

VIRASORO CONSTRAINTS FOR QUANTUM
COHOMOLOGY

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In [8], Eguchi, Hori and Xiong proposed a conjecture that the partition function of topological sigma model coupled to gravity is annihilated by infinitely many differential operators which form a half branch of the Virasoro algebra. A similar conjecture was also proposed by S. Katz [16] (See also [9]). Assuming this conjecture is true, they were able to reproduce certain instanton numbers of some projective spaces known before (cf. the above cited references and [10] for details). This conjecture is also referred to as the Virasoro conjecture by some authors. The main purpose of this paper is to give a proof of this conjecture for the genus zero part.

The theory of topological sigma model coupled to gravity has been extensively studied recently by both mathematicians and physicists. This theory is built on the intersection theory of moduli spaces of stable maps from Riemann surfaces to a fixed manifold V^{2d} , which is a smooth projective variety (or more generally, a symplectic manifold). To each cohomology class of V (denoted by \mathcal{O}) and a non-negative integer n , there is associated a quantum field theory operator, denoted by $\tau_n(\mathcal{O})$. When $n = 0$, the corresponding operator is simply denoted by \mathcal{O} and is called a *primary field*. For $n > 0$, $\tau_n(\mathcal{O})$ is called the n -th (*gravitational*) *descendent* of \mathcal{O} . The so called k -point genus- g *correlators* in topological

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field theory can be defined via the *Gromov-Witten invariants* as follows:

$$\begin{aligned} & \langle \tau_{n_1}(\mathcal{O}_1)\tau_{n_2}(\mathcal{O}_2)\cdots\tau_{n_k}(\mathcal{O}_k) \rangle_g \\ & := \sum_{A \in H_2(V, \mathbb{Z})} q^A \int_{[\overline{\mathcal{M}}_{g,k}(V,A)]^{\text{virt}}} c_1(E_1)^{n_1} \cup \text{ev}_1^*(\mathcal{O}_1) \cup \\ & \quad \cdots \cup c_1(E_k)^{n_k} \cup \text{ev}_k^*(\mathcal{O}_k), \end{aligned}$$

where q^A belongs to the Novikov ring (i.e., the multiplicative ring spanned by monomials $q^A = q_1^{a_1} \cdots q_r^{a_r}$ over the ring of rational numbers, where $\{q_1, \dots, q_r\}$ is a fixed basis of $H_2(V, \mathbb{Z})$ and $A = \sum_{i=1}^r a_i q_i$), $[\overline{\mathcal{M}}_{g,k}(V, A)]^{\text{virt}}$ is the virtual moduli space of degree A stable maps from k -marked genus- g curves to V (cf. [20]), $c_1(E_i)$ is the first Chern class of the tautological line bundle E_i over $[\overline{\mathcal{M}}_{g,k}(V, A)]^{\text{virt}}$ whose fiber over each stable map is defined by the cotangent space of the underlying curve at the i -th marked point, and ev_i is the evaluation map from $[\overline{\mathcal{M}}_{g,k}(V, A)]^{\text{virt}}$ to V defined by evaluating each stable map at the i -th marked point. We also refer to [24] for more discussions in the case of semi-positive symplectic manifolds, which include all Fano Manifolds and Calabi-Yau manifolds as special cases.

All genus- g correlators can be assembled into a generating function, called the genus- g *free energy function*, in the following way:

$$\begin{aligned} F_g(T) & := \left\langle \exp \sum_{n,\alpha} t_n^\alpha \tau_n(\mathcal{O}_\alpha) \right\rangle_g \\ & = \sum_{\{k_{n,\alpha}\}} \left(\prod_{n,\alpha} \frac{(t_n^\alpha)^{k_{n,\alpha}}}{k_{n,\alpha}!} \right) \left\langle \prod_{n,\alpha} (\tau_n(\mathcal{O}_\alpha))^{k_{n,\alpha}} \right\rangle_g, \end{aligned}$$

where $\mathcal{O}_1, \dots, \mathcal{O}_N$ form a basis of $H^*(V, \mathbb{Q})$, α ranges from 1 to N , n ranges over the set of all non-negative integers \mathbb{Z}_+ , and $\{k_{n,\alpha}\}$ ranges over the set of all collections of non-negative integers, almost all (except finite number) of them are zero, labeled by n and α , and

$$T = \{t_n^\alpha \mid n \in \mathbb{Z}_+, \alpha = 1, \dots, N\}$$

is an infinite set of parameters. The space of all parameters T is called the *big phase space*. Its subspace

$$\{T \mid t_n^\alpha = 0 \text{ for all } n > 0\}$$

is called the *small phase space*. The genus zero free energy F_0 restricted to the small phase space is the potential function of the *Quantum cohomology* of V , whose third derivatives define the quantum ring structure on $H^*(V, \mathbb{Q})$. The generating function of all free energy functions, i.e.,

$$Z(T; \lambda) := \exp \sum_{g \geq 0} \lambda^{2g-2} F_g(T),$$

is called the *partition function*, and λ is called the *genus expansion parameter*.

It is widely expected that the partition function Z has many interesting properties. For example, it always satisfies the (generalized) string equation and dilaton equation (cf. [27], [5]). When V is just a point, it was conjectured by Witten [27] and proved by Kontsevich [17] (and also by Witten [28]) that Z is a τ -function of the KdV hierarchy. On the other hand, it is well known that the τ -function of the KdV hierarchy which satisfies the string equation is annihilated by a sequence of differential operators, which form a half branch of the Virasoro algebra (cf. [4], [11], and [15]). For general manifold V , it is not clear at this stage what kind of integrable system might govern Z . However it seems very promising that an analogue of the Virasoro constraints could still exist. In [8], Eguchi, Hori and Xiong constructed a sequence of linear differential operators, denoted by L_n with $n \in \mathbb{Z}$, on the big phase space (see Section 2 for the precise form of these operators). They checked that these operators define a representation of the Virasoro algebra with the central charge equal to the Euler characteristic number of V , i.e., the commutators of these operators satisfy the following relation

$$(1) \quad [L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n} \frac{m(m^2 - 1)}{12} \cdot \chi(V),$$

if the following condition is satisfied:

$$(2) \quad \frac{1}{4} \sum_{\alpha=1}^N b_\alpha (1 - b_\alpha) = \frac{1}{24} \left(\frac{3-d}{2} \chi(V) - \int_V c_1(V) \wedge c_{d-1}(V) \right),$$

where d is equal to a half of the (real) dimension of V , $b_\alpha = \frac{1}{2} \dim(\mathcal{O}_\alpha) - \frac{1}{2}(d-1)$, and $c_i(V)$ is the i -th Chern class of V . Condition (2) is needed in order that $[L_{-1}, L_1] = L_0$. The following conjecture was proposed for Fano manifolds with only even dimensional cohomology classes (See also [9] for a more general conjecture)

Conjecture 0.1 (*Eguchi-Hori-Xiong, Katz*). $L_n Z \equiv 0$ for all $n \geq -1$.

We will call equation $L_n Z = 0$ the L_n constraint. The L_{-1} constraint is the string equation. The L_0 constraint is a combination of the selection rule, the divisor equation and the dilaton equation. All these equations hold for general manifold V (cf. [24] [27], as well as [13]). Moreover, due to the Virasoro type relation (1), if L_1 and L_2 constraints are true, then L_n constraint is true for all $n > 0$.

If we write $(L_n Z)/Z$ as a Laurent series in λ , where λ is the genus expansion parameter, then each L_n constraint gives a sequence of differential equations for the free energy functions F_g , corresponding to the coefficients of different powers of λ . Notice that these differential equations are no longer linear when $n > 0$ since they contain some quadratic terms. The coefficient of λ^{-2} gives a differential equation which only involves genus-0 free energy F_0 . We call this equation the **genus-0 L_n constraint**. If this equation holds, we also say that F_0 satisfies the L_n constraint. The main result of this paper can be stated as

Theorem 0.2. *If V has only even dimensional cohomology classes (or if we only consider even dimensional cohomology classes in the topological sigma model), then the genus-0 free energy F_0 satisfies the L_1 and L_2 constraints.*

Remark. (1) In this theorem, we do not assume that V is Fano. In fact, we even do not assume that V is algebraic. All what are needed in the proof of this theorem are the string equation, the dilaton equation, the genus-0 topological recursion relation, and Hori's L_0 constraint, which in turn follows from the selection rule and the divisor equation (see Sections 1.2 and 1.3 for precise forms of these equations). Therefore this theorem should be true for all manifolds where these equations hold, e.g. smooth projective varieties and semi-positive symplectic manifolds.

(2) In this theorem, we also do not assume condition (2), which is needed to guarantee that $[L_{-1}, L_1] = -2L_0$. The reason behind this is that the constant term in the L_0 operator does not affect the genus-0 constraints. As it was pointed out in [3], if V has only even dimensional cohomology classes, then condition (2) is equivalent to $h^{p,q}(V) = 0$ for $p \neq q$, where $h^{p,q}(V)$ is the Hodge number of V .

(3) As we mentioned above, as long as the Virasoro relation (1) holds for $m, n > 0$, this theorem implies the genus-0 L_n constraint for all $n > 0$. Consequently, the genus-0 part of Conjecture 0.1 is true.

(4) The assumption that V has only even dimensional cohomology classes is not essential. The general case may be treated by the same method. However, in this paper, we only consider this case for simplicity.

This theorem is a combination of Proposition 3.5 and Proposition 4.5, which will be proved in Section 3 and Section 4 respectively. The main idea of our proof can be described as follows: If we consider the first derivative part of Eguchi, Hori, and Xiong's L_n operator as a vector field, denoted by \mathcal{L}_n , on the big phase space, then we have the equation:

$$(3) \quad \sum_{\sigma, \rho=1}^N \langle\langle \mathcal{L}_n (\mathcal{L}_0 - (n+1)\mathcal{D}) \mathcal{O}_\sigma \rangle\rangle_0 \eta^{\sigma\rho} \langle\langle \mathcal{O}_\rho \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle\rangle_0 \\ = \sum_{\sigma, \rho=1}^N \langle\langle \mathcal{L}_n \tau_k(\mathcal{O}_\mu) \mathcal{O}_\sigma \rangle\rangle_0 \eta^{\sigma\rho} \langle\langle \mathcal{O}_\rho (\mathcal{L}_0 - (n+1)\mathcal{D}) \tau_l(\mathcal{O}_\nu) \rangle\rangle_0,$$

where \mathcal{D} is the dilaton vector field defined in Section 1.2, $\langle\langle \cdot \cdot \cdot \rangle\rangle_0$ is the 3-point genus-0 correlation function which is a symmetric tensor on the big phase space defined by the third derivatives of F_0 , and $(\eta^{\sigma\rho})$ is the inverse matrix of the intersection form on $H^*(V, \mathbb{Q})$. This is a simple corollary of the *generalized WDVV equation*:

$$\sum_{\sigma, \rho=1}^N \frac{\partial^3 F_0}{\partial t_m^\alpha \partial t_n^\beta \partial t_0^\sigma} \eta^{\sigma\rho} \frac{\partial^3 F_0}{\partial t_0^\rho \partial t_k^\mu \partial t_l^\nu} = \sum_{\sigma, \rho=1}^N \frac{\partial^3 F_0}{\partial t_m^\alpha \partial t_k^\mu \partial t_0^\sigma} \eta^{\sigma\rho} \frac{\partial^3 F_0}{\partial t_0^\rho \partial t_n^\beta \partial t_l^\nu},$$

which is satisfied by the genus-0 free energy function (cf. [27]). If F_0 satisfies the L_n constraint, then we can compute both sides of equation (3) by using the genus-0 L_0 and L_n constraints and the dilaton equation. The result for the left-hand side is an expression which contains infinitely many terms, while the result for the right-hand side only contains finitely many terms. Our crucial observation is that the difference of these two expressions is the second derivative, i.e., $\frac{\partial}{\partial t_l^\nu} \frac{\partial}{\partial t_k^\mu}$, of a function which does not depend on $\tau_k(\mathcal{O}_\mu)$ and $\tau_l(\mathcal{O}_\nu)$. Moreover, up to some linear terms, this function is just the coefficient of λ^{-2} in the Laurent expansion of $(L_{n+1}Z)/Z$. Vanishing of this function is the genus-0 L_{n+1} constraint. This observation provides us with a general strategy for proving the genus-0 Virasoro constraints, which will be described in more detail in Section 2. Such a strategy could be easily adapted to prove many other constraints, as it will be demonstrated in Section 5.

We would like to mention that in [8], a heuristic argument for deriving the genus-0 L_1 constraint for $\mathbb{C}P^n$ was given. This argument is based on a recursion formula, called fundamental recursion relation, which was discovered in [7] (see also equation (8)). However it seems that there is a serious gap in this argument. What was really proved in [8] is the following:

$$\frac{\partial}{\partial t_0^\alpha} \left(b_\alpha^2 \tilde{\Psi}_{0,1} + b_\alpha \tilde{\Psi}_{0,1} + \Psi_{0,1} \right) = 0,$$

for every $\alpha = 1, \dots, N$, where $\Psi_{0,1}$ and $\tilde{\Psi}_{0,1}$ are two functions on the big phase space which do not depend on α . Integrating this equation with respect to t_0^α and assuming that the integration constant is zero, one obtains

$$b_\alpha^2 \tilde{\Psi}_{0,1} + b_\alpha \tilde{\Psi}_{0,1} + \Psi_{0,1} = 0.$$

If $\dim H^*(V, \mathbb{Q}) \geq 3$, this equation would imply $\Psi_{0,1} = 0$, which is equivalent to the genus-0 L_1 constraint, and $\tilde{\Psi}_{0,1} = 0$ a new constraint called the \tilde{L}_1 constraint (which will also be proved in Section 5 of this paper). However, in this procedure, it is not clear why the integration constant, which still depends on infinitely many other parameters, should be zero. It seems that to prove the vanishing of the integration constant is as difficult as to prove the L_1 constraint itself. We also note that in the derivation of [8], the two sequences of constraints (L_n constraints and \tilde{L}_n constraints) are always mingled together. It is not clear how to separate these two sequences using the original arguments in [8]. In this paper, we note for the first time that these two sequences can be treated independently.

This paper is organized as follows. In Section 1, we first define the basic notation used in this paper. We then review some well known facts about correlation functions and derive some simple but very useful applications of these facts. Virasoro operators of Eguchi, Hori, and Xiong are introduced in Section 2. We then give the precise interpretation of Conjecture 0.1 for free energy functions. At the end of Section 2, we describe a general strategy for using the generalized WDVV equation to prove the genus-0 part of Conjecture 0.1. This strategy is carried out for L_1 and L_2 constraints in Section 3 and Section 4 respectively. In Section 5, we prove two other genus-0 constraints, called \tilde{L}_1 and \tilde{L}_2 constraints, which were also conjectured in [8]. We will discuss higher genus cases in a forthcoming paper.

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1. Relations among different correlation functions

In this section, we review some well known formulas for correlation functions and derive some of their immediate consequences. We will always identify quantum field theory operators $\tau_m(\mathcal{O}_\alpha)$ with the tangent vector fields $\frac{\partial}{\partial t_m^\alpha}$ and view the *genus- g correlation functions*, denoted by $\langle\langle \rangle\rangle_g$, as symmetric tensors on the big phase space defined by

$$\langle\langle \tau_{m_1}(\mathcal{O}_{\alpha_1})\tau_{m_2}(\mathcal{O}_{\alpha_2})\cdots\tau_{m_k}(\mathcal{O}_{\alpha_k}) \rangle\rangle_g := \frac{\partial^k}{\partial t_{m_1}^{\alpha_1}\partial t_{m_2}^{\alpha_2}\cdots\partial t_{m_k}^{\alpha_k}} F_g,$$

where F_g is the genus- g free energy function.

