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RIEMANNIAN MANIFOLDS WITH GEODESIC SYMMETRIES OF ORDER 3

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To S. S. Chern on his 60th birthday

1. Introduction

When is it possible to classify all compact simply connected Riemannian manifolds satisfying a given curvature condition? On the one hand Cartan [3], succeeded in classifying symmetric spaces. In contrast to this it seems hopeless, at least with our present knowledge, to classify compact simply connected Riemannian manifolds with parallel Ricci tensor. In the first place it is not known if such manifolds must be homogeneous (probably not); secondly, Wolf [22] has given an extremely large number of examples of compact homogeneous Einstein manifolds.

In the present paper it is shown that there is a different curvature condition for which a classification can be effected. Specifically, the following result is proved. Let $\mathfrak{X}(M)$ denote the Lie algebra of vector fields on a differentiable manifold M.

Theorem (1.1). *Let M be a C^∞ compact simply connected almost Hermitian manifold with almost complex structure J, Riemannian connection ∇, and curvature tensor R. Assume that*

(*i*) $\nabla_X(J)X = 0$ *for all* $X \in \mathfrak{X}(M)$ (*that is, M is nearly Kählerian, see* § 2),

(*ii*) $\nabla_X(R)_{XJXXJX} = 0$ *for all* $X \in \mathfrak{X}(M)$.

Then M may be decomposed as a Riemannian product $M = M_1 \times \cdots \times M_r$ where $M_1, \cdots, M_r$ are listed in Tables V, VI, *and* VII, *in* § 6. *Furthermore if $M = M_1 \times \cdots \times M_r$ is such a product, then $M, M_1, \cdots, M_r$ satisfy* (*i*) *and* (*ii*), *and are Einstein manifolds.*

The basic idea behind this theorem involves the notion of *Riemannian* 3-*symmetric space*. Roughly, such a manifold is described as follows. For each point $p \in M$ there is an isometry $\theta_p : M \to M$ with p as an isolated fixed point such that $\theta_p^3 = 1$. Furthermore in § 4 we define *Riemannian locally* 3-*symmetric space*. Any such manifold has a naturally defined almost complex structure on it. See § 4 for precise definitions.

Theorem (1.1) is proved by giving formulas which characterize the curvature

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tensor of a Riemannian locally 3-symmetric space. This is done in § 4. Our result is similar to that of [1] and [21]. In these papers an identity which characterizes the curvature operator of a general homogeneous space is given. See also [9], [15], [19], [20].

Any Hermitian symmetric space satisfies the hypotheses of Theorem (1.1). Therefore Riemannian 3-symmetric spaces can be regarded as natural generalizations of Hermitian symmetric spaces.

Actually, we prove Theorem (1.1) in a great deal more generality. Instead of supposing that M is compact with positive definite metric tensor, we only assume that M is pseudo-Riemannian and the group $\mathscr{I}(M)$ of holomorphic isometries of M is a reductive Lie group. These manifolds are classified and Theorem (1.1) is obtained as a special case. Furthermore, an even more general classification results when the condition $\nabla_X(J)X = 0$ is weakened.

This paper is primarily concerned with the geometry, as opposed to the Lie group theory, of Riemannian 3-symmetric spaces. The latter is given in [23]. The classification theorems of [23] are used in § 6.

The theory of 3-symmetric spaces parallels that of ordinary symmetric spaces to a great extent. However, there are important exceptions. As already noted, 3-symmetric spaces are automatically almost complex manifolds. Furthermore, the notion of "dual symmetric space" is muddled for 3-symmetric spaces. The "dual" of a 3-symmetric space exists, but it may not be unique, and its metric frequently has a different signature.

It is clear, though, that for any particular feature of the theory of ordinary symmetric spaces, onc can try to find the corresponding theorems for 3-symmetric spaces. Such questions are interesting, but the author has decided not to treat them comprehensively here in order to keep the paper of reasonable length.

The methods of [23] and the present paper can be combined to work out a characterization of the curvature operator of Riemannian manifolds with geodesic symmetries of any odd order and the classification of such spaces. However, the computations quickly become formidable.

This brings up another point. Wolf [22] has classified all compact simply connected homogeneous spaces whose isotropy representation is irreducible. These include the irreducible symmetric spaces and several 3-symmetric spaces. It should be possible to characterize the other spaces in Wolf's list by the identities satisfied by their curvature operators. Of particular interest are the manifolds $0(n)/\mathrm{Ad}(G)$ where G is a compact simple Lie group of dimension n and the homogeneous almost complex manifolds with irreducible isotropy representation. Another class of manifolds for which there should be an interesting characterization in terms of curvature operators is the spaces G/K where G is compact, K has maximal rank, G/K has a G-invariant almost complex structure but K is not the centralizer of a torus. These manifolds have been classified in [23].

In a different direction, Ambrose and Singer [1] and Singer [21] have suggested classifying homogeneous spaces by certain curvature identities given in [1] and [21] related to their characterization of the curvature operator of a homogeneous space. Thus there are two possible ways to proceed. On the one hand, one can take a class of homogeneous spaces, determine identities satisfied by their curvature operators, and classify all manifolds satisfying these identities. Alternatively, one can try to classify manifolds satisfying the curvature identities of [1] and [21] together with additional identities.

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2. Nearly Kähler manifolds

Let M be a C^∞ pseudo-Riemannian manifold with metric tensor $\langle\ ,\ \rangle$. Denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields of M, and by ∇_X and $R_{WXYZ}(W, X, Y, Z \in \mathfrak{X}(M))$ the Riemannian connection and curvature tensor of M, respectively. We say that M is *almost Hermitian* provided that M has an almost complex structure J such that $\langle JX, JY\rangle = \langle X, Y\rangle$ for all $X,\ Y \in \mathfrak{X}(M)$. The *Kähler form* of M is the 2-form F defined by $F(X, Y) = \langle JX, Y\rangle$ for all $X,\ Y \in \mathfrak{X}(M)$.

An almost Hermitian manifold is said to be *nearly Kählerian* if $\nabla_X(J)X = 0$ for all $X \in \mathfrak{X}(M)$. This is strictly weaker than the condition $\nabla_X(J)Y = 0$ for all $X,\ Y \in \mathfrak{X}(M)$, which is the defining property for Kähler manifolds. The almost complex structure of a non-Kähler nearly Kähler manifold is never integrable. In [6], [7], [8] it is shown that many theorems about the topology and geometry of Kähler manifolds have generaizations to nearly Kähler manifolds.

The key fact which allows one to prove interesting theorems about nearly Kähler manifolds is that the curvature operator of a nearly Kähler manifold satisfies certain identities. These identities are only slightly more complicated than the corresponding ones for Kähler manifolds.

The following result was proved in [8] for the case of positive definite metric.

Proposition (2.1). *The curvature tensor R of a nearly Kähler manifold satisfies the identity*

$$R_{WXYZ} - R_{WXJYJZ} = \langle \nabla_W(J)X, \nabla_Y(J)Z\rangle\ , \quad \textit{for all } W, X, Y, Z \in \mathfrak{X}(M)\ . \tag{2.1}$$

Corollary (2.2). *For a nearly Kähler manifold we have*

$$R_{WXYZ} = R_{JWJXJYJZ} = R_{JWJXYZ} + R_{JWXJYZ} + R_{JWXYJZ}\ , \quad \textit{for all } W, X, Y, Z \in \mathfrak{X}(M)\ .$$

It is natural to ask whether there exist similar formulas for the covariant derivative of a nearly Kähler manifold. In fact such formulas exist, but first we prove a preliminary result which will be needed both in this section and in § 3. Throughout the rest of this section M denotes a nearly Kähler manifold. Let $\mathfrak{S}$ denote the cyclic sum.

Proposition (2.3). *For $W, X, Y, Z \in \mathfrak{X}(M)$ we have*

$$2\nabla_{WX}^2(F)(Y,Z) = \underset{XYZ}{\mathfrak{S}} \langle \nabla_W(J)X, J\nabla_Y(J)Z\rangle = \underset{XYZ}{\mathfrak{S}} R_{WJXYZ} \,. \tag{2.2}$$

Proof. We have the general formula

$$\begin{aligned} &\nabla_{WX}^2(F)(Y,Z) - \nabla_{XW}^2(F)(Y,Z) \\ &\qquad = -(R_{WX}\cdot F)_{YZ} = -R_{WXJYZ} - R_{WXYJZ} \,. \end{aligned} \tag{2.3}$$

Since M is nearly Kählerian, (2.3) implies that

$$\begin{aligned} \nabla_{WW}^2(F)(Y,Z) &= -\nabla_{WY}^2(F)(W,Z) + \nabla_{YW}^2(F)(W,Z) \\ &= -\langle \nabla_W(J)Y, J\nabla_W(J)Z\rangle \,. \end{aligned} \tag{2.4}$$

Linearization of (2.4) yields

$$\begin{aligned} &\nabla_{WX}^2(F)(Y,Z) + \nabla_{XW}^2(F)(Y,Z) \\ &\qquad = -\langle \nabla_W(J)Y, J\nabla_X(J)Z\rangle + \langle \nabla_W(J)Z, J\nabla_X(J)Y\rangle \,. \end{aligned} \tag{2.5}$$

From (2.3) and (2.5) we obtain the first part of (2.2). The second part follows from the first part, Proposition (2.1) and the first Bianchi identity.

Proposition (2.3) makes it possible to derive an identity for nearly Kähler manifolds relating the curvature tensor to its covariant derivative.

