, the Stiefel-Whitney numbers $\langle w_{2i_1}{}^2 \cdots w_{2i_r}{}^2, [N]\rangle$ define homomorphisms of Ω_*/Tor into Z_2, or of $(\Omega_*/\text{Tor}) \otimes Z_2$ into Z_2 agreeing with the mod 2 reductions of the Pontrjagin numbers.

There exists a universal polynomial $s_i(\sigma_1, \cdots, \sigma_i)$ with integral coefficients which expresses the symmetric function $\sum_\alpha t_\alpha^i$ in terms of the elementary symmetric functions σ_j of the variables t_α. A manifold N of dimension $4k$ is an indecomposable in $(\Omega_*/\text{Tor}) \otimes Z_2$ if and only if $\langle s_k(\mathcal{P}_1, \cdots, \mathcal{P}_k), [N]\rangle$ is odd, or equivalently $\langle s_k(w_2{}^2, \cdots, w_{2k}{}^2), [N]\rangle \neq 0$ in Z_2.

Now the index defines a ring homomorphism $I\colon \Omega_* \to Z$ sending the torsion subgroup to zero, and reducing mod 2 defines a homomorphism $I_2\colon (\Omega_*/\mathrm{Tor}) \otimes Z_2 \to Z_2$. Since the index of CP($2i$) is 1, the kernel of I_2 is the ideal generated by the classes CP($2i$) $-$ (CP(2))^i, and may also be described as the ideal generated by any set of indecomposables $Z_{4i}, i > 1$, of $(\Omega_*/\mathrm{Tor}) \otimes Z_2$, which lie in the kernel of I_2.

One now wishes to exhibit such indecomposables lying in the image of D. Let CP(1) be the complex projective space consisting of complex lines in C^2, and let λ be the complex line bundle over CP(1) whose total space consists of pairs (α, x), where α is a complex line in C^2 and x is a vector in the line α. Form the real projective space bundle RP$(\lambda \oplus 4(k-1))$ consisting of the real one-dimensional subspaces of the fibers of the Whitney sum of λ and a trivial real bundle of dimension $4(k-1), k > 1$. This is a closed oriented manifold of dimension $4k-1$. Considering $\lambda \oplus 4(k-1)$ as $\{\lambda \oplus (4k-5)\} \oplus 1$ one has an involution given by multiplication by 1 in the fibers of $\lambda \oplus (4k-5)$ and by -1 in the fibers of the last summand 1. This induces an orientation reversing involution π on RP$(\lambda \oplus 4(k-1))$.

The manifold RP$(\lambda \oplus 4(k-1))$ bounds. In fact, one may fiber RP$(\lambda \oplus 4(k-1))$ over the complex projective space bundle CP$(\lambda \oplus 2(k-1))$ consisting of complex lines in the fibers of the Whitney sum of λ and a trivial complex bundle of dimension $2(k-1)$ with fiber S^1. If η is the complex line bundle over CP$(\lambda \oplus (2k-1))$, whose total space consists of pairs (α, x) where α is a complex line in a fiber and $x \in \alpha$, then RP$(\lambda \oplus 4(k-1))$ is in fact the unit sphere bundle of the complex line bundle $\eta \otimes_c \eta$, (see [8, p. 297]). Thus RP$(\lambda \oplus 4(k-1))$ is the boundary of the oriented manifold $D(\eta \otimes_c \eta)$ given by the unit disc bundle.

Lemma. *The image of* $(D(\eta \otimes_c \eta), \pi)$ *under the homomorphism* D *is an indecomposable element in* $(\Omega_*/\mathrm{Tor}) \otimes Z_2$ *of dimension* $4k$ *for each* $k > 1$.

Before proceeding to prove this, one may note that this gives Proposition 2′. The image of qD is clearly an ideal in Ω_*/Tor containing $2(\Omega_*/\mathrm{Tor})$ and a generator for $(\Omega_*/\mathrm{Tor}) \otimes Z_2$ in each dimension $4k, k > 1$. Since it is contained in the kernel of the homomorphism to Z_2 induced by the index, this shows equality, which is the result desired.