1.1 Convention of notation

We will use the following convention of notation throughout the paper unless otherwise stated. We will use d to denote one half of the real dimension of V . N is the dimension of the space of cohomology classes $H^*(V, \mathbb{Q})$. Lower case Greek letters, e.g. $\alpha, \beta, \gamma, \dots$, etc., will be used to index the cohomology classes. The range of these indices is from 1 to N . Lower case English letters, e.g. i, j, k, m, n, \dots , etc., will be used to index the level of gravitational descendents. Their range is the set of all non-negative integers, i.e., \mathbb{Z}_+ . All summations are over the entire ranges of the indices unless otherwise indicated. We fix a basis $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_N$ of $H^*(V, \mathbb{Q})$ which is arranged in such an order that the dimension of \mathcal{O}_α is non-decreasing with respect to α . In particular, \mathcal{O}_1 is equal to the identity element of the ordinary cohomology ring. Gravitational descendents are denoted by $\tau_m(\mathcal{O}_\alpha)$ whose corresponding parameters are t_m^α , where $m \in \mathbb{Z}_+$ and $\alpha = 1, \dots, N$. $\tau_0(\mathcal{O}_\alpha)$ is always identified with \mathcal{O}_α . We also consider $\tau_m(\mathcal{O}_\alpha)$ with $m < 0$ as a zero operator. Let $\eta_{\alpha\beta} = \int_V \mathcal{O}_\alpha \cup \mathcal{O}_\beta$ be the intersection form on $H^*(V, \mathbb{Q})$. We will use $\eta = (\eta_{\alpha\beta})$ and $\eta^{-1} = (\eta^{\alpha\beta})$ to lower and raise indices. Let $\mathcal{C} = (\mathcal{C}_\alpha^\beta)$ be the matrix of multiplication by the first Chern class $c_1(V)$ in the ordinary cohomology ring, i.e.,

$$(4) \quad c_1(V) \cup \mathcal{O}_\alpha = \sum_\beta \mathcal{C}_\alpha^\beta \mathcal{O}_\beta.$$

Since we are dealing with even dimensional cohomology classes only, both η and $\mathcal{C}\eta$ are symmetric matrices, where the entries of $\mathcal{C}\eta$ are given by $\mathcal{C}_{\alpha\beta} = \int_V c_1(V) \cup \mathcal{O}_\alpha \cup \mathcal{O}_\beta$. Let $q_\alpha = (1/2)\dim(\mathcal{O}_\alpha)$ and

$$(5) \quad b_\alpha = q_\alpha - \frac{1}{2}(d-1).$$

The following simple observations will be used throughout the calculations without mentioning: If $\eta^{\alpha\beta} \neq 0$ or $\eta_{\alpha\beta} \neq 0$, then $b_\alpha = 1 - b_\beta$. $\mathcal{C}_\alpha^\beta \neq 0$ implies $b_\beta = 1 + b_\alpha$, and $\mathcal{C}_{\alpha\beta} \neq 0$ implies $b_\beta = -b_\alpha$.

Instead of coordinates $\{t_m^\alpha \mid m \in \mathbb{Z}_+, \alpha = 1, \dots, N\}$, it is very convenient to use the following shifted coordinates on the big phase space

$$(6) \quad \tilde{t}_m^\alpha = t_m^\alpha - \delta_{m,1}\delta_{\alpha,1} = \begin{cases} t_m^\alpha - 1, & \text{if } m = \alpha = 1, \\ t_m^\alpha, & \text{otherwise.} \end{cases}$$

Notice that these two coordinate systems have different origins.

1.2 Some special vector fields on the big phase space

The first vector field, which will be used extensively later, is the following

$$\mathcal{S} := - \sum_{m,\alpha} \tilde{t}_m^\alpha \frac{\partial}{\partial t_{m-1}^\alpha}.$$

We call this vector field the **string vector field**. The famous *string equation* (cf. [24] and [27]) can be expressed as

$$\langle\langle \mathcal{S} \rangle\rangle_g = \mathcal{S}F_g = \frac{1}{2}\delta_{g,0} \sum_{\alpha,\beta} \eta_{\alpha\beta} t_0^\alpha t_0^\beta.$$

This equation is equivalent to Eguchi, Hori, and Xiong's L_{-1} constraint. Using this equation and the fact that $[\mathcal{S}, \frac{\partial}{\partial t_m^\alpha}] = \frac{\partial}{\partial t_{m-1}^\alpha}$, we can show the following:

Lemma 1.1.

- (1) $\langle\langle \mathcal{S} \rangle\rangle_0 = \frac{1}{2} \sum_{\alpha,\beta} \eta_{\alpha\beta} t_0^\alpha t_0^\beta.$
- (2) $\langle\langle \mathcal{S}\tau_m(\mathcal{O}_\alpha) \rangle\rangle_0 = \langle\langle \tau_{m-1}(\mathcal{O}_\alpha) \rangle\rangle_0 + \delta_{m,0} \sum_{\beta} \eta_{\alpha\beta} t_0^\beta.$
- (3) $\langle\langle \mathcal{S}\tau_m(\mathcal{O}_\alpha)\tau_n(\mathcal{O}_\beta) \rangle\rangle_0 = \langle\langle \tau_m(\mathcal{O}_\alpha)\tau_{n-1}(\mathcal{O}_\beta) \rangle\rangle_0 + \langle\langle \tau_{m-1}(\mathcal{O}_\alpha)\tau_n(\mathcal{O}_\beta) \rangle\rangle_0 + \delta_{m,0}\delta_{n,0}\eta_{\alpha\beta}.$

Another special vector field is

$$\mathcal{D} := - \sum_{m,\alpha} \tilde{t}_m^\alpha \frac{\partial}{\partial t_m^\alpha}.$$

We call \mathcal{D} the **Dilaton vector field**. Notice that some authors call $\tau_1(\mathcal{O}_1)$ the dilaton operator which is different from \mathcal{D} . The name for \mathcal{D} is justified by the so called *dilaton equation*, which can be expressed as

$$\langle\langle \mathcal{D} \rangle\rangle_g = \mathcal{D}F_g = (2g - 2)F_g + \frac{1}{24} \chi(V) \delta_{g,1}.$$

Using this equation and the fact that $[\mathcal{D}, \frac{\partial}{\partial t_m^\alpha}] = \frac{\partial}{\partial t_m^\alpha}$, we can show:

Lemma 1.2.

- (1) $\langle\langle \mathcal{D} \rangle\rangle_0 = -2F_0.$
- (2) $\langle\langle \mathcal{D} \tau_m(\mathcal{O}_\alpha) \rangle\rangle_0 = - \langle\langle \tau_m(\mathcal{O}_\alpha) \rangle\rangle_0.$
- (3) $\langle\langle \mathcal{D} \tau_m(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \equiv 0.$

Probably the most important vector field in deriving the Virasoro constraints is

$$\mathcal{X} := - \sum_{m,\alpha} \left(m + b_\alpha - \frac{3-d}{2} \right) \tilde{t}_m^\alpha \frac{\partial}{\partial t_m^\alpha} - \sum_{m,\alpha,\beta} C_{\alpha}^{\beta} \tilde{t}_m^\alpha \frac{\partial}{\partial t_{m-1}^\beta},$$

where \mathcal{C} is the matrix of multiplication by the first Chern class defined by (4), and b_α is defined by (5). When restricted to the small phase space, \mathcal{X} is the Euler vector field of the Frobenius manifold defined by the restriction of the genus-0 free energy F_0 (cf. [6]). Therefore we also call \mathcal{X} itself the **Euler vector field**. It seems that the significance of this vector field on the big phase space was first noticed in [7] where it is called the perturbed first Chern class. As noted in [7], the divisor equation for the first Chern class $c_1(V)$ together with the selection rule implies the following:

Lemma 1.3.

$$\begin{aligned} \langle\langle \mathcal{X} \rangle\rangle_g = \mathcal{X}F_g &= (3-d)(1-g)F_g + \frac{1}{2} \delta_{g,0} \sum_{\alpha,\beta} C_{\alpha\beta} t_0^\alpha t_0^\beta \\ &\quad - \frac{1}{24} \delta_{g,1} \int_V c_1(V) \cup c_{d-1}(V). \end{aligned}$$

Adopting the language of Frobenius manifolds, we call this equation the *quasi-homogeneity equation*. Using this equation and the fact that

$$[\mathcal{X}, \frac{\partial}{\partial t_m^\alpha}] = \left(m + b_\alpha - \frac{3-d}{2}\right) \frac{\partial}{\partial t_m^\alpha} + \sum_\beta C_\alpha^\beta \frac{\partial}{\partial t_{m-1}^\beta},$$

we can show

Lemma 1.4.

- (1) $\langle\langle \mathcal{X} \rangle\rangle_0 = (3-d)F_0 + \frac{1}{2} \sum_{\alpha,\beta} C_{\alpha\beta} t_0^\alpha t_0^\beta.$
- (2) $\langle\langle \mathcal{X} \tau_m(\mathcal{O}_\alpha) \rangle\rangle_0 = \left(m + b_\alpha + \frac{3-d}{2}\right) \langle\langle \tau_m(\mathcal{O}_\alpha) \rangle\rangle_0 + \sum_\beta C_\alpha^\beta \langle\langle \tau_{m-1}(\mathcal{O}_\beta) \rangle\rangle_0 + \delta_{m,0} \sum_\beta C_{\alpha\beta} t_0^\beta.$
- (3) $\langle\langle \mathcal{X} \tau_m(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \rangle\rangle_0 = \delta_{m,0} \delta_{n,0} C_{\alpha\beta} + (m+n+b_\alpha+b_\beta) \langle\langle \tau_m(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \rangle\rangle_0 + \sum_\gamma C_\alpha^\gamma \langle\langle \tau_{m-1}(\mathcal{O}_\gamma) \tau_n(\mathcal{O}_\beta) \rangle\rangle_0 + \sum_\gamma C_\beta^\gamma \langle\langle \tau_m(\mathcal{O}_\alpha) \tau_{n-1}(\mathcal{O}_\gamma) \rangle\rangle_0.$

Let $\mathcal{L}_0 := -\mathcal{X} - \frac{3-d}{2}\mathcal{D}$. Then the dilaton equation and the quasi-homogeneity equation imply

$$(7) \quad \begin{aligned} \langle\langle \mathcal{L}_0 \rangle\rangle_g = \mathcal{L}_0 F_g &= -\frac{1}{2} \delta_{g,0} \sum_{\alpha,\beta} C_{\alpha\beta} t_0^\alpha t_0^\beta \\ &\quad - \frac{1}{24} \delta_{g,1} \left(\frac{3-d}{2} \chi(V) - \int_V c_1(V) \cup c_{d-1}(V) \right). \end{aligned}$$

This equation was first discovered by Hori [14]. It is equivalent to Eguchi, Hori and Xiong’s L_0 constraint for the partition function.

1.3 Genus-0 topological recursion relation and its applications

Topological recursion relations make it possible to express many correlation functions involving gravitational descendents by those only involve primary fields. Such relations have been proven to exist in genus-0 (cf [24] and [27]) and genus 1 and 2 (cf. [12], [13] and [2]). In this paper we

only consider the genus-0 case. Genus-0 *topological recursion relation* has the following form:

$$\langle\langle \tau_m(\mathcal{O}_\alpha)\tau_n(\mathcal{O}_\beta)\tau_k(\mathcal{O}_\gamma) \rangle\rangle_0 = \sum_{\sigma} \langle\langle \tau_{m-1}(\mathcal{O}_\alpha)\mathcal{O}_\sigma \rangle\rangle_0 \langle\langle \mathcal{O}^\sigma \tau_n(\mathcal{O}_\beta)\tau_k(\mathcal{O}_\gamma) \rangle\rangle_0,$$

for $m > 0$. In this formula, we used the convention that the indices of primary fields are raised by η^{-1} . Therefore \mathcal{O}^σ should be understood as $\sum_{\rho} \eta^{\sigma\rho} \mathcal{O}_\rho$. As noted by Witten ([27]), this recursion relation implies the *generalized WDVV equation*:

$$\begin{aligned} \sum_{\sigma} \langle\langle \tau_m(\mathcal{O}_\alpha)\tau_n(\mathcal{O}_\beta)\mathcal{O}_\sigma \rangle\rangle_0 \langle\langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu)\tau_l(\mathcal{O}_\nu) \rangle\rangle_0 \\ = \sum_{\sigma} \langle\langle \tau_m(\mathcal{O}_\alpha)\tau_k(\mathcal{O}_\mu)\mathcal{O}_\sigma \rangle\rangle_0 \langle\langle \mathcal{O}^\sigma \tau_n(\mathcal{O}_\beta)\tau_l(\mathcal{O}_\nu) \rangle\rangle_0. \end{aligned}$$

When restricted to the space of primary fields, this equation implies the associativity of the algebra defined by the third derivatives of F_0 and η^{-1} . However when gravitational descendents are involved, the exact algebraic structure hidden in this equation seems not very clear. As we will see later in this paper, the genus zero Virasoro constraints are actually disguised in this equation.

Genus-0 topological recursion relation is a recursion formula for 3-point functions. It can be used to derive recursion formulas for 2-point functions when combined with other equations. For example, applying the topological recursion relation to the 3-point functions in Lemma 1.4 (3), we get

$$\begin{aligned} \sum_{\mu,\nu} \{ \delta_{m,0} \delta_{\mu,\alpha} + \langle\langle \tau_{m-1}(\mathcal{O}_\alpha)\mathcal{O}^\mu \rangle\rangle_0 \} \langle\langle \mathcal{O}_\mu \mathcal{X} \mathcal{O}_\nu \rangle\rangle_0 \\ \cdot \{ \langle\langle \mathcal{O}^\nu \tau_{n-1}(\mathcal{O}_\beta) \rangle\rangle_0 + \delta_{n,0} \delta_{\nu,\beta} \} \\ (8) \quad = \delta_{m,0} \delta_{n,0} \mathcal{C}_{\alpha\beta} + (m+n+b_\alpha+b_\beta) \langle\langle \tau_m(\mathcal{O}_\alpha)\tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\ + \sum_{\sigma} \mathcal{C}_\alpha^\sigma \langle\langle \tau_{m-1}(\mathcal{O}_\sigma)\tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\ + \sum_{\sigma} \mathcal{C}_\beta^\sigma \langle\langle \tau_m(\mathcal{O}_\alpha)\tau_{n-1}(\mathcal{O}_\sigma) \rangle\rangle_0. \end{aligned}$$

Notice that, by Lemma 1.4 (3),

$$(9) \quad \langle\langle \mathcal{O}_\mu \mathcal{X} \mathcal{O}_\nu \rangle\rangle_0 = \mathcal{C}_{\mu\nu} + (b_\mu + b_\nu) \langle\langle \mathcal{O}_\mu \mathcal{O}_\nu \rangle\rangle_0,$$

which only involves primary fields. Therefore (8) is really a recursion relation if $m+n+b_\alpha+b_\beta \neq 0$. This recursion relation was first noticed

in [7], where it was called the fundamental recursion relation. It was also used in [8] to give a heuristic argument (with some serious gaps) to the genus-0 Virasoro constraints for $\mathbb{C}P^n$.

Applying the topological recursion relation to the 3-point functions in Lemma 1.1 (3) and notice that

$$\langle\langle \mathcal{O}_\mu \mathcal{S} \mathcal{O}_\nu \rangle\rangle_0 = \eta_{\mu\nu},$$

we get another recursion formula:

$$\begin{aligned} (10) \quad & \langle\langle \tau_m(\mathcal{O}_\alpha) \tau_{n-1}(\mathcal{O}_\beta) \rangle\rangle_0 + \langle\langle \tau_{m-1}(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\ & = \delta_{m,0} \langle\langle \mathcal{O}_\alpha \tau_{n-1}(\mathcal{O}_\beta) \rangle\rangle_0 + \delta_{n,0} \langle\langle \tau_{m-1}(\mathcal{O}_\alpha) \mathcal{O}_\beta \rangle\rangle_0 \\ & \quad + \sum_{\sigma} \langle\langle \tau_{m-1}(\mathcal{O}_\alpha) \mathcal{O}_\sigma \rangle\rangle_0 \langle\langle \mathcal{O}^\sigma \tau_{n-1}(\mathcal{O}_\beta) \rangle\rangle_0. \end{aligned}$$

In this paper, this formula will mainly be used to shift the level of descendents from one primary field to another. It is also interesting to observe that sometimes it is very effective to use this formula to reduce the level of descendents. For example, for $m = n > 0$ and $\alpha = \beta$, this formula takes the following simple form:

$$\langle\langle \tau_m(\mathcal{O}_\alpha) \tau_{m-1}(\mathcal{O}_\alpha) \rangle\rangle_0 = \frac{1}{2} \sum_{\sigma} \langle\langle \tau_{m-1}(\mathcal{O}_\alpha) \mathcal{O}_\sigma \rangle\rangle_0 \langle\langle \mathcal{O}^\sigma \tau_{m-1}(\mathcal{O}_\alpha) \rangle\rangle_0.$$

2. Virasoro operators

In this section, we first give the constructions of Virasoro operators by Eguchi, Hori, and Xiong. We then describe the relationship between these operators and the generalized WDVV equation. This provide us with a general strategy to prove the genus-0 part of the Virasoro constraints. We will use the normalizations in [9], which are more consistent with [24] and [27].