Proposition (2.4). *Let $U, W, X, Y, Z \in \mathfrak{X}(M)$. Then*

$$\begin{aligned} &\nabla_U(R)_{WXYZ} - \nabla_U(R)_{WXJYJZ} \\ &\quad = R_{WX\nabla_U(J)YJZ} + R_{WXJY\nabla_U(J)Z} + \tfrac{1}{2}\{R_{UJWX\nabla_Y(J)Z} + R_{JXUW\nabla_Y(J)Z} \\ &\qquad + R_{WXUJ\nabla_Y(J)Z} + R_{UJYZ\nabla_W(J)X} + R_{JZUY\nabla_W(J)X} + R_{YZUJ\nabla_W(J)X}\} \,. \end{aligned} \tag{2.6}$$

Proof. We take the covariant derivative of (2.1) and obtain

$$\begin{aligned} &\nabla_U(R)_{WXYZ} - \nabla_U(R)_{WXJYJZ} - R_{WX\nabla_U(J)YJZ} - R_{WXJY\nabla_U(J)Z} \\ &\qquad = \nabla_{UW}^2(F)(X,\nabla_Y(J)Z) + \nabla_{UY}^2(F)(Z,\nabla_W(J)X) \,. \end{aligned} \tag{2.7}$$

Then (2.6) follows from (2.7) and Proposition (2.3).

Corollary (2.5). *We have, for $W, X \in \mathfrak{X}(M)$,*

$$\nabla_W(R)_{WXWX} - \nabla_W(R)_{WXJWJX} = R_{WXJW\nabla_W(J)X} \,.$$

Finally we obtain an identity involving ∇R alone.

Proposition (2.6). *For $U, W, X, Y, Z \in \mathfrak{X}(M)$ we have*

$$(2.8) \quad \nabla_U(R)_{WXYZ} + \nabla_U(R)_{JWJXJYJZ} = \nabla_U(R)_{JWJXYZ} + \nabla_U(R)_{WXJYJZ} \, .$$

Proof. We substitute JW, JX, JY, JZ for W, X, Y, Z in (2.6) and add the result to (2.6). Using (2.1) and the first Bianchi identity we obtain (2.8).

3. Local diffeomorphisms associated with almost complex structures

We begin with the following observation.

Proposition (3.1). *Let M be a C^∞ almost complex structure J. Then for each $p \in M$ there exist a neighborhood $U(p)$ and a diffeomorphism $\theta_p: U(p) \to U(p)$ such that*

(*i*) $\theta_p{}^3 = 1$,

(*ii*) *p is the unique fixed point of θ_p.*

Proof. Put $\Theta = -\frac{1}{2}I + (\sqrt{3}/2)J$ where I is the identity. Then $\Theta^3 = I$, and for each $p \in M$ there exist a neighborhood $U(p)$ and a diffeomorphism $\theta_p: U(p) \to U(p)$ such that $\theta_{p*} = \Theta_p$. Furthermore we may choose θ_p so that (on a possibly smaller neighborhood) we have $\theta_p{}^3 = 1$, and p is the only fixed point.

This proposition suggests the following.

Definition. A *family of local cubic diffeomorphisms* on a C^∞ manifold M is a differentiable function $p \to \theta_p$ which assigns to each $p \in M$ a diffeomorphism θ_p on a neighborhood $U(p)$ of p which satisfies (*i*) and (*ii*) of Proposition (3.1).

Next we prove the converse of Proposition (3.1).

Proposition (3.2). *Let M be a C^∞ manifold, and assume $p \to \theta_p$ is a family of local cubic diffeomorphisms on M. Then there is a C^∞ almost complex structure J on M.*

Proof. Let $\theta_{p*}: M_p \to M_p$ denote the induced tangent map of θ_p at p, and write $\theta_{p*} = -\frac{1}{2}I_p + (\sqrt{3}/2)J_p$ where $I_p: M_p \to M_p$ is the identity. Since $\theta_{p*}{}^3 = I_p$ and θ_{p*} has no real eigenvalues, we have $J_p{}^2 = -I_p$. Furthermore $p \to J_p$ is differentiable because $p \to \theta_p$ is. Hence the proposition follows.

Definition. Let $p \to \theta_p$ be a family of local cubic diffeomorphisms on a manifold M. Then the *canonical almost complex structure* J of the family is the one defined by $\theta_{p*} = -\frac{1}{2}I_p + (\sqrt{3}/2)J_p$ for all $p \in M$.

In the sequel the relationship between J and $\Theta = -\frac{1}{2}I + (\sqrt{3}/2)J$ will be important. We shall need a linear algebra result relating the two. For this, let V be an even dimensional vector space over $\boldsymbol{R}$ with almost complex structure J. Let ϕ be a tensor on V, covariant of degree r and contravariant of degree s. We shall denote the value of ϕ on $x_1, \cdots, x_r \in V$, $\omega_1, \cdots, \omega_s \in V^*$ by $\phi(x_1, \cdots, x_r, \omega_1, \cdots, \omega_s)$ or $\phi_{x_1 \cdots x_r \omega_1 \cdots \omega_s}$.

If $L: V \to V$ is a linear isomorphism, then $L(\phi)$ is defined by $L(\phi)(x_1, \cdots, x_r, \omega_1, \cdots, \omega_s) = \phi(L^{-1}x_1, \cdots, L^{-1}x_r, L^{-1}\omega_1, \cdots, L^{-1}\omega_s)$. In the particular case where the contravariant degree of ϕ is 1 we may regard ϕ as a multilinear map such that $\phi(x_1, \cdots, x_q) \colon V \to V$ is a linear map for all $x_1, \cdots, x_q$. Then we have $L(\phi)(x_1, \cdots, x_q) = L \cdot \phi(L^{-1}x_1, \cdots, L^{-1}x_q)$.

Definition. We say that L *preserves* ϕ if $L(\phi) = \phi$.

Proposition (3.3). *Let α, β, γ, η, ζ be covariant tensors on V of degrees* $1, \cdots, 5$, *respectively. Then the following are necessary and sufficient conditions that Θ preserve them:*

(*i*) $\alpha = 0$,

(*ii*) $\beta_{JxJy} = \beta_{xy}$ *for all* $x, y \in V$,

(*iii*) $\gamma_{xyz} = -\gamma_{JxJyz} = -\gamma_{JxyJz} = -\gamma_{xJyJz}$, *for all* $x, y, z \in V$,

(*iv*) $\eta_{wxyz} = \eta_{JwJxyz} + \eta_{JwxJyz} + \eta_{JwxyJz}$, *etc.*, *for all* $w, x, y, z \in V$,

(*v*) $\zeta_{vwxyz} = -\frac{1}{2}\{\zeta_{JvJwxyz} + \cdots + \zeta_{vwxJyJz}\}$, *for all* $v, w, x, y, z \in V$.

Proof. (*i*)–(*v*) all have similar proofs, so for example we prove (*iii*). Suppose $\Theta(\gamma) = \gamma$. Then for $x, y, z \in V$ we have

$$\begin{aligned}\gamma_{xyz} = \gamma_{\theta x \theta y \theta z} = \gamma_{\theta^2 x \theta^2 y \theta^2 z} &= -\frac{1}{8}\gamma_{xyz} \pm \frac{\sqrt{3}}{8}\{\gamma_{Jxyz} + \gamma_{xJyz} + \gamma_{xyJz}\} \\ &\quad -\frac{3}{8}\{\gamma_{JxJyz} + \gamma_{JxyJz} + \gamma_{xJyJz}\} \pm \frac{3\sqrt{3}}{8}\gamma_{JxJyJz} \,. \end{aligned} \tag{3.1}$$

From (3.1) it follows that

$$3\gamma_{xyz} + \gamma_{JxJyz} + \gamma_{JxyJz} + \gamma_{xJyJz} = 0 \,. \tag{3.2}$$

Replacing x and y by Jx and Jy in (3.2) we obtain

$$\gamma_{xyz} + 3\gamma_{JxJyz} - \gamma_{JxyJz} - \gamma_{xJyJz} = 0 \,. \tag{3.3}$$

From (3.2) and (3.3) we have that $\gamma_{xyz} = -\gamma_{JxJyz}$; similarly, $\gamma_{xyz} = -\gamma_{JxyJz} = -\gamma_{xJyJz}$.

Finally reversing these steps we see that (*iii*) implies $\Theta(\gamma) = \gamma$.

Corollary (3.4). *Let η and ζ be covariant tensors of degrees* 4 *and* 5 *respectively.*

(*i*) *If $\Theta(\eta) = \eta$, then $\eta_{wxyz} = \eta_{JwJxJyJz}$.*

(*ii*) *$\Theta(\zeta) = \zeta$ if and only if*

$$\zeta_{vwxyz} = -\frac{1}{3}\{\zeta_{JvJwJxJyz} + \cdots + \zeta_{vJwJxJyJz}\} \,.$$

Corollary (3.5). *Let M be an almost complex manifold with almost complex structure J, and denote by S the torsion tensor of J. Let $\langle\, , \rangle$ be a pseudo-Riemannian metric on M. Then*

(*i*) *Θ preserves J and S,*

(*ii*) *Θ preserves $\langle\, , \rangle$ if and only if M is almost Hermitian,* (*iii*) *Θ preserves ∇J if and only if M is quasi-Kählerian.*

Proof. (*i*) and (*ii*) are obvious. For (*iii*) we recall that M is *quasi-Kählerian* if and only if $\nabla_X(J)(Y) + \nabla_{JX}(J)(JY) = 0$ for all $X, Y \in \mathfrak{X}(M)$, [6]. Then it is easy to check (*iii*).

Let θ_p be a local cubic diffeomorphism. Although θ_p preserves the canonical almost complex structure J at p it is in general false that θ_p is a holomorphic map in a neighborhood of p. If θ_p is an isometry, the next proposition gives a necessary and sufficient condition for θ_p to be a holomorphic isometry.

Proposition (3.6). *Let M be a C^∞ almost Hermitian manifold, and assume there is a family of local cubic diffeomorphisms $p \to \theta_p$, each of which is an isometry. Suppose the canonical almost complex structure J of the family is the same as that of M. Then the following two conditions are equivalent:*

(*i*) *Each θ_p is a holomorphic isometry,*

(*ii*) *Θ preserves ∇J and $\nabla^2 J$.*

Proof. That (*i*) implies (*ii*) is obvious, because at p we have $\Theta_p = \theta_{p*}$. Conversely, suppose (*ii*) holds. Then for all $p \in M$ we have $\theta_{p*}((\nabla^k\Theta)_p) = (\nabla^k\Theta)_p$ for $k = 0,1,2$. According to [5, Theorem 4.11] this implies that θ_p is a holomorphic isometry.

