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APPLICATION OF THE HIGHER OSCULATING SPACES TO THE SPHERICAL PRINCIPAL SERIES

NOLAN R. WALLACH

1. Introduction

The purpose of this paper is to use a geometric construction (analogous to the higher osculating spaces and fundamental forms of immersions) to study certain infinite dimensional Banach representations of semisimple Lie groups (the spherical principal series). Few of our results are new, most can be found in Kostant [5] or Helgason [4]. However, the proofs are new and quite elementary (in comparison to those of Kostant and Helgason).

In § 2 we define the spherical principal series and study duality in the series. In § 3 we study cyclic vectors for these representations and prove several results on finite dimensional class 1 representations including the fact that every finite dimensional class 1 representation is realized as a canonically defined subspace of a principal series representation. In § 4 the geometric construction alluded to above is given. § 5 is devoted to applications of the results of §§ 1–4. We prove in particular that almost all of the elements of the spherical principal series (not necessarily unitary) are irreducible. This is the weakest possible way of stating the result of Kostant [5]. In § 6 we give Kostant's complete solution to which elements of the spherical principal series for Lorentz groups are irreducible.

2. The spherical principle series

Let G be a connected semisimple Lie group with finite center, $G = KAN$ an Iwasawa decomposition of G, K a maximal compact subgroup of G, AN an Iwaaswa subgroup of G, N the unipotent radical of AN, and A a maximal split torus of G acting semisimply on N. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}$, be respectively the Lie algebras of G, K, A, N, and M the centralizer of A in K. Set $B = MAN$. Then $G/B = K/M$ under the map $kanB \mapsto kM$ for $k \in K, a \in A, n \in N$.

Let dx be the K-invariant normalized measure on K/M, and $\mathfrak{a}'$ and $\mathfrak{a}'_C$ respectively the spaces of real valued and complex valued linear forms on $\mathfrak{a}$. If $\lambda \in \mathfrak{a}'_C$, we define a Banach representation (π_λ, X^λ) of G as follows:

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i) X^λ is the Hilbert space of all measurable functions $f\colon G \to C$ so that $f(gman) = e^{-\lambda(\log a)}f(g)(\log\colon A \to \mathfrak{a}$ is the inverse map to $\exp\colon \mathfrak{a} \to A)$, $g \in G$, $m \in M, a \in A, n \in N$, and such that $\int_{K/M} |f(x)|^2\, dx < \infty(|f(km)| = |f(k)|$ thus $|f(x)|$ is well defined for $x \in K/M)$. We set $\langle f_1, f_2\rangle = \int_{K/M} f_1(x)\overline{f_2(x)dx}$, for $f_1, f_2 \in X^\lambda$.

ii) $(\pi_\lambda(g_0)\cdot f)(g) = f(g_0^{-1}g)$ for $g_0, g \in G$. It is not hard to check (cf. Harish-Chandra [1]) that (π_λ, X^λ) is a continuous Banach representation of G.

For each $h \in \mathfrak{a}$ define $\rho(h) = (1/2)tr(adh|_{\mathfrak{n}})$. If $\lambda \in \mathfrak{a}'_C$, then $\lambda = \lambda_1 + \sqrt{-1}\lambda_2$, for $\lambda_1, \lambda_2 \in \mathfrak{a}'$. Define $\bar{\lambda} = \lambda_1 - \sqrt{-1}\lambda_2$.

We now define two G-invariant pairings; a sesquilinear pairing of X^λ and $X^{2\rho-\bar{\lambda}}$ and a bilinear pairing between X^λ and $X^{2\rho-\lambda}$. If $f_1 \in X^\lambda$ and $f_2 \in X^{2\rho-\bar{\lambda}}$, define

$$(1) \qquad (f_1, f_2) = \int_K (f_1(k)\overline{f_2(k)}dk\ ,$$

where dk is normalized Haar measure on K. If $f_1 \in X^\lambda$ and $f_2 \in X^{2\rho-\lambda}$, define

$$(2) \qquad \{f_1, f_2\} = \int_K f_1(k)f_2(k)dk\ .$$

Lemma 2.1. (a) *If $f_1 \in X^\lambda, f_2 \in X^{2\rho-\bar{\lambda}}, g \in G$, then $(\pi_\lambda(g)\cdot f_1, \pi_{2\rho-\bar{\lambda}}(g)\cdot f_2) = (f_1, f_2)$.*

