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FUNCTIONS OF TRANSITION FOR CERTAIN KÄHLER MANIFOLDS

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1. Introduction

In [1] Adler has shown that Kähler metrics can be classified by geometric conditions of the image of an isometry into certain Grassmannians. In this paper, we find a necessary condition on the isometry which will guarantee that the original metric was in fact a Hodge metric. (The cohomology class of the fundamental form of the metric belongs to an integral cohomology class.)

Some standard conventions are observed. Differentiable will mean differentiable of class C^∞. If φ is a mapping, φ_* will denote the induced map in tangent spaces. Lower case letters will denote the Lie algebra, upper case letters the Lie group. For example, $o(n)$ will denote the Lie algebra of the orthogonal group $0(n)$. Finally, if g is an element of a matrix group, g^t will denote the transpose of g.

The results of this paper are part of the author's Ph. D. thesis, which was written at Purdue University under A. W. Adler.

The following material can be found in [1]. We include it here for the sake of completeness.

By a modification of Nash's theorem on isometric imbeddings in Euclidean space, it can be shown that every k-dimensional Riemannian manifold M can be isometrically imbedded in S^{k+p-1} (the unit sphere in E^{k+p}) where p is a large positive integer depending on K but not on M.

Let $B^+_{0(2n)} = 0(2n + p)/0(2n) \times 0(p - 1)$. Then $B^+_{0(2n)}$ can be considered as the set of all pairs (P_1, P_2), where P_1 is a $2n$-plane in $E(2n + p)$ through the origin, and P_2 is a vector in $E(2n + p)$ orthogonal to P_1. Let F be an isometric imbedding of a $2n$-dimensional Riemannian manifold M into S^{2n+p-1}. Each point $F(m)$ of $F(M)$ defines an element of $B^+_{0(2n)}$ (i.e., a pair (P_1, P_2)) as follows: P_1 is to be the tangent space of $F(M)$ at $F(m)$ translated to the origin in $E(2n + p)$, and P_2 is to be the position vector of $F(m)$ The mapping $\pi\colon F(M) \to B^+_{0(2n)}$ defined by $\pi(m) = (P_1, P_2)$ is called the spherical image mapping; on composition with F, it determines a map f of M into $B^+_{0(2n)}$.

Let B' be the bundle of orthonormal bases over M. Then B' is the space of all $(2n+1)$-tuples $(m, e_1, \cdots, e_{2n})$, where m is a point in M, and $e_1, \cdots, e_{2n}$ is an orthonormal basis for M_m. Define a mapping

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$$g\colon B' \to V^+_{2n,p} = 0(2n+p)/0(p-1)$$

by

$$g(m\colon e_1, \cdots, e_{2n}) = [F(m), F'_*(e_1), \cdots, F'_*(e_{2n})]\ ,$$

where $F'_*(e_i)$ denotes the vector derived from $F_*(e_i)$ by parallel translation to the origin in E^{2n+p}. The following diagram is the commutative

$$\begin{tikzcd}
B' \arrow[r, "g"] \arrow[d, "\tau"'] & V^+_{2n,p} \arrow[d, "\lambda"] \\
M \arrow[r, "f"] \arrow[dr, "F"'] & B^+_{0(2n)} \arrow[d, "\alpha"] \\
& S^{2n+p-1}
\end{tikzcd}$$

where λ, τ, and α are the natural mappings.

Let M be a hermitian manifold. Then the bundle B' of orthonormal bases of M is reducible to a principal $U(n)$-bundle B over M, and the condition that M be Kähler is equivalent to the existence of a torsionless connection on B. Let $B^+_{U(n)} = 0(2n+p)/U(n) \times 0(p-1)$. Then there is a mapping $\bar{f}$ of M into $B^+_{U(n)}$ such that the following diagram commutes:

$$\begin{tikzcd}
& 0(2n+p) \arrow[d, "\sigma"] \\
B \arrow[r, "g"] \arrow[d, "\tau^1"'] & V^+_{2n,p} \arrow[d, "\tau"] \\
M \arrow[r, "\bar{f}"] \arrow[dr, "f"] \arrow[ddr, "F"'] & B_{U(n)} \arrow[d, "\partial"] \\
& B^+_{0(2n)} \arrow[d, "\alpha"] \\
& S^{2n+p-1}
\end{tikzcd}$$

where τ, τ^1, ∂ are the natural mappings.

