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COMPACT FOUR-DIMENSIONAL EINSTEIN MANIFOLDS

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1. Introduction

There are few known examples of compact four-dimensional Einstein manifolds, namely,

(a) flat riemannian manifolds,

(b) compact symmetric spaces S^4, $S^2 \times S^2$, CP^2,

(c) manifolds whose universal coverings are the corresponding noncompact symmetric spaces (see Borel [5]).

On the other hand, there are few examples of four-manifolds which do *not* admit an Einstein metric. Berger [3] proved that a four-dimensional Einstein manifold X must have Euler characteristic $\chi \geq 0$ with equality iff X is flat, and so for example $T^4 \# T^4$ and $S^1 \times S^3$ do not admit Einstein metrics. However, if X is simply connected, then χ is necessarily positive, and this led Eells and Sampson to pose the following question [8]: Are there simply connected compact manifolds which do not carry an Einstein metric?

Theorem 1 gives an inequality between the signature τ and Euler characteristic χ of a four-dimensional Einstein manifold which allows us to answer this question.

Theorem 1. *Let X be a compact four-dimensional Einstein manifold with signature τ and Euler characteristic χ. Then*

$$|\tau| \leq \tfrac{2}{3}\chi \,. \tag{1.1}$$

Furthermore, if equality occurs then $\pm X$ is either flat or its universal covering is a $K3$ surface. If the universal covering of X is a $K3$ surface, then X is a $K3$ surface ($\pi_1 = 1$), an Enriques surface ($\pi_1 = Z_2$) or the quotient of an Enriques surface by a free antiholomorphic involution with $\pi_1 = Z_2 \times Z_2$.

(A $K3$ surface is a complex surface with first Betti number $b_1 = 0$ and first Chern class $c_1 = 0$, and an Enriques surface is a complex surface with $b_1 = 0$ and $2c_1 = 0$. Note that all $K3$ surfaces are diffeomorphic to a quartic surface in CP^3; see Kodaira [9].)

The examples (a), (b) and (c) above all have a further property: their sectional curvatures are nonnegative or nonpositive. With this as an additional

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hypothesis we have the stronger inequality of Theorem 2.

Theorem 2. *Let X be a compact four-dimensional Einstein manifold with nonnegative (or nonpositive) sectional curvature. Then*

$$|\tau| \leq (\tfrac{2}{3})^{3/2}\chi \ . \tag{1.2}$$

Since $(\frac{2}{3})^{3/2}$ is irrational, clearly equality can only occur if X is flat.

2. Remarks

1. Let $X = nCP^2$ (the connected sum of n copies of CP^2). Then X is simply connected, and $\tau = n, \chi = n + 2$; so applying Theorem 1 we obtain that if X is Einsteinian, then $n < \frac{2}{3}(n + 2)$, i.e., $n < 4$. Hence nCP^2 $(n \geq 4)$ is a simply connected compact manifold which does not carry an Einstein metric.

2. It is not known (to the author) whether there exist Enriques surfaces with free antiholomorphic involutions but, more importantly, it is not known whether a $K3$ surface actually admits an Einstein metric. This seems an interesting question, especially in view of the following equivalent formulations for a $K3$ surface:

(i) X admits a quaternionic Kähler structure,
(ii) X admits a Ricci-flat Kähler structure,
(iii) X admits an Einstein metric,
(iv) X admits a riemannian metric of zero scalar curvature.

A quaternionic Kähler structure is a reduction of the holonomy group from $SO(4)$ to $Sp(1)$. Since $Sp(1) = SU(2)$, (i) $\Rightarrow$ (ii). If the Ricci tensor is zero, then X is Einsteinian, so (ii) $\Rightarrow$ (iii). We shall see in the proof of Theorem 1 that any Einstein metric on a $K3$ surface must have zero scalar curvature, hence (iii) $\Rightarrow$ (iv). To show (iv) $\Rightarrow$ (i) we use the vanishing theorem of Lichnerowicz for harmonic spinors [10]. A $K3$ surface is a spin manifold with nonzero $\hat{A}$-genus, and hence if X admits a metric of zero scalar curvature, then from [10] there exists a parallel spinor. This implies a reduction of the holonomy group to the isotropy subgroup of the spin representation Δ^+ or Δ^-. The isotropy subgroup of Δ^+ is $SU(2) = Sp(1)$, and so (modulo a change of orientation) (iv) $\Rightarrow$ (i). Note that Calabi's conjecture implies (ii).

3. Let us apply Theorem 2 to $X = nCP^2$. If X admits an Einstein metric of nonpositive or nonnegative sectional curvature, then $n \leq (\frac{2}{3})^{3/2}(n + 2) < \frac{5}{9}(n + 2)$ and so $n \leq 2$. We know CP^2 admits an Einstein metric of nonnegative sectional curvature; Cheeger [6] has constructed a (non-Einstein) metric of nonnegative sectional curvature on $2CP^2$.