Turning to the proof of the lemma, it suffices to show that the Stiefel-Whitney number

$$\langle s_k(w_2{}^2, \cdots, w_{2k}{}^2), [D(D(\eta \otimes_c \eta), \pi)]\rangle$$

is nonzero, but this Stiefel-Whitney number is an invariant of the unoriented cobordism class of the manifold. Then $\rho D(D(\eta \otimes_c \eta), \pi) = D\rho(D(\eta \otimes_c \eta), \pi) = D'\partial\rho(D(\eta \otimes_c \eta), \pi) = D'(\mathrm{RP}(\lambda \oplus 4(k-1)), \pi)$, and one needs only to consider the Stiefel-Whitney number of

$$X = \mathrm{RP}(\lambda \oplus 4(k-1)) \times S^1/(\pi \times a) .$$

Now the bundle $(R \times S^1)/((x, z) \sim (-x, -z)) \to S^1/(z \sim -z)$ is the nontrivial line bundle θ over RP(1), and so X may be identified as the real projective space bundle of lines in the fibers of $\lambda \oplus (4k-5) \oplus \theta$ over $\mathrm{CP}(1) \times \mathrm{RP}(1)$; $X = \mathrm{RP}(\lambda \oplus (4k-5) \oplus \theta)$.

The mod 2 cohomology structure of such a projective space bundle is well-known, as in [3, p. 61] for example. Letting $p: X \to \mathrm{CP}(1) \times \mathrm{RP}(1)$ be the projection, one sees that $H^*(X; Z_2)$ is the free module over $H^*(\mathrm{CP}(1) \times \mathrm{RP}(1); Z_2)$ via p^* on $1, c, \cdots, c^{4k-3}$ where c is a class in $H^1(X; Z_2)$ with $\langle c^{4k-3}p^*(x), [X]\rangle = \langle x, [\mathrm{CP}(1) \times \mathrm{RP}(1)]\rangle$ for $x \in H^3(\mathrm{CP}(1) \times \mathrm{RP}(1); Z_2)$. Letting $\alpha = w_2(\lambda) \in H^2(\mathrm{CP}(1); Z_2)$ and $\beta = w_1(\theta) \in H^1(\mathrm{RP}(1); Z_2)$, and ignoring all pullback homomorphisms in notation, one finds that c satisfies the relation

$$c^{4k-2} + \beta c^{4k-3} + \alpha c^{4k-4} + \alpha\beta c^{4k-5} = 0 ,$$

and the Stiefel-Whitney class of X is

$$w(X) = (1 + c^2 + \alpha)(1 + c)^{4k-5}(1 + c + \beta) .$$

Squaring the Stiefel-Whitney class of X and noting $\alpha^2 = \beta^2 = 0$ give

$$\begin{aligned} w(X)^2 &= 1 + w_1(X)^2 + \cdots + w_{4k}(X)^2 \\ &= (1 + c^2)^2(1 + c)^{2(4k-5)}(1 + c)^2 \\ &= (1 + c)^{8k-4} = (1 + c^4)^{2k-1} , \end{aligned}$$

so that $w_{2i}{}^2$ is the i-th elementary symmetric function in the $(2k-1)$ variables c^4. Thus $s_k(w_2{}^2, \cdots, w_{2k}{}^2)$ is $(2k-1)c^{4k}$.

Now

$$\begin{aligned} c^{4k} &= c^2(\beta c^{4k-3} + \alpha c^{4k-4} + \alpha\beta c^{4k-5}) \\ &= \beta c(c^{4k-2}) + \alpha c^{4k-2} + \alpha\beta c^{4k-3} \\ &= \beta c(\beta c^{4k-3} + \alpha c^{4k-4} + \alpha\beta c^{4k-5}) \\ &\quad + \alpha(\beta c^{4k-3} + \alpha c^{4k-4} + \alpha\beta c^{4k-5}) + \alpha\beta c^{4k-3} \\ &= 3\alpha\beta c^{4k-3} , \end{aligned}$$

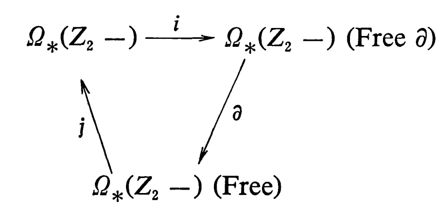
and so

$$\begin{aligned} \langle s_k(w_2^2, \cdots, w_{2k}^2), [X]\rangle &= \langle 3(2k-1)\alpha\beta c^{4k-3}, [X]\rangle \\ &= \langle \alpha\beta, [\mathrm{CP}(1) \times \mathrm{RP}(1)]\rangle = 1 . \end{aligned}$$

3. Calculation of the groups

Before ending, it seems desirable to compute the groups $\Omega_*(Z_2\,-)$ and $\Omega_*(Z_2\,-\partial)$. Lee and Wasserman [6] have analyzed $\Omega_*(Z_2\,-)$, but their results stop just short of a clean answer. One follows the procedure established by Conner and Floyd [4].