(b) *If $f_1 \in X^\lambda, f_2 \in X^{2\rho-\lambda}, g \in G$, then*

$$\{\pi_\lambda(g)f_1, \pi_{2\rho-\lambda}(g)f_2\} = \{f_1, f_2\}\ .$$

Proof. If $f \in X^{2\rho-\lambda}$, then $\bar{f} \in X^{2\rho-\bar{\lambda}}$. Thus if $f_1 \in X^\lambda, f_2 \in X^{2\rho-\lambda}$, then $\{f_1, f_2\} = (f_1, \bar{f_2})$. It is therefore sufficient to prove (a).

Let $g \in G$. Then $g = kan, k \in k, a \in A, n \in N$. Set $k(g) = k, H(g) = \log a$. Then $k\colon G \to K, H\colon G \to \mathfrak{a}$ are C^∞ mappings. Let F be a continuous function on K. If $g \in G$, then (cf. Helgason [3, p. 51])

$$(3) \qquad \int_K F(k)dk = \int_K F(k(gk))e^{-2\rho(H(gk))}dk\ .$$

Now let $f_1 \in X^\lambda, f_2 \in X^{2\rho-\bar{\lambda}}$. Then

$$\begin{aligned}(\pi_\lambda(g)f_1, \pi_{2\rho-\bar{\lambda}}(g)f_2) &= \int_K f_1(g^{-1}k)\overline{f_2(g^{-1}k)}dk \\ &= \int_K f_1(k(g^{-1}k))e^{-\lambda(H(g^{-1}k))}\overline{f_2(k(g^{-1}k))}e^{(-2\rho+\lambda)(H(g^{-1}k))}dk\end{aligned}$$

$$= \int_K f_1(k(g^{-1}k))\overline{f_2(k(g^{-1}k))}e^{-2\rho(H(g^{-1}k))}dk = (f_1, f_2)\ , \quad \text{by (3).} \quad \text{q.e.d.}$$

For each $\lambda \in \mathfrak{a}'_C$, (π_λ, X^λ) is a unitary representation of K which is unitarily equivalent with $L^2(K/M)$). Let $\hat{K}_0$ be the set of all equivalence classes of irreducible, finite dimensional, continuous K-modules which have a nonzero M-fixed vector. Let $V^M = \{v \in V \mid m\cdot v = v \text{ for all } m \in M\}$ for a K-module V, and let $V_\gamma \in \gamma$ be fixed for each $\gamma \in \hat{K}_0$.

$f \in X^\lambda$ is said to be K-finite if the linear hull of $\pi_\lambda(K)f$ is finite dimensional. Let X^λ_F be the space of all K-finite elements of X^λ. Then the Peter-Weyl theorem (applied to $L^2(K/M)$) implies that X^λ_F is dense X^λ, and Frobenious reciprocity that, as a K-module, $X^\lambda_F = \sum_{\gamma \in \hat{K}_0} \dim(V^M_\gamma)V_\gamma$.

3. Cyclic elements of $\mathfrak{a}'_C$

Let $f_\lambda \in X^\lambda$ be defined by $f_\lambda(g) = e^{-\lambda(H(g))}$ for each $\lambda \in \mathfrak{a}'_C$. We say that $f \in X^\lambda$ is a cyclic vector for X^λ if X^λ is the smallest closed invariant subspace of X^λ containing f, and that $\lambda \in \mathfrak{a}'_C$ is cyclic if f_λ is a cyclic vector for X^λ. Let X^λ_e be the smallest closed invariant subspace of X^λ containing f_λ.