Let x be a tangent vector to $0(2n+p)$, and denote by X the element of $0(2n+p)$ defined by x. Define $w(x)$ to be the projection of X into $o(2n)$. Since the 1-form w is horizontal over $V^+_{2n,p}$ and right invariant under the action of $0(p-1)$ there is a $o(2n)$-valued 1-form w^1 on $V^+_{2n,p}$ such that $\sigma^*(w^1) = w$. A vector x on $B^+_{U(n)}$ is said to be H-horizontal if there is a vector y on $V^+_{2n,p}$ with $x = \tau^*(y)$ and $w^1(y) = 0$.

The importance of the notion of H-horizontality is seen in the following theorem:

Theorem 1. *A $2n$-dimensional Riemannian manifold M is a Kähler manifold if and only if it admits a mapping g into $B^+_{U(n)}$ such that:*

(a) *$g(M)$ is H-horizontal,*

(b) *the projection of $g(M)$ into $B^+_{0(2n)}$ is the spherical image of the projection of $g(M)$ into S^{2n+p-1}.*

A $2n$-dimensional submanifold M of $B^+_{U(n)}$ is said to be a K-manifold if it satisfies the following two conditions:

(a) M is H-horizontal, that is, every tangent vector of M is H-horizontal.

(b) The projection of M into $B^+_{0(2n)}$ is the image under the spherical map of the projection of M into S^{2n+p-1}.

By Theorem 1, every K-manifold can be identified with a Kähler manifold. In fact, it can be shown that every K-manifold $\underline{M}$ induces a partial Hermitian metric h in $B^+_{U(n)}$. Let $\hat{\Omega}$ denote the fundamental form of h, and Ω be the restriction of $\hat{\Omega}$ to $\underline{M}$. Then Ω is the fundamental form of the natural Kähler metric and complex structure on $\underline{M}$, and we have

Theorem 2. *Let M be a $2n$-dimensional manifold with Riemannian metric r.*

1. *r is the real part of a Kähler metric of an almost complex structure on M if and only if M admits a differentiable isometric imbedding $\underline{f}$ onto a K-manifold. In case such an $\underline{f}$ exists, it is in fact a homeomorphic isometry with respect to the natural Kähler metric and complex structure of $\underline{f}(M)$.*

2. *If r is the real part of a Kähler metric of a complex analytic structure on M, then $\underline{f}^*(\hat{\underline{\Omega}})$ is the fundamental form of the metric.*

Let x be a tangent vector of $0(2n+p)$, and denote by X the element of $o(2n+p)$ defined by x. Define 1-forms w_0 and w' as follows:

$w_0(x)$ is to be the projection of X into $u(n)$, the Lie algebra of $U(n)$, and $w'(x)$ is to be the projection of X into $o(2n+p-1)$. Identify $o(2n+p)$ with the space of $(2n+p)\times(2n+p)$ skew symmetric real matrices, and denote by $w_{i,j}$ the 1-form which assigns to each matrix its (i,j)th entry, $1 \leq i, j \leq 2n+p$. Let

$$\text{trace Im}\, X = \sum_{k=1}^{n} w_{k,n+k}(w_0(X)) = \sum_{k=1}^{n} w_{k,n+k}(X) .$$

Note that trace Im is invariant under the action of $U(n)\times 0(p-1)$. Finally let Ω' denote the curvature form of w'.

Let M be a compact complex analytic manifold with a Kähler metric h, and $\underline{f}$ be a differential isometric imbedding of M into a k-manifold. Then the 2-forms trace Im dw_0, trace Im $w'\wedge w'$, and trace Im Ω' are horizontal over $\underline{f}(M)$. Since they are also invariant under the right action of $U(n)\times 0(p-1)$, they induce 2-forms on $\underline{f}(M)$. Denote these 2-forms by Trace Im dw_0, Trace Im $w'\wedge w'$, and Trace Im Ω', respectively.

Proposition 1. (a) $(1/2\pi)\underline{f}^*(\text{Trace Im } dw_0)$ *is the first Chern form of the Kähler metric on* M.

(b) $(1/2\pi)\underline{f}^*(\text{Trace Im } \Omega')$ *is the fundamental form of the Kähler metric on* M.