Define

$$(11) \quad L_{-1} := \sum_{m,\alpha} \tilde{t}_m^\alpha \frac{\partial}{\partial t_{m-1}^\alpha} + \frac{1}{2\lambda^2} \sum_{\alpha,\beta} \eta_{\alpha\beta} t_0^\alpha t_0^\beta,$$

$$\begin{aligned}
 L_0 &:= \sum_{m,\alpha} (m + b_\alpha) \tilde{t}_m^\alpha \frac{\partial}{\partial t_m^\alpha} + \sum_{m,\alpha,\beta} C_\alpha^\beta \tilde{t}_m^\alpha \frac{\partial}{\partial t_{m-1}^\beta} \\
 (12) \quad &+ \frac{1}{2\lambda^2} \sum_{\alpha,\beta} C_{\alpha\beta} t_0^\alpha t_0^\beta \\
 &+ \frac{1}{24} \left(\frac{3-d}{2} \chi(V) - \int_V c_1(V) \cup c_{d-1}(V) \right),
 \end{aligned}$$

and for $n \geq 1$,

$$\begin{aligned}
 L_n &:= \sum_{m,\alpha,\beta} \sum_{j=0}^{m+n} A_\alpha^{(j)}(m, n) (C^j)_{\alpha\beta} \tilde{t}_m^\alpha \frac{\partial}{\partial t_{m+n-j}^\beta} \\
 (13) \quad &+ \frac{\lambda^2}{2} \sum_{\alpha,\beta,\gamma} \sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} B_\alpha^{(j)}(k, n) (C^j)_{\alpha\beta} \eta^{\alpha\gamma} \frac{\partial}{\partial t_k^\gamma} \frac{\partial}{\partial t_{n-k-1-j}^\beta} \\
 &+ \frac{1}{2\lambda^2} \sum_{\alpha,\beta} (C^{n+1})_{\alpha\beta} t_0^\alpha t_0^\beta,
 \end{aligned}$$

where C^j is the j -th power of the matrix C , $(C^{n+1})_{\alpha\beta}$ are entries of the matrix $C^{n+1}\eta$, $A_\alpha^{(j)}(m, n)$ and $B_\alpha^{(j)}(k, n)$ are constants defined in terms of Gamma function by

$$A_\alpha^{(j)}(m, n) := \frac{\Gamma(b_\alpha + m + n + 1)}{\Gamma(b_\alpha + m)} \sum_{m \leq l_1 < l_2 < \dots < l_j \leq m+n} \left(\prod_{i=1}^j \frac{1}{b_\alpha + l_i} \right),$$

and

$$\begin{aligned}
 B_\alpha^{(j)}(m, n) &:= \frac{\Gamma(m + 2 - b_\alpha) \Gamma(n - m + b_\alpha)}{\Gamma(1 - b_\alpha) \Gamma(b_\alpha)} \\
 &\sum_{-m-1 \leq l_1 < l_2 < \dots < l_j \leq n-m-1} \left(\prod_{i=1}^j \frac{1}{b_\alpha + l_i} \right).
 \end{aligned}$$

When $j = 0$, the last factors in $A_\alpha^{(j)}(m, n)$ and $B_\alpha^{(j)}(m, n)$ should be understood as equal to 1. Any term which contains t_m^α with $m < 0$ should be understood as zero. Eguchi, Hori, and Xiong also construct L_{-n} for $n > 0$. However, the significance of these operators is not clear and we do not deal with them in this paper.

It is well known that $L_n Z(T; \lambda) \equiv 0$ for $n = -1$ or 0 , where $T = \{t_m^\alpha \mid m \in \mathbb{Z}_+, \alpha = 1, \dots, N\}$ and $Z(T; \lambda)$ is the partition function

defined in the introduction. The first equation (i.e., for $n = -1$) is the string equation. The second equation (i.e., for $n = 0$) is equivalent to (7). The analogous equation for $n \geq 1$ is the content of Conjecture 0.1. Let $\Psi_{g,n}(T)$ be the coefficient of λ^{2g-2} in the Laurent expansion of $(L_n Z(T; \lambda))/Z(T; \lambda)$. In other words, $\Psi_{g,n}$ is defined by

$$(14) \quad L_n Z(T; \lambda) = \left\{ \sum_{g \geq 0} \Psi_{g,n} \lambda^{2g-2} \right\} Z(T; \lambda).$$

We call the equation $L_n Z = 0$ the L_n -constraint for the partition function; it is equivalent to $\Psi_{g,n} = 0$ for all g . The equation $\Psi_{g,n} = 0$ will be called **genus- g L_n -constraint**. For $n = -1$ or 0 , this is a first order linear differential equation for the genus- g free energy F_g . When $n \geq 1$, it is a second order non-linear differential equation involving all free energy functions $F_{g'}$ with $0 \leq g' \leq g$. The genus-0 constraints are special in the sense that only F_0 is involved in these equations. It is straightforward to check the following fact:

Lemma 2.1. *Suppose that the L_n operators satisfy the Virasoro relation*

$$[L_m, L_n] = (m - n)L_{m+n} \quad \text{for } m, n \geq 1.$$

Given $m, n \geq 1$ and $m \neq n$, if $\Psi_{g',m} = \Psi_{g',n} \equiv 0$ for all g' satisfying $0 \leq g' \leq g$, then $\Psi_{g,m+n} \equiv 0$.

In this paper, we are only interested in the genus-0 constraints $\Psi_{0,n} = 0$. We first observe that to prove the genus-0 L_n constraints, it suffices to show that all second derivatives of $\Psi_{0,n}$ vanish. In fact, Lemma 1.2 (2) and (3) at the origin trivially imply the following:

$$\left. \frac{\partial^2}{\partial t_1^1 \partial t_k^\mu} \Psi_{0,n} \right|_{T=0} = - \left. \frac{\partial}{\partial t_k^\mu} \Psi_{0,n} \right|_{T=0} \quad \text{and} \quad \left. \frac{\partial}{\partial t_1^1} \Psi_{0,n} \right|_{T=0} = -2\Psi_{0,n}|_{T=0}.$$

(Same formulas also hold for $\tilde{\Psi}_{0,n}$ defined in Section 5.) Therefore once we know that all the second derivatives of $\Psi_{0,n}$ are zero, $\Psi_{0,n}$ and all of its first derivatives have to vanish at the origin. Consequently $\Psi_{0,n}$ is constantly equal to zero.

It is also of interest to observe that all the vector fields introduced in Section 1.2, i.e., \mathcal{S} , \mathcal{D} , and \mathcal{X} , vanish at a very special point

$$\tilde{T}_0 = \{\tilde{t}_m^\alpha = 0 \mid m \in \mathbb{Z}_+, \alpha = 1, \dots, N\}.$$

It follows from Lemma 1.2 (2) that all 1-point genus-0 correlation functions vanish at this point, i.e.,

$$(15) \quad \langle\langle \tau_m(\mathcal{O}_\alpha) \rangle\rangle_0 |_{\tilde{T}_0} = 0$$

for all m and α . Consequently, $\Psi_{0,n}$ and all of its first partial derivatives vanish at \tilde{T}_0 since each term of these functions contains either \tilde{t}_m^α for some α and m , or a 1-point genus-0 correlation function. However, there is a little problem with this argument since the genus-0 energy function is just a formal power series at the origin and may not converge at \tilde{T}_0 (we would like to thank Getzler for pointing out this to us). Although one might expect that such a nice function should converge, rigorously speaking, we need to use the arguments in the last paragraph, which are simply obtained by applying Lemma 1.2 at another point.

In the rest of this paper, we will show that all the second derivatives of $\Psi_{0,n}$ vanish by using the generalized WDVV equation as described in the following strategy. Write the first derivative part of the operator L_n as a vector field \mathcal{L}_n on the big phase space. We already saw two of these vector fields in Section 1.2, i.e., $\mathcal{L}_{-1} = -\mathcal{S}$ and $\mathcal{L}_0 = -\mathcal{X} - \frac{3-d}{2}\mathcal{D}$. For any two operators $\tau_k(\mathcal{O}_\mu)$ and $\tau_l(\mathcal{O}_\nu)$, the generalized WDVV equation implies

$$\begin{aligned} \sum_{\alpha} \langle\langle \mathcal{L}_n(\mathcal{L}_0 - (n+1)\mathcal{D})\mathcal{O}_\alpha \rangle\rangle_0 \langle\langle \mathcal{O}^\alpha \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle\rangle_0 \\ = \sum_{\alpha} \langle\langle \mathcal{L}_n \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \rangle\rangle_0 \langle\langle \mathcal{O}^\alpha (\mathcal{L}_0 - (n+1)\mathcal{D}) \tau_l(\mathcal{O}_\nu) \rangle\rangle_0. \end{aligned}$$

Compute both sides of this equation by using the genus-0 L_n constraint (which is assumed to be true). It can be shown that the difference of the resulting expressions is equal to $\frac{\partial^2}{\partial t_l^\nu \partial t_k^\mu} \Psi_{0,n+1}$. Therefore the generalized WDVV equation implies that all the second derivatives of $\Psi_{0,n+1}$ are zero. As noted above, this proves the genus-0 L_{n+1} constraint. Although the computation involved in this process is a little tedious, it is in fact quite straightforward. The only subtleties here, if there is any, are when and where to use the recursion formula (10) and Lemma 1.4. In the rest of the paper, we carry out this strategy for the L_1 and L_2 constraints in full details. Due to the existence of the Virasoro type relations between L_n operators, this implies all the genus-0 Virasoro constraints.

3. L_1 constraint for genus zero free energy function

As explained in Section 2, the genus-0 L_1 constraint is equivalent to the equation $\Psi_{0,1} = 0$, where

$$\begin{aligned}
 \Psi_{0,1} = & \sum_{m,\alpha} (m + b_\alpha)(m + b_\alpha + 1) \tilde{t}_m^\alpha \langle \langle \tau_{m+1}(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 & + \sum_{m,\alpha,\beta} (2m + 2b_\alpha + 1) C_\alpha^\beta \tilde{t}_m^\alpha \langle \langle \tau_m(\mathcal{O}_\beta) \rangle \rangle_0 \\
 (16) \quad & + \sum_{m,\alpha,\beta} (C^2)_\alpha^\beta \tilde{t}_m^\alpha \langle \langle \tau_{m-1}(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & + \frac{1}{2} \sum_\alpha b_\alpha (1 - b_\alpha) \langle \langle \mathcal{O}_\alpha \rangle \rangle_0 \langle \langle \mathcal{O}^\alpha \rangle \rangle_0 \\
 & + \frac{1}{2} \sum_{\alpha,\beta} (C^2)_{\alpha\beta} t_0^\alpha t_0^\beta.
 \end{aligned}$$

As noted at the end of Section 2, to prove $\Psi_{0,1} = 0$, it suffices to show that all the second partial derivatives of $\Psi_{0,1}$ are equal to zero. We will see that this fact actually follows from the generalized WDVV equation. According to the general strategy described at the end of Section 2, we should compute 3-point correlation functions involving two vector fields:

$$\mathcal{L}_0 = -\mathcal{X} - \frac{3-d}{2} \mathcal{D} = \sum_{m,\alpha} (m + b_\alpha) \tilde{t}_m^\alpha \frac{\partial}{\partial t_m^\alpha} + \sum_{m,\alpha,\beta} C_\alpha^\beta \tilde{t}_m^\alpha \frac{\partial}{\partial t_{m-1}^\beta}$$

and

$$\mathcal{L}_0 - \mathcal{D} = \sum_{m,\alpha} (m + b_\alpha + 1) \tilde{t}_m^\alpha \frac{\partial}{\partial t_m^\alpha} + \sum_{m,\alpha,\beta} C_\alpha^\beta \tilde{t}_m^\alpha \frac{\partial}{\partial t_{m-1}^\beta}.$$

We first compute the following 3-point correlation function:

Lemma 3.1.

$$\begin{aligned}
 & \langle \langle \mathcal{L}_0(\mathcal{L}_0 - \mathcal{D})\tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 &= - \sum_{n,\sigma} (n + b_\sigma)(n + b_\sigma + 1) \tilde{t}_n^\sigma \langle \langle \tau_n(\mathcal{O}_\sigma)\tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 &\quad - \sum_{n,\sigma,\rho} (2n + 2b_\sigma + 1) \mathcal{C}_\sigma^\rho \tilde{t}_n^\sigma \langle \langle \tau_{n-1}(\mathcal{O}_\rho)\tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 &\quad - \sum_{n,\sigma,\rho} (\mathcal{C}^2)_\sigma^\rho \tilde{t}_n^\sigma \langle \langle \tau_{n-2}(\mathcal{O}_\rho)\tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 &\quad + (m + b_\alpha)(m + b_\alpha - 1) \langle \langle \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 &\quad + \sum_\sigma (b_\alpha + b_\sigma + 2m - 2) \mathcal{C}_\alpha^\sigma \langle \langle \tau_{m-1}(\mathcal{O}_\sigma) \rangle \rangle_0 \\
 &\quad + \sum_\sigma (\mathcal{C}^2)_\alpha^\sigma \langle \langle \tau_{m-2}(\mathcal{O}_\sigma) \rangle \rangle_0 \\
 &\quad + \delta_{m,0} \left\{ \sum_\sigma (2b_\alpha - 1) \mathcal{C}_{\alpha\sigma} t_0^\sigma - \sum_\sigma (\mathcal{C}^2)_{\alpha\sigma} \tilde{t}_1^\sigma \right\} \\
 &\quad + \delta_{m,1} \sum_\sigma (\mathcal{C}^2)_{\alpha\sigma} t_0^\sigma.
 \end{aligned}$$

Proof. By Lemma 1.2 (3),

$$\begin{aligned}
 & \langle \langle \mathcal{L}_0\tau_n(\mathcal{O}_\beta)\tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 = - \langle \langle \mathcal{X}\tau_n(\mathcal{O}_\beta)\tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 &= -\delta_{m,0}\delta_{n,0}\mathcal{C}_{\alpha\beta} - (m + n + b_\alpha + b_\beta) \langle \langle \tau_n(\mathcal{O}_\beta)\tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 &\quad - \sum_\gamma \mathcal{C}_\alpha^\gamma \langle \langle \tau_n(\mathcal{O}_\beta)\tau_{m-1}(\mathcal{O}_\gamma) \rangle \rangle_0 - \sum_\gamma \mathcal{C}_\beta^\gamma \langle \langle \tau_{n-1}(\mathcal{O}_\gamma)\tau_m(\mathcal{O}_\alpha) \rangle \rangle_0.
 \end{aligned}$$

Hence

$$\begin{aligned}
 & \langle \langle \mathcal{L}_0(\mathcal{L}_0 - \mathcal{D})\tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 &= \sum_{n,\beta} (n + b_\beta + 1) \tilde{t}_n^\beta \langle \langle \mathcal{L}_0\tau_n(\mathcal{O}_\beta)\tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 &\quad + \sum_{n,\beta,\sigma} \mathcal{C}_\beta^\sigma \tilde{t}_n^\beta \langle \langle \mathcal{L}_0\tau_{n-1}(\mathcal{O}_\sigma)\tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 &= - \sum_{n,\beta} (n + b_\beta + 1)(m + n + b_\alpha + b_\beta) \tilde{t}_n^\beta \langle \langle \tau_n(\mathcal{O}_\beta)\tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 &\quad - \sum_{n,\beta,\gamma} (n + b_\beta + 1) \mathcal{C}_\alpha^\gamma \tilde{t}_n^\beta \langle \langle \tau_n(\mathcal{O}_\beta)\tau_{m-1}(\mathcal{O}_\gamma) \rangle \rangle_0
 \end{aligned}$$

$$\begin{aligned}
& - \sum_{n,\beta,\gamma} (n + b_\beta + 1) C_\beta^\gamma \tilde{t}_n^\beta \langle \langle \tau_{n-1}(\mathcal{O}_\gamma) \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
& - \delta_{m,0} \delta_{n,0} \sum_{n,\beta} (n + b_\beta + 1) C_{\alpha\beta} \tilde{t}_n^\beta \\
& - \sum_{n,\beta,\sigma} (m + n + b_\alpha + b_\sigma - 1) C_\beta^\sigma \tilde{t}_n^\beta \langle \langle \tau_{n-1}(\mathcal{O}_\sigma) \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
& - \sum_{n,\beta,\sigma,\gamma} C_\beta^\sigma C_\alpha^\gamma \tilde{t}_n^\beta \langle \langle \tau_{n-1}(\mathcal{O}_\sigma) \tau_{m-1}(\mathcal{O}_\gamma) \rangle \rangle_0 \\
& - \sum_{n,\beta,\sigma,\gamma} C_\beta^\sigma C_\sigma^\gamma \tilde{t}_n^\beta \langle \langle \tau_{n-2}(\mathcal{O}_\gamma) \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
(17) \quad & - \delta_{m,0} \delta_{n,1} \sum_{n,\beta,\sigma} C_\beta^\sigma C_{\alpha\sigma} \tilde{t}_n^\beta \\
= & - \sum_{n,\beta} (n + b_\beta)(n + b_\beta + 1) \tilde{t}_n^\beta \langle \langle \tau_n(\mathcal{O}_\beta) \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
& - \sum_{n,\beta,\gamma} (2n + 2b_\beta + 1) C_\beta^\gamma \tilde{t}_n^\beta \langle \langle \tau_{n-1}(\mathcal{O}_\gamma) \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
& - \sum_{n,\beta,\gamma} (C^2)_\beta^\gamma \tilde{t}_n^\beta \langle \langle \tau_{n-2}(\mathcal{O}_\gamma) \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
& - \delta_{m,0} \sum_\beta (b_\beta + 1) C_{\alpha\beta} \tilde{t}_0^\beta \\
& - \delta_{m,0} \sum_\beta (C^2)_{\alpha\beta} \tilde{t}_1^\beta \\
& - \sum_{n,\beta} (n + b_\beta + 1)(m + b_\alpha) \tilde{t}_n^\beta \langle \langle \tau_n(\mathcal{O}_\beta) \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
& - \sum_{n,\beta,\sigma} (m + b_\alpha) C_\beta^\sigma \tilde{t}_n^\beta \langle \langle \tau_{n-1}(\mathcal{O}_\sigma) \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
& - \sum_{n,\beta,\gamma} (n + b_\beta + 1) C_\alpha^\gamma \tilde{t}_n^\beta \langle \langle \tau_n(\mathcal{O}_\beta) \tau_{m-1}(\mathcal{O}_\gamma) \rangle \rangle_0 \\
& - \sum_{n,\beta,\sigma,\gamma} C_\beta^\sigma C_\alpha^\gamma \tilde{t}_n^\beta \langle \langle \tau_{n-1}(\mathcal{O}_\sigma) \tau_{m-1}(\mathcal{O}_\gamma) \rangle \rangle_0.
\end{aligned}$$

On the other hand, by Lemma 1.2 (2) and Lemma 1.4 (2) we have

$$\begin{aligned}
 & \sum_{n,\beta} (n + b_\beta + 1) \tilde{t}_n^\beta \langle \langle \tau_n(\mathcal{O}_\beta) \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 & \quad + \sum_{n,\beta,\sigma} C_\beta^\sigma \tilde{t}_n^\beta \langle \langle \tau_{n-1}(\mathcal{O}_\sigma) \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 (18) \quad & = \langle \langle (\mathcal{L}_0 - \mathcal{D}) \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 & = - \langle \langle \mathcal{X} \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 - \frac{5-d}{2} \langle \langle \mathcal{D} \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 & = - (m + b_\alpha - 1) \langle \langle \tau_m(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 & \quad - \sum_\sigma C_\alpha^\sigma \langle \langle \tau_{m-1}(\mathcal{O}_\sigma) \rangle \rangle_0 - \delta_{m,0} \sum_\sigma C_{\alpha\sigma} t_0^\sigma.
 \end{aligned}$$

The lemma then follows by applying (18) to the last 4 terms in (17).

q.e.d.