Proposition (3.7). *Let M be a nearly Kähler manifold with almost complex structure J. Then $\Theta = -\frac{1}{2}I + (\sqrt{3}/2)J$ preserves $\nabla^k J$ for $k = 0, 1, 2, \cdots$.*

Proof. This is obvious for $k = 0$. Furthermore a nearly Kähler manifold is quasi-Kählerian [6], and so Θ preserves ∇J by Corollary (3.5) (*iii*). That Θ preserves $\nabla^k J$ for $k \geq 2$ follows from Proposition (2.3) and induction.

Corollary (2.2) and Proposition (3.3) suggest the study of almost Hermitian manifolds satisfying either or both of the conditions $\Theta(R) = R$, $\Theta(\nabla R) = \nabla R$. For future reference we collect here some results about these manifolds.

Proposition (3.8). *Let M be an almost Hermitian manifold with curvature operator R.*

(*i*) *$\Theta(R) = R$ if and only if for all $W, X, Y, Z \in \mathfrak{X}(M)$ we have $R_{WXYZ} = R_{JWJXYZ} + R_{JWXJYZ} + R_{JWXYJZ}$.*

(*ii*) *If $\Theta(R) = R$, then $R_{WXYZ} = R_{JWJXJYJZ}$ for all $W, X, Y, Z \in \mathfrak{X}(M)$.*

(*iii*) *If M is nearly Kählerian, then $\Theta(R) = R$.*

Proof. These are immediate from Corollary (2.2), Proposition (3.3) and Corollary (3.4).

We remark that the condition $\Theta(R) = R$ by itself does not imply that M is either quasi-Kählerian or nearly Kählerian. In fact any contractible even dimensional Riemannian manifold with constant nonzero curvature has an integrable almost complex structure J such that $\Theta(R) = R$. Such a manifold is not quasi-Kählerian if its dimension is greater than or equal to 4.

Next we obtain some conditions on ∇R.

Proposition (3.9). *Let M be an almost Hermitian manifold.*

(*i*) *If $R_{WXYZ} = R_{JWJXJYJZ}$ for all $W, X, Y, Z \in \mathfrak{X}(M)$, then for all V, $W, X, Y, Z \in \mathfrak{X}(M)$*

$$\nabla_V(R)_{WXYZ} - \nabla_V(R)_{JWJXJYJZ} = R_{\nabla_V(J)WJXJYJZ} + \cdots + R_{JWJXJY\nabla_V(J)Z} .$$

(*ii*) *If* $\Theta(R) = R$, *then for all* $V, W, X, Y, Z \in \mathfrak{X}(M)$

$$\begin{aligned}\nabla_V(R)_{WXYZ} - \nabla_V(R)_{JWJXYZ} - \nabla_V(R)_{JWXJYZ} - \nabla_V(R)_{JWXYJZ} \\ = R_{\nabla_V(J)WJXJYJZ} + R_{JW\nabla_V(J)XYZ} + R_{JWX\nabla_V(J)YZ} + R_{JWXY\nabla_V(J)Z} .\end{aligned}$$

(*iii*) *If* $\Theta(R) = R$, *then for all* $V, W, X, Y, Z \in \mathfrak{X}(M)$

$$\nabla_V(R)_{WXYZ} + \nabla_V(R)_{JWJXJYJZ} = \nabla_V(R)_{JWJXYZ} + \cdots + \nabla_V(R)_{WXJYJZ} .$$

(*iv*) *If* $\Theta(R) = R$ *and* M *is quasi-Kählerian, then for all* $V, W, X, Y \in \mathfrak{X}(M)$

$$\nabla_V(R)_{WXYZ} - \nabla_V(R)_{JWJXJYJZ} = -\nabla_{JV}(R)_{JWXYZ} - \cdots - \nabla_{JV}(R)_{WXYJZ} .$$

Proof. (*i*) and (*ii*) are proved by taking the covariant derivatives of appropriate identities involving R, and (*iii*) follows from (*ii*). For (*iv*) we have from (*i*) and (*ii*) that for all $V, W, X, Y, Z \in \mathfrak{X}(M)$

$$\begin{aligned}&\nabla_{JV}(R)_{JWXYZ} + \cdots + \nabla_{JV}(R)_{WXYJZ} \\ &\quad = R_{\nabla_{JV}(J)JWJXJYJZ} - R_{W\nabla_{JV}(J)XYZ} - R_{WX\nabla_{JV}(J)YZ} - R_{WXY\nabla_{JV}(J)Z} \\ &\quad = -R_{\nabla_V(J)WJXJYJZ} - R_{JW\nabla_V(J)XJYJZ} - R_{JWJX\nabla_V(J)YJZ} - R_{JWJXJY\nabla_V(J)Z} \\ &\quad = -\nabla_V(R)_{WXYZ} + \nabla_V(R)_{JWJXJYJZ} .\end{aligned}$$

Next we give a characterization of ∇R which will be important in § 4.

Proposition (3.10). *Let* M *be an almost Hermitian manifold with* $\Theta(R)=R$. *Then the following conditions are equivalent*:

(*i*) $\Theta(\nabla R) = \nabla R$,

(*ii*) $\nabla_V(R)_{WXYZ} + \nabla_V(R)_{JWJXJYJZ} = 0$ *for all* $V, W, X, Y, Z \in \mathfrak{X}(M)$,

(*iii*) $-2\nabla_V(R)_{WXYZ} = \nabla_{JV}(R)_{JWXYZ} + \cdots + \nabla_{JV}(R)_{WXYJZ}$ *for all* $V, W, X, Y, Z \in \mathfrak{X}(M)$.

Proof. Suppose (*i*) holds. From Proposition (3.3) (*v*) we have

$$\begin{aligned}\nabla_V(R)_{WXYZ} = -\tfrac{1}{2}\{&\nabla_{JV}(R)_{JWXYZ} + \cdots + \nabla_{JV}(R)_{WXYJZ} \\ &+ \nabla_V(R)_{JWJXYZ} + \cdots + \nabla_V(R)_{WXJYJZ}\} .\end{aligned} \tag{3.4}$$

We substitute JW, JX, JY, JZ for W, X, Y, Z in (3.4), add the result to (3.4), and obtain

$$\begin{aligned}&\nabla_V(R)_{WXYZ} + \nabla_V(R)_{JWJXJYJZ} + \nabla_V(R)_{JWJXYZ} + \cdots + \nabla_V(R)_{WXJYJZ} \\ &\quad = -\tfrac{1}{2}\{\nabla_{JV}(R)_{JWXYZ} - \nabla_{JV}(R)_{WJXJYJZ} + \cdots + \nabla_{JV}(R)_{WXYJZ} \\ &\qquad - \nabla_{JV}(R)_{JWJXJYZ}\} .\end{aligned} \tag{3.5}$$

In (3.5) we replace W and X by JW and JX and add the resulting equation to (3.5). We obtain

$$\begin{aligned}\nabla_V(R)_{WXYZ} + \nabla_V(R)_{JWJXJYJZ} + \nabla_V(R)_{JWJXYZ} \\ + \nabla_V(R)_{WXJYJZ} = 0 \,. \end{aligned} \tag{3.6}$$

Then (3.6) and Proposition (3.9) (*iii*) imply (*ii*).

Next suppose (*ii*) holds. We apply (*ii*) and the second Bianchi identity to the right hand side of (*iii*). After some calculation we obtain (*iii*).

Finally suppose (*iii*) holds. Applying (*iii*) twice we have

$$\begin{aligned}&\nabla_V(R)_{WXYZ} + \nabla_V(R)_{JWJXJYJZ} \\ &\quad = -\tfrac{1}{2}\{\nabla_{JV}(R)_{JWXYZ} - \nabla_{JV}(R)_{WJXJYJZ} + \cdots \\ &\qquad\quad + \nabla_{JV}(R)_{WXYJZ} - \nabla_{JV}(R)_{JWJXJYZ}\} \\ &\quad = \nabla_V(R)_{WXYZ} + \nabla_V(R)_{JWJXJYJZ} - \nabla_V(R)_{JWJXYZ} \\ &\qquad - \cdots - \nabla_V(R)_{WXJYJZ} \,. \end{aligned} \tag{3.7}$$

Thus $\nabla_V(R)_{JWJXYZ} + \cdots + \nabla_V(R)_{WXJYJZ} = 0$, and so (*iii*) implies (*i*).

Corollary (3.11). *If M is an almost Herminian manifold with $\Theta(R) = R$ and $\Theta(\nabla R) = \nabla R$, then the Ricci curvature of M is parallel.*

Proof. This follows from Proposition (3.10) (*ii*).

Finally we note the following result.

Proposition (3.12). *Let M be a nearly Kähler manifold. Then the following conditions are equivalent:*

(*i*) $\Theta(\nabla R) = \nabla R$,

(*ii*) $\nabla_Y(R)_{YJYYJY} = 0$ *for all* $Y \in \mathfrak{X}(M)$.

Proof. It is clear that for any almost Hermitian manifold, (*i*) implies (*ii*). Conversely, assume that M is nearly Kählerian and (*ii*) holds. M is quasi-Kählerian, so from Proposition (3.9) (*iv*) and the second Bianchi identity we have

$$\nabla_X(R)_{YJYYJY} = \nabla_Y(R)_{YJXYJY} + \nabla_{JY}(R)_{JYJXYJY} \,. \tag{3.8}$$

On the other hand, linearization of (*ii*) yields

$$\nabla_X(R)_{YJYYJY} + 2\nabla_Y(R)_{XJYYJY} + 2\nabla_Y(R)_{YJXYJY} = 0 \,. \tag{3.9}$$

We replace Y by JY in (3.9), add the result to (3.9) and use the second Bianchi identity. We obtain

$$4\nabla_X(R)_{YJYYJY} + 2\nabla_Y(R)_{YJXYJY} + 2\nabla_{JY}(R)_{JYJXYJY} = 0 \,. \tag{3.10}$$

Then (3.8) and (3.10) imply

$$(3.11) \qquad \nabla_X(R)_{YJYYJY} = 0$$

for all $X, Y \in \mathfrak{X}(M)$. We linearize (3.11) twice and make use of Proposition (2.6). The result is that (3.11) implies condition (*ii*) of Proposition (3.10). Hence $\Theta(\nabla R) = \nabla R$.