(c) $\text{Trace Im } \Omega' = \text{Trace Im } dw_0 + \text{Trace Im } w' \wedge w'$.

The fact that $w = w_0$ on $f^{-1}[\underline{f}(M)]$ implies that $\text{Trace Im } w' \wedge w' = \sum_{i=1}^{n} \sum_{\alpha=2+1}^{2n+p-1} w_{i\alpha} \wedge w_{i+n\alpha}$. We will denote this form by $\Omega^{\perp}$.

2. A condition

A K-manifold M' contained in $B^{+}_{U(n)}$ will be said to be special if each m' of M' has a neighborhood $V(m')$ which admits a cross-section σ_v into $\delta^{-1}(V)$ such that $d(\sigma_v{}^* w^{\perp}) = 0$, where $w^{\perp}$ denotes the $o(2n + p - 1)$ valued 1-form $w' - w$.

A complex analytic manifold M together with a Kähler metric $K(\ ,\ )$ on M will be said to be a special Kähler manifold if it admits an isometric imbedding F into S^{2n+p-1} (for some p) such that $\underline{f}(M)$ is a special K-manifold. Let D denote covariant differentiation with respect to the connection w' on $O(2n + p)$ as a bundle over S^{2n+p-1}.

Proposition 2. *Let* $(M, K(\ ,\ ))$ *be a special Kähler manifold,* F *be as prescribed, and* $m \in M$. *Then there is an orthonormal basis* $e_1, \cdots, e_{2n+p-1}$ *of vector fields tangent to* S^{2n+p-1} *and defined on some neighborhood* $F(U(m))$ *of* $F(m)$ *such that:*

(a) $e_1, \cdots, e_{2n}$ *is a basis for the tangent space of* $F(U)$,

(b) $e_{2n+1}, \cdots, e_{2n+p-1}$ *is a basis for the orthonormal complement to the tangent space of* $F(U)$,

(c) $dw_{i,\alpha}^{u,e} = 0$ *for* $i = 1, \cdots, 2n$: $\alpha = 2n + 1, \cdots, 2n + p - 1$ *where* $w_{i,\alpha}^{u,e}(x) = \langle D_x e_i, e_\alpha \rangle$.

The converse is also true.

Proof. Given a cross-section σ_u on a neighborhood $\underline{f}(U)$ of a point $\underline{f}(m)$, one gets an orthonormal basis for vector fields tangent to S^{2n+p-1} and defined in a neighborhood $F(U)$ of $F(m)$ satisfying (a) and (b). Conversely, such an orthonormal basis $e_1, \cdots, e_{2n+p-1}$ gives a cross-section $\sigma_u(\underline{f}(m)) = \{F(m), e_1, \cdots, e_{2n+p-1}\}$ defined on the neighborhood $\underline{f}(U)$ of $\underline{f}(m)$. So it suffices to show that $d(\sigma_u{}^*(w^{\perp})) = 0$ if and only if $dw_{i,\alpha}^{u,e} = 0$ for all $i = 1, \cdots, 2n$; $\alpha = 2n + 1, \cdots 2n + p - 1$. But this is immediate since, in fact, $w_{i,\alpha}^{u,e} = \beta^* \sigma_u{}^*((w^{\perp})_{i\alpha})$, this last statement being the equivalence of the Çartan and bundle definitions of a connection.