Setting $m = 0$ in Lemma 3.1 , multiplying both sides of the equation by $\langle \langle \mathcal{O}^\alpha \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0$, and summing over α , then applying the genus-0 topological recursion relation, we get

$$\begin{aligned}
 & \sum_\alpha \langle \langle \mathcal{L}_0(\mathcal{L}_0 - \mathcal{D}) \mathcal{O}_\alpha \rangle \rangle_0 \langle \langle \mathcal{O}^\alpha \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & = - \sum_{n,\sigma} (n + b_\sigma)(n + b_\sigma + 1) \tilde{t}_n^\sigma \langle \langle \tau_{n+1}(\mathcal{O}_\sigma) \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 (19) \quad & - \sum_{n,\sigma,\rho} (2n + 2b_\sigma + 1) C_\sigma^\rho \tilde{t}_n^\sigma \langle \langle \tau_n(\mathcal{O}_\rho) \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & - \sum_{n,\sigma,\rho} (C^2)_\sigma^\rho \tilde{t}_n^\sigma \langle \langle \tau_{n-1}(\mathcal{O}_\rho) \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & + \sum_\alpha b_\alpha (b_\alpha - 1) \langle \langle \mathcal{O}_\alpha \rangle \rangle_0 \langle \langle \mathcal{O}^\alpha \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0.
 \end{aligned}$$

Notice that the ranges of summations may change when using the topological recursion relation. Hence some scattered terms may be absorbed into a big summation after using the topological recursion relation.

On the other hand, using Lemma 1.2 (3) and Lemma 1.4 (3), we

have

$$\begin{aligned}
 & \sum_{\alpha, \beta} \langle \langle \mathcal{L}_0 \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \rangle \rangle_0 \eta^{\alpha\beta} \langle \langle \mathcal{O}_\beta (\mathcal{L}_0 - \mathcal{D}) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &= \sum_{\alpha, \beta} \langle \langle \mathcal{X} \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \rangle \rangle_0 \eta^{\alpha\beta} \langle \langle \mathcal{O}_\beta \mathcal{X} \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 (20) \quad &= \sum_{\alpha, \beta} \left\{ \delta_{k,0} C_{\mu\alpha} + (k + b_\mu + b_\alpha) \langle \langle \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \rangle \rangle_0 \right. \\
 &\quad \left. + \sum_{\sigma} C_{\mu}^{\sigma} \langle \langle \tau_{k-1}(\mathcal{O}_\sigma) \mathcal{O}_\alpha \rangle \rangle_0 \right\} \eta^{\alpha\beta} \\
 &\quad \cdot \left\{ \delta_{l,0} C_{\nu\beta} + (l + b_\nu + b_\beta) \langle \langle \tau_l(\mathcal{O}_\nu) \mathcal{O}_\beta \rangle \rangle_0 \right. \\
 &\quad \left. + \sum_{\rho} C_{\nu}^{\rho} \langle \langle \tau_{l-1}(\mathcal{O}_\rho) \mathcal{O}_\beta \rangle \rangle_0 \right\}.
 \end{aligned}$$

The generalized WDVV equation implies that the left-hand sides of equations (19) and (20) are equal. However, the right-hand sides of these two equations appear very different from each other. One obvious distinction between them is that the right-hand side of (20) has only finitely many terms, while the right-hand side of (19) has infinitely many terms due to the existence of infinitely many gravitational descendants. In the rest of this section, we will show that the difference of these two expressions is $\frac{\partial}{\partial t_\nu} \frac{\partial}{\partial t_k} \Psi_{0,1}$.

We first prove two lemmas which express certain quadratic functions of 2-point correlation functions in terms of linear functions of correlation functions.

Lemma 3.2.

$$\begin{aligned}
 & \sum_{\alpha} \{ b_\alpha (k + b_\mu - l - b_\nu) - (k + b_\mu)(l + b_\nu + 1) \} \\
 &\quad \cdot \langle \langle \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \rangle \rangle_0 \langle \langle \mathcal{O}^\alpha \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &= (k + b_\mu - l - b_\nu) \sum_{\alpha} \{ C_{\nu}^{\alpha} \langle \langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 &\quad \quad \quad - C_{\mu}^{\alpha} \langle \langle \tau_k(\mathcal{O}_\alpha) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \} \\
 &\quad - (k + b_\mu)(k + b_\mu + 1) \langle \langle \tau_{k+1}(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &\quad - (l + b_\nu)(l + b_\nu + 1) \langle \langle \tau_k(\mathcal{O}_\mu) \tau_{l+1}(\mathcal{O}_\nu) \rangle \rangle_0.
 \end{aligned}$$

Proof. Let

$$\begin{aligned}
 f &:= \sum_{\alpha} \{b_{\alpha}(k + b_{\mu} - l - b_{\nu}) - (k + b_{\mu})(l + b_{\nu} + 1)\} \\
 &\quad \cdot \langle\langle \tau_k(\mathcal{O}_{\mu})\mathcal{O}_{\alpha} \rangle\rangle_0 \langle\langle \mathcal{O}^{\alpha}\tau_l(\mathcal{O}_{\nu}) \rangle\rangle_0 \\
 &= (k + b_{\mu} - l - b_{\nu}) \sum_{\alpha} (b_{\alpha} - b_{\nu} - l - 1) \langle\langle \tau_k(\mathcal{O}_{\mu})\mathcal{O}_{\alpha} \rangle\rangle_0 \langle\langle \mathcal{O}^{\alpha}\tau_l(\mathcal{O}_{\nu}) \rangle\rangle_0 \\
 &\quad - (b_{\nu} + l)(b_{\nu} + l + 1) \sum_{\alpha} \langle\langle \tau_k(\mathcal{O}_{\mu})\mathcal{O}_{\alpha} \rangle\rangle_0 \langle\langle \mathcal{O}^{\alpha}\tau_l(\mathcal{O}_{\nu}) \rangle\rangle_0.
 \end{aligned}$$

Application of Lemma 1.4 (3) to the first term and of the recursion formula (10) to the second term leads to

$$\begin{aligned}
 f &= (k + b_{\mu} - l - b_{\nu}) \sum_{\alpha} \langle\langle \tau_k(\mathcal{O}_{\mu})\mathcal{O}_{\alpha} \rangle\rangle_0 \\
 &\quad \cdot \left\{ - \langle\langle \mathcal{O}^{\alpha}\mathcal{X}\tau_l(\mathcal{O}_{\nu}) \rangle\rangle_0 + \sum_{\sigma} c_{\nu}^{\sigma} \langle\langle \mathcal{O}^{\alpha}\tau_{l-1}(\mathcal{O}_{\sigma}) \rangle\rangle_0 + \delta_{l,0}c_{\nu}^{\alpha} \right\} \\
 &\quad - (b_{\nu} + l)(b_{\nu} + l + 1) \{ \langle\langle \tau_{k+1}(\mathcal{O}_{\mu})\tau_l(\mathcal{O}_{\nu}) \rangle\rangle_0 + \langle\langle \tau_k(\mathcal{O}_{\mu})\tau_{l+1}(\mathcal{O}_{\nu}) \rangle\rangle_0 \}.
 \end{aligned}$$

Using the genus-0 topological recursion relation to the first term and formula (10) to the second term, we have

$$\begin{aligned}
 f &= - (k + b_{\mu} - l - b_{\nu}) \langle\langle \tau_{k+1}(\mathcal{O}_{\mu})\mathcal{X}\tau_l(\mathcal{O}_{\nu}) \rangle\rangle_0 \\
 &\quad + (k + b_{\mu} - l - b_{\nu}) \sum_{\sigma} c_{\nu}^{\sigma} \{ \langle\langle \tau_{k+1}(\mathcal{O}_{\mu})\tau_{l-1}(\mathcal{O}_{\sigma}) \rangle\rangle_0 \\
 &\quad \quad + \langle\langle \tau_k(\mathcal{O}_{\mu})\tau_l(\mathcal{O}_{\sigma}) \rangle\rangle_0 - \delta_{l,0} \langle\langle \tau_k(\mathcal{O}_{\mu})\mathcal{O}_{\sigma} \rangle\rangle_0 \} \\
 &\quad + \delta_{l,0}(k + b_{\mu} - l - b_{\nu}) \sum_{\alpha} \langle\langle \tau_k(\mathcal{O}_{\mu})\mathcal{O}_{\alpha} \rangle\rangle_0 c_{\nu}^{\alpha} \\
 &\quad - (b_{\nu} + l)(b_{\nu} + l + 1) \{ \langle\langle \tau_{k+1}(\mathcal{O}_{\mu})\tau_l(\mathcal{O}_{\nu}) \rangle\rangle_0 + \langle\langle \tau_k(\mathcal{O}_{\mu})\tau_{l+1}(\mathcal{O}_{\nu}) \rangle\rangle_0 \}.
 \end{aligned}$$

Applying Lemma 1.4 (3) to the first term, we obtain

$$\begin{aligned}
 f = & -(k + b_\mu - l - b_\nu) \left\{ (k + b_\mu + l + b_\nu + 1) \langle \langle \tau_{k+1}(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \right. \\
 & + \sum_{\sigma} C_\mu^\sigma \langle \langle \tau_k(\mathcal{O}_\sigma) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & \left. + \sum_{\sigma} C_\nu^\sigma \langle \langle \tau_{k+1}(\mathcal{O}_\mu) \tau_{l-1}(\mathcal{O}_\sigma) \rangle \rangle_0 \right\} \\
 & + (k + b_\mu - l - b_\nu) \sum_{\sigma} C_\nu^\sigma \left\{ \langle \langle \tau_{k+1}(\mathcal{O}_\mu) \tau_{l-1}(\mathcal{O}_\sigma) \rangle \rangle_0 \right. \\
 & \left. + \langle \langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\sigma) \rangle \rangle_0 \right\} \\
 & - (b_\nu + l)(b_\nu + l + 1) \left\{ \langle \langle \tau_{k+1}(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \right. \\
 & \left. + \langle \langle \tau_k(\mathcal{O}_\mu) \tau_{l+1}(\mathcal{O}_\nu) \rangle \rangle_0 \right\}.
 \end{aligned}$$

Simplifying this expression gives the desired formula. q.e.d.

Lemma 3.3.

$$\begin{aligned}
 & \sum_{\alpha, \beta} (k + b_\alpha + b_\mu) C_\nu^\beta \langle \langle \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \rangle \rangle_0 \langle \langle \mathcal{O}^\alpha \tau_{l-1}(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & + \sum_{\alpha, \beta, \sigma} C_\mu^\alpha C_\nu^\beta \langle \langle \tau_{k-1}(\mathcal{O}_\alpha) \mathcal{O}_\sigma \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_{l-1}(\mathcal{O}_\beta) \rangle \rangle_0 \\
 = & \sum_{\alpha} (k + b_\mu + l + b_\nu + 1) C_\nu^\alpha \langle \langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 & + \sum_{\alpha, \beta} C_\mu^\alpha C_\nu^\beta \langle \langle \tau_{k-1}(\mathcal{O}_\alpha) \tau_l(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & + \sum_{\alpha} (C^2)_\nu^\alpha \langle \langle \tau_k(\mathcal{O}_\mu) \tau_{l-1}(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 & - \delta_{k,0} \sum_{\alpha, \beta} C_\mu^\alpha C_\nu^\beta \langle \langle \mathcal{O}_\alpha \tau_{l-1}(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & - \delta_{l,0} \sum_{\alpha} C_\nu^\alpha \langle \langle \mathcal{X} \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \rangle \rangle_0 + \delta_{k,0} \delta_{l,0} (C^2)_{\mu\nu}.
 \end{aligned}$$

Proof. Let

$$\begin{aligned} f &:= \sum_{\alpha,\beta} (k + b_\alpha + b_\mu) C_\nu^\beta \langle\langle \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \rangle\rangle_0 \langle\langle \mathcal{O}^\alpha \tau_{l-1}(\mathcal{O}_\beta) \rangle\rangle_0 \\ &\quad + \sum_{\alpha,\beta,\sigma} C_\mu^\alpha C_\nu^\beta \langle\langle \tau_{k-1}(\mathcal{O}_\alpha) \mathcal{O}_\sigma \rangle\rangle_0 \langle\langle \mathcal{O}^\sigma \tau_{l-1}(\mathcal{O}_\beta) \rangle\rangle_0 \\ &= \sum_{\alpha,\beta} C_\nu^\beta \left\{ (k + b_\alpha + b_\mu) \langle\langle \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \rangle\rangle_0 \right. \\ &\quad \left. + \sum_{\sigma} C_\mu^\sigma \langle\langle \tau_{k-1}(\mathcal{O}_\sigma) \mathcal{O}_\alpha \rangle\rangle_0 \right\} \langle\langle \mathcal{O}^\alpha \tau_{l-1}(\mathcal{O}_\beta) \rangle\rangle_0. \end{aligned}$$

By Lemma 1.4 (3), we have

$$f = \sum_{\alpha,\beta} C_\nu^\beta \{ \langle\langle \mathcal{X} \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \rangle\rangle_0 - \delta_{k,0} C_{\mu\alpha} \} \langle\langle \mathcal{O}^\alpha \tau_{l-1}(\mathcal{O}_\beta) \rangle\rangle_0.$$

Applying the topological recursion relation to the first term yields

$$\begin{aligned} f &= \sum_{\beta} C_\nu^\beta \{ \langle\langle \mathcal{X} \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\beta) \rangle\rangle_0 - \delta_{l,0} \langle\langle \mathcal{X} \tau_k(\mathcal{O}_\mu) \mathcal{O}_\beta \rangle\rangle_0 \} \\ &\quad - \delta_{k,0} \sum_{\alpha,\beta} C_\nu^\beta C_{\mu\alpha} \langle\langle \mathcal{O}^\alpha \tau_{l-1}(\mathcal{O}_\beta) \rangle\rangle_0. \end{aligned}$$

Using Lemma 1.4 (3) again to the first term, we get

$$\begin{aligned} f &= \sum_{\beta} C_\nu^\beta \left\{ \delta_{k,0} \delta_{l,0} C_{\mu\beta} + (k + b_\mu + l + b_\beta) \langle\langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\beta) \rangle\rangle_0 \right. \\ &\quad \left. + \sum_{\sigma} C_\mu^\sigma \langle\langle \tau_{k-1}(\mathcal{O}_\sigma) \tau_l(\mathcal{O}_\beta) \rangle\rangle_0 \right. \\ &\quad \left. + \sum_{\sigma} C_\beta^\sigma \langle\langle \tau_k(\mathcal{O}_\mu) \tau_{l-1}(\mathcal{O}_\sigma) \rangle\rangle_0 \right\} \\ &\quad - \delta_{l,0} \sum_{\beta} C_\nu^\beta \langle\langle \mathcal{X} \tau_k(\mathcal{O}_\mu) \mathcal{O}_\beta \rangle\rangle_0 \\ &\quad - \delta_{k,0} \sum_{\alpha,\beta} C_\nu^\beta C_{\mu\alpha} \langle\langle \mathcal{O}^\alpha \tau_{l-1}(\mathcal{O}_\beta) \rangle\rangle_0. \end{aligned}$$

The lemma then follows from the fact that $C_\nu^\beta \neq 0$ implies $b_\beta = b_\nu + 1$.
 q.e.d.