4. Berger [3] has shown that for a four-dimensional Einstein manifold of strictly positive sectional curvature, $\chi \leq 10$ (in fact closer examination shows that strict inequality must hold, so $\chi \leq 9$). Applying Theorem 2 we see that $|\tau| \leq 4$. This reduces slightly the number of possible homotopy types of such manifolds.

5. It is shown in [12] that if a compact four-dimensional manifold has abelian fundamental group, then the inequality $|\tau| \leq \chi$ holds. It would be interesting, in view of Theorem 1, to know if admitting an Einstein metric implies anything about the fundamental group.

3. Proof of Theorem 1

We use the normal form for the curvature tensor at each point of a four-dimensional Einstein manifold given in Berger [2] and Singer-Thorpe [11].

We regard the curvature tensor R as a symmetric linear transformation of the bundle Λ^2 of 2-forms defined by

$$R(e_i \wedge e_j) = \Omega^i_j = \tfrac{1}{2}\Sigma R_{ijkl} e_k \wedge e_l$$

relative to a local orthonormal basis $\{e_i\}$ of the 1-forms. The theorem on the normal form of R then states that there exists such an orthonormal basis such that relative to the corresponding basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_3 \wedge e_4, e_4 \wedge e_2, e_2 \wedge e_3\}$ of Λ^2, R takes the form

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

where

$$A = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_2 \end{bmatrix}, \qquad B = \begin{bmatrix} \mu_1 & & 0 \\ & \mu_2 & \\ 0 & & \mu_3 \end{bmatrix}.$$

The Bianchi identity implies that $\Sigma\mu_i = 0$. Moreover, $\Sigma\lambda_i = \frac{1}{2}\,\mathrm{trace}\, R = \frac{1}{4}X$ scalar curvature. It will be convenient to regard $(\lambda_1, \lambda_2, \lambda_3)$ and (μ_1, μ_2, μ_3) as vectors $\lambda, \mu \in \boldsymbol{R}^3$ in what follows.

Now by the Gauss-Bonnet theorem, the Euler characteristic χ of X is given by integrating the following form over X:

$$\begin{aligned} \frac{1}{2^4\pi^2 2!}\Sigma\varepsilon_{ijkl}\Omega^i_j \wedge \Omega^k_l &= \frac{1}{2^5\pi^2}\Sigma\varepsilon_{ijkl}R(e_i \wedge e_j) \wedge R(e_k \wedge e_l) \\ &= \frac{1}{4\pi^2}(\Sigma\lambda_i^2 + \mu_i^2) * 1 = \frac{1}{4\pi^2}(|\lambda|^2 + |\mu|^2) * 1 . \end{aligned}$$

(Since $|\lambda|^2 + |\mu|^2 \geq 0$ with equality iff $\lambda = \mu = 0$, we have here Berger's result.)

The first Pontrjagin class p_1 of X is given by integrating the following form over X:

$$-\frac{1}{4\pi^2 2!}\,\mathrm{trace}\,\Omega^2 = \frac{1}{8\pi^2}\Sigma R(e_i \wedge e_j) \wedge R(e_i \wedge e_j)$$

$$= \frac{1}{\pi^2}(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3) * 1 = \frac{1}{\pi^2}(\lambda, \mu) * 1 .$$

We have the inequality $|\lambda|^2 + |\mu|^2 \geq 2(\lambda, \mu)$ with equality iff $\lambda = \mu$, and so from the above two expressions we get $\chi \geq \frac{1}{2}p_1$ by integration. X with opposite orientation is still an Einstein manifold, so we also have $\chi \geq -\frac{1}{2}p_1$. By the Hirzebruch signature formula we obtain the signature $\tau = \frac{1}{3}p_1$ and hence the inequality (1.1) of Theorem 1.

Now let us consider the case where equality occurs, say $-\tau = \frac{2}{3}\chi$. In this case $\lambda = -\mu$ and so $\Sigma\lambda_i = -\Sigma\mu_i = 0$ by the Bianchi identity; hence X has zero scalar curvature. Since X is Einsteinian, the Ricci tensor $(=kg_{ij})$ vanishes.

Suppose X is not flat. Then by Berger's result we have $\chi > 0$. We claim that the fundamental group $\pi_1(X)$ is finite. First we show that $b_1 = 0$. Suppose not, then by Hodge theory there exists a harmonic 1-form on X. Since the Ricci tensor is zero, every harmonic 1-form must be parallel by the vanishing theorem of Bochner and Myers [4]. In particular we have a nonvanishing vector field on X. Since $\chi \neq 0$, this is impossible and so $b_1 = 0$.