Proposition 3.1. (a) *If X^λ_e is completely reducible, then X^λ_e is irreducible.*

(b) *If $\dim X^\lambda_e < \infty$, then X^λ_e is irreducible, and λ is the lowest restricted weight of X^λ_e relative to the Weyl chamber of $\mathfrak{A}$ determined by N.*

(c) *Let (π, V) be a finite dimensional, continuous, irreducible representation of G which has a nonzero K-fixed vector, and λ the lowest restricted weight of V. Then X^λ_e is equivalent to (π, V).*

Before proving Proposition 3.1 we need an elementary lemma of Kostant. Let $U(\mathfrak{g})$ be the complexified universal enveloping algebra of $\mathfrak{g}$, $(X^\lambda_F)^*$ be the contragradient $U(\mathfrak{g})$-module to X^λ_F, and $\xi_\lambda \in (X^\lambda_F)^*$ be defined by $\xi_\lambda(f) = f(e)$. Set $Z^\lambda = U(\mathfrak{g})\xi_\lambda$.

Lemma 3.1. *If $f \in X^\lambda_F$ is such that $\alpha(f) = 0$ for all $\alpha \in Z^\lambda$, then $f = 0$.*

Proof. If $f \in X^\lambda_F$, then $f\colon G \to C$ is analytic. Thus if $(gf)(e) = 0$ for each $g \in U(\mathfrak{g})$, then $f = 0$. By the definition of the contragradient action, the element $\alpha_g \in (X^\lambda_F)^*$ defined by $\alpha_g(f) = (g\cdot f)(e)$ is in Z^λ for each $g \in U(\mathfrak{g})$. This proves the lemma.

We now prove Proposition 3.1. We suppose that X^λ_e is completely reducible. If $X^\lambda_e = U_1 \oplus U_2$, U_i being a closed nonzero invariant subspace of X^λ_e for $i = 1, 2$, then letting $P_i\colon X^\lambda_e \to U_i$ be the corresponding projection for $i = 1, 2$, we see that $P_i f_\lambda \neq 0$ for $i = 1, 2$. Thus $f_\lambda \in U_1 \cap U_2$ since the multiplicity of the trivial representation of K in X^λ is one. This contradiction implies (a).

We now prove (b). Assume that $\dim X^\lambda_e < \infty$. Then as a G-module X^λ_e is completely reducible. Hence (a) implies that X^λ_e is irreducible. Now Lemma 3.1 implies that $Z^\lambda|_{X^\lambda_e} = (X^\lambda_e)^*$. If $n \in \mathfrak{n}$ and $f \in X^\lambda_F$, we see that

$$(n\cdot\xi_\lambda)(f) = -\xi_\lambda(n\cdot f) = -\frac{d}{dt}f(\exp(-tn))|_{t=0} = 0\ .$$

Thus $n\cdot\xi_\lambda = 0$ for all $n \in \mathfrak{n}$. If $h \in \mathfrak{a}$ and $f \in X_F^\lambda$, then

$$(h\cdot\xi_\lambda)(f) = -\xi_\lambda(h\cdot f) = -\frac{d}{dt}(f(\exp -th))|_{t=0} = -\frac{d}{dt}e^{\lambda(th)}f(e)|_{t=0}$$
$$= -\lambda(h)\xi_\lambda(f)\ .$$

Thus the highest restricted weight of $(X_e^\lambda)^*$ is $-\lambda$; this proves (b)

We now prove (c). Let (π, V) be a finite dimensional, continuous, irreducible representation of G with K-fixed vector $v_0 \neq 0$, (π^*, V^*) the contragradient representation, ϕ an element of the highest restricted weight space, the weight being λ, of V^*. Define the function $f_v\colon G \to C$ for each $v \in V$ via $f_v(g) = \phi(g^{-1}v)$. Then $f_{\pi(g_0)\cdot v}(g) = \phi(g^{-1}g_0v) = f_v(g_0^{-1}g)$. Thus the map $\eta\colon V \to C^\infty(G)$, $\eta(v) = f_v$, is a G-module homomorphism. Since (π, V) is irreducible and $\eta \neq 0$, η is injective. Now let $k \in k$, $a \in A$, $n \in N$. Then

$$f_{v_0}(kan) = \phi(n^{-1}a^{-1}k^{-1}v_0) = \phi(n^{-1}a^{-1}v_0)$$
$$= \phi(a^{-1}v_0) = (a\cdot\phi)(v_0) = e^{\lambda(\log a)}\phi(v_0)\ .$$

Thus $f_{v_0} = \phi(v_0)f_{-\lambda}$. This clearly implies that $\eta\colon V \to X_e^{-\lambda}$ is a G-module isomorphism.