To begin one introduces $\Omega_*(Z_2\,-)$(Free), the bordism group of free orientation reversing involutions, and the relative group $\Omega_*(Z_2\,-)$(Free ∂) of orientation reversing involutions on compact manifolds with boundary, which are free on the boundary. One then has an exact sequence



with maps induced by inclusion of types of manifolds and by taking boundaries.

Being given (M, π) with π an orientation reversing free involution, one has a double cover $p\colon M \to M/Z_2$, and this may be easily seen to be the orientation double cover of M/Z_2 since π reverses the orientation. Thus assigning to (M, π) the class of M/Z_2 gives an isomorphism $\Omega_n(Z_2\,-)(\text{Free}) \cong \mathcal{N}_n$.

Further, being given (M, π), $M \times [-1, 1]$ has a fixed point free orientation preserving involution $\pi \times (-1)$ so that $M \times [-1, 1]/\pi \times (-1)$ is an oriented manifold with boundary M. The involution induced by $\pi \times 1$ or $1 \times (-1)$ is then orientation reversing and extends π. In fact, $(M \times [-1, 1])/(\pi \times (-1))$ is just the disc bundle of the cover ρ, and the involution is just multiplication by -1 in the fibers of this bundle. Assigning to (M, π) the class of $(M \times [-1, 1])/(\pi \times (-1))$ with involution induced by $\pi \times 1$ defines a splitting homomorphism

$$\sigma\colon \Omega_n(Z_2\,-)(\text{Free}) \to \Omega_{n+1}(Z_2\,-)(\text{Free }\partial)$$

with $\partial\sigma = 1$.

Now turning to $\Omega_n(Z_2\,-)$(Free ∂), consider an n-dimensional manifold V with orientation reversing involution π whose restriction to the boundary of V is free. The fixed point set of π is then a disjoint union of closed submanifolds F^k (k-dimensional part) imbedded in the interior of V. If ν^{n-k} is the normal bundle of F^k in V, then a neighborhood of $F = \bigcup F^k$ in V may be identified with the disjoint union of the disc bundles $D(\nu^{n-k})$ with the action of π being identified as multiplication by -1 in the fibers of these disc bundles. Now each disc bundle $D(\nu^{n-k})$ is an oriented manifold, and hence there is a chosen isomorphism of the determinant bundles $\det \tau_F k$ and $\det \nu^{n-k}$ of the tangent bundle of F^k and ν^{n-k} (note: $\tau_{D(\nu)} \cong$ pullback of $\tau_F \oplus$ pullback of ν). Further, since π reverses orientation, and multiplication by -1 in the fibers of ν^{n-k} has degree $(-1)^{n-k}$, one must have $n-k$ odd. Since $n-k$ is odd, the bundle $(\det \tau_F k) \otimes \nu^{n-k}$ is then an oriented bundle ξ^{n-k} over F^k.

Assigning to (V, π) the classes of (F^{n-2j-1}, ξ^{2j+1}) defines a homomorphism

$$f\colon \Omega_n(Z_2 -)(\text{Free } \partial) \to \bigoplus_{j=0}^{[n/2]} \mathcal{N}_{n-2j-1}(\text{BSO}_{2j+1}) ,$$

which is in fact an isomorphism. The inverse to f assigns to (F, ξ) the manifold $D((\det \tau_F) \otimes \xi)$ with involution given by multiplication by -1 in the fibers of the disc bundle. Calling this function g, fg is clearly 1, while $gf(V, \pi)$ is represented by a neighborhood of F in V, and since the action is free on the complement of the neighborhood, the two actions are cobordant.

Note. Lee and Wasserman classify the normal bundles ν^{n-k} and extend to $(D(\nu^{n-k}), S(\nu^{n-k})) \to (\text{MO}_{n-k,\infty})$ to get $\Omega_n(Z_2 -)(\text{Free } \partial) \cong \oplus \tilde{\Omega}_n(\text{MO}_{2j+1})$. Since $\tilde{H}_*(\text{MO}_{2j+1})$ has every element of order 2, these groups may be computed, but it is far simpler to compute $\mathcal{N}_*(\text{BSO}_{2j+1})$.