3. The isomorphism between de Rahm and Čech cohomology for special K-manifolds

Let M be a special K-manifold contained in $B^{+}_{U(n)}$, m a point of M, and

$U(m)$ a neighborhood of m in M admitting a cross-section $\sigma_{u(m)}$ into $0(2n+p)$ such that $\sigma^*_{u(m)}w^\perp$ is closed. Then $\mathscr{U} = \{U(m); m \in M\}$ is an open covering of M. Let $\mathscr{V} = \{V_s; s \in S\}$ be a locally finite (differentiably) simple refinement of the covering $\mathscr{U}$, [3]. Since $\mathscr{V}$ is a refinement of $\mathscr{U}$, each V_s is contained in some member of the covering $\mathscr{U}$. Hence, for each s in S, there is a cross-section σ_s defined on V_s such that $d(\sigma_s{}^*w^\perp) = 0$. Since each V_s is simply connected, there are functions $\underline{h}^s_{i\alpha}$, $i = 1, \cdots, 2n$; $\alpha = 2n+1, \cdots, 2n+p-1$, such that $d\underline{h}^s_{i\alpha} = \sigma_s{}^*w_{i\alpha}$ on V_s. Let $\underline{h}^s$ be the skew symmetric $(2n+p)\times(2n+p)$ matrix whose (i,α)th entry is $\underline{h}^s_{i\alpha}$ for $i = 1, \cdots, 2n$; $\alpha = 2n+1, \cdots, 2n+p-1$ and whose remaining entries above the diagonal are zero. Let h^s be the skew symmetric matrix defined on $\delta^{-1}(V_s)$ by $h^s = \underline{h}^s \circ \delta$, where δ is the natural projection of $0(2n+p)$ onto $B^+_{u(n)}$.

Lemma 1. *On $\sigma_s(V_s)$, $dh^s = w^\perp$.*

Proof. $\sigma^{s*}dh^s = \sigma_s{}^*(d(\underline{h}^s \circ \delta) = \sigma_s{}^*(\delta^* d\underline{h}^s) = (\delta \circ \sigma_s)^*(d\underline{h}^s) = \sigma_s{}^*w^\perp$.

Let R_g denote right translation along the fiber for $\delta^{-1}(V_s)$ by an element g of $U(n) \times 0(p-1)$. Then $\delta \circ R_g = \delta$, for all g in $U(n) \times 0(p-1)$. For each b in $\delta^{-1}(V_s)$ define $g_s(b)$ to be the element of $U(n) \times 0(p-1)$ such that $R_{g_s(b)}(b) = \sigma_s \circ \delta(b)$, that is, $g_s(b)$ is to be the element of $U(n) \times 0(p-1)$ such that right translation by $g_s(b)$ carries b to the point of the cross-section $\sigma_s(V_s)$ lying in the same fiber as b.

Lemma 2. *Let b be any point of $\delta^{-1}(V_s)$. Then*

$$w_b{}^\perp = g_s{}^t(b)(R_{g_s(b)})^*(dh^s(\delta_s \circ \delta(b)))g_s(b) \ .$$

Proof. Since $b \in \delta^{-1}(V_s)$, b can be written as $(m, f_1, \cdots, f_{2n+p-1})$, where $m = \alpha \circ \partial \circ \delta(b), f_1, \cdots, f_{2n}$ is an orthonormal basis for the vector fields tangent to $\alpha \circ \partial(M)$ on some neighborhood of the point m, and $f_{2n+1}, \cdots, f_{2n+p-1}$ is an orthonormal basis for the orthogonal complement to the tangent space of M on this neighborhood. Here, as before, $\alpha \circ \partial$ denotes the natural projection of $B^+_{U(n)}$ onto S^{2n+p-1}. Let $e_1, \cdots, e_{2n+p-1}$ be the orthonormal vector fields defined by the section σ_b. Then by definition of $g_s(b)$, we have

$$(f_1(m), \cdots, f_{2n+p-1}(m)) = (e_1{}^{(m)}, \cdots, e^{(m)}_{2n+p-1})g_s(b) \ ,$$

where $m = \alpha \circ \partial \circ \delta(b)$. Let x be any tangent vector of $\delta^{-1}(V_s)$ at the point b and $\underline{x} = (\alpha \circ \partial \circ \delta)_* x$. Then

$$\begin{aligned}
(w_{i\alpha}(x))_b &= \langle f_i, D_{\underline{x}} f_\alpha \rangle = \left\langle \sum_{k=1}^{2n} g_{ki} e_k, D_{\underline{x}} \sum_{\beta=2n-1}^{2n+p-1} g_{\beta\alpha} e_\beta \right\rangle \\
&= \sum_{k=1}^{2n} \sum_{\beta=2n+1}^{2n+p-1} g_{ki} \langle e_k, D_{\underline{x}}(g_{\beta\alpha} e_\beta) \rangle = \sum_{k=1}^{2n} \sum_{\beta=2n+1}^{2n+p-1} g_{ki} \langle e_k, \underline{x}(g_{\beta\alpha}) e_\beta + g_{\beta\alpha} D_{\underline{x}} e_\beta \rangle \\
&= \sum_{k=1}^{2n} \sum_{\beta=2n+1}^{2n+p-1} g_{ki} \langle e_k, D_{\underline{x}} e_\beta \rangle g_{\beta\alpha} = \sum_{k=1}^{2n} \sum_{\beta=2n+1}^{2n+p-1} g_{ki} [w_{k\beta}(R^*_{g_s(b)}(x)]_{\sigma_s \delta(b)} g_{\beta\alpha} \ .
\end{aligned}$$