Lemma 3.2. *$\lambda \in \mathfrak{a}_C'$ is cyclic if and only if any nonzero closed invariant subspace of $X^{2\rho-\bar\lambda}$* (resp. $X^{2\rho-\lambda}$) *contains $f_{2\rho-\bar\lambda}$* (resp. $f_{2\rho-\lambda}$). *In particular, if $\lambda \in \mathfrak{a}_C'$ is cyclic, then $X_e^{2\rho-\bar\lambda}$ and $X_e^{2\rho-\lambda}$ are irreducible.*

Proof. Suppose $\lambda \in \mathfrak{a}_C'$ is cyclic. We prove the result for $X^{2\rho-\bar\lambda}$. The proof for $X^{2\rho-\lambda}$ is exactly the same by substituting $\{\ ,\ \}$ for $(\ ,\ )$.

Suppose that $U \subset X^{2\rho-\bar\lambda}$ is nonzero, closed and invariant. There is $g \in G$ such that $(\pi_\lambda(g)f_\lambda, U) \neq (0)$. Since λ is cyclic, $(f_\lambda, U) \neq (0)$. This implies that the trivial representation of K appears in U. Since the multiplicity of the trivial representation of K in $X^{2\rho-\bar\lambda}$ is one, $f_{2p-\bar\lambda} \in U$.

Conversely, suppose that every nonzero closed invariant subspace of $X^{2\rho-\bar\lambda}$ contains $f_{2\rho-\bar\lambda}$, and let $f \in X^{2\rho-\bar\lambda}$. Then the smallest closed invariant subspace of $X^{2\rho-\bar\lambda}$ containing f contains $f_{2\rho-\bar\lambda}$. Hence there is $g \in G$ such that $(\pi_\lambda(g)f_\lambda, f) \neq 0$; this clearly implies that $X_e^\lambda = X^\lambda$. q.e.d.

The following are immediate consequences of Lemma 3.2.

Corollary 3.2. *Let $\lambda \in \mathfrak{a}_C'$. Then (π_λ, X^λ) is irreducible if and only if λ and $2\rho - \bar\lambda$* (resp. *λ and $2\rho - \lambda$*) *are cyclic.*

Corollary 3.3. *Let $\lambda \in \mathfrak{a}_C'$. Then λ is cyclic if and only if $\bar\lambda$ is cyclic, and X^λ is irreducible if and only lf $X^{\bar\lambda}$ is irreducible.*

Before proceeding we need

Lemma 3.4. *2ρ is cyclic.*

Proof. Suppose that f is a K-finite element of $L^2(K/M)$ such that

$$\int_K (\pi_{2\rho}(g)f_{2\rho})(k)f(k)dk = 0 , \qquad \text{for all } g \in G .$$

Then

$$\int_K (e^{-2\rho(H(g^{-1}k))})f(k)dk = 0 , \qquad \text{for all } g \in G ,$$

which implies that $f \equiv 0$ (see Helgason [3]). Thus $X_e^{2\rho}$ is dense in $X^{2\rho}$, and hence $X^{2\rho} = X_e^{2\rho}$.

4. The map B

We now abstract the situation in §§ 2 and 3. Let X be a complex Hilbert space, and π a continuous representation of G on X. We assume:

(i) The space of K-fixed vectors in X is one dimensional and consists of analytic vectors.

(ii) $\pi(k)$ is unitary for $k \in K$.

(iii) Each irreducible K-submodule of X appears with finite multiplicity.

We note that the representations (π_λ, X^λ) satisfy (i), (ii), (iii).

Let $\mathfrak{p}$ be the orthogonal compliment to $\mathfrak{k}$ in $\mathfrak{g}$ relative to the killing form of $\mathfrak{g}$, and note that $\mathfrak{a} \subset \mathfrak{p}$. Let K act on $\mathfrak{p}$ by the adjoint action, and $S(\mathfrak{p}_C)$ be the symmetric algebra on $\mathfrak{p}_C = \mathfrak{p} \otimes_R C$. Then $S(\mathfrak{p}_C)$ is naturally a K-module.