Note that this is also true for any finite covering $\bar{X}$ of degree k; for we can pull back the Einstein metric on X to $\bar{X}$, and then $\bar{X}$ again satisfies the hypotheses of our theorem since $\bar{\tau} = k\tau$, $\bar{\chi} = k\chi$. We now apply the Cheeger-Gromoll splitting theorem [7, Theorem 3]: X has nonnegative Ricci curvature, and so either π_1 is finite or there is a finite covering of X with $b_1 \neq 0$. Since $b_1 = 0$ for all finite coverings, we see that π_1 is finite.

We now consider the bundle of 2-forms Λ^2. Λ^2 splits as a direct sum $\Lambda^+ \oplus \Lambda^-$ where $\Lambda^\pm$ are the eigenspaces of the Hodge star operator. In our case we shall show that Λ^+ is a flat bundle relative to the connection induced by the riemannian connection. The decomposition above corresponds to the decomposition of the second exterior power representation λ^2 of $SO(4)$ into irreducible subspaces $\lambda^2 = \lambda^+ \oplus \lambda^-$. The representation λ^+ defines a homomorphism $l^+ \colon SO(4) \to SO(3)$ with kernel $SU(2)$ under the standard inclusion. We can identify the Lie algebra of $SO(4)$ with λ^2, and then the kernel of dl^+ is just λ^-.

Now the curvature tensor R of X is a section of $\Lambda^- \otimes \Lambda^2$, and so to show that Λ^+ is flat we must show that R is a section of $\Lambda^- \otimes \Lambda^2$. From the normal form of the curvature tensor we have $\lambda = -\mu$. In terms of the given local orthonormal basis we then get

$$R = \lambda_1(e_1 \wedge e_2 - e_3 \wedge e_4) \otimes (e_1 \wedge e_2 - e_3 \wedge e_4) + \text{similar terms.}$$

In particular, R is a section of $\Lambda^- \otimes \Lambda^2$, and so the bundle Λ^+ has zero curvature and is therefore flat. Take the universal covering $\bar{X}$ of X. Since $\pi_1(X)$ is finite, $\bar{X}$ is compact and simply connected, and the bundle Λ^+ on $\bar{X}$ has three linearly independent parallel sections. These reduce the holonomy group from $SO(4)$ to the kernel of l^+, that is, $SU(2)$. Hence $\bar{X}$ is a compact Ricci-flat, two-dimensional Kähler manifold. The first Chern class c_1 is represented in the de Rham cohomology by the Ricci form, and so $c_1 = 0$ since $\bar{X}$ is simply connected. We thus have a complex surface with $b_1 = 0$ and $c_1 = 0$, so that it is a $K3$ surface.

Now to get the nonsimply connected manifolds we have to consider the possible free actions of a finite group G of isometries of $\bar{X}$. First, the order of such a group must divide the signature and Euler characteristic of $\bar{X}$. For a $K3$ surface, $\tau = -16$, $\chi = 24$ so the order of G must divide 8.

Suppose the order of $G = 8$. Then for $X = \bar{X}/G$ we have $\tau = -2, \chi = 3$. Since $b_1 = 0$ this means $b_2 = 1$, but then $|\tau| > b_2$ which is impossible, so G must be of order 2 or 4, that is, $G = \boldsymbol{Z}_2$, $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ or $\boldsymbol{Z}_4$.

As mentioned in Remark 2, our $K3$ surface $\bar{X}$ is a quaternionic Kähler manifold, i.e., it has three almost complex structures I, J, K, parallel with respect to the riemannian connection and such that $IJ = -JI$, etc. In fact, by duality we can regard these as the three linearly independent 2-forms which parallelize Λ^+. Note that $aI + bJ + cK$ is also a complex structure where a, b,c are constants and $a^2 + b^2 + c^2 = 1$, so that any parallel Λ^+ form on $\bar{X}$ defines (after normalization) a complex structure.

The dimension b_2^+ of the space of harmonic 2-forms in Λ^+ on a four-manifold with $b_1 = 0$ is given from Hodge theory by

$$b_2^+ = \tfrac{1}{2}(\tau + \chi - 2) .$$

For $\bar{X}$, $b_2^+ = \frac{1}{2}(-16 + 24 - 2) = 3$, so every harmonic Λ^+ form is parallel. For $X = \bar{X}/\boldsymbol{Z}_2, b_2^+ = \frac{1}{2}(-8 + 12 - 2) = 1$, so that the $\boldsymbol{Z}_2$ action on $\bar{X}$ leaves fixed one harmonic (and therefore parallel) Λ^+ form L. $L/\|L\|$ is then a complex structure left fixed by $\boldsymbol{Z}_2$, so X is a complex surface with $b_1 = 0$ and $2c_1 = 0$, i.e., an Enriques surface.