where $1 \leq k \leq 2n$, $2n - 1 \leq \beta \leq 2n + p - 1$. Since by Lemma 1, $\sigma_s{}^*dh^s \equiv \sigma_s{}^*w^\perp$, the assertion now follows.

Define an operator T on pairs A_1, A_2 of $(2n + p) \times (2n + p)$ matrices by $T(A_1, A_2) =$ trace Im $[A_1, A_2]$ where $[A_1, A_2]$ denotes the matrix $A_1A_2 - A_2A_1$. Finally, define a 1-form α_s on $\delta^{-1}(V_s)$ by

$$\alpha_s(X)_b = T[g_s{}^t(b)h^s(b)g_s(b), w_b{}^\perp(X)]$$

for any tangent vector X of $\delta^{-1}(V_s)$ at the point b. Recall that $\Omega^\perp$ denotes the 2-form $\sum_{\alpha=2n+1}^{2n+p-1} \sum_{i=1}^{n} w_{i\alpha} \wedge w_{i+n\alpha} =$ trace Im $w^\perp \wedge w^\perp$.

Proposition 3. *On $\delta^{-1}(V_s)$, $d\alpha_s/2 = \Omega^\perp$.*

Proof. Let b be any point of $\delta^{-1}(V_s)$. Then

$$w_b{}^\perp = g_s{}^t(b)(R^*_{g_s(b)}(dh^s{}_{\sigma_s\delta(b)}))g_s(b) .$$

Since trace Im is right invariant under the action of $U(n) \times 0(p - 1)$, we have

$$\begin{aligned}\alpha_s &= T(g^th^sg_s, w^\perp) = \text{trace Im}\,(g_s{}^th^sg_sw^\perp - w^\perp g_s{}^th^sg_s)\\ &= \text{trace Im}\,(g_s{}^th^sg_sg_s{}^t(R^*_{g_s}dh^s)g_s - g_s{}^t(R^*_{g_s}dh^s)g_sg_s{}^th^sg_s)\\ &= \text{trace Im}\,(h^s(b)R^*_{g_s}dh^s{}_{\sigma_s\delta(b)} - R^*_{g_s}dh^s{}_{\sigma_s\delta(b)}h^s) ,\end{aligned}$$

which becomes, in consequence of $h^s(\sigma_s\delta(b)) = h^s(b)$,

$$\alpha_s = \text{trace Im}\,(h^s(b)dh_b{}^s - dh_b{}^sh^s(b)) .$$

Thus

$$\begin{aligned}d\alpha_s &= \text{trace Im}\,(dh_b{}^s \wedge dh_b{}^s + dh_b{}^sdh_b{}^s)\\ &= 2\,\text{trace Im}\,(R^*_{g_s(b)}(dh^s{}_{\sigma_s\delta(b)} \wedge dh^s{}_{\sigma_s\delta(b)})\\ &= 2R^*_{g_s(b)}\,\text{trace Im}\,(w^\perp_{\sigma_s\delta(b)} \wedge w^\perp_{\sigma_s\delta(b)}) = 2\Omega_b{}^\perp ,\end{aligned}$$

since the 2-form $\Omega_b{}^\perp$ is invariant under the right action of $U(n) \times 0(p - 1)$.

Now let r and s be elements of the index set S such that $V_r \cap V_s$ is not empty. For each m in $V_r \cap V_s$, let $g_{rs}(m)$ be the element of $U(n) \times 0(p - 1)$ such that

$$\sigma_s(m) = R_{g_{rs}(m)}(\sigma_r(m)) .$$

If b is any point in $\delta^{-1}(V_r \cap V_s)$, then $g_s(b)g_r{}^t(b) = g_{rs}(\delta(b))$.

