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THE FOUR-VERTEX THEOREM IN HYPERBOLIC SPACE

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Let $e_i, i = 1, 2, 3$, be the natural frame field on Minkowski 3-space and $'D$ be the connection such that $'D_V W = (VW^i)e_i$. Using the metric $\langle , \rangle$ of the 3-space which has one minus sign, the hyperbolic plane is represented by $\langle x, x\rangle = -1$. Thus, x is a unit normal of the latter surface and we see that $'D_V x = V$. Denoting by D the induced connection on the hyperbolic plane we have for its tangent vectors

$$(1) \qquad 'D_V W = D_V W + \langle V, W\rangle x \, .$$

On account of (1) we find $R(U, V)W = -\langle V, W\rangle U + \langle U, W\rangle V$, and hence the curvature of our surface is indeed -1.

If T and N designate the unit tangent and normal of a curve in the hyperbolic plane we know that $D_T T = \kappa N$ and $D_T N = -\kappa T$. Now because of (1) $'D_T T = \kappa N + x$. But $'D_T T = {'\kappa}'N$, where $'\kappa$ is the space curvature and $'N$ the space normal to the curve. We therefore record for later reference

$$(2) \qquad ('\kappa)^2 = \kappa^2 - 1 \, .$$

Also, if s stands for arc length we infer from (1) that

$$(3) \qquad 'D_T N = N'(s) = D_T N = -\kappa T = -\kappa x'(s) \, .$$

Through the two vertices which an oval necessarily has we draw a straight line whose equation is $\langle c, x\rangle = 0$. Then with all integrals taken around the oval we conclude in the usual manner with the aid of (3) that

$$\oint \langle c, x\rangle \kappa'(s) ds = -\oint \langle c, x'(s)\rangle \kappa \, ds = \oint \langle c, N'(s)\rangle ds = 0 \, .$$

This establishes the essence of the proof due to Herglotz [2, p. 201].

We now apply our methods to hyperbolic 3-space. In the imbedding Minkowski 4-space we see that $('\kappa)^2 = \langle T'(s), T'(s)\rangle$ is equivalent to

$$(4) \qquad ('\kappa)^2 = (\langle x', x'\rangle\langle x'', x''\rangle - \langle x', x''\rangle^2)/\langle x', x'\rangle^3 \, ,$$

where the primes indicate differentiation with respect to some parameter u.

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Following Gericke [1] we consider a curve which is the rim of a Möbius Band and also lies on a torus. Let r be the radius of the rotating circle, and R be the radius of the locus described by its center. Setting $p = \cosh R \sinh r$ and $a = \tanh R \coth r$ the curve in question is parametrized as follows, $0 \leq u < 4\pi$,

$$\begin{aligned} x^1 &= p[a - \sin(u/2)] \cos u \,, \\ x^2 &= p[a - \sin(u/2)] \sin u \,, \\ x^3 &= p \operatorname{sech} R \cos u/2 \,, \\ x^4 &= p[\coth r - \tanh R \sin(u/2)] \,. \end{aligned}$$

Because of (2) which remains valid for a space curve, we find the maxima and minima of κ differentiating $('\kappa)^2$ which itself is computed by the use of (4). We state the result of the lengthy computation for the given curve.

$$\begin{aligned} &2p^2\langle x', x'\rangle^4[('\kappa)^2]' \\ &\quad = \cos(u/2)\{[3a^5/2 + 3a^3 + (-3 \operatorname{sech}^4 R + 36 \operatorname{sech}^2 R)/32] \\ &\qquad - [21a^4/2 + (9 \operatorname{sech}^2 R + 72)a^2/8 + (9/8) \operatorname{sech}^2 R] \sin(u/2) \\ &\qquad + [24a^3 + (9 \operatorname{sech}^2 R + 72)a/8] \sin^2(u/2) - (24a^3 + 3) \sin^3(u/2) \\ &\qquad + (21/2)a \sin^4(u/2) - (3/2) \sin^5(u/2)\} \,. \end{aligned}$$

Now $\sin(u/2)$ is bounded and the leading term a^5 can be made so large as to make the expression in braces positive. This is accomplished by making r sufficiently small. In this case then vertices occur only at $u = \pi$ and $u = 3\pi$.

References

[1] H. Gericke, *Beispiel einer geschlossenen Raumkurve mit nur zwei Scheiteln*, Jber. Deutsch. Math.-Verein **47** (1937) 22–24.

[2] D. Laugwitz, *Differential and Riemannian geometry*, Academic Press, New York, 1965.

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