4. Pseudo-Riemannian locally 3-symmetric spaces

We begin by defining the main objects of study of this paper.

Definition. A *pseudo-Riemannian locally* 3-*symmetric space* M is a C^∞ pseudo-Riemannian manifold M together with a family of local cubic diffeomorphisms $p \to \theta_p$ such that each θ_p is a holomorphic isometry in a neighborhood of p with respect to the canonical almost complex structure of the family.

From now on the only almost complex structure we shall consider on a pseudo-Riemannian locally 3-symmetric space is the canonical one. We shall call θ_p a *local* (and later a *global*) *cubic holomorphic isometry*.

The notion of pseudo-Riemannian locally 3-symmetric space is a special case of a more general concept due to Graham and Ledger [5]:

Definition. An *affine locally s-regular manifold* is a C^∞ affine manifold M together with a family of maps $\{s_p\}$ with the following properties:

(*i*) For each $p \in M$ there exist neighborhoods $U(p)$ and $V(p)$ such that $s_p\colon U(p) \to V(p)$ is an affine transformation with p as an isolated fixed point.

(*ii*) Let S be the tensor field of type (1,1) defined by $S_p = (s_{p*})_p$ for all $p \in M$. We then require that

$$(S_{p*})_q \circ S_q = S_{s_p(q)} \circ (S_{p*})_q \,.$$

An important result of Graham and Ledger [5, Theorem 4.12] is that an affine locally s-regular manifold M together with its affine connection and tensor field S are necessarily analytic. Furthermore, if M has a compatible pseudo-Riemannian metric, then it is also analytic. The idea of the proof is to introduce a new connection $\bar{\nabla}$ with parallel torsion and curvature such that all the old structure of M is parallel with respect to the new connection. A theorem of Kobayashi and Nomizu [12, Vol. 1, Theorem 7.4, p. 261] then implies everything is analytic.

Specialization of these results to the case at hand yields

Proposition (4.1). *A pseudo-Riemannian locally* 3-*symmetric space is an analytic manifold. Each local cubic isometry θ_p is analytic and the map $p \to \theta_p$ is analytic.*

For a nearly Kähler manifold to be a pseudo-Riemannian locally 3-symmetric space it is possible to weaken the conditions on the loca lcubic diffeomorphism.

Proposition (4.2). *Let $p \to \theta_p$ be a family of local cubic isometries on a pseudo-Riemannian manifold such that the canonical almost complex structure* *is nearly Kählerian. Then M is a pseudo-Riemannian locally* 3*-symmetric space.*

Proof. This is immediate from Propositions (4.1), (3.6) and (3.7).

Definition. A *pseudo-Riemannian* 3*-symmetric space* is a connected pseudo-Riemannian locally 3-symmetric space in which the domain of definition of each local cubic isometry is all of M, i.e., the cubic holomorphic isometries are global.

We drop the prefix "pseudo" in this and the preceding definitions if the metric is positive definite.

An ordinary pseudo-Riemannian locally symmetric space is characterized by the fact that its curvature tensor is parallel. We now obtain an analogous characterization of pseudo-Riemannian locally 3-symmetric spaces. Such spaces are almost complex manifolds by Proposition (3.2), and so it is natural to expect the canonical almost complex structure J to play a role in the characterization.

First we obtain some necessary conditions.

Proposition (4.3). *Let M be a pseudo-Riemannian locally* 3*-symmetric space. Then*

(*i*) $R_{WXYZ} = R_{JWJXYZ} + R_{JWXJYZ} + R_{JWXYJZ}$ *for* $W, X, Y, Z \in \mathfrak{X}(M)$,

(*ii*) $$\nabla_V(R)_{WXYZ} = (\sqrt{3}/4)\{R_{\nabla_V(J)WXYZ} + \cdots + R_{WXY\nabla_V(J)Z}\} - (3/4)\{R_{J\nabla_V(J)WXYZ} + \cdots + R_{WXYJ\nabla_V(J)Z}\}$$

for $V, W, X, Y, Z \in \mathfrak{X}(M)$,

(*iii*) $\nabla^k_{V_1\cdots V_k}(R)_{WXYZ} = (-1)^k \nabla^k_{V_1\cdots V_k}(R)_{JWJXJYJZ}$ *for* $V_1, \cdots, V_k, W, X, Y, Z \in \mathfrak{X}(M)$,

(*iv*) *the Ricci curvature of M is parallel.*

Proof. We have $\Theta(R) = R$ and $\Theta(\nabla R) = \nabla R$. Hence (*i*) and (*iv*) follow immediately from Proposition (3.8) and Corollary (3.11). Furthermore, (*ii*) is a consequence of [5, Lemma 4.4]. Finally (*iii*) follows either from (*ii*) or from Proposition (3.10) and the fact that $R_{WXYZ} = R_{JWJXJYJZ}$ for $W, X, Y, Z \in \mathfrak{X}(M)$.

Corollary (4.4). *Let M be a pseudo-Riemannian locally* 3*-symmetric space. Then the following conditions are equivalent*:

(*i*) *M is Kählerian,*

(*ii*) *M is locally Hermitian symmetric.*

Proof. If M is Kählerian, then

$$\nabla_V(R)_{WXYZ} = \nabla_V(R)_{JWJXJYJZ}$$

for all $V, W, X, Y, Z \in \mathfrak{X}(M)$. This and Proposition (4.3) (*ii*) imply that M is locally Hermitian symmetric. The converse is obvious.

We now prove the converse of Proposition (4.3).

Theorem (4.5). *Let M be a C^∞ pseudo-Riemannian manifold satisfying the following conditions*:

(*i*) *M is almost Hermitian with almost complex structure J,*

(*ii*) *Θ preserves ∇J and $\nabla^2 J$, where* $\Theta = -\frac{1}{2}I + (\sqrt{3}/2)J$, (*iii*) $R_{WXYZ} = R_{JWJXYZ} + R_{JWXJYZ} + R_{JWXYJZ}$ *for all* $W, X, Y, Z \in \mathfrak{X}(M)$,

(*iv*) $\nabla_V(R)_{WXYZ} + \nabla_V(R)_{JWJXJYJZ} = 0$ *for all* $V, W, X, Y, Z \in \mathfrak{X}(M)$.

Then M is a pseudo-Riemannian locally 3-*symmetric space, and J is the canonical almost complex structure determined by the local cubic isometries of M.*

Proof. By Propositions (3.8) and (3.10) we have $\Theta(R) = R$ and $\Theta(\nabla R) = \nabla R$. Furthermore it follows from Proposition (3.9) (*i*) that

$$(4.2) \qquad 2\nabla_V(R)_{WXYZ} = R_{\nabla_V(J)WJXJYJZ} + \cdots + R_{JWJXJY\nabla_V(J)Z} \,.$$

From (4.2) and induction it follows that for any $n \geq 1$, $\nabla^n R$ can be expressed in terms of $\nabla^k R$ and $\nabla^l J$ where k and $l - 1$ are strictly less than n. Using hypothesis (*ii*) and induction we conclude that

$$\Theta(\nabla^n R) = \nabla^n R \qquad \text{for } n = 0, 1, 2, \cdots .$$

Now let $p \in M$. According to [12, Vol. 1, Theorem 7.2, p. 259] there exist a neighborhood $U(p)$ and an affine map $\theta_p \colon U(p) \to U(p)$ such that the tangent map θ_p on M_p coincides with Θ on M_p. Since Θ is a linear isometry on M_p, θ_p is an isometry on $U(p)$. Furthermore from (4.1) it follows that $\theta_p{}^3 = 1$. That each θ_p is holomorphic follows from hypothesis (*ii*) and Proposition (3.6).

Theorem (4.5) can be significantly simplified for nearly Kähler manifolds.

Theorem (4.6). *Let M be an analytic pseudo-Riemannian manifold satisfying the following conditions*:

(*i*) *M is nearly Kählerian with almost complex structure J*,

(*ii*) $\nabla_X(R)_{XJXXJX} = 0$ *for all* $X \in \mathfrak{X}(M)$.

Then M is a pseudo-Riemannian locally 3-*symmetric space, and J is the canonical almost complex structure determined by the local cubic isometries of M.*

Proof. By means of Proposition (3.7), Corollary (2.2), and Proposition (3.12), the hypotheses of Theorem (4.6) imply those of Theorem (4.5). Hence the result follows.

We next prove some results for pseudo-Riemannian 3-symmetric spaces for which the analogous results are known to be true for ordinary pseudo-Riemannian symmetric spaces. We shall need these in the next sections.

Theorem (4.7). *A complete connected simply connected pseudo-Riemannian locally* 3-*symmetric space is a pseudo-Riemannian* 3-*symmetric space.*

Proof. By [12, Vol. 1, Corollary 6.2, p. 255] each local cubic isometry θ_p can be extended to a global affine diffeomorphism $\tilde{\theta}_p$ of M. Since $\theta_p{}^3 = 1$ and $\tilde{\theta}_p$ map geodesics into geodesics, we have $\tilde{\theta}_p{}^3 = 1$. Moreover $\tilde{\theta}_p$ is an isometry at p and affine everywhere, so it is a global isometry. Finally, if J denotes the canonical almost complex structure of M, then $\tilde{\theta}_p \cdot J$ is also an almost complex structure on M. They are both analytic and coincide on an open neighborhood of p, and so they coincide everywhere. This completes the proof.