Lemma 3. *On $V_r \cap V_s$, the 1-form*

$$T(h^sg_{sr}, h^rdg_{rs}^t) + T(h^sdg_{rs}, h^rg_{rs}^t)$$

is closed.

Proof. By an application of Lemma 3 we have $d\underline{h}^r = g_{rs}^t d\underline{h}^s g_{rs}$. Thus $dg_{rs}^t d\underline{h}^s g_{rs} = g_{rs}^t d\underline{h}^s dg_{rs}$, and

$$\begin{aligned} d(T(\underline{h}^s g_{rs}, \underline{h}^r dg_{rs}^t) &+ T(\underline{h}^s dg_{rs}, \underline{h}^r g_{rs}^t)) \\ &= d \operatorname{trace} \operatorname{Im} (\underline{h}^s g_{rs} \underline{h}^r dg_{rs}^t - \underline{h}^r dg_{rs}^t \underline{h}^s g_{rs}) \\ &\quad + d \operatorname{trace} \operatorname{Im} (-\underline{h}^r g_{rs}^t \underline{h}^s dg_{rs} + \underline{h}^s dg_{rs} \underline{h}^r g_{rs}^t) \\ &= \operatorname{trace} \operatorname{Im} (d(\underline{h}^s g_{rs} \underline{h}^r dg_{rs}^t + \underline{h}^s dg_{rs} \underline{h}^r g_{rs}^t) \\ &\quad - \operatorname{trace} \operatorname{Im} (d(\underline{h}^r dg_{rs}^t \underline{h}^s g_{rs} + \underline{h}^r g_{rs}^t \underline{h}^s dg_{rs})) . \end{aligned}$$

The first term above is equal to

$$\begin{aligned} \operatorname{trace} \operatorname{Im} (d\underline{h}^s g_{rs} \underline{h}^r dg_{rs}^t &+ \underline{h}^s g_{rs} d\underline{h}^r dg_{rs}^t + d\underline{h}^s dg_{rs} \underline{h}^r g_{rs}^t - \underline{h}^s dg_{rs} d\underline{h}^r g_{rs}^t) \\ &= \operatorname{trace} \operatorname{Im} ((-dg_{rs}^t d\underline{h}^s g_{rs} + g_{rs}^t d\underline{h}^s dg_{rs})(\underline{h}^r)) \\ &\quad + \operatorname{trace} \operatorname{Im} (\underline{h}^s (g_{rs} d\underline{h}^r dg_{rs}^t - dg_{rs} d\underline{h}^r g_{rs}^t)) = 0 . \end{aligned}$$

Similarly,

$$\operatorname{trace} \operatorname{Im} (d(\underline{h}^r dg_{rs}^t \underline{h}^s g_{rs} + \underline{h}^r g_{rs}^t \underline{h}^s dg_{rs})) = 0 .$$

Since $V_r \cap V_s$ is simply connected, there exists a function f_{rs}^1 such that

$$df_{rs}^1 = T(\underline{h}^s g_{rs}, \underline{h}^r dg_{rs}^t) + T(\underline{h}^s dg_{rs}, \underline{h}^r g_{rs}^t)$$

on $V_r \cap V_s$. Define a function f_{rs}^2 on $V_r \cap V_s$ by $f_{rs}^2 = T(\underline{h}^s g_{rs}, \underline{h}^r g_{rs}^t)$ and let $\underline{\alpha}_s$ (resp. $\underline{\alpha}_r$) be the 1-form $\sigma_s^*(\alpha_s)$(resp. $\sigma_r^*(\underline{\alpha})$).

Proposition 4. *On $V_r \cap V_s$, $\underline{\alpha}_s - \underline{\alpha}_r = d(f_{rs}^2 - f_{rs}^1)$.*

Proof.
$$\begin{aligned} \alpha_s - \alpha_r &= T(\underline{h}^s, d\underline{h}^s) - T(\underline{h}^r, d\underline{h}^r) \\ &= T(\underline{h}^s, g_{rs} d\underline{h}^r g_{rs}^t) - T(\underline{h}^r, g_{rs}^t d\underline{h}^s g_{rs}) \\ &= T(\underline{h}^s, g_{rs} d\underline{h}^r g_{rs}^t) + T(g_{rs}^t d\underline{h}^s g_{rs}, \underline{h}^r) . \end{aligned}$$