Now from (20) we can deduce the following:

Lemma 3.4.

$$\begin{aligned}
& \sum_{\alpha} \langle \langle \mathcal{L}_0 \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\alpha} \rangle \rangle_0 \langle \langle \mathcal{O}^{\alpha}(\mathcal{L}_0 - \mathcal{D}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
&= (k + b_{\mu})(k + b_{\mu} + 1) \langle \langle \tau_{k+1}(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
&\quad + (l + b_{\nu})(l + b_{\nu} + 1) \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_{l+1}(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
&\quad + \sum_{\alpha} (2k + 2b_{\mu} + 1) \mathcal{C}_{\mu}^{\alpha} \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
&\quad + \sum_{\alpha} (2l + 2b_{\nu} + 1) \mathcal{C}_{\nu}^{\alpha} \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\alpha}) \rangle \rangle_0 \\
&\quad + \sum_{\alpha} (\mathcal{C}_{\mu}^2)^{\alpha} \langle \langle \tau_{k-1}(\mathcal{O}_{\alpha}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
&\quad + \sum_{\alpha} (\mathcal{C}_{\nu}^2)^{\alpha} \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_{l-1}(\mathcal{O}_{\alpha}) \rangle \rangle_0 \\
&\quad + \sum_{\alpha} b_{\alpha}(1 - b_{\alpha}) \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\alpha} \rangle \rangle_0 \langle \langle \mathcal{O}^{\alpha} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
&\quad + \delta_{k,0} \delta_{l,0} (\mathcal{C}^2)_{\mu\nu}.
\end{aligned}$$

Proof. By (20),

$$\begin{aligned}
& \sum_{\alpha, \beta} \langle \langle \mathcal{L}_0 \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\alpha} \rangle \rangle_0 \eta^{\alpha\beta} \langle \langle \mathcal{O}_{\beta}(\mathcal{L}_0 - \mathcal{D}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
&= - \sum_{\alpha} \left\{ b_{\alpha}(k + b_{\mu} - l - b_{\nu}) \right. \\
&\quad \left. - (k + b_{\mu})(l + b_{\nu} + 1) \right\} \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\alpha} \rangle \rangle_0 \langle \langle \mathcal{O}^{\alpha} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
&\quad + \sum_{\alpha} b_{\alpha}(1 - b_{\alpha}) \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\alpha} \rangle \rangle_0 \langle \langle \mathcal{O}^{\alpha} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
&\quad + \sum_{\alpha, \rho} (k + b_{\mu} + b_{\alpha}) \mathcal{C}_{\nu}^{\rho} \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\alpha} \rangle \rangle_0 \langle \langle \mathcal{O}^{\alpha} \tau_{l-1}(\mathcal{O}_{\rho}) \rangle \rangle_0 \\
&\quad + \sum_{\alpha, \sigma, \rho} \mathcal{C}_{\mu}^{\sigma} \mathcal{C}_{\nu}^{\rho} \langle \langle \tau_{k-1}(\mathcal{O}_{\sigma}) \mathcal{O}_{\alpha} \rangle \rangle_0 \langle \langle \mathcal{O}^{\alpha} \tau_{l-1}(\mathcal{O}_{\rho}) \rangle \rangle_0 \\
&\quad + \sum_{\beta, \sigma} (l + b_{\nu} + b_{\beta}) \mathcal{C}_{\mu}^{\sigma} \langle \langle \tau_{k-1}(\mathcal{O}_{\sigma}) \mathcal{O}^{\beta} \rangle \rangle_0 \langle \langle \mathcal{O}_{\beta} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{\beta, \sigma, \rho} C_\mu^\sigma C_\nu^\rho \left\langle \left\langle \tau_{k-1}(\mathcal{O}_\sigma) \mathcal{O}^\beta \right\rangle \right\rangle_0 \left\langle \left\langle \mathcal{O}_\beta \tau_{l-1}(\mathcal{O}_\rho) \right\rangle \right\rangle_0 \\
 & - \sum_{\beta, \sigma, \rho} C_\mu^\sigma C_\nu^\rho \left\langle \left\langle \tau_{k-1}(\mathcal{O}_\sigma) \mathcal{O}^\beta \right\rangle \right\rangle_0 \left\langle \left\langle \mathcal{O}_\beta \tau_{l-1}(\mathcal{O}_\rho) \right\rangle \right\rangle_0 \\
 & + \delta_{k,0} \sum_{\beta} C_\mu^\beta \left\{ (l + b_\nu + b_\beta) \left\langle \left\langle \tau_l(\mathcal{O}_\nu) \mathcal{O}_\beta \right\rangle \right\rangle_0 \right. \\
 & \qquad \qquad \qquad \left. + \sum_{\rho} C_\nu^\rho \left\langle \left\langle \tau_{l-1}(\mathcal{O}_\rho) \mathcal{O}_\beta \right\rangle \right\rangle_0 \right\} \\
 & + \delta_{l,0} \sum_{\alpha} C_\nu^\alpha \left\{ (k + b_\mu + b_\alpha) \left\langle \left\langle \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \right\rangle \right\rangle_0 \right. \\
 & \qquad \qquad \qquad \left. + \sum_{\sigma} C_\mu^\sigma \left\langle \left\langle \tau_{k-1}(\mathcal{O}_\sigma) \mathcal{O}_\alpha \right\rangle \right\rangle_0 \right\} \\
 & + \delta_{k,0} \delta_{l,0} (C^2)_{\mu\nu}.
 \end{aligned}$$

Applying Lemma 3.2 to the first term, Lemma 3.3 to the third and fourth terms, an analogue of Lemma 3.3 with (μ, k) interchanged with (ν, l) to the fifth and sixth terms, and formula (10) to the seventh term, we obtain

$$\begin{aligned}
 & \sum_{\alpha, \beta} \left\langle \left\langle \mathcal{L}_0 \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \right\rangle \right\rangle_0 \eta^{\alpha\beta} \left\langle \left\langle \mathcal{O}_\beta (\mathcal{L}_0 - \mathcal{D}) \tau_l(\mathcal{O}_\nu) \right\rangle \right\rangle_0 \\
 & = - (k + b_\mu - l - b_\nu) \sum_{\alpha} \left\{ C_\nu^\alpha \left\langle \left\langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\alpha) \right\rangle \right\rangle_0 \right. \\
 & \qquad \qquad \qquad \left. - C_\mu^\alpha \left\langle \left\langle \tau_k(\mathcal{O}_\alpha) \tau_l(\mathcal{O}_\nu) \right\rangle \right\rangle_0 \right\} \\
 & + (k + b_\mu)(k + b_\mu + 1) \left\langle \left\langle \tau_{k+1}(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \right\rangle \right\rangle_0 \\
 & + (l + b_\nu)(l + b_\nu + 1) \left\langle \left\langle \tau_k(\mathcal{O}_\mu) \tau_{l+1}(\mathcal{O}_\nu) \right\rangle \right\rangle_0 \\
 & + \sum_{\alpha} b_\alpha (1 - b_\alpha) \left\langle \left\langle \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \right\rangle \right\rangle_0 \left\langle \left\langle \mathcal{O}^\alpha \tau_l(\mathcal{O}_\nu) \right\rangle \right\rangle_0 \\
 & + \sum_{\alpha} (k + b_\mu + l + b_\nu + 1) C_\nu^\alpha \left\langle \left\langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\alpha) \right\rangle \right\rangle_0 \\
 & + \sum_{\alpha, \beta} C_\mu^\alpha C_\nu^\beta \left\langle \left\langle \tau_{k-1}(\mathcal{O}_\alpha) \tau_l(\mathcal{O}_\beta) \right\rangle \right\rangle_0 \\
 & + \sum_{\alpha} (C^2)_\nu^\alpha \left\langle \left\langle \tau_k(\mathcal{O}_\mu) \tau_{l-1}(\mathcal{O}_\alpha) \right\rangle \right\rangle_0
 \end{aligned}$$

$$\begin{aligned}
 & -\delta_{k,0} \sum_{\alpha,\beta} C_\mu^\alpha C_\nu^\beta \langle\langle \mathcal{O}_\alpha \tau_{l-1}(\mathcal{O}_\beta) \rangle\rangle_0 \\
 & -\delta_{l,0} \sum_{\alpha} C_\nu^\alpha \langle\langle \mathcal{X} \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \rangle\rangle_0 \\
 & +\delta_{k,0} \delta_{l,0} (C^2)_{\mu\nu} \\
 & + \sum_{\alpha} (k + b_\mu + l + b_\nu + 1) C_\mu^\alpha \langle\langle \tau_l(\mathcal{O}_\nu) \tau_k(\mathcal{O}_\alpha) \rangle\rangle_0 \\
 & + \sum_{\alpha,\beta} C_\nu^\alpha C_\mu^\beta \langle\langle \tau_{l-1}(\mathcal{O}_\alpha) \tau_k(\mathcal{O}_\beta) \rangle\rangle_0 \\
 & + \sum_{\alpha} (C^2)_\mu^\alpha \langle\langle \tau_l(\mathcal{O}_\nu) \tau_{k-1}(\mathcal{O}_\alpha) \rangle\rangle_0 \\
 & -\delta_{l,0} \sum_{\alpha,\beta} C_\nu^\alpha C_\mu^\beta \langle\langle \mathcal{O}_\alpha \tau_{k-1}(\mathcal{O}_\beta) \rangle\rangle_0 \\
 & -\delta_{k,0} \sum_{\alpha} C_\mu^\alpha \langle\langle \mathcal{X} \tau_l(\mathcal{O}_\nu) \mathcal{O}_\alpha \rangle\rangle_0 \\
 & +\delta_{k,0} \delta_{l,0} (C^2)_{\mu\nu} \\
 & - \sum_{\sigma,\rho} C_\mu^\sigma C_\nu^\rho \{ \langle\langle \tau_k(\mathcal{O}_\sigma) \tau_{l-1}(\mathcal{O}_\rho) \rangle\rangle_0 + \langle\langle \tau_{k-1}(\mathcal{O}_\sigma) \tau_l(\mathcal{O}_\rho) \rangle\rangle_0 \\
 & \quad -\delta_{k,0} \langle\langle \mathcal{O}_\sigma \tau_{l-1}(\mathcal{O}_\rho) \rangle\rangle_0 - \delta_{l,0} \langle\langle \tau_{k-1}(\mathcal{O}_\sigma) \mathcal{O}_\rho \rangle\rangle_0 \} \\
 & +\delta_{k,0} \sum_{\beta} C_\mu^\beta \left\{ (l + b_\nu + b_\beta) \langle\langle \tau_l(\mathcal{O}_\nu) \mathcal{O}_\beta \rangle\rangle_0 \right. \\
 & \quad \left. + \sum_{\rho} C_\nu^\rho \langle\langle \tau_{l-1}(\mathcal{O}_\rho) \mathcal{O}_\beta \rangle\rangle_0 \right\} \\
 & +\delta_{l,0} \sum_{\alpha} C_\nu^\alpha \left\{ (k + b_\mu + b_\alpha) \langle\langle \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \rangle\rangle_0 \right. \\
 & \quad \left. + \sum_{\sigma} C_\mu^\sigma \langle\langle \tau_{k-1}(\mathcal{O}_\sigma) \mathcal{O}_\alpha \rangle\rangle_0 \right\} \\
 & +\delta_{k,0} \delta_{l,0} (C^2)_{\mu\nu}.
 \end{aligned}$$

Simplifying this expression and applying Lemma 1.4 (3) to the two terms containing the Euler vector field \mathcal{X} , we arrive at the desired formula.

q.e.d.

Now it is straightforward to check that the difference between the right-hand side of Lemma 3.4 and the right-hand side of equation (19)

is $\frac{\partial}{\partial t_l^\nu} \frac{\partial}{\partial t_k^\mu} \Psi_{0,1}$. Hence the generalized WDVV equation implies that $\frac{\partial}{\partial t_l^\nu} \frac{\partial}{\partial t_k^\mu} \Psi_{0,1} = 0$. Therefore we have proved that all the second derivatives of $\Psi_{0,1}$ vanish. As mentioned at the end of Section 2, this implies the following.

Proposition 3.5. *The genus-0 free energy function F_0 satisfies the L_1 constraint.*

The proof of other genus-0 Virasoro constraints has the similar flavor, as we will see in Section 4 for the case of L_2 constraint.

4. L_2 constraint for genus zero free energy function

The genus-0 L_2 constraint is equivalent to the equation $\Psi_{0,2} = 0$, where

$$\begin{aligned}
 \Psi_{0,2} = & \sum_{m,\alpha} (m + b_\alpha)(m + b_\alpha + 1)(m + b_\alpha + 2) \tilde{t}_m^\alpha \langle \langle \tau_{m+2}(\mathcal{O}_\alpha) \rangle \rangle_0 \\
 & + \sum_{m,\alpha,\beta} \{3(m + b_\alpha)^2 + 6(m + b_\alpha) + 2\} \mathcal{C}_\alpha^\beta \tilde{t}_m^\alpha \langle \langle \tau_{m+1}(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & + \sum_{m,\alpha,\beta} 3(m + b_\alpha + 1) (\mathcal{C}^2)_\alpha^\beta \tilde{t}_m^\alpha \langle \langle \tau_m(\mathcal{O}_\beta) \rangle \rangle_0 \\
 (21) \quad & + \sum_{m,\alpha,\beta} (\mathcal{C}^3)_\alpha^\beta \tilde{t}_m^\alpha \langle \langle \tau_{m-1}(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & - \sum_\alpha (b_\alpha - 1) b_\alpha (b_\alpha + 1) \langle \langle \tau_1(\mathcal{O}_\alpha) \rangle \rangle_0 \langle \langle \mathcal{O}^\alpha \rangle \rangle_0 \\
 & - \frac{1}{2} \sum_{\alpha,\beta} (3b_\alpha^2 - 1) \mathcal{C}_\alpha^\beta \langle \langle \mathcal{O}_\beta \rangle \rangle_0 \langle \langle \mathcal{O}^\alpha \rangle \rangle_0 \\
 & + \frac{1}{2} \sum_{\alpha,\beta} (\mathcal{C}^3)_{\alpha\beta} t_0^\alpha t_0^\beta.
 \end{aligned}$$

As in the proof of the L_1 constraint, we only need to show that all the second derivatives of $\Psi_{0,2}$ are equal to zero. This time we need to

compute 3-point correlation functions involving vector fields

$$\begin{aligned}
 \mathcal{L}_1 &:= \sum_{m,\alpha} (m+b_\alpha)(m+b_\alpha+1) \tilde{t}_m^\alpha \frac{\partial}{\partial t_{m+1}^\alpha} \\
 (22) \quad &+ \sum_{m,\alpha,\beta} (2m+2b_\alpha+1) C_\alpha^\beta \tilde{t}_m^\alpha \frac{\partial}{\partial t_m^\beta} \\
 &+ \sum_{m,\alpha,\beta} (C^2)_\alpha^\beta \tilde{t}_m^\alpha \frac{\partial}{\partial t_{m-1}^\beta}
 \end{aligned}$$

and

$$\mathcal{L}_0 - 2\mathcal{D} = \sum_{m,\alpha} (m+b_\alpha+2) \tilde{t}_m^\alpha \frac{\partial}{\partial t_m^\alpha} + \sum_{m,\alpha,\beta} C_\alpha^\beta \tilde{t}_m^\alpha \frac{\partial}{\partial t_{m-1}^\beta}.$$

The genus-0 L_1 constraint can be reformulated as

$$\begin{aligned}
 \langle\langle \mathcal{L}_1 \rangle\rangle_0 &= -\frac{1}{2} \sum_\alpha b_\alpha (1-b_\alpha) \langle\langle \mathcal{O}_\alpha \rangle\rangle_0 \langle\langle \mathcal{O}^\alpha \rangle\rangle_0 \\
 (23) \quad &\quad -\frac{1}{2} \sum_{\alpha,\beta} (C^2)_{\alpha\beta} t_0^\alpha t_0^\beta.
 \end{aligned}$$

Using this equation and the fact that

$$\begin{aligned}
 [\mathcal{L}_1, \frac{\partial}{\partial t_m^\alpha}] &= -(m+b_\alpha)(m+b_\alpha+1) \frac{\partial}{\partial t_{m+1}^\alpha} \\
 (24) \quad &- \sum_\beta (2m+2b_\alpha+1) C_\alpha^\beta \frac{\partial}{\partial t_m^\beta} \\
 &- \sum_\beta (C^2)_\alpha^\beta \frac{\partial}{\partial t_{m-1}^\beta},
 \end{aligned}$$

we can prove the following.