Let $v_0 \in X$ be a K-fixed vector, and define a map B: $S(\mathfrak{p}_C) \to X$ inductively. Let $S^j(\mathfrak{p}_C)$ be the j^{th} symmetric power of $\mathfrak{p}_C$, and define B_0: $S^0(\mathfrak{p}_C) = C \to X$ via $B_0(c) = cv_0$. Set $V_0 = B_0(S^0(\mathfrak{p}_C))$. Suppose that we have defined $B_0, \cdots, B_n$ and $V_j = B_j(S^j(\mathfrak{p}_C)), j = 0, \cdots, n$. Let $v \mapsto v^{N_n}$ be the projection of X onto the orthogonal compliment of $V_0 + \cdots + V_n$. Define and note that

$$\tilde{B}_{n+1}(X_1, \cdots, X_{n+1}) = ((X_1, \cdots, X_{n+1})v_0)^{N_n},$$

$$\begin{aligned}\tilde{B}_{n+1}(X_1, \cdots, X_i, X_{i+1}, \cdots, X_{n+1}) &- \tilde{B}_{n+1}(X_1, \cdots, X_{i+1}, X_i, \cdots, X_{n+1}) \\ &= (X_1, \cdots, [X_i, X_{i+1}], \cdots, X_{n+1}v_0)^{N_n} = 0 .\end{aligned}$$

Thus $\tilde{B}_{n+1}$ is symmetric, and $\tilde{B}_{n+1}$ induces a linear map B_{n+1}: $S^{n+1}(\mathfrak{p}_C) \to X$. Set $V_{n+1} = B_{n+1}(S^{n+1}(\mathfrak{p}_C))$, and note that

$$B_{n+1}(X_1, \cdots, X_{n+1}) = (X_1 \cdot B_n(X_2, \cdots, X_{n+1}))^{N_n} .$$

Set $B = \sum_{j=0}^{\infty} B_j$.

Lemma 4.1. *B: $S(\mathfrak{p}_C) \to X$ is a K-module homomorphism.*

Proof. Let $X \in \mathfrak{p}$, and let X^j be the j^{th} symmetric power of X. If $k \in K$, then

$k \cdot X^j = (Ad(k)X)^j$. Now

$$B_j(k \cdot X^j) = ((Ad(k) \cdot X)^j v_0)^{N j-1}$$
$$= (kX^j k^{-1} v_0)^{N j-1} = (k \cdot X^j v_0)^{N j-1} = k \cdot (X^j v_0)^{N j-1},$$

which proves the lemma since K acts unitarily.

Let X_e be the smallest closed invariant subspace of X containing v_0.

Lemma 4.2. *X_e is the closure of $B(S(\mathfrak{p}_C))$ in X.*

Proof. Clearly $B(S(\mathfrak{p}_C)) \subset X_e$. We therefore need only to show that $\pi(g)v_0 \in \overline{B(S(\mathfrak{p}_C))}$ for all $g \in G$. If $g \in G$, then $g = \exp X \cdot k$, for $k \in K$, $X \in \mathfrak{p}$. Thus we need only to show that $\pi(\exp X) \cdot v_0 \in \overline{B(S(\mathfrak{p}_C))}$ for each $X \in \mathfrak{p}$. But clearly $X^j \cdot v_0 \in B(S(\mathfrak{p}_C))$ for all j. Since $\pi(\exp X)v_0 = \sum_{j=0}^{\infty} X^j v_0/j!$, v_0 being an analytic vector we have $\pi(\exp X)v_0 \in \overline{B(S(\mathfrak{p}_C))}$, and the lemma is proved.

Let J be the space of all K-fixed elements of $S(\mathfrak{p}_C)$ and let $J^+ = (\sum_{j=1}^{\infty} S^j(\mathfrak{p}_C)) \cap J$.

Lemma 4.3. (a) *Ker B is a homogeneous ideal in $S(\mathfrak{p}_C)$.*

(b) $J^+ \subset \operatorname{Ker} B$.

Proof. (a) is clear from the fact $B_{j+1}(Xu) = (X \cdot B_j(u))^{Nj}$ for $u \in S^j(\mathfrak{p}_C)$, $X \in \mathfrak{p}$.