For $X = \bar{X}/G$ where G is of order 4, $b_2^+ = \frac{1}{2}(-4 + 6 - 2) = 0$. We can regard X as the quotient of an Enriques surface by a free $\boldsymbol{Z}_2$ action. Since $b_2^+ = 0$, the involution cannot leave fixed the complex structure on the Enriques surface and must therefore take it into its conjugate. In other words, X is the quotient of an Enriques surface by a free antiholomorphic involution.

It remains to rule out the case $G = \boldsymbol{Z}_4$. Let P be the principal $SO(4)$ bundle of orthonormal frames of $\bar{X}$. Since a $K3$ surface is a spin manifold, P has a double covering $\hat{P}$ which is a principal Spin (4) bundle. Let f be a generator of G. Then f acts on P, and is covered by an action $\hat{f}$ on $\hat{P}$ (see Atiyah and Bott [1]).

Suppose $\hat{f}^4 = 1$. Then G acts on $\hat{P}$, and $X = \bar{X}/G$ is a spin manifold with principal spin bundle $\hat{P}/G$. If $\hat{f}^4 = -1$, then we can define an action of G on the principal Spin^c (4) bundle $\hat{P} \times_{\boldsymbol{Z}_2} S^1$ as follows:

$$f(p, e^{i\theta}) = (\hat{f}p, e^{i(\theta + \pi/4)}) .$$

Thus X is a Spin^c manifold with principal Spin^c bundle $\hat{P} \times_{Z_2} S^1/G$. In either case we can define a Dirac operator on X which has index equal to $\hat{A}(X)$. But $\hat{A}(X) = -\frac{1}{8}\tau(X) = \frac{1}{2}$ which is not an integer. Hence Z_4 cannot act freely on a $K3$ surface. Note that we cannot use the same argument for $G = Z_2 \times Z_2$ since it may act on $\hat{P}$ as the non-abelian group of quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$.

Finally, if $\tau = \frac{2}{3}\chi$, then X with opposite orientation has $-\tau = \frac{2}{3}\chi$ and we can apply the above arguments.

4. Proof of Theorem 2

In the normal form of the curvature tensor, the numbers $\lambda_1, \lambda_2, \lambda_3$ are critical values of the sectional curvature function, so suppose X has nonnegative sectional curvature. Then the vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ lies in the region $\{(x_1, x_2, x_3) \in R^3\colon x_i \geq 0,\ 1 \leq i \leq 3\}$. On the other hand, μ is constrained by the Bianchi identity to lie in the plane $\Sigma x_i = 0$, so that if λ, μ are nonzero, then the angle θ between them must satisfy $\cos\theta \leq \sqrt{2/3}$. Therefore

$$(\lambda, \mu) \leq \sqrt{2/3}\, |\lambda| \cdot |\mu| \,,$$

and this holds even if λ or μ vanishes. From the inequality $|\lambda|^2 + |\mu|^2 \geq 2|\lambda| \cdot |\mu|$, we get

$$|\lambda|^2 + |\mu|^2 \geq 2\sqrt{3/2}(\lambda, \mu) \,.$$

Integrating and using the expressions for Euler characteristic and signature in the proof of Theorem 1, we obtain the inequality (1.2). The case of nonpositive sectional curvature is similar.

Added in proof. Concerning Remark 2, we can in fact find Enriques surfaces with free antiholomorphic involutions; the author is grateful to M. F. Atiyah for the following idea. Let A and B be real 3×3 matrices, x and $y \in C^3$, and consider the algebraic variety X in P^5 given by the equations $\sum_j A_{ij}x_j^2 + B_{ij}y_j^2 = 0$ $1 \leq i \leq 3$. For generic A and B this is a complete intersection of three nonsingular hyperquadrics. By the Lefschets theorem $b_1 = 0$ and by an easy calculation $c_1 = 0$, so X is a $K3$ surface. We define the commuting involutions τ and σ on P^5 by $\tau(x, y) = (x, -y)$, $\sigma(x, y) = (\bar{x}, \bar{y})$ and since A and B are real, both τ and σ act on X.

At a fixed point of τ on X, $\sum A_{ij}x_j^2 = 0$ and $\sum B_{ij}y_j^2 = 0$, so if A and B are invertible, then τ is free and holomorphic. At a fixed point of σ on X, $\sum A_{ij}x_j\bar{x}_j + B_{ij}y_j\bar{y}_j = 0$, so if $A_{1j}, B_{1j} > 0$ for all j, then σ is free. At a fixed point of $\sigma\tau$ on X, $\sum A_{ij}x_j\bar{x}_j - B_{ij}y_j\bar{y}_j = 0$, so if $A_{2j}, -B_{2j} > 0$ for all j, then $\sigma\tau$ is free. Thus choosing A and B appropriately, σ and τ generate the required free $Z_2 \times Z_2$ action on X.

Finally we should point out that the inequality (1.1) has also been found by A. Gray.

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