Lemma 4.1.

$$\begin{aligned}
 & \langle\langle \mathcal{L}_1 \tau_m(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\
 &= -(m + b_\alpha)(m + b_\alpha + 1) \langle\langle \tau_{m+1}(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\
 &\quad - \sum_\sigma (2m + 2b_\alpha + 1) \mathcal{C}_\alpha^\sigma \langle\langle \tau_m(\mathcal{O}_\sigma) \tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\
 &\quad - \sum_\sigma (\mathcal{C}^2)_\alpha^\sigma \langle\langle \tau_{m-1}(\mathcal{O}_\sigma) \tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\
 &\quad - (n + b_\beta)(n + b_\beta + 1) \langle\langle \tau_m(\mathcal{O}_\alpha) \tau_{n+1}(\mathcal{O}_\beta) \rangle\rangle_0 \\
 &\quad - \sum_\sigma (2n + 2b_\beta + 1) \mathcal{C}_\beta^\sigma \langle\langle \tau_m(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\sigma) \rangle\rangle_0 \\
 &\quad - \sum_\sigma (\mathcal{C}^2)_\beta^\sigma \langle\langle \tau_m(\mathcal{O}_\alpha) \tau_{n-1}(\mathcal{O}_\sigma) \rangle\rangle_0 \\
 &\quad - \sum_\sigma b_\sigma(1 - b_\sigma) \langle\langle \tau_m(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \mathcal{O}_\sigma \rangle\rangle_0 \langle\langle \mathcal{O}^\sigma \rangle\rangle_0 \\
 &\quad - \sum_\sigma b_\sigma(1 - b_\sigma) \langle\langle \tau_m(\mathcal{O}_\alpha) \mathcal{O}_\sigma \rangle\rangle_0 \langle\langle \mathcal{O}^\sigma \tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\
 &\quad - \delta_{m,0} \delta_{n,0} (\mathcal{C}^2)_{\alpha\beta}.
 \end{aligned}$$

Proof.

$$\begin{aligned}
 & \langle\langle \mathcal{L}_1 \tau_m(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\
 &= \mathcal{L}_1 \frac{\partial}{\partial t_m^\alpha} \frac{\partial}{\partial t_n^\beta} F_0 \\
 &= \left\{ \frac{\partial}{\partial t_m^\alpha} \mathcal{L}_1 + [\mathcal{L}_1, \frac{\partial}{\partial t_m^\alpha}] \right\} \frac{\partial}{\partial t_n^\beta} F_0 \\
 &= \frac{\partial}{\partial t_m^\alpha} \left\{ \frac{\partial}{\partial t_n^\beta} \mathcal{L}_1 + [\mathcal{L}_1, \frac{\partial}{\partial t_n^\beta}] \right\} F_0 + [\mathcal{L}_1, \frac{\partial}{\partial t_m^\alpha}] \frac{\partial}{\partial t_n^\beta} F_0 \\
 &= \frac{\partial}{\partial t_m^\alpha} \frac{\partial}{\partial t_n^\beta} \langle\langle \mathcal{L}_1 \rangle\rangle_0 + \frac{\partial}{\partial t_m^\alpha} [\mathcal{L}_1, \frac{\partial}{\partial t_n^\beta}] F_0 + [\mathcal{L}_1, \frac{\partial}{\partial t_m^\alpha}] \frac{\partial}{\partial t_n^\beta} F_0.
 \end{aligned}$$

The lemma then follows from (23) and (24). q.e.d.

We can now compute

Lemma 4.2.

$$\begin{aligned}
& \langle \langle \mathcal{L}_1(\mathcal{L}_0 - 2\mathcal{D})\tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
&= - \sum_{m,\alpha} (m + b_\alpha)(m + b_\alpha + 1)(m + b_\alpha + 2) \\
&\quad \cdot \tilde{t}_m^\alpha \langle \langle \tau_{m+1}(\mathcal{O}_\alpha)\tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
&- \sum_{m,\alpha,\sigma} \{3(m + b_\alpha)^2 + 6(m + b_\alpha) + 2\} \\
&\quad \cdot \mathcal{C}_\alpha^\sigma \tilde{t}_m^\alpha \langle \langle \tau_m(\mathcal{O}_\sigma)\tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
&- \sum_{m,\alpha,\sigma} 3(m + b_\alpha + 1)(\mathcal{C}^2)_\alpha^\sigma \tilde{t}_m^\alpha \langle \langle \tau_{m-1}(\mathcal{O}_\sigma)\tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
&- \sum_{m,\alpha,\sigma} (\mathcal{C}^3)_\alpha^\sigma \tilde{t}_m^\alpha \langle \langle \tau_{m-2}(\mathcal{O}_\sigma)\tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
&+ (n + b_\beta)(n + b_\beta + 1)(n + b_\beta - 1) \langle \langle \tau_{n+1}(\mathcal{O}_\beta) \rangle \rangle_0 \\
&+ \sum_\sigma \{3(n + b_\beta)^2 - 1\} \mathcal{C}_\beta^\sigma \langle \langle \tau_n(\mathcal{O}_\sigma) \rangle \rangle_0 \\
&+ \sum_\sigma 3(n + b_\beta)(\mathcal{C}^2)_\beta^\sigma \langle \langle \tau_{n-1}(\mathcal{O}_\sigma) \rangle \rangle_0 \\
&+ \sum_\sigma (\mathcal{C}^3)_\beta^\sigma \langle \langle \tau_{n-2}(\mathcal{O}_\sigma) \rangle \rangle_0 \\
&- \sum_\sigma (b_\sigma - 1)b_\sigma(n + b_\beta - 1) \langle \langle \mathcal{O}^\sigma \rangle \rangle_0 \langle \langle \mathcal{O}_\sigma\tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
&- \sum_{\sigma,\rho} (b_\sigma - 1)b_\sigma \mathcal{C}_\beta^\rho \langle \langle \tau_{n-1}(\mathcal{O}_\rho)\mathcal{O}_\sigma \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \rangle \rangle_0 \\
&- \delta_{n,0} \left\{ \sum_\sigma b_\beta(b_\beta + 1)\mathcal{C}_{\beta\sigma} \langle \langle \mathcal{O}^\sigma \rangle \rangle_0 \right. \\
&\quad \left. - 3b_\beta \sum_\sigma (\mathcal{C}^2)_{\beta\sigma} t_0^\sigma + \sum_\sigma (\mathcal{C}^3)_{\beta\sigma} \tilde{t}_1^\sigma \right\} \\
&+ \delta_{n,1} \sum_\sigma (\mathcal{C}^3)_{\beta\sigma} t_0^\sigma.
\end{aligned}$$

Proof. Using Lemma 4.1, we have

$$\begin{aligned}
 & \langle\langle \mathcal{L}_1(\mathcal{L}_0 - 2\mathcal{D})\tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\
 &= \sum_{m,\alpha} (m + b_\alpha + 2) \tilde{t}_m^\alpha \langle\langle \mathcal{L}_1\tau_m(\mathcal{O}_\alpha)\tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\
 & \quad + \sum_{m,\alpha,\sigma} C_\alpha^\sigma \tilde{t}_m^\alpha \langle\langle \mathcal{L}_1\tau_{m-1}(\mathcal{O}_\sigma)\tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\
 &= - \sum_{m,\alpha} (m + b_\alpha)(m + b_\alpha + 1)(m + b_\alpha + 2) \tilde{t}_m^\alpha \\
 & \quad \cdot \langle\langle \tau_{m+1}(\mathcal{O}_\alpha)\tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\
 & \quad - \sum_{m,\alpha,\sigma} (2m + 2b_\alpha + 1)(m + b_\alpha + 2) C_\alpha^\sigma \tilde{t}_m^\alpha \\
 & \quad \cdot \langle\langle \tau_m(\mathcal{O}_\sigma)\tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\
 & \quad - \sum_{m,\alpha,\sigma} (m + b_\alpha + 2) (C^2)_\alpha^\sigma \tilde{t}_m^\alpha \langle\langle \tau_{m-1}(\mathcal{O}_\sigma)\tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\
 & \quad - \sum_{m,\alpha} (m + b_\alpha + 2)(n + b_\beta)(n + b_\beta + 1) \tilde{t}_m^\alpha \\
 & \quad \cdot \langle\langle \tau_m(\mathcal{O}_\alpha)\tau_{n+1}(\mathcal{O}_\beta) \rangle\rangle_0 \\
 & \quad - \sum_{m,\alpha,\sigma} (m + b_\alpha + 2)(2n + 2b_\beta + 1) C_\beta^\sigma \tilde{t}_m^\alpha \\
 & \quad \cdot \langle\langle \tau_m(\mathcal{O}_\alpha)\tau_n(\mathcal{O}_\sigma) \rangle\rangle_0 \\
 & \quad - \sum_{m,\alpha,\sigma} (m + b_\alpha + 2) (C^2)_\beta^\sigma \tilde{t}_m^\alpha \langle\langle \tau_m(\mathcal{O}_\alpha)\tau_{n-1}(\mathcal{O}_\sigma) \rangle\rangle_0 \\
 & \quad - \sum_{m,\alpha,\sigma} (m + b_\alpha + 2) b_\sigma (1 - b_\sigma) \tilde{t}_m^\alpha \\
 & \quad \cdot \langle\langle \tau_m(\mathcal{O}_\alpha)\tau_n(\mathcal{O}_\beta)\mathcal{O}_\sigma \rangle\rangle_0 \langle\langle \mathcal{O}^\sigma \rangle\rangle_0 \\
 & \quad - \sum_{m,\alpha,\sigma} (m + b_\alpha + 2) b_\sigma (1 - b_\sigma) \tilde{t}_m^\alpha \\
 & \quad \cdot \langle\langle \tau_m(\mathcal{O}_\alpha)\mathcal{O}_\sigma \rangle\rangle_0 \langle\langle \mathcal{O}^\sigma \tau_n(\mathcal{O}_\beta) \rangle\rangle_0 \\
 & \quad - \delta_{n,0} \sum_\alpha (b_\alpha + 2) (C^2)_{\alpha\beta} \tilde{t}_0^\alpha \\
 & \quad - \sum_{m \geq 1} \sum_{\alpha,\sigma} (m + b_\sigma - 1)(m + b_\sigma) C_\alpha^\sigma \tilde{t}_m^\alpha \langle\langle \tau_m(\mathcal{O}_\sigma)\tau_n(\mathcal{O}_\beta) \rangle\rangle_0
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{m,\alpha,\sigma,\rho} (2m + 2b_\sigma - 1) C_\alpha^\sigma C_\sigma^\rho \tilde{t}_m^\alpha \langle \langle \tau_{m-1}(\mathcal{O}_\rho) \tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & - \sum_{m,\alpha,\sigma,\rho} C_\alpha^\sigma (C^2)_\sigma^\rho \tilde{t}_m^\alpha \langle \langle \tau_{m-2}(\mathcal{O}_\rho) \tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & - \sum_{m,\alpha,\sigma} (n + b_\beta)(n + b_\beta + 1) C_\alpha^\sigma \tilde{t}_m^\alpha \langle \langle \tau_{m-1}(\mathcal{O}_\sigma) \tau_{n+1}(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & - \sum_{m,\alpha,\sigma,\rho} (2n + 2b_\beta + 1) C_\alpha^\sigma C_\beta^\rho \tilde{t}_m^\alpha \langle \langle \tau_{m-1}(\mathcal{O}_\sigma) \tau_n(\mathcal{O}_\rho) \rangle \rangle_0 \\
 & - \sum_{m,\alpha,\sigma,\rho} C_\alpha^\sigma (C^2)_\beta^\rho \tilde{t}_m^\alpha \langle \langle \tau_{m-1}(\mathcal{O}_\sigma) \tau_{n-1}(\mathcal{O}_\rho) \rangle \rangle_0 \\
 & - \sum_{m,\alpha,\sigma,\rho} b_\rho (1 - b_\rho) C_\alpha^\sigma \tilde{t}_m^\alpha \langle \langle \tau_{m-1}(\mathcal{O}_\sigma) \tau_n(\mathcal{O}_\beta) \mathcal{O}_\rho \rangle \rangle_0 \langle \langle \mathcal{O}^\rho \rangle \rangle_0 \\
 & - \sum_{m,\alpha,\sigma,\rho} b_\rho (1 - b_\rho) C_\alpha^\sigma \tilde{t}_m^\alpha \langle \langle \tau_{m-1}(\mathcal{O}_\sigma) \mathcal{O}_\rho \rangle \rangle_0 \langle \langle \mathcal{O}^\rho \tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & - \delta_{n,0} \sum_{\alpha,\sigma} C_\alpha^\sigma (C^2)_\sigma \tilde{t}_1^\alpha.
 \end{aligned}$$

The fourth term and the thirteenth term can be combined together to produce a correlation function involving $\mathcal{L}_0 - 2\mathcal{D}$. The same is true when combining together the fifth term and the fourteenth term, the sixth term and the fifteenth term, the seventh term and the sixteenth term, the eighth term and the seventeenth term. Using the fact that $b_\sigma = b_\alpha + 1$ if $C_\alpha^\sigma \neq 0$, we can simplify the above expression as follows:

$$\begin{aligned}
 & \langle \langle \mathcal{L}_1(\mathcal{L}_0 - 2\mathcal{D}) \tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & = - \sum_{m,\alpha} (m + b_\alpha)(m + b_\alpha + 1)(m + b_\alpha + 2) \tilde{t}_m^\alpha \\
 & \quad \cdot \langle \langle \tau_{m+1}(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & - \sum_{m,\alpha,\sigma} \{3(m + b_\alpha)^2 + 6(m + b_\alpha) + 2\} C_\alpha^\sigma \tilde{t}_m^\alpha \\
 & \quad \cdot \langle \langle \tau_m(\mathcal{O}_\sigma) \tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & + \sum_{\alpha,\sigma} b_\alpha (b_\alpha + 1) C_\alpha^\sigma \tilde{t}_0^\alpha \langle \langle \mathcal{O}_\sigma \tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
 (25) \quad & - \sum_{m,\alpha,\sigma} 3(m + b_\alpha + 1) (C^2)_\alpha^\sigma \tilde{t}_m^\alpha \langle \langle \tau_{m-1}(\mathcal{O}_\sigma) \tau_n(\mathcal{O}_\beta) \rangle \rangle_0
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{m,\alpha,\rho} (\mathcal{C}^3)_\alpha^{\rho} \tilde{t}_m^\alpha \langle \langle \tau_{m-2}(\mathcal{O}_\rho) \tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & - (n + b_\beta)(n + b_\beta + 1) \langle \langle (\mathcal{L}_0 - 2\mathcal{D}) \tau_{n+1}(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & - \sum_{\sigma} (2n + 2b_\beta + 1) \mathcal{C}_\beta^\sigma \langle \langle (\mathcal{L}_0 - 2\mathcal{D}) \tau_n(\mathcal{O}_\sigma) \rangle \rangle_0 \\
 & - \sum_{\sigma} (\mathcal{C}^2)_\beta^\sigma \langle \langle (\mathcal{L}_0 - 2\mathcal{D}) \tau_{n-1}(\mathcal{O}_\sigma) \rangle \rangle_0 \\
 & - \sum_{\sigma} b_\sigma(1 - b_\sigma) \langle \langle (\mathcal{L}_0 - 2\mathcal{D}) \tau_n(\mathcal{O}_\beta) \mathcal{O}_\sigma \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \rangle \rangle_0 \\
 & - \sum_{\sigma} b_\sigma(1 - b_\sigma) \langle \langle (\mathcal{L}_0 - 2\mathcal{D}) \mathcal{O}_\sigma \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & - \delta_{n,0} \left\{ \sum_{\alpha} (b_\alpha + 2) (\mathcal{C}^2)_{\alpha\beta} \tilde{t}_0^\alpha + \sum_{\alpha} (\mathcal{C}^3)_{\alpha\beta} \tilde{t}_1^\alpha \right\}.
 \end{aligned}$$

By Lemma 1.2 (2) and Lemma 1.4 (2),

$$\begin{aligned}
 & \langle \langle (\mathcal{L}_0 - 2\mathcal{D}) \tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & = -(n + b_\beta - 2) \langle \langle \tau_n(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & \quad - \sum_{\sigma} \mathcal{C}_\beta^\sigma \langle \langle \tau_{n-1}(\mathcal{O}_\sigma) \rangle \rangle_0 - \delta_{n,0} \sum_{\sigma} \mathcal{C}_{\beta\sigma} t_0^\sigma,
 \end{aligned}$$

and by Lemma 1.2 (3) and Lemma 1.4 (3),

$$\begin{aligned}
 & \langle \langle (\mathcal{L}_0 - 2\mathcal{D}) \tau_n(\mathcal{O}_\beta) \mathcal{O}_\rho \rangle \rangle_0 \\
 & = -(n + b_\beta + b_\rho) \langle \langle \tau_n(\mathcal{O}_\beta) \mathcal{O}_\rho \rangle \rangle_0 \\
 & \quad - \sum_{\gamma} \mathcal{C}_\beta^\gamma \langle \langle \tau_{n-1}(\mathcal{O}_\gamma) \mathcal{O}_\rho \rangle \rangle_0 - \delta_{n,0} \mathcal{C}_{\beta\rho}.
 \end{aligned}$$

Applying these two formulas to the right-hand side of (26) and simplifying, we obtain the desired formula. q.e.d.