We prove (b). Let $u \in J^+$. Then $u = \sum_{j=2}^{n} u_j$, for $u_j \in S^j(\mathfrak{p}_C)$, $u_j \in J^+$. Now $B_j(u_j)$ is K-fixed, thus by assumption (i), $B_j(u_j) = cv_0$. But by definition of B_j for $j \geq 2$, $\langle B_j(u_j), v_0 \rangle = 0$. Thus $c = 0$. q.e.d.

In the special case where $(\pi, X) = (\pi_\lambda, X^\lambda)$ and $v_0 = f_\lambda$, we denote B by B^λ.

5. The space H

Extend the restriction of the Killing form on p to be a Hermitian inner product on $S(\mathfrak{p}_C)$ in the canonical manner. Let $I = \operatorname{Ker} B^{2\rho}$, and set $H = I^\perp$ in $S(\mathfrak{p}_C)$.

Lemma 5.1. *Suppose that* $\dim \mathfrak{a} = 1$. *Let $e_1, \cdots, e_n$ be an orthonormal basis of $\mathfrak{p}$, and identify $S(\mathfrak{p}_C)$ with the polynomial mappings of $\mathfrak{p} \to C$ using the complex bilinear extension of the Killing form restricted to $\mathfrak{p}$. If $v \in \mathfrak{p}$, then define $(\partial_v f)(x) = \dfrac{d}{dt}(f(x + tv))|_{t=0}$ for $f \in S(\mathfrak{p}_C)$. Let $\tilde{H} = \operatorname{Ker}\left(\sum_{i=1}^{n} \partial_{e_i}^2\right)$ in $S(\mathfrak{p}_C)$. Then $H = \tilde{H}$.*

Proof. Let $r = \sum_{i=1}^{n} e_j^2 \in S^2(\mathfrak{p})$, and let $x \in \mathfrak{a}$ be a unit vector. Since $\dim \mathfrak{a} = 1$, we see that $Ad(K) \cdot x$ is the unit sphere S of $\mathfrak{p}$, and the isotropy group for this action on S is M. Thus $S = K/M$.

Using the fact $\sum_{i=1}^{n} \partial_{e_i}^2 r = 2n$. It is easy to see that $S(\mathfrak{p}_C) = \tilde{H} \oplus S(\mathfrak{p}_C)r$. Since

$r \in J^+$, from Lemma 4.3 it follows that $\tilde{H} \subset H$. Furthermore, $S(\mathfrak{p}_C)|_S = \tilde{H}|_S$. The Stone-Weierstrauss theorem now implies hat $\tilde{H}|_S$ is the space of all K-finite elements of $L^2(K/M)$. Thus by Frobenious reciprocity, $\tilde{H}|_S = \sum_{\gamma \in \hat{K}_0} (\dim V_\gamma^M)V_\gamma$. Since Lemma 3.4 implies that $H = \sum_{\gamma \in \hat{K}_0} (\dim V_\gamma^M)V_\gamma$. We have $H = \tilde{H}$. q.e.d.

In general, using Lemma 3.4 we see that $H = \sum_{\gamma \in \hat{K}_0} \dim (V_\gamma^M)V_\gamma$ as a K-module, and that $B^{2\rho}\colon H \to L^2(K/M)_F$ is an isomorphism of K-modules, where $L^2(K/M)_F$ is the space of all K-finite elements of $L^2(K/M)$.

Let $H_\gamma = \dim (V_\gamma^M)V_\gamma \subset H$. For each $\lambda \in \mathfrak{a}_C'$ we now define a K-homomorphism $P_\gamma^\lambda\colon H_\gamma \to H_\gamma$ as follows.

Define $\eta_j^\lambda\colon S^j(\mathfrak{p}_C) \to X^\lambda$ by $\eta_j^\lambda(X^j) = X^j \cdot f_\lambda$ for each j and $X \in \mathfrak{p}_C$, and $\eta^\lambda\colon S(\mathfrak{p}_C) \to L^2(K/M)_F$ to be $\sum_{j=0}^{\infty} \eta_j^\lambda$. Here we look upon $\eta^\lambda(u)$ as a function on K/M for $u \in S(\mathfrak{p}_C)$. Then define $P_\gamma^\lambda = (B^{2\rho})^{-1} \circ \eta^\lambda|_{H_\gamma}$, and let $p_\gamma(\lambda) = \det (P_\gamma^\lambda \,|\, H_0^M)$.