Theorem (4.8). *Let M be a pseudo-Riemannian* 3-*symmetric space. Then* *the group $\mathscr{I}(M)$ of holomorphic isometries of M acts transitively on M.*

Proof. Let $p \in M$, and choose a neighborhood V of p which is a normal neighborhood of each of its points. Then there exists a neighborhood $W \subset V$ of p such that $\theta_q(p) \in V$ for all $q \in W$. Since V is a normal neighborhood of each of its points, $\exp_q^{-1}$ is defined on V for all $q \in V$. Define $\psi: W \to V$ by $\psi(q) = \theta_q(p)$. Since $p \to \theta_p$ is differentiable so is ψ. From (4.1) it follows that the tangent map ψ_{*p} of ψ at p is given by

$$\psi_{*p} = I_p - \Theta_p \tag{4.3}$$

(see [16]). Now 1 is not an eigenvalue of Θ_p and so ψ_{*p} is nonsingular. Hence ψ is a diffeomorphism on some neighborhood U of p with $U \subset W$. Then $\psi(U)$ is a neighborhood of p contained in the $\mathscr{I}(M)$-orbit of p, because each $q \in \psi(U)$ is the image of p under the holomorphic isometry θ_q. The $\mathscr{I}(M)$-orbit of p is thus open. Furthermore, the complement of the $\mathscr{I}(M)$-orbit of p is also open, because it is the union of $\mathscr{I}(M)$-orbits. Since M is connected, the theorem follows.

This proof is patterned after that of [17, Theorem 2].

Remark. In [5] the theorems corresponding to Theorems (4.7) and (4.8) for s-regular manifolds are proved by different methods.

Theorem (4.9). *A Riemannian 3-symmetric space is complete.*

Proof. This follows from Theorem (4.8) and the following fact about Riemannian manifolds: if a Riemannian manifold M has a transitive group of isometries, then M is complete.

The author does not know if a pseudo-Riemannian 3-symmetric space need be complete. The usual proof (see [12, Vol. 2, p. 223] does not immediately generalize.

We next turn to the problem of decomposing a Riemannian 3-symmetric space into the Riemannian product of pseudo-Riemannian 3-symmetric spaces which are in some sense irreducible. It turns out that such a decomposition exists; it is slightly different from the de Rham decomposition.

Definition. Let M be a pseudo-Riemannian 3-symmetric space. We say that M is *indecomposable* if and only if M is not flat and whenever M is the Riemannian product of pseudo-Riemannian 3-symmetric spaces M_1 and M_2, then either $M = M_1$ or $M = M_2$.

Let M be a simply connected pseudo-Riemannian manifold. Recall [11], [12] that M is *weakly irreducible* provided that the holonomy group at any point has no invariant nondegenerate subspaces.

Proposition (4.10). *Let M be a indecomposable simply connected pseudo-Riemannian 3-symmetric space. Then either M is weakly irreducible or M is a Riemannian product $M = N \times N$ where N is weakly irreducible.*

Proof. Suppose M is not weakly irreducible. Then $M = N \times N_1$ where N is weakly irreducible. Let $p \in M$, and consider the global cubic holomorphic isometry θ_p. Since M is indecomposable we must have $\theta_p(N) \neq N$. Furthermore the tangent space M_p is spanned by the subspaces N_p and $\theta_p(N)_p$. It follows that M is isometric to $N \times N$.

For example, we shall show in the next section that if G is any compact simple Lie group, then $G \times G$ is a Riemannian 3-symmetric space which is indecomposable.

Theorem (4.11). *Let M be a simply connected pseudo-Riemannian* 3-*symmetric space. Then M is a Riemannian product $M = M_0 \times M_1 \times \cdots \times M_r$ where M_0 is an even dimensional Euclidean space and $M_1, \cdots, M_r$ are indecomposable pseudo-Riemannian* 3-*symmetric spaces.*

Proof. We use the de Rham decomposition theorem for pseudo-Riemannian manifolds [11], [23] and get $M = M_0 \times N_1 \times \cdots \times N_k$ where M_0 is a Euclidean space and $N_1, \cdots, N_k$ are weakly irreducible. Let θ_p be the global cubic holomorphic isometry at $p \in M$; then θ_p preserves this decomposition. We must have $\theta_p(M_0) = M_0$, and so M_0 is even dimensional. If $\theta_p(N_i) \neq N_i$ for some i, then there exists N_j isometric to N_i such that $\theta_p(N_i \times N_j) = N_i \times N_j$. Therefore we obtain the decomposition

$$M = M_0 \times M_1 \times \cdots \times M_r ,$$

where $\theta_p(M_i) = M_i$ for $i = 0, \cdots, r$, and for $i = 1, \cdots, r$ either M_i is irreducible or $M_i = N_i \times N_i$ where N_i is irreducible. The restriction of θ_p to M_i for each $p \in M_i$ makes M_i into a pseudo-Riemannian 3-symmetric space.

5. Pseudo-Riemannian 3-symmetric spaces as coset manifolds

Some of the theorems of this section are special cases of results of [5]. Let M be a pseudo-Riemannian 3-symmetric space, G the largest connected group of holomorphic isometries of M, and H the isotropy subgroup of G at a point $p \in M$. Denote by θ_p the (global) cubic holomorphic isometry of M at p, and put $t(g) = \theta_p \circ g \circ \theta_p^{-1}$ for $g \in G$. Let $G^t = \{g \in G \mid t(g) = g\}$, and let $G_0{}^t$ be the identity component of G^t.

Proposition (5.1). *Let M be a pseudo-Riemannian* 3-*symmetric space. Then*

(*i*) *t is an automorphism of G and $t^3 = 1$,*

(*ii*) *we have $G_0{}^t \subseteq H \subseteq G^t$,*

(*iii*) *there are a G-invariant pseudo-Riemannian metric and a G-invariant almost complex structure on the coset space G/H so that G/H is holomorphically isometric to M.*

Proof. For (*i*) let $g \in G$. Then $\theta_p \in \mathscr{I}(M)$ and so $t(g) \in G$. Easy calculations prove the rest of (*i*).

Next let $h \in H$. Then at p the tangent maps of h, $t(h)$, and θ_p satisfy

$$t(h)_* = \theta_{p*} \circ h_* \circ \theta_{p*}^{-1} = \left(-\frac{1}{2}I + \frac{\sqrt{3}}{2}J\right)_p \circ h_* \circ \left(-\frac{1}{2}I - \frac{\sqrt{3}}{2}J\right)_p .$$

Since h is a holomorphic isometry we have $h_* \circ J_p = J_p \circ h_*$ and so $t(h)_* = h_*$. Two isometries with the same tangent map coincide, and so $t(h) = h$. This proves $H \subseteq G^t$.

Let $s \to g_s$ be a 1-parameter subgroup of G^t. Then $t(g_s) = g_s$, and so

$$(\theta_p \circ g_s)(p) = (g_s \circ \theta_p)(p) = g_s(p) .$$

Thus the orbit $\{g_s(p) \mid s \in \boldsymbol{R}\}$ is fixed by θ_p. Since p is an isolated fixed point of θ_p, we have $g_s(p) = p$ for all s. Thus $g_s \in H$, and we have proved $G_0{}^t \subseteq H$.

Finally for (*iii*) we note that there exists a diffeomorphism $\psi: M \to G/H$. We then just require that ψ be a holomorphic isometry.

Proposition (5.2). *Let G be a Lie group, $t: G \to G$ an automorphism with $t^3 = 1$, and H a subgroup of G with $G_0{}^t \subseteq H \subseteq G^t$. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively, and denote by t_* the automorphism of $\mathfrak{g}$ induced by t. Then*

(*i*) *we have $\mathfrak{h} = \{X \in \mathfrak{g} \mid t_*X = X\}$,*

(*ii*) *G/H is a reductive homogeneous space.*

Proof. (*i*) is obvious. For (*ii*) we decompose $\mathfrak{g} \otimes \boldsymbol{C}$ as

$$\mathfrak{g} \otimes \boldsymbol{C} = (\mathfrak{h} \otimes \boldsymbol{C}) \oplus \mathfrak{m}^+ \oplus \mathfrak{m}^- ,$$

where $\mathfrak{h} \otimes \boldsymbol{C}$, $\mathfrak{m}^+$, $\mathfrak{m}^-$ are the eigenspaces of t_* corresponding to the eigenvalues 1, $\frac{1}{2}(-1 + \sqrt{-3})$, $\frac{1}{2}(-1 - \sqrt{-3})$, respectively. Let $\mathfrak{m} = (\mathfrak{m}^+ \oplus \mathfrak{m}^-) \cap \mathfrak{g}$ so that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} .$$

Now let $X \in \mathfrak{m}^\pm$ and $h \in H$. Then we have

$$t_*(\mathrm{Ad}\,(h)X) = \mathrm{Ad}\,(t(h))t_*X = \mathrm{Ad}\,(h)t_*X = \tfrac{1}{2}(-1 \pm \sqrt{-3})\,\mathrm{Ad}\,(h)X .$$

Hence $\mathrm{Ad}\,(h)(\mathfrak{m}^\pm) \subseteq \mathfrak{m}^\pm$, and so $\mathrm{Ad}\,(h)(\mathfrak{m}) \subseteq \mathfrak{m}$. Therefore G/H is a reductive homogeneous space.

Corollary (5.3). *A pseudo-Riemannian* 3*-symmetric space is a reductive homogeneous space.*

We can now prove the converse of Theorem (5.1). As usual with a reductive homogeneous space G/H, we write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H respectively. Then $\mathfrak{m}$ may be identified with the tangent space to G/H at the coset H.

Theorem (5.4). *Let G be a connected Lie group, and $t: G \to G$ an automorphism of order* 3. *Let H be a subgroup with $G_0{}^t \subseteq H \subseteq G^t$, and write*

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ *as in Theorem* (5.2). *If* $\langle\ ,\ \rangle$ *is any* (*not necessarily positive definite*) *metric on* $\mathfrak{m}$ *which is both* Ad (H)*-and* t_**-invariant, then* $\langle\ ,\ \rangle$ *induces a* G*-invariant metric on* G/H *which makes* G/H *into a pseudo-Riemannian* 3*-symmetric space. Furthermore, if we write* $t_* \mid \mathfrak{m} = -\frac{1}{2}I + (\sqrt{3}/2)J$, *then* J *induces the canonical almost complex structure on* G/H, *and* $\langle\ ,\ \rangle$ *is almost Hermitian with respect to* J.