Setting $n = 0$ in Lemma 4.2, multiplying both sides of the equation by $\langle \langle \mathcal{O}^\beta \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0$, summing over β , and then applying the genus-0 topological recursion relation, we get

$$\begin{aligned}
 & \sum_{\beta} \langle \langle \mathcal{L}_1(\mathcal{L}_0 - 2\mathcal{D})\mathcal{O}_{\beta} \rangle \rangle_0 \langle \langle \mathcal{O}^{\beta} \tau_k(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 &= - \sum_{m, \alpha} (m + b_{\alpha})(m + b_{\alpha} + 1)(m + b_{\alpha} + 2) \\
 & \quad \cdot \tilde{t}_m^{\alpha} \langle \langle \tau_{m+2}(\mathcal{O}_{\alpha}) \tau_k(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & - \sum_{m, \alpha, \sigma} \{3(m + b_{\alpha})^2 + 6(m + b_{\alpha}) + 2\} \\
 & \quad \cdot \mathcal{C}_{\alpha}^{\sigma} \tilde{t}_m^{\alpha} \langle \langle \tau_{m+1}(\mathcal{O}_{\sigma}) \tau_k(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 (26) \quad & - \sum_{m, \alpha, \sigma} 3(m + b_{\alpha} + 1) (\mathcal{C}^2)_{\alpha}^{\sigma} \tilde{t}_m^{\alpha} \langle \langle \tau_m(\mathcal{O}_{\sigma}) \tau_k(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & - \sum_{m, \alpha, \sigma} (\mathcal{C}^3)_{\alpha}^{\sigma} \tilde{t}_m^{\alpha} \langle \langle \tau_{m-1}(\mathcal{O}_{\sigma}) \tau_k(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & + \sum_{\beta} (b_{\beta} - 1) b_{\beta} (b_{\beta} + 1) \langle \langle \tau_1(\mathcal{O}_{\beta}) \rangle \rangle_0 \langle \langle \mathcal{O}^{\beta} \tau_k(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & + \sum_{\beta, \sigma} (3b_{\beta}^2 - 1) \mathcal{C}_{\beta}^{\sigma} \langle \langle \mathcal{O}_{\sigma} \rangle \rangle_0 \langle \langle \mathcal{O}^{\beta} \tau_k(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & - \sum_{\beta, \sigma} (b_{\sigma} - 1) b_{\sigma} (b_{\sigma} - 1) \langle \langle \mathcal{O}^{\sigma} \rangle \rangle_0 \\
 & \quad \cdot \langle \langle \mathcal{O}_{\sigma} \mathcal{O}_{\beta} \rangle \rangle_0 \langle \langle \mathcal{O}^{\beta} \tau_k(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & - \sum_{\beta, \sigma} b_{\beta} (b_{\beta} + 1) \mathcal{C}_{\beta\sigma} \langle \langle \mathcal{O}^{\sigma} \rangle \rangle_0 \langle \langle \mathcal{O}^{\beta} \tau_k(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0.
 \end{aligned}$$

Notice that the ranges of summations may change when using the topological recursion relation. Hence some scattered terms may be absorbed into a big summation after using the topological recursion relation.

By Lemma 1.2 (3) and the definition of \mathcal{L}_0 ,

$$\langle \langle (\mathcal{L}_0 - 2\mathcal{D})\tau_m(\mathcal{O}_{\alpha})\tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 = - \langle \langle \mathcal{X}\tau_m(\mathcal{O}_{\alpha})\tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0,$$

for any m, n, α, β . Thus by Lemma 4.1 we have

$$\begin{aligned}
 & \sum_{\beta} \langle \langle \mathcal{L}_1 \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\beta} \rangle \rangle_0 \left\langle \left\langle \mathcal{O}^{\beta} (\mathcal{L}_0 - 2\mathcal{D}) \tau_l(\mathcal{O}_{\nu}) \right\rangle \right\rangle_0 \\
 &= \sum_{\beta} (k + b_{\mu})(k + b_{\mu} + 1) \langle \langle \tau_{k+1}(\mathcal{O}_{\mu}) \mathcal{O}_{\beta} \rangle \rangle_0 \left\langle \left\langle \mathcal{O}^{\beta} \chi_{\tau_l}(\mathcal{O}_{\nu}) \right\rangle \right\rangle_0 \\
 &+ \sum_{\beta, \sigma} (2k + 2b_{\mu} + 1) \mathcal{C}_{\mu}^{\sigma} \langle \langle \tau_k(\mathcal{O}_{\sigma}) \mathcal{O}_{\beta} \rangle \rangle_0 \left\langle \left\langle \mathcal{O}^{\beta} \chi_{\tau_l}(\mathcal{O}_{\nu}) \right\rangle \right\rangle_0 \\
 &+ \sum_{\beta, \sigma} (\mathcal{C}^2)_{\mu}^{\sigma} \langle \langle \tau_{k-1}(\mathcal{O}_{\sigma}) \mathcal{O}_{\beta} \rangle \rangle_0 \left\langle \left\langle \mathcal{O}^{\beta} \chi_{\tau_l}(\mathcal{O}_{\nu}) \right\rangle \right\rangle_0 \\
 &+ \sum_{\beta} b_{\beta} (b_{\beta} + 1) \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\beta}) \rangle \rangle_0 \left\langle \left\langle \mathcal{O}^{\beta} \chi_{\tau_l}(\mathcal{O}_{\nu}) \right\rangle \right\rangle_0 \\
 (27) \quad &+ \sum_{\beta, \sigma} (2b_{\beta} + 1) \mathcal{C}_{\beta}^{\sigma} \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\sigma} \rangle \rangle_0 \left\langle \left\langle \mathcal{O}^{\beta} \chi_{\tau_l}(\mathcal{O}_{\nu}) \right\rangle \right\rangle_0 \\
 &+ \sum_{\beta, \sigma} b_{\sigma} (1 - b_{\sigma}) \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\beta} \mathcal{O}_{\sigma} \rangle \rangle_0 \langle \langle \mathcal{O}^{\sigma} \rangle \rangle_0 \left\langle \left\langle \mathcal{O}^{\beta} \chi_{\tau_l}(\mathcal{O}_{\nu}) \right\rangle \right\rangle_0 \\
 &+ \sum_{\beta, \sigma} b_{\sigma} (1 - b_{\sigma}) \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\sigma} \rangle \rangle_0 \langle \langle \mathcal{O}^{\sigma} \mathcal{O}_{\beta} \rangle \rangle_0 \left\langle \left\langle \mathcal{O}^{\beta} \chi_{\tau_l}(\mathcal{O}_{\nu}) \right\rangle \right\rangle_0 \\
 &+ \delta_{k,0} \sum_{\beta} (\mathcal{C}^2)_{\mu\beta} \left\langle \left\langle \mathcal{O}^{\beta} \chi_{\tau_l}(\mathcal{O}_{\nu}) \right\rangle \right\rangle_0.
 \end{aligned}$$

From Lemma 1.4 (3), we know that the right-hand side of (27) has only finitely many terms. As in the proof of the L_1 constraint, we will show that the difference of the right-hand sides of (27) and (26) is equal to $\frac{\partial}{\partial t_l^{\nu}} \frac{\partial}{\partial t_k^{\mu}} \Psi_{0,2}$. For this purpose, we need to express the products of correlation functions in (27) as summations of correlation functions.

Using the generalized WDVV equation to the sixth term and the topological recursion relation to the first three terms and the seventh term on the right-hand side of (27), we obtain

$$\begin{aligned}
 & \sum_{\beta} \langle \langle \mathcal{L}_1 \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\beta} \rangle \rangle_0 \left\langle \left\langle \mathcal{O}^{\beta} (\mathcal{L}_0 - 2\mathcal{D}) \tau_l(\mathcal{O}_{\nu}) \right\rangle \right\rangle_0 \\
 &= (k + b_{\mu})(k + b_{\mu} + 1) \langle \langle \tau_{k+2}(\mathcal{O}_{\mu}) \chi_{\tau_l}(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 &+ \sum_{\sigma} (2k + 2b_{\mu} + 1) \mathcal{C}_{\mu}^{\sigma} \langle \langle \tau_{k+1}(\mathcal{O}_{\sigma}) \chi_{\tau_l}(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 &+ \sum_{\sigma} (\mathcal{C}^2)_{\mu}^{\sigma} \langle \langle \tau_k(\mathcal{O}_{\sigma}) \chi_{\tau_l}(\mathcal{O}_{\nu}) \rangle \rangle_0
 \end{aligned}$$

$$\begin{aligned}
 & -\delta_{k,0} \sum_{\sigma} (\mathcal{C}^2)_{\mu}^{\sigma} \langle \langle \mathcal{O}_{\sigma} \mathcal{X} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & + \sum_{\beta} b_{\beta} (b_{\beta} + 1) \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_1(\mathcal{O}_{\beta}) \rangle \rangle_0 \langle \langle \mathcal{O}^{\beta} \mathcal{X} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & + \sum_{\beta, \sigma} (2b_{\beta} + 1) \mathcal{C}_{\beta}^{\sigma} \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\sigma} \rangle \rangle_0 \langle \langle \mathcal{O}^{\beta} \mathcal{X} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & + \sum_{\beta, \sigma} b_{\sigma} (1 - b_{\sigma}) \langle \langle \mathcal{X} \mathcal{O}_{\beta} \mathcal{O}_{\sigma} \rangle \rangle_0 \langle \langle \mathcal{O}^{\sigma} \rangle \rangle_0 \\
 & \quad \cdot \langle \langle \mathcal{O}^{\beta} \tau_k(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & + \sum_{\sigma} b_{\sigma} (1 - b_{\sigma}) \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}^{\sigma} \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_{\sigma}) \mathcal{X} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & + \delta_{k,0} \sum_{\beta} (\mathcal{C}^2)_{\mu\beta} \langle \langle \mathcal{O}^{\beta} \mathcal{X} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0.
 \end{aligned}$$

We then apply Lemma 1.4 (3) to get rid of the Euler vector fields in the above expression. After simplification, we will find many terms which also appear in $\frac{\partial}{\partial t_i^{\nu}} \frac{\partial}{\partial t_k^{\mu}} \Psi_{0,2}$ or on the right-hand side of (26). We call those terms *good terms*. There are also many terms which appear neither in $\frac{\partial}{\partial t_i^{\nu}} \frac{\partial}{\partial t_k^{\mu}} \Psi_{0,2}$, nor on the right-hand side of (26); we call those terms *bad terms*. Grouping good terms together and bad terms together gives the following:

$$\begin{aligned}
 & \sum_{\beta} \langle \langle \mathcal{L}_1 \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\beta} \rangle \rangle_0 \langle \langle \mathcal{O}^{\beta} (\mathcal{L}_0 - 2\mathcal{D}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & = (k + b_{\mu})(k + b_{\mu} + 1)(k + b_{\mu} + 2) \langle \langle \tau_{k+2}(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & \quad + \sum_{\sigma} \{3(k + b_{\mu})^2 + 6(k + b_{\mu}) + 2\} \mathcal{C}_{\mu}^{\sigma} \langle \langle \tau_{k+1}(\mathcal{O}_{\sigma}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & \quad + \sum_{\sigma} 3(k + b_{\mu} + 1) (\mathcal{C}^2)_{\mu}^{\sigma} \langle \langle \tau_k(\mathcal{O}_{\sigma}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & \quad + \sum_{\sigma} (\mathcal{C}^3)_{\mu}^{\sigma} \langle \langle \tau_{k-1}(\mathcal{O}_{\sigma}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & \quad + \sum_{\beta} (1 - b_{\beta}) b_{\beta} (1 + b_{\beta}) \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_1(\mathcal{O}_{\beta}) \rangle \rangle_0 \langle \langle \mathcal{O}^{\beta} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 (28) \quad & + \sum_{\alpha, \beta} b_{\alpha} (1 - b_{\alpha}) (b_{\alpha} + b_{\beta}) \langle \langle \mathcal{O}^{\alpha} \rangle \rangle_0 \langle \langle \mathcal{O}_{\alpha} \mathcal{O}_{\beta} \rangle \rangle_0 \\
 & \quad \cdot \langle \langle \mathcal{O}^{\beta} \tau_k(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{\alpha, \beta} b_\alpha (1 - b_\alpha) C_{\alpha\beta} \langle \langle \mathcal{O}^\alpha \rangle \rangle_0 \langle \langle \mathcal{O}^\beta \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & + \sum_{\alpha} b_\alpha (1 - b_\alpha) (b_\alpha + 1) \langle \langle \tau_k(\mathcal{O}_\mu) \mathcal{O}^\alpha \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_\alpha) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & - \sum_{\alpha, \beta} (3b_\alpha^2 - 1) C_\alpha^\beta \langle \langle \tau_k(\mathcal{O}_\mu) \mathcal{O}_\beta \rangle \rangle_0 \langle \langle \mathcal{O}^\alpha \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & + \delta_{k,0} \delta_{l,0} (C^3)_{\mu\nu} \\
 & + (k + b_\mu)(k + b_\mu + 1)(l + b_\nu) \langle \langle \tau_{k+2}(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & + \sum_{\sigma} (2k + 2b_\mu + 1)(l + b_\nu) C_\mu^\sigma \langle \langle \tau_{k+1}(\mathcal{O}_\sigma) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & + \sum_{\sigma} (k + b_\mu)(k + b_\mu + 1) C_\nu^\sigma \langle \langle \tau_{k+2}(\mathcal{O}_\mu) \tau_{l-1}(\mathcal{O}_\sigma) \rangle \rangle_0 \\
 & + \sum_{\sigma} (l + b_\nu) (C^2)_\mu^\sigma \langle \langle \tau_k(\mathcal{O}_\sigma) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & + \sum_{\alpha, \beta} (2k + 2b_\mu + 1) C_\mu^\alpha C_\nu^\beta \langle \langle \tau_{k+1}(\mathcal{O}_\alpha) \tau_{l-1}(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & + \sum_{\alpha, \beta} (C^2)_\mu^\alpha C_\nu^\beta \langle \langle \tau_k(\mathcal{O}_\alpha) \tau_{l-1}(\mathcal{O}_\beta) \rangle \rangle_0 \\
 & + \sum_{\beta} b_\beta (b_\beta + 1)(l + b_\nu) \langle \langle \tau_k(\mathcal{O}_\mu) \tau_1(\mathcal{O}_\beta) \rangle \rangle_0 \langle \langle \mathcal{O}^\beta \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & + \sum_{\beta} b_\beta (1 - b_\beta)(l + b_\nu) \langle \langle \tau_k(\mathcal{O}_\mu) \mathcal{O}^\beta \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_\beta) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & + \sum_{\alpha, \beta} (2b_\alpha + 1)(l + b_\nu) C_\alpha^\beta \langle \langle \tau_k(\mathcal{O}_\mu) \mathcal{O}_\beta \rangle \rangle_0 \langle \langle \mathcal{O}^\alpha \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & + \delta_{l,0} \left\{ \sum_{\beta} b_\beta (b_\beta + 1) C_\nu^\beta \langle \langle \tau_k(\mathcal{O}_\mu) \tau_1(\mathcal{O}_\beta) \rangle \rangle_0 \right. \\
 & \quad \left. + \sum_{\beta} (2b_\nu + 3) (C^2)_\nu^\beta \langle \langle \tau_k(\mathcal{O}_\mu) \mathcal{O}_\beta \rangle \rangle_0 \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{\beta, \sigma} b_\beta(b_\beta + 1)C_\nu^\sigma \langle \langle \tau_k(\mathcal{O}_\mu)\tau_1(\mathcal{O}_\beta) \rangle \rangle_0 \langle \langle \mathcal{O}^\beta \tau_{l-1}(\mathcal{O}_\sigma) \rangle \rangle_0 \\
 & + \sum_{\beta, \sigma} b_\beta(1 - b_\beta)C_\nu^\sigma \langle \langle \tau_k(\mathcal{O}_\mu)\mathcal{O}^\beta \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_\beta)\tau_{l-1}(\mathcal{O}_\sigma) \rangle \rangle_0 \\
 & + \sum_{\alpha, \beta, \sigma} (2b_\beta + 1)C_\beta^\alpha C_\nu^\sigma \langle \langle \tau_k(\mathcal{O}_\mu)\mathcal{O}_\alpha \rangle \rangle_0 \langle \langle \mathcal{O}^\beta \tau_{l-1}(\mathcal{O}_\sigma) \rangle \rangle_0,
 \end{aligned}$$

where the first 10 terms are good terms and rest terms are bad terms. In order to transform bad terms into good terms, we need more properties for correlation functions.