Lemma 5.2. (a) *If $p_\gamma(\lambda) \neq 0$ for each $\gamma \in \hat{K}_0$, then λ is cyclic.*

(b) *Suppose* $\dim \mathfrak{a} = 1$. *Then λ is cyclic if and only if $p_\gamma(\lambda) \neq 0$ for all $\gamma \in \hat{K}_0$.*

(c) *Let $m(\gamma) = \max\{j \,|\, V_\gamma \subset H \cap S^j(\mathfrak{p}_C)\}$. Then $p_\gamma\colon \mathfrak{a}_C' \to C$ is a polynomial mapping of degree at most $m(\gamma)(\dim V_\gamma^M)^2$.*

Proof. (a) Suppose that $p_\gamma(\lambda) \neq 0$. Then $P_\gamma^\lambda\colon H_\gamma^M \to H_\gamma^M$ is injective, and the multiplicity of V_γ in $\operatorname{Im} P_\gamma^\lambda$ is $\dim V_\gamma^M$ so that $P_\gamma^\lambda\colon H_\gamma \to H_\gamma$ is injective. Thus, if $p_\gamma(\lambda) \neq 0$ for all $\gamma \in \hat{K}_0$ we see that $\eta^\lambda(S(\mathfrak{p}_C)) = X_F^\lambda$. Clearly $\eta^\lambda(S(\mathfrak{p}_C)) \subset X_e^\lambda$. Hence, if $p_\gamma(\lambda) \neq 0$ for all $\gamma \in \hat{K}_0$, then λ is cyclic.

(b) follows directly from Lemma 5.1.

We now prove (c). Let $\xi\colon S(\mathfrak{p}_C) \to C^\infty(K/M;\mathfrak{a}_C)$(= the C^∞ maps from K/M to $\mathfrak{a}_C$) be defined by

$$\xi(X^j)(k) = \frac{d^j}{dt^j} H(\exp(-tX)k)|_{t=0} \qquad \text{for } X \in \mathfrak{p}\ .$$

(Recall that $H(kan) = \log a$ for $k \in K, a \in A, n \in N$.) Let for each n a nonnegative integer P_n be a polynomial in n-variables recursively defined by:

(i) $P_0 = 1$.

(ii) $P_{n+1}(x_1, \cdots, x_{n+1}) = \sum_{j=1}^{n} \frac{\partial P_n}{\partial x_j} x_{j+1} + P_n x_1$.

It is straightforward to check that $\deg P_n = n$ and that $\eta_j^\lambda(X^j) = P_j(\lambda(\xi(X)), \cdots, \lambda(\xi(X^i)))$ for $X \in \mathfrak{p}$. Now suppose that $H_\gamma \cap S^j(\mathfrak{p}_C) \neq (0)$ for $j = i_1, \cdots, i_m$ and $i_1 < \cdots < i_m = m(\gamma)$. Let X_{j,i_e} be elements of $\mathfrak{p}$ so that $X_{1,i_e}^{i_e}, \cdots, X_{k_e,i_e}^{i_e}$ ($k_e = \dim_R S^{i_e}(\mathfrak{p})$) is a basis of $S^{i_e}(\mathfrak{p}_C)$. If $u \in H_\gamma$, then $u = \sum a_{m,n} X_{m,i_n}^{i_n}$ for $a_{m,n} \in C$, and

$$P_\gamma^\lambda(u) = (B^{2\rho})^{-1}\left(\sum a_{m,n} P_{i_n}(\lambda(\xi(X_{m,i_n}^{i_n})), \cdots, \lambda(\xi(X_{m,i_n}^{i_n}))\right) .$$

Thus $\lambda \mapsto P_\gamma^\lambda$ depends polynomially with λ and has coefficients of degree at most $m(\gamma)$ relative to a basis matrix, and hence (c) follows.