Now note that the term $\mathcal{N}_{n-1}(\text{BSO}_1)$ is just $\mathcal{N}_{n-1}$ (BSO_1 is contractible) and is mapped by ∂ isomorphically onto $\Omega_{n-1}(Z_2 -)(\text{Free})$, with the splitting s sending $\Omega_{n-1}(Z_2 -)(\text{Free})$ isomorphically to $\mathcal{N}_{n-1}(\text{BSO}_1)$.

Being given $(F^{n-2j-1}, \xi^{2j+1}) = x, \sigma\partial x$ is represented by the disc bundle of the double cover $S((\det \tau_F) \otimes \xi) \to \text{RP}((\det \tau_F) \otimes \xi)$ which has the same boundary as $D((\det \tau_F) \otimes \xi)$. Joining these along the common boundary is the same as dividing out the antipodal involution on the boundary of $D((\det \tau_F) \otimes \xi)$, which forms the real projective space bundle $\text{RP}(((\det \tau_F) \otimes \xi) \oplus 1)$, and the involution may be identified with that induced by $(-1) \times 1$ (or equivalently $1 \times (-1)$) in the fibers of the Whitney sum.

Note. If ρ is a line bundle, $\text{RP}(\rho \otimes \eta) \cong \text{RP}(\eta)$ by assigning to a line l in η the line $\rho \otimes l$ in $\rho \otimes \eta$. Thus $\text{RP}(((\det \tau_F) \otimes \xi) \oplus 1)$ is also $\text{RP}(\xi \oplus \det \tau_F)$ and the involution may be identified as that induced by -1×1 or 1×-1.

Thus one has:

Proposition 3. *Taking the fixed data and assigning to (F, ξ) the class of* $\text{RP}(\xi \oplus \det \tau_F)$ *with involution induced by* 1×-1 *give isomorphisms*

$$\Omega_n(Z_2 -) \cong \bigoplus_{j=1}^{[n/2]} \mathcal{N}_{n-2j-1}(\text{BSO}_{2j+1}) .$$

One may now understand the exact sequence for $\Omega_*(Z_2 - \partial)$. For this one has a result of Rosenzweig [10, p. 5].

Proposition 4. *The homomorphism $\varepsilon\colon \Omega_*(Z_2 -) \to \Omega_*$ has image precisely the torsion subgroup.*

Proof. If (M, π) is an orientation reversing involution, $1 \cup \pi$ is a diffeomorphism of $2M = M \cup M$ and $M \cup -M = \partial(M \times [0, 1])$, so every element in image ε is torsion (*Note*: $1 \cup \pi$ is equivariant so every class in $\Omega_*(Z_2 -)$ has order 2). On the other hand, P. G. Anderson [1] has shown that every class in Tor Ω_* is a sum of classes of the manifolds $RP(\det \tau_F \oplus (2k-1))$, which admits the orientation reversing involution induced by -1×1. q.e.d.

Thus one has a short exact sequence

$$0 \to \Omega_*/\mathrm{Tor} \to \Omega_*(Z_2 - \partial) \to \mathrm{Ker}\, \varepsilon \to 0 .$$

Since ker ε is a Z_2 vector space, which has a computable dimension and since the extensions involved are known from the previous section (i.e., in dimension $4k-1$ of ker ε, the number of Z_2's which are involved in nontrivial extensions is the number of partitions of k minus 1, by looking at im qD: $\Omega_*(Z_2 - \partial) \to \Omega_*/\mathrm{Tor}$), one has:

Proposition 5. *$\Omega_*(Z_2 - \partial)$ is a direct sum of copies of Z and Z_2 with the number of summands being computable (but horrible).*

To close, one may note that D: $\Omega_*(Z_2 - \partial) \to \Omega_*$ does not map onto the torsion subgroup, so that working with Ω_*/Tor was necessary. In fact, one has $\Omega_4(Z_2 -) \cong \mathscr{N}_1(BSO_3) = 0$, so

$$\Omega_5(Z_2 -) \xrightarrow{\text{epic}} \Omega_5 \longrightarrow \Omega_5(Z_2 - \partial) \longrightarrow \Omega_4(Z_2 -) = 0$$

giving $\Omega_5(Z_2 - \partial) = 0$, but $\Omega_5 \cong Z_2$ and D cannot be epic.

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