Since $T = \operatorname{trace} \operatorname{Im} ([\ ,\ ])$, and trace Im is invariant under the right action of $U(n) \times 0(p-1)$, we have:

$$\begin{aligned} \alpha_s - \alpha_r &= T(\underline{h}^s g_{rs}, d\underline{h}^r g_{rs}^t) + T(d\underline{h}^s g_{rs}, \underline{h}^r g_{rs}^t) \\ &= d(T(\underline{h}^s g_{rs}, \underline{h}^r g_{rs}^t)) - T(\underline{h}^s g_{rs}, \underline{h}^r dg_{rs}^t) - T(\underline{h}^s dg_{rs}, \underline{h}^r g_{rs}^t) \\ &= d(f_{rs}^2 - f_{rs}^1) . \end{aligned}$$

Now let r, s, and t be any elements of the index set S such that $V_r \cap V_s \cap V_t$ is not empty, and let

$$a_{rst} = (f_{rs}^2 - f_{rt}^2 + f_{st}^2) - (f_{rs}^1 - f_{rt}^1 + f_{st}^1) .$$

Let $\{a\}$ denote the cohomology class of $\hat{H}^2(M,R)$ of which $[(1/4\pi)a_{rst}]$ is a representative, and $\{\text{trace Im } w^\perp \wedge w^\perp\}$ the cohomology class of the 2-form trace Im $w^\perp \wedge w^\perp$ in $H^2(M)$.

Theorem 3. *Let M be a special K-manifold, and $[a]$ as defined above. If ϕ denotes the isomorphism of $H^2(M)$ onto $H^2(V,R)$, then*

$$\phi(\{\text{trace Im } w^\perp \wedge w^\perp\}) = \{a\} .$$

Proof. The assertion follows immediately from Propositions 3 and 4.

4. A sufficient condition for a special Kähler manifold to be a Hodge manifold

Theorem 4. *Let $(M, K(\ ,\ ))$ be a special Kähler manifold. Suppose moreover that the matrices h^s can be chosen so that $dg_{rs}h^r g^t_{rs} + g_{rs}h^r dg^t_{rs}$ vanishes whenever r and s are elements of the index set S such that $V_r \cap V_s$ is not empty. Then $K(\ ,\ )$ is a Hodge metric.*

The remainder of this section is devoted to the proof of this theorem.

Lemma 1. *Under the conditions of Theorem 4 the functions f^1_{rs} can be chosen to be identically zero.*

Proof. By definition,

$$\begin{aligned} df^1_{rs} &= T(h^s g_{rs}, h^r dg^t_{rs}) + T(h^s dg_{rs}, h^r g^t_{rs}) \\ &= T(h^s, g_{rs}h^r dg^t_{rs} + dg_{rs}h^r g^t_{rs}) = 0 \end{aligned}$$

on $V_r \cap V_s$. Thus f^1_{rs} is a constant and, in fact, can be chosen to be zero.

Define constant matrices c_{rs} by $c_{rs} = h^s - g_{rs}h^r g^t_{rs}$ for all r, s in the index set S such that $V_r \cap V_s$ is not empty.

Lemma 2. *If r, s, t are elements of the index set S such that $V_r \cap V_s \cap V_t$ is not empty, then*

$$c_{rt} + g_{rt}c_{sr}g^t_{rt} - c_{st} = 0 .$$

Proof. We have

$$\begin{aligned} h^r &= g_{sr}h^s g^t_{sr} + c_{sr} , \\ h^t &= g_{st}h^s g^t_{st} + c_{st} , \\ h^t &= g_{rt}h^r g^t_{rt} + d_{rt} , \end{aligned}$$

so,

$$\begin{aligned} c_{rt} + g_{rt}c_{sr}g^t_{rt} - c_{st} &= g_{st}h^s g^t_{st} - g_{rt}h^r g^t_{rt} + g_{rt}c_{rs}g^t_{rt} \\ &= g_{rt}(g_{sr}h^s g^t_{rs} - h^r)g^t_{rt} + g_{rt}c_{sr}g^t_{rt} \\ &= -g_{rt}(c_{sr})g^t_{rt} + g_{rt}c_{sr}g^t_{rt} = 0 . \end{aligned}$$

Now fix any element s of the index set S and for each r in S such that $V_r \cap V_s$ is not empty, define $\hat{h}^r = h^r - c_{sr}$.