Proof. Let $M = G/H$. Then we have a projection $\pi: G \to M$; let $p = \pi(e)$, where e is the identity of G. Define $\theta_p: M \to M$ by $\theta_p(g(p)) = t(g)(p)$. Then θ_p is well-defined, it is a diffeomorphism of M, and $\theta_p{}^3 = 1$.

To show that p is an isolated fixed point of θ_p, suppose $\theta_p(g(p)) = g(p)$ for some $g \in G$. Then $t(g)(p) = g(p)$, and so putting $h = g^{-1}t(g)$ we have $h \in H$. Hence $t(h) = h$, and so

$$(5.1) \qquad h^3 = ht(h)t^2(h) = g^{-1}t(g)t(g^{-1}t(g))t^2(g^{-1}t(g)) = e\ .$$

If g is near e, so is h, and (5.1) shows that $h = e$. Thus $t(g) = g$; moreover, $g \in G_0^t$, and we have $g \in H$. Therefore p is an isolated fixed point of θ_p.

For any other point $q \in M$, let $g \in G$ be such that $g(p) = q$. We define θ_q by $\theta_q = g \circ \theta_p \circ g^{-1}$. Then θ_q is independent of the choice of g, it is a diffeomorphism of M with $\theta_q{}^3 = 1$, and q is an isolated fixed point of θ_q.

We get a G-invariant metric on G/H by translating the metric $\langle\ ,\ \rangle$ on $\mathfrak{m}$ via G. Since $\langle\ ,\ \rangle$ is t_*-invariant, θ_p is an isometry at p. Thus θ_p preserves $\langle\ ,\ \rangle$ and all of its covariant derivatives at p. Everything is analytic, and so θ_p is an isometry of M. Then each $g \in G$ is an isometry of M, and so each θ_q is an isometry of M.

We also translate the almost complex structure J on $\mathfrak{m}$ to each tangent space of $M = G/H$ via G. In order to verify that each θ_q is holomorphic, it suffices to prove that θ_p is holomorphic, because each $g \in G$ is holomorphic by definition of J.

First, we note that since M is homogeneous, the Riemannian connection ∇ of M is given at p by the formula

$$(5.2) \quad 2\langle \nabla_X Y, Z\rangle = -\langle X, [Y,Z]\rangle - \langle Y, [X,Z]\rangle + \langle Z, [X,Y]\rangle\ ,$$

for $X, Y, Z \in \mathfrak{m}$. From (5.2) it follows that $\langle \nabla^k{}_{X_1 \cdots X_k}(J)(Y), Z\rangle$ can be expressed in terms of $\langle\ ,\ \rangle$, J, and $[\ ,\ ]_{\mathfrak{m}}$. Now θ_{p*} preserves all of these, and $\theta_{p*} = (-\frac{1}{2}I + (\sqrt{3}/2)J)_p$. It follows from Proposition (3.6) that θ_p is holomorphic. This completes the proof.

Almost complex structures which are derived from an automorphism of order 3 are characterized as follows.

Proposition (5.5). *Let* $M = G/H$ *be a homogeneous space where* $\mathfrak{h}$ *is the fixed point set of an automorphism of* $\mathfrak{g}$ *of order* 3. *Then the canonical almost complex structure on* M *satisfies*

(*i*) $[JX, Y]_{\mathfrak{m}} = -J[X, Y]_{\mathfrak{m}}$, (*ii*) $[X, Y]_{\mathfrak{h}} = [JX, JY]_{\mathfrak{h}}$
for all $X, Y \in \mathfrak{m}$. Conversely, if G is simply connected and G/H has a G-invariant almost complex structure satisfying (*i*) *and* (*ii*), *then there exists an automorphism $t\colon G \to G$ such that $G_0^t \subseteq H \subseteq G^t$.*

This is proved in [23].

So far we have not used the hypothesis that $M = G/H$ be nearly Kählerian. We do this after we show that in our situation the property of being nearly Kählerian is equivalent to the well-known notion of natural reductivity.

Recall [14] that a homogeneous space $M = G/H$ with a G-invariant pseudo-Riemannian metric $\langle\, ,\, \rangle$ is said to be *naturally reductive* if it admits an $\mathrm{Ad}\,(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ satisfying the condition

$$(5.3) \qquad \langle [X, Y]_{\mathfrak{m}}, Z\rangle = \langle X, [Y, Z]_{\mathfrak{m}}\rangle \qquad \text{for all } X, Y, Z \in \mathfrak{m}\ .$$

Proposition (5.6). *Let $M = G/H$ be a pseudo-Riemannian* 3-*symmetric space. Then the following conditions are equivalent*:

(*i*) *M is naturally reductive,*

(*ii*) *the canonical almost complex structure of M is nearly Kählerian.*

Proof. The Riemannian connection of M given by (5.2). This, together with Theorem (5.5) (*i*), implies that

$$(5.4) \qquad \langle \nabla_X(J)X, JY\rangle = 2\langle X, [X, Y]\rangle \qquad \text{for } X, Y \in \mathfrak{X}(M)\ .$$

From (5.3) and (5.4) it follows that (*i*) and (*ii*) are equivalent.

6. The classification

We can now prove our main classification theorem.

Theorem (6.1). *Let M be a simply connected pseudo-Riemannian* 3-*symmetric space such that the group $\mathcal{I}(M)$ of holomorphic isometries of M is a reductive Lie group. Then M may be decomposed as a Riemannian product $M = M_0 \times M_1 \times \cdots \times M_r$ where M_0 is a complex Euclidean space and $M_i = G_i/H_i$ $(1 \leq i \leq r)$ is one of the spaces listed in Tables* I, II, III, *and* IV. *Each M_i $(1 \leq i \leq r)$ is an almost complex manifold in a natural way. Any homogeneous metric on M_i which is compatible with the almost complex structure makes M_i into an indecomposable* (*quasi-Kählerian*) *pseudo-Riemannian* 3-*symmetric space. Of all such metrics there is one, unique up to a scalar multiple, that is nearly Kählerian and makes M_i naturally reductive.*

Proof. According to Theorem (4.11) M can be decomposed as $M = M_0 \times M_1 \times \cdots \times M_r$ where M_0 is a complex Euclidean space and M_i $(1 \leq i \leq r)$ is an indecomponsable pseudo-Riemannian 3-symmetric space. Hence without loss of generality M is indecomposable. Write $M = G/H$ as in Theorem (5.1), and decompose the Lie algebra of G as $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. We may assume that G is a connected reductive Lie group acting effectively. Then $\mathfrak{h}$ is the fixed point set of an automorphism of $\mathfrak{g}$ of order 3. All such spaces have been classified [23, Theorem 7.17]. Then Tables I, II, III, and IV are the same as the corresponding tables of [23, Theorem 7.17]. By Theorem (5.4) each of these spaces is a pseudo-Riemannian 3-symmetric space with respect to any homogeneous metric compatible with the canonical almost complex structure.

Furthermore it is clear that each of the spaces $M = G/H$ in Tables I, II, III, and IV is naturally reductive with respect to a metric induced from a biinvariant metric on the Lie algebra of G. Conversely, any other naturally reductive metric must be a scalar multiple of the biinvariant one on some component of the isotropy representation of H. Then (5.3) implies that it is a scalar-multiple on the other components. By Theorem (5.6), $M = G/H$ is nearly Kählerian with respect to this naturally reductive metric.

Note. In the following and subsequent tables we adopt the notational conventions of [23]. See [23,p.118]. Also k denotes the dimension of the cone of homogeneous metrics on each homogeneous space compatible with the canonical almost complex structure.

Table I

G: Centerless classical simple

H: Centralizer of a compact total subgroup

G	H	Conditions	k
$SU^m(n)/Z_n$	$S\{U^{s_1}(r_1) \times U^{s_2}(r_2) \times U^{s_3}(r_3)\}/Z_n$	$n = r_1 + r_2 + r_3$	1 if $r_1 = 0$
$SL(n, R)/Z_2$	$\left\{SL\left(\frac{1}{n}, C\right) \times T^1\right\}/Z_{n/2},\ n \equiv 0(2)$	$m = s_1 + s_2 + s_3$	
$SL\left(\frac{n}{2}, Q\right)/Z_2$		$0 < r_1 < r_2 < r_3$	
$SL(n, C)/Z_n$	$S\{GL(r_1, C) \times GL(r_2, C) \times GL(r_3, C)\}/Z_n$	$1 \leq r_2$	3 if $r_1 > 0$
$SO^{2s+2t}(2n+1)$	$U^s(r) \times SO^{2t}(2n-2r+1)$	$1 \leq r \leq n$	1 if $r = 1$
$SO(2n+1, C)$	$GL(r, C) \times SO(2n-2r-1, C)$		2 if $r > 1$
$Sp^{s+t}(n)/Z_2$	$\{U^s(r) \times Sp^t(n-r)\}/Z_2$	$1 \leq r \leq n$	1 if $r = n$
$Sp(n, R)/Z_2$	$\{U^s(r) \times Sp(n-r, R)\}/Z_2$		
$Sp(n, C)/Z_2$	$\{GL(r, C) \times Sp(n-r, C)\}/Z_2$		2 if $r < n$
$SO^{2s+t}(2n)/Z_2$	$\{U^s(r) \times SO^t(2n-2r)\}/Z_2$	$1 \leq r \leq n$	1 if $r = 1$ or n
$SO^*(2n)/Z_2$	$\{U^s(r) \times SO^*(2n-2r)\}/Z_2$		
$SO(2n, C)/Z_2$	$\{GL(r, C) \times SO(2n-2r, C)\}/Z_2$		2 if $1 < r < n$