Corollary 5.1. (a) *Let $Z \subset \mathfrak{a}'_C$ be the set of all λ which are not cyclic. Then Z is contained in a countable number of complex algebraic hypersurfaces of $\mathfrak{a}'_C$.*

(b) *If* $\dim \mathfrak{a} = 1$, *then Z is countable.*

In particular almost every element of $\mathfrak{a}'_C$ is cyclic. Thus for almost every element λ of $\mathfrak{a}'_C$, (π_λ, X^λ) is irreducible.

6. The generalized Lorentz group

We note that $p_\gamma(2\rho) = 1$ for all $\gamma \in \hat{K}_0$ by the definition of the $p_\gamma(\lambda)$, and recall that $SO_e(n, 1)$ is the connected open subgroup of the subgroup of $GL(n+1, R)$ leaving the quadratic form $\sum_{i=1}^{n} x_i^2 - x_{n+1}^2$ invariant. Denote by the Universal covering group of $SO_e(n, 1)$ by Spin $(n, 1)$, and let $G = SO_e(n, 1)$ or Spin $(n, 1)$.

Let α be the positive restricted root of $\mathfrak{g}$ relative to $\mathfrak{a}$ and the choice of N. Then $\rho = \left(\frac{n-1}{2}\right)\alpha$. Since $K = SO(n)$ or Spin (n), and the action of K on $\mathfrak{p}$ is the usual action of K on $\boldsymbol{R}^n$, H^j is an irreducible K-module for each j. Furthermore by the classical theory of spherical harmonics, H^j and H^k are inequivalent K-modules if $j \neq k$. Let $p_j(\lambda) = P_\gamma(\lambda\alpha)$ for $\gamma \in \hat{K}_0$ such that $H^j \in \gamma$.

Lemma 6.1. *Let $G = SO_e(n, 1)$ or* Spin$(n, 1)$. *Then $\lambda \in \mathfrak{a}'_C$ is cyclic if and only if* $\dim X_e^\lambda = \infty$.

Proof. The necessity is clear. If $\dim X_e^\lambda = \infty$, then we must have $B_j^\lambda \neq 0$ for each j. But then $X_e^\lambda \supset B_j^\lambda(H^j)$ for each j. Thus $X_e^\lambda \supset X_F^\lambda$, and hence $X^\lambda = X_e^\lambda$.

Theorem 6.1. *Let $G = SO_e(n, 1)$ or* Spin $(n, 1)$. *Then* $p_j(\lambda) = \prod_{k=0}^{j-1} (\lambda + k) / \prod_{k=0}^{j-1} (n + 1 + k)$.

Proof. By the above, $\dim V_\gamma^M = 1$ for each $\gamma \in \hat{K}_0$. Furthermore if $H^j \in \gamma$, then $m(\gamma) = j$, and therefore degree $p_j(\lambda) \leq j$. Furthermore if $p_j(\lambda) = 0$, then $\dim X_e^{\lambda\alpha} < \infty$. Thus by Proposition 3.1, $\lambda\alpha$ is the lowest restricted weight of $X_e^{\lambda\alpha}$. Since the lowest restricted weight μ of a finite dimensional class 1 representation of G must satisfy $\langle \mu, \alpha \rangle / \langle \alpha, \alpha \rangle = -k$, k being a nonnegative integer, by the classical theorem of Cartan and Helgason, $\lambda = -k$ if $\dim X_e^{\lambda\alpha} < \infty$. Let W^k be the irreducible class 1, G-module with lowest restricted weight $-k\alpha$. Then, as a K-module, $W^k = \sum_{j=0}^{k} H^j$, and $p_j(-k) = 0$ for $j > k$. This implies that p_j is a scalar multiple of $\prod_{j=0}^{k} (\lambda + j)$, which is found from the identity $p_j(n + 1) = 1$.

Corollary 6.1. (*Kostant, Helgason*). *Let $G = SO_e(n, 1)$ or* Spin $(n, 1)$.

(a) *$\lambda \in \mathfrak{a}_C'$ is cyclic if and only if $\lambda \neq -k\alpha$, k being a nonpositive integer.*

(b) *(π_λ, X^λ) is irreducible if and anly if $\lambda \neq k\alpha$ with k an integer and $k \leq 0$ or $k \geq n-1$.*

Proof. (a) follows immediately from the proof of Theorem 6.1, and (b) directly from Corollary 3.2.

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