Lemma 3. *Under the same hypothesis as in Lemma* 2,

$$g_{rt}\hat{h}^r g_{rt} = \hat{h}^t .$$

Proof.
$$\begin{aligned} g_{rt}\hat{h}^r g_{rt}^t &= g_{rt}(h^r - c_{sr})g_{rt}^t = g_{rt}h^r g_{rt}^t - g_{rt}c_{sr}g_{rt}^t \\ &= g_{rt}h^r g_{rt}^t + c_{rt} - c_{st} = h^t - c_{st} = \hat{h}^t . \end{aligned}$$

Let $\tilde{S}$ denote all elements r of the index set S such that $V_r \cap V_s$ is not empty. If u is an element of S such that $V_u \cap V_r$ is not empty for some r in $\tilde{S}$, define $\hat{h}^u = h^u - \hat{c}_{ru}$, where $\hat{c}_{ru}$ is defined by $\hat{c}_{ru} = h^u - g_{ru}\hat{h}^r g_{ru}^t$.

Lemma 4. *$\hat{h}^u$ is well defined.*

Proof. We must show that if r and t are elements of $\tilde{S}$ such that $V_u \cap V_r$ and $V_u \cap V_t$ are not empty, then $h^u - \hat{c}_{ru} = h^u - \hat{c}_{tu}$, that is, $\hat{c}_{ru} = \hat{c}_{tu}$. But, as before, $\hat{c}_{ru} + g_{ru}\hat{c}_{tr}g_{ru}^t - \hat{c}_{tu} = 0$, and, by Lemma 3, $\hat{c}_{tr} = 0$. Hence the lemma follows.

Continuing this process defines matrices $\hat{h}^r$ for each r in the index set S in the same connected component as V_s such that $g_{rt}\hat{h}^r g_{rt}^t = \hat{h}^t$ for all r and t with $V_r \cap V_t$ not empty. Doing this for every connected component gives matrices $\hat{h}^r$ for each r in S such that $g_{rt}\hat{h}^r g_{rt}^t = \hat{h}^t$ for all r and t such that $V_r \cap V_t$ is not empty.

Lemma 5. *The cohomology class* $[a]$ *vanishes.*

Proof. Since $d\hat{h}^s = dh^s$ for all s, and trace Im is invariant under the right action of $U(n) \times 0(p-1)$, a representative of $[a]$ is

$$\begin{aligned} a_{rst} &= T(\hat{h}^s g_{rs}, \hat{h}^r g_{rs}^t) - T(\hat{h}^t g_{rt}, \hat{h}^r g_{rt}') + T(\hat{h}^t g_{st}, \hat{h}^s g_{st}^t) \\ &= T(g_{rs}^t\hat{h}^s g_{rs}, \hat{h}^r) - T(g_{rt}^t\hat{h}^t g_{rt}, \hat{h}^r) + T(g_{st}^t\hat{h}^t g_{st}, \hat{h}^s) . \end{aligned}$$

Since $T(\hat{h}^r, \hat{h}^r) = \text{trace Im}([\hat{h}^r, \hat{h}^r]) = 0$ for all r in S, we have

$$\begin{aligned} a_{rst} &= T(g_{rs}^t\hat{h}^s g_{rs} - \hat{h}^r, \hat{h}^r) - T(g_{rt}^t\hat{h}^t g_{rt} - \hat{h}^r, \hat{h}^r) + T(g_{st}^t\hat{h}^t g_{st} - \hat{h}^s, \hat{h}^s) \\ &= 0 \end{aligned}$$

by the definition of h. Thus the cohomology class $[a]$ vanishes.

Lemma 6. *If* $\{a\}$ *vanishes, then* $K(\, , \,)$ *is a Hodge metric.*

Proof. By Proposition 1, $(1/2\pi)\Omega^{\perp} = \Omega - c_1$, where c_1 is the first Chern form of M, and Ω is the fundamental form of the metric $K(\, , \,)$. By Theorem 3, $\phi(\Omega^{\perp}) = \{a\}$. Thus $\{a\}$ vanishes, so that the first Chern form and the fundamental form of the metric are cohomologous. Since the first Chern form is integral, the assertion follows.

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