Table II

G: Centerless exceptional group

H: Centralizer of a compact toral subgroup

G	H	Conditions	k
G_2	$U(2)$	—	2
$G_2^* = G_{2,A_1A_1}$	$U(2), U^1(2)$	—	
G_2^C	$GL(2,C)$	—	
F_4	$\{\mathrm{Spin}(7) \times T^1\}/Z_2$, $\{Sp(3) \times T^1\}/Z_2$	—	2
F_{4,B_4}	$\{\mathrm{Spin}^r(7) \times T^1\}/Z_2$, $\{Sp^1(3) \times T^1\}/Z_2$	$r = 0,1$	
F_{4,C_3C_1}	$\{\mathrm{Spin}^r(7) \times T^1\}/Z_2$, $\{Sp^t(3) \times T^1\}/Z_2$, $\{Sp(3,R) \times T^1\}/Z_2$	$r = 2,3; t = 0,1$	
F_4^C	$\{\mathrm{Spin}(7,C) \times C^*\}/Z_2$, $\{Sp(3,C) \times C^*\}/Z_2$	—	
E_6/Z_3	$\{SO(10) \times SO(2)\}/Z_2$	—	1
	$\{S(U(5) \times U(1)) \times SU(2)\}/Z_2$, $\{[SU(6)/Z_3] \times T^1\}/Z_2$	—	2
	$\{[SO(8) \times SO(2)] \times SO(2)\}/Z_2$	—	3
E_{6,A_1A_5}	$\{SO^*(10) \times SO(2)\}/Z_2$, $\{SO^4(10) \times SO(2)\}/Z_2$	—	1
	$\{S(U^r(5) \times U(1)) \times SU^s(2)\}/Z_2$	$(s,r)=(0,0)$, $(0,1),(0,2),(1,2)$	2
	$\{[SU^r(6)/Z_3] \times T^1\}/Z_2$	$r = 0,2,3$	2
	$\{[SO^*(8) \times SO(2)\}/Z_2$ $\{[SO^r(8) \times SO(2)] \times SO(2)\}/Z_2$	$r = 2,4$	3
E_{6,D_5T^1}	$\{SO^r(10) \times SO(2)\}/Z_2$, $\{SO^*(10) \times SO(2)\}/Z_2$	$r = 0,2$	1
	$\{S(U^r(5) \times U(1)) \times SU^s(2)\}/Z_2$	$(s,r) = (1,0)$, $(0,1),(1,1),(0,2)$	2
	$\{[SU^r(6)/Z_3] \times T^1\}/Z_2$	$r = 1,2$	2
	$\{[SO^*(8) \times SO(2)] \times SO(2)\}/Z_2$, $\{[SO^r(8) \times SO(2)] \times SO(2)\}/Z_2$	$r = 0,2$	3
E_6^C/Z_3	$\{SO(10,C) \times C^*\}/Z_2$	—	1
	$\{S(GL(5,C) \times C^*) \times SL(2,C)\}/Z_2$, $\{[SL(6,C)/Z_3] \times C^*\}/Z_2$	—	2
	$\{[SO(8,C) \times C^*] \times C^*\}/Z_2$	—	3

Table II—Continued

G: Centerless exceptional group

H: Centralizer of a compact toral subgroup

G	H	Conditions	k
E_7/Z_2	$\{E_6 \times T^1\}/Z_3$	—	1
	$\{SU(2) \times [SO(10) \times SO(2)]\}/Z_2$, $\{SO(2) \times SO(12)\}/Z_2$, $S(U(7) \times U(1))/Z_4$	—	2
E_{7,A_7}	$\{E_{6,A_1A_5} \times T^1\}/Z_2$	—	1
	$\{SU(2) \times [SO^*(10) \times SO(2)]\}/Z_2$, $\{SU^1(2) \times [SO^4(10) \times SO(2)]\}/Z_2$, $\{SO(2) \times SO^*(12)\}/Z_2$, $\{SO(2) \times SO^6(12)\}/Z_2$, $S(U^r(7) \times U(1))/Z_4$	$r = 0,3$	2
E_{7,A_1D_6}	$\{E_{6,D_5T^1} \times T^1\}/Z_2$, $E_{6,A_1A_5} \times T^1\}/Z_2$	—	1
	$\{SU^t(2) \times [SO^r(10) \times SO(2)]\}/Z_2$, $\{SU^1(2) \times [SO^*(10) \times SO(2)]\}/Z_2$, $\{SO(2) \times SO^p(12)\}/Z_2$, $S(U^s(7) \times U(1))/Z_4$	$(t,r) = (0,0)$, $(0,2),(1,2),(0,4)$ $p = 0,4$ $s = 1,2,3$	2
E_{7,E_6T^1}	$\{E_6 \times T^1\}/Z_3$, $\{E_{6,D_5T^1} \times T^1\}/Z_2$	—	1
	$\{SU^1(2) \times [SO(10) \times SO(2)]\}/Z_2$, $\{SU(2) \times [SO^*(10) \times SO(2)]\}/Z_2$, $\{SO(2) \times SO^*(12)\}/Z_2$, $\{SO(2) \times SO^6(12)\}/Z_2$, $S(U^r(7) \times U(1))/Z_4$	$r = 1,2$	2
E_7^C/Z_2	$\{E_6^C \times C^*\}/Z_3$	—	1
	$\{SL(2,C) \times [SO(10,C) \times C^*]\}/Z_2$, $\{C^* \times SO(12,C)\}/Z_2$, $S\{GL(7,C) \times C^*\}/Z_4$	—	2
E_8	$SO(14) \times SO(2)$, $\{E_7 \times T^1\}/Z_2$	—	2
E_{8,D_8}	$SO(14) \times SO(2)$, $SO^6(14) \times SO(2)$, $SO^*(14) \times SO(2)$, $\{E_{7,A_1D_6} \times T^1\}/Z_2$, $\{E_{7,A_7} \times T^1\}Z_2$	—	2
E_{8,A_1E_7}	$SO^2(14) \times SO(2)$, $SO^4(14) \times SO(2)$, $SO^*(14) \times SO(2)$, $\{E_7 \times T^1\}/Z_2$, $\{E_{7,E_6T_1} \times T^1\}/Z_2$, $\{E_{7,A_1D_6} \times T^1\}/Z_2$	—	2
E_8^C	$SO(14,C) \times C^*$, $\{E_7^C \times C^*\}/Z_2$	—	2

Table III

G: Centerless simple
H: Not the centralizer of a torus, rank H = rank G
($k = 1$, G is exceptional, H has center of order 3)

G_2	$SU(3)$
$G_2^* = G_{2,A_1A_1}$	$SU^1(3)$
G_2^C	$SL(3,C)$
F_4	$\{SU(3) \times SU(3)\}/Z_3$
F_{4,B_4}	$\{SU^1(3) \times SU(3)\}/Z_3$
F_{4,C_3C_1}	$\{SU(3) \times SU^1(3)\}/Z_3$, $\{SU^1(3) \times SU^1(3)\}/Z_3$
F_4^C	$\{SL(3,C) \times SL(3,C)\}/Z_3$
E_6/Z_3	$\{SU(3) \times SU(3) \times SU(3)\}/\{Z_3 \times Z_3\}$
E_{6,A_1A_5}	$\{SU^1(3) \times SU(3) \times SU(3)\}/\{Z_3 \times Z_3\}$ $\{SU^1(3) \times SU^1(3) \times SU^1(3)\}/\{Z_3 \times Z_3\}$
E_{6,D_5T^1}	$\{SU^1(3) \times SU^1(3) \times SU(3)\}/\{Z_3 \times Z_3\}$
E_{6,F_4}	$\{SL(3,C) \times SU(3)\}/Z_3$
E_{6,C_4}	$\{SL(3,C) \times SU^1(3)\}/Z_3$
E_6^C/Z_3	$\{SL(3,C) \times SL(3,C) \times SL(3,C)\}/\{Z_3 \times Z_3\}$
E_7/Z_2	$\{SU(3) \times [SU(6)/Z_2]\}/Z_3$
E_{7,A_7}	$\{SU(3) \times [SU^1(6)/Z_2]\}/Z_3$, $\{SU^1(3) \times [SU^3(6)/Z_2]\}/Z_3$
E_{7,A_1D_6}	$\{SU^1(3) \times [SU(6)/Z_2]\}/Z_3$, $\{SU(3) \times [SU^2(6)/Z_2]\}/Z_3$, $\{SU^1(3) \times [SU^2(6)/Z_2]\}/Z_3$
E_{7,E_6T^1}	$\{SU^1(3) \times [SU^1(6)/Z_2]\}/Z_3$, $\{SU(3) \times [SU^3(6)/Z_2]\}/Z_3$
E_7^C	$\{SL(3,C) \times [SL(6,C)/Z_2]\}/Z_3$
E_8	$\{SU(3) \times E_6\}/Z_3$, $SU(9)/Z_3$
E_{8,D_8}	$\{SU(3) \times E_{6,D_5T^1}\}/Z_3$, $\{SU^1(3) \times E_{6,A_1A_5}\}/Z_3$, $SU^1(9)/Z_3$, $SU^4(9)/Z_3$
E_{8,A_1E_7}	$\{SU^1(3) \times E_6\}/Z_3$, $\{SU^1(3) \times E_{6,D_5T^1}\}/Z_3$, $\{SU(3) \times E_{6,A_1A_5}\}/Z_3$, $SU^2(9)/Z_3$, $SU^3(9)/Z_3$
E_8^C	$\{SL(3,C) \times E_6^C\}/Z_3$, $SL(9,C)/Z_3$

Table IV
rank $G >$ rank H
$(k = 1)$

G	H	Conditions
Spin (8)	$SU(3)/Z_3$	—
$SO^4(8)$	$SU^1(3)/Z_3$	
Spin $(8, C)$	$SL(3, C)/Z_3$	
Spin (8), $\text{Spin}^1(8)$	G_2	—
$\text{Spin}^3(8)$, $\text{Spin}^4(8)$	G_2^*	
Spin $(8, C)$	G_2^C	
$\{L^* \times L^* \times L^*\}/\delta Z^*$	$\delta L^*/\delta Z^*$	Note (1)
$\{L^C \times L^*\}/\delta Z^*$	$\delta L^*/\delta Z^*$	Note (2)
$\{L^C \times L^C \times L^C\}/\delta Z^*$	$\delta L^C/\delta Z$	Note (1)
vector group R^2	$\{0\}$	—
Note 1	$\mathscr{L}$ is an arbitrary compact simple Lie algebra. $\mathscr{L}^*$ is an arbitrary real form of $\mathscr{L} \otimes C$. L^* and L^C denote the connected simply connected Lie groups with Lie algebras $\mathscr{L}^*$ and $\mathscr{L}^C$; Z^* and Z denote their centers. $\delta(x) = (x, x, x)$.	
Note 2	$\delta(x) = (\pi(x), x)$ where π: $L^* \to L^C$ gives the universal covering of the R-analytic subgroup of L^C with Lie algebra $\mathscr{L}^*$.	

Corollary (6.2). *Let M satisfy the hypotheses of Theorem* (6.1), *and in addition assume that M is Riemannian. Then M may be decomposed as a Riemannian product $M = M_0 \times M_1 \times \cdots \times M_r$ where M_0 is a complex Euclidean space and each M_i $(1 \leq i \leq r)$ is an indecomposable Riemannian* 3-*symmetric space. If M_i is compact, it is listed in Tables* V, VI, *and* VII; *and if M_i is noncompact, it is listed in Table* VIII.

Table V

G: Compact centerless simple
H: Centralizer of a torus

G	H	k
$SU(n)/Z_n$ $n \geq 2$	$S\{U(r_1) \times U(r_2) \times U(r_3)\}/Z_n$ $0 \leq r_1 \leq r_2 \leq r_3,\ 0 < r_2,\ r_1 + r_2 + r_3 = n$	1 if $r_1 = 0$ 2 if $r_1 > 0$
$SO(2n+1)$ $n \geq 1$	$U(r) \times SO(2n-2r+1),\ 1 \leq r \leq n$	1 if $r = 1$ 2 if $r > 1$
$Sp(n)/Z_2$	$\{U(r)\,Sp(n-r)\}/Z_2,\ 1 \leq r \leq n$	1 if $r = n$ 2 if $r < n$
$SO(2n)/Z_2$ $n \geq 3$	$\{U(r) \times SO(2n-2r)\}/Z_2,\ 1 \leq r \leq n$	1 if $r = 1$ or n 2 if $1 < r < n$
G_2	$U(2)$	2
F_4	$\{\mathrm{Spin}(7) \times T^1\}/Z_2$	2
	$\{Sp(3) \times T^1\}/Z_2$	2
E_6/Z_3	$\{SO(10) \times SO(2)\}/Z_2$	1
	$\{S(U(5) \times U(1)) \times SU(2)\}/Z_2$	2
	$\{[SU(6)/Z_3] \times T^1\}/Z_2$	2
	$\{[SO(8) \times SO(2)] \times SO(2)\}/Z_2$	3
E_7/Z_2	$\{E_6 \times T^1\}/Z_3$	1
	$\{SU(2) \times [SO(10) \times SO(2)]\}/Z_2$	2
	$\{SO(2) \times SO(12)\}/Z_2$	2
	$S\{U(7) \times U(1)\}/Z_4$	2
E_8	$SO(14) \times SO(2)$	2
	$\{E_7 \times T^1\}/Z_2$	2

Table VI

G: Compact centerless simple
H: Semisimple with center of order 3
(G is exceptional, $k = 1$)

G	H
G_2	$SU(3)$
F_4	$\{SU(3) \times SU(3)\}/Z_3$
E_6/Z_3	$\{SU(3) \times SU(3) \times SU(3)\}/\{Z_3 \times Z_3\}$
E_7/Z_2	$\{SU(3) \times [SU(6)/Z_2]\}/Z_3$
E_8	$\{SU(3) \times E_6\}/Z_3$
	$SU(9)/Z_3$

Table VII

G: Compact, rank $G >$ rank H

$(k = 1)$

G	H
Spin (8)	$SU(3)/Z_3$
	G_2
$\{L \times L \times L\}/Z$ where L is compact simple and simply connected and Z is its center embedded diagonally.	L/Z where L is embedded diagonally in $L \times L \times L$ and Z is its center.

Table VIII

G: Noncompact centerless simple

H: Compact

G	H	conditions	k
$SU^{r_i}(n)/Z_n$	$S(U(r_1) \times U(r_2) \times U(r_3))/Z_n$	$0 \leq r_1 \leq r_2 \leq r_3$ $1 \leq r_2$	1 if $r_1 = 0$ 3 if $r_1 > 0$
$SO^{2r}(2n+1)$	$U(r) \times SO(2n-2r+1)$	$1 \leq r \leq n$	1 if $r = 1$ 2 if $r > 1$
$Sp^r(n)/Z_2$	$(U(r) \times Sp(n-r))/Z_2$	$1 \leq r \leq n$	1 if $r = n$ 2 if $r < n$
	$(U(n-r) \times Sp(r))/Z_2$		
$SO^{2r}(2n)/Z_2$	$(U(r) \times SO(2n-2r))/Z_2$	$1 \leq r \leq n$	1 if $r = 1$ or n 2 if $1 < r < n$
	$(U(n-r) \times SO(2r))/Z_2$		
$G_2^* = G_{2,A_1A_1}$	$U(2)$	—	2
F_{4,C_3C_1}	$(Sp(3) \times T^1)/Z_2$	—	2
E_{6,A_1A_5}	$((S(U(5) \times U(1))) \times SU(2))/Z_2$	—	2
	$(SU(6)/Z_3 \times T_1)/Z_2$	—	2
E_{6,D_5T_1}	$SO(10) \times SO(2)/Z_2$	—	1
E_{7,A_7}	$S(U(7) \times T^1)/Z_4$	—	2
E_{7,A_1D_6}	$(SU(2) \times SO(10) \times T^1)/Z_2$	—	2
	$(SO(12) \times T^1)/Z_2$	—	2

Table VIII—Continued

G: Noncompact centerless simple

H: Compact

G	H	conditions	k
E_{7,E_6T^1}	$(E_6 \times T^1)/Z_3$	—	1
E_{8,D_8}	$SO(14) \times SO(2)$	—	2
E_{8,A_1E_7}	$(E_7 \times T^1)/Z_2$	—	2
$\mathrm{Spin}^1(8)$	G_2	—	Note (1) in Table IV
vector group R^2	$\{0\}$	—	Note (2) in Table IV

Corollary (6.3). *Let M be a simply connected pseudo-Riemannian Hermitian symmetric space such that the group $\mathscr{J}(M)$ of holomorphic isometries is a reductive Lie group. Then M may be decomposed as a Riemannian product $M = M_0 \times M_1 \times \cdots \times M_r$ where M_0 is a complex Euclidean space and $M_i = G_i/H_i$ is one of the spaces listed in Table IX. The metric on M_i is unique up to a scalar multiple.*

Table IX: Reductive pseudo-Riemannian Hermitian symmetric spaces

G: Centerless simple

H: Centralizer of a torus

(rank G = rank K, $k = 1$)

G	H	conditions
$SU^{u+v}(p+q)/Z_{p+q}$	$S(U^u(p) \times U^v(q))/Z_{p+q}$	$1 \leq p \leq q$, $2u \leq p$, $2v \leq q$
$SL(2n, R)/Z_2$	$(SL(n, C) \times T^1)/Z_n$	$n > 1$
$SL(n, Q)/Z_2$	$(SL(n, C) \times T^1)/Z_n$	$n > 1$
$SL(p+q, C)/Z_{p+q}$	$S(GL(p, C) \times GL(q, C) \times C^*)/Z_{p+q}$	$1 \leq p \leq q$
$SO^{t+s}(2n+1)$	$SO^t(2n-2) \times T^1$	$0 \leq t \leq n-1$, $s = 0, 2$
$SO(2n+1, C)$	$SO(2n-1) \times C^*$	—
$Sp^t(n)/Z_2$	$U^t(n)/Z_2$	$0 \leq 2t \leq n$
$Sp(n, R)$	$U^t(n)/Z_2$	$0 \leq 2t \leq n$
$Sp(n, C)$	$GL(n, C)/Z_2$	—

Table IX: Reductive pseudo-Riemannian Hermitian symmetric spaces—continued

G: Centerless simple

H: Centralizer of a torus

(rank G = rank K, $k = 1$)

G	H	conditions
$SO^{t+s}(2n)/Z_2$	$(SO^t(2n-2) \times T^1)/Z_2$	$0 \leq t \leq n-2,\ s = 0,2$
$SO^{2t}(2n)/Z_2$	$U^t(n)/Z_2$	$0 \leq 2t \leq n$
$SO^*(2n)/Z_2$	$(SO^*(2n-2) \times T^1)/Z_2$	—
	$U^t(n)/Z_2$	$0 \leq 2t \leq r$
$SO(2n, C)/Z_2$	$(SO(2n-2, C) \times C^*)/Z_2$	—
	$GL(n, C)/Z_2$	—
E_6/Z_3	$(SO(10) \times T^1)/Z_2$	—
E_{6,A_1A_5}	$(SO^*(10)) \times T^1)/Z_2$	—
	$(SO^4(10) \times T^1)/Z_2$	—
E_{6,D_5T^1}	$(SO(10) \times T^1)/Z_2$	—
	$(SO^2(10) \times T^1)/Z_2$	—
	$(SO^*(10) \times T^1)/Z_2$	—
E_6^C/Z_3	$(SO(10, C) \times C^*)/Z_2$	—
E_7/Z_2	$E_6 \times T^1/Z_3$	—
E_{7,A_7}	$(E_{6,A_1A_5} \times T^1)/Z_2$	—
E_{7,A_1D_6}	$(E_{6,D_5T^1} \times T^1)/Z_2$	—
	$(E_{6,A_1A_5} \times T^1)/Z_2$	—
E_{7,E_6T^1}	$(E_6 \times T^1)/Z_3$	—
	$(E_{6,D_5T^1} \times T^1/Z_2$	—
E_7^C/Z_2	$(E_6^C \times C^*)/Z_3$	—

Table IX can also be deduced from Berger's classification of affine symmetric spaces G/H with G simple [2].

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