Lemma 4.3.

$$\begin{aligned}
 (i) \quad & \sum_{\beta} \langle \langle \mathcal{X}_{\tau_k(\mathcal{O}_\mu)\tau_1(\mathcal{O}_\beta)} \rangle \rangle_0 \langle \langle \mathcal{O}^\beta \mathcal{X}_{\tau_l(\mathcal{O}_\nu)} \rangle \rangle_0 \\
 & = \sum_{\beta} \langle \langle \mathcal{X}_{\tau_k(\mathcal{O}_\mu)\mathcal{O}^\beta} \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_\beta)\mathcal{X}_{\tau_l(\mathcal{O}_\nu)} \rangle \rangle_0, \\
 (ii) \quad & \sum_{\beta} \langle \langle \tau_{k-1}(\mathcal{O}_\mu)\mathcal{O}^\beta \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_\beta)\mathcal{X}_{\tau_l(\mathcal{O}_\nu)} \rangle \rangle_0 \\
 & = \sum_{\beta} \langle \langle \tau_{k-1}(\mathcal{O}_\mu)\tau_1(\mathcal{O}_\beta) \rangle \rangle_0 \langle \langle \mathcal{O}^\beta \mathcal{X}_{\tau_l(\mathcal{O}_\nu)} \rangle \rangle_0 \\
 & \quad + \langle \langle \tau_{k+1}(\mathcal{O}_\mu)\mathcal{X}_{\tau_l(\mathcal{O}_\nu)} \rangle \rangle_0 \\
 & \quad - \delta_{k,0} \langle \langle \tau_1(\mathcal{O}_\mu)\mathcal{X}_{\tau_l(\mathcal{O}_\nu)} \rangle \rangle_0 - \delta_{k,-1} \langle \langle \mathcal{O}_\mu \mathcal{X}_{\tau_l(\mathcal{O}_\nu)} \rangle \rangle_0, \\
 (iii) \quad & \sum_{\beta} \langle \langle \tau_k(\mathcal{O}_\mu)\mathcal{O}^\beta \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_\beta)\tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & = \sum_{\beta} \langle \langle \tau_k(\mathcal{O}_\mu)\tau_1(\mathcal{O}_\beta) \rangle \rangle_0 \langle \langle \mathcal{O}^\beta \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & \quad + \langle \langle \tau_{k+2}(\mathcal{O}_\mu)\tau_l(\mathcal{O}_\nu) \rangle \rangle_0 - \langle \langle \tau_k(\mathcal{O}_\mu)\tau_{l+2}(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & \quad - \delta_{k,-1} \langle \langle \tau_1(\mathcal{O}_\mu)\tau_l(\mathcal{O}_\nu) \rangle \rangle_0 - \delta_{k,-2} \langle \langle \mathcal{O}_\mu \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & \quad + \delta_{l,-1} \langle \langle \tau_k(\mathcal{O}_\mu)\tau_1(\mathcal{O}_\nu) \rangle \rangle_0 + \delta_{l,-2} \langle \langle \tau_k(\mathcal{O}_\mu)\mathcal{O}_\nu \rangle \rangle_0.
 \end{aligned}$$

Proof. (i) follows by applying topological recursion relation to both sides of the equation for the terms which contain τ_1 .

To prove (ii), we first use topological recursion relation, and then

use formula (10) as follows:

$$\begin{aligned} & \sum_{\beta} \langle \langle \tau_{k-1}(\mathcal{O}_{\mu}) \mathcal{O}^{\beta} \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_{\beta}) \mathcal{X} \tau_1(\mathcal{O}_{\nu}) \rangle \rangle_0 \\ &= \sum_{\alpha, \beta} \langle \langle \tau_{k-1}(\mathcal{O}_{\mu}) \mathcal{O}^{\beta} \rangle \rangle_0 \langle \langle \mathcal{O}_{\beta} \mathcal{O}_{\alpha} \rangle \rangle_0 \langle \langle \mathcal{O}^{\alpha} \mathcal{X} \tau_1(\mathcal{O}_{\nu}) \rangle \rangle_0 \\ &= \sum_{\alpha} \left\{ \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\alpha} \rangle \rangle_0 + \langle \langle \tau_{k-1}(\mathcal{O}_{\mu}) \tau_1(\mathcal{O}_{\alpha}) \rangle \rangle_0 \right. \\ & \quad \left. - \delta_{k,0} \langle \langle \mathcal{O}_{\mu} \mathcal{O}_{\alpha} \rangle \rangle_0 \right\} \langle \langle \mathcal{O}^{\alpha} \mathcal{X} \tau_1(\mathcal{O}_{\nu}) \rangle \rangle_0. \end{aligned}$$

Applying topological recursion relation again to the first and the third terms, we obtain (ii).

We now prove (iii). Let

$$f := \sum_{\beta} \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}^{\beta} \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_{\beta}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0.$$

By (10),

$$f = \sum_{\beta} \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}^{\beta} \rangle \rangle_0 \left\{ - \langle \langle \mathcal{O}_{\beta} \tau_{l+1}(\mathcal{O}_{\nu}) \rangle \rangle_0 + \delta_{l,-1} \langle \langle \mathcal{O}_{\beta} \mathcal{O}_{\nu} \rangle \rangle_0 \right. \\ \left. + \sum_{\alpha} \langle \langle \tau_l(\mathcal{O}_{\nu}) \mathcal{O}^{\alpha} \rangle \rangle_0 \langle \langle \mathcal{O}_{\alpha} \mathcal{O}_{\beta} \rangle \rangle_0 \right\}.$$

Using (10) reversely to each term gives

$$\begin{aligned} f = & - \langle \langle \tau_{k+1}(\mathcal{O}_{\mu}) \tau_{l+1}(\mathcal{O}_{\nu}) \rangle \rangle_0 - \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_{l+2}(\mathcal{O}_{\nu}) \rangle \rangle_0 \\ & + \delta_{k,-1} \langle \langle \mathcal{O}_{\mu} \tau_{l+1}(\mathcal{O}_{\nu}) \rangle \rangle_0 + \delta_{l,-2} \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\nu} \rangle \rangle_0 \\ & + \delta_{l,-1} \left\{ \langle \langle \tau_{k+1}(\mathcal{O}_{\mu}) \mathcal{O}_{\nu} \rangle \rangle_0 + \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_1(\mathcal{O}_{\nu}) \rangle \rangle_0 - \delta_{k,-1} \langle \langle \mathcal{O}_{\mu} \mathcal{O}_{\nu} \rangle \rangle_0 \right\} \\ & + \sum_{\alpha} \langle \langle \tau_l(\mathcal{O}_{\nu}) \mathcal{O}^{\alpha} \rangle \rangle_0 \left\{ \langle \langle \tau_1(\mathcal{O}_{\alpha}) \tau_k(\mathcal{O}_{\mu}) \rangle \rangle_0 \right. \\ & \quad \left. + \langle \langle \mathcal{O}_{\alpha} \tau_{k+1}(\mathcal{O}_{\mu}) \rangle \rangle_0 - \delta_{k,-1} \langle \langle \mathcal{O}_{\alpha} \mathcal{O}_{\mu} \rangle \rangle_0 \right\}. \end{aligned}$$

Applying (10) again to the last two terms and simplifying, we obtain (iii). q.e.d.

Using Lemma 4.3, we can show

Lemma 4.4.

$$\begin{aligned}
 & \sum_{\beta} b_{\beta}(b_{\beta} + 1) \langle \langle \tau_k(\mathcal{O}_{\mu})\tau_1(\mathcal{O}_{\beta}) \rangle \rangle_0 \langle \langle \mathcal{O}^{\beta}\tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & \quad + \sum_{\beta} b_{\beta}(1 - b_{\beta}) \langle \langle \tau_k(\mathcal{O}_{\mu})\mathcal{O}^{\beta} \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_{\beta})\tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & \quad + \sum_{\alpha, \beta} (2b_{\beta} + 1)C_{\alpha\beta} \langle \langle \tau_k(\mathcal{O}_{\mu})\mathcal{O}^{\alpha} \rangle \rangle_0 \langle \langle \mathcal{O}^{\beta}\tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 = & -(k + b_{\mu})(k + b_{\mu} + 1) \langle \langle \tau_{k+2}(\mathcal{O}_{\mu})\tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & + (l + b_{\nu} + 1)(l + b_{\nu} + 2) \langle \langle \tau_k(\mathcal{O}_{\mu})\tau_{l+2}(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & - \sum_{\alpha} (2k + 2b_{\mu} + 1)C_{\mu}^{\alpha} \langle \langle \tau_{k+1}(\mathcal{O}_{\alpha})\tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & + \sum_{\alpha} (2l + 2b_{\nu} + 3)C_{\nu}^{\alpha} \langle \langle \tau_k(\mathcal{O}_{\mu})\tau_{l+1}(\mathcal{O}_{\alpha}) \rangle \rangle_0 \\
 & - \sum_{\alpha} (C^2)_{\mu}^{\alpha} \langle \langle \tau_k(\mathcal{O}_{\alpha})\tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & + \sum_{\alpha} (C^2)_{\nu}^{\alpha} \langle \langle \tau_k(\mathcal{O}_{\mu})\tau_l(\mathcal{O}_{\alpha}) \rangle \rangle_0 \\
 & - \delta_{l,-1} \left\{ b_{\nu}(b_{\nu} + 1) \langle \langle \tau_k(\mathcal{O}_{\mu})\tau_1(\mathcal{O}_{\nu}) \rangle \rangle_0 \right. \\
 & \quad \left. + \sum_{\alpha} (2b_{\nu} + 1)C_{\nu}^{\alpha} \langle \langle \tau_k(\mathcal{O}_{\mu})\mathcal{O}_{\alpha} \rangle \rangle_0 \right\}.
 \end{aligned}$$

Proof. By Lemma 1.4 (3),

$$\begin{aligned}
 & \sum_{\beta} b_{\beta}(b_{\beta} + 1) \langle \langle \tau_k(\mathcal{O}_{\mu})\tau_1(\mathcal{O}_{\beta}) \rangle \rangle_0 \langle \langle \mathcal{O}^{\beta}\tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 = & - \sum_{\beta} \left\{ (k + b_{\mu} + b_{\beta} + 1) \langle \langle \tau_k(\mathcal{O}_{\mu})\tau_1(\mathcal{O}_{\beta}) \rangle \rangle_0 \right. \\
 & \quad \left. - (k + b_{\mu}) \langle \langle \tau_k(\mathcal{O}_{\mu})\tau_1(\mathcal{O}_{\beta}) \rangle \rangle_0 \right\} \\
 & \quad \cdot \left\{ (l + b_{\nu} + 1 - b_{\beta}) \langle \langle \mathcal{O}^{\beta}\tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \right. \\
 (29) \quad & \quad \left. - (l + b_{\nu} + 1) \langle \langle \mathcal{O}^{\beta}\tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= - \sum_{\beta} \left\{ \langle \langle \mathcal{X} \tau_k(\mathcal{O}_{\mu}) \tau_1(\mathcal{O}_{\beta}) \rangle \rangle_0 - \sum_{\sigma} c_{\mu}^{\sigma} \langle \langle \tau_{k-1}(\mathcal{O}_{\sigma}) \tau_1(\mathcal{O}_{\beta}) \rangle \rangle_0 \right. \\
 &\quad \left. - \sum_{\sigma} c_{\beta}^{\sigma} \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\sigma} \rangle \rangle_0 - (k + b_{\mu}) \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_1(\mathcal{O}_{\beta}) \rangle \rangle_0 \right\} \\
 &\quad \cdot \left\{ \langle \langle \mathcal{X} \mathcal{O}^{\beta} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 - \sum_{\rho} c_{\nu}^{\rho} \langle \langle \mathcal{O}^{\beta} \tau_{l-1}(\mathcal{O}_{\rho}) \rangle \rangle_0 - \delta_{l,0} c_{\nu}^{\beta} \right. \\
 &\quad \left. - (l + b_{\nu} + 1) \langle \langle \mathcal{O}^{\beta} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \right\}.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 &\sum_{\beta} b_{\beta}(1 - b_{\beta}) \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}^{\beta} \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_{\beta}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 &= \sum_{\beta} \left\{ (k + b_{\mu} + 1 - b_{\beta}) \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}^{\beta} \rangle \rangle_0 \right. \\
 &\quad \left. - (k + b_{\mu}) \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}^{\beta} \rangle \rangle_0 \right\} \\
 &\quad \cdot \left\{ (l + b_{\nu} + 1 + b_{\beta}) \langle \langle \tau_1(\mathcal{O}_{\beta}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \right. \\
 (30) \quad &\quad \left. - (l + b_{\nu} + 1) \langle \langle \tau_1(\mathcal{O}_{\beta}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \right\} \\
 &= \sum_{\beta} \left\{ \langle \langle \mathcal{X} \tau_k(\mathcal{O}_{\mu}) \mathcal{O}^{\beta} \rangle \rangle_0 \right. \\
 &\quad \left. - \sum_{\sigma} c_{\mu}^{\sigma} \langle \langle \tau_{k-1}(\mathcal{O}_{\sigma}) \mathcal{O}^{\beta} \rangle \rangle_0 - \delta_{k,0} c_{\mu}^{\beta} \right. \\
 &\quad \left. - (k + b_{\mu}) \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}^{\beta} \rangle \rangle_0 \right\}
 \end{aligned}$$

$$\left\{ \begin{aligned} & \langle \langle \mathcal{X}\tau_1(\mathcal{O}_\beta)\tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\ & - \sum_{\rho} c_{\nu}^{\rho} \langle \langle \tau_1(\mathcal{O}_\beta)\tau_{l-1}(\mathcal{O}_\rho) \rangle \rangle_0 \\ & - \sum_{\rho} c_{\beta}^{\rho} \langle \langle \mathcal{O}_\rho\tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\ & - (l + b_{\nu} + 1) \langle \langle \tau_1(\mathcal{O}_\beta)\tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \end{aligned} \right\}.$$

Expanding both (29) and (30), summing them together, then using Lemma 4.3 to simplify, we obtain that, for $k \geq 0$ and $l \geq -1$,

$$\begin{aligned} & \sum_{\beta} b_{\beta}(b_{\beta} + 1) \langle \langle \tau_k(\mathcal{O}_\mu)\tau_1(\mathcal{O}_\beta) \rangle \rangle_0 \langle \langle \mathcal{O}^{\beta}\tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\ & + \sum_{\beta} b_{\beta}(1 - b_{\beta}) \langle \langle \tau_k(\mathcal{O}_\mu)\mathcal{O}^{\beta} \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_\beta)\tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\ = & - \sum_{\sigma} c_{\mu}^{\sigma} \{ \langle \langle \tau_{k+1}(\mathcal{O}_\sigma)\mathcal{X}\tau_l(\mathcal{O}_\nu) \rangle \rangle_0 - \delta_{k,0} \langle \langle \tau_1(\mathcal{O}_\sigma)\mathcal{X}\tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \} \\ & - (k + b_{\mu}) \langle \langle \tau_{k+2}(\mathcal{O}_\mu)\mathcal{X}\tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\ & + \sum_{\sigma} c_{\nu}^{\sigma} \left\{ \langle \langle \tau_k(\mathcal{O}_\mu)\mathcal{X}\tau_{l+1}(\mathcal{O}_\sigma) \rangle \rangle_0 \right. \\ & \quad - \delta_{l,0} \langle \langle \tau_k(\mathcal{O}_\mu)\mathcal{X}\tau_1(\mathcal{O}_\sigma) \rangle \rangle_0 \\ & \quad \left. - \delta_{l,-1} \langle \langle \tau_k(\mathcal{O}_\mu)\mathcal{X}\mathcal{O}_\sigma \rangle \rangle_0 \right\} \\ & - \sum_{\sigma,\rho} c_{\mu}^{\sigma} c_{\nu}^{\rho} \left\{ \langle \langle \tau_{k-1}(\mathcal{O}_\sigma)\tau_{l+1}(\mathcal{O}_\rho) \rangle \rangle_0 \right. \\ & \quad - \langle \langle \tau_{k+1}(\mathcal{O}_\sigma)\tau_{l-1}(\mathcal{O}_\rho) \rangle \rangle_0 \\ & \quad + \delta_{k,0} \langle \langle \tau_1(\mathcal{O}_\sigma)\tau_{l-1}(\mathcal{O}_\rho) \rangle \rangle_0 \\ & \quad - \delta_{l,0} \langle \langle \tau_{k-1}(\mathcal{O}_\sigma)\tau_1(\mathcal{O}_\rho) \rangle \rangle_0 \\ & \quad \left. - \delta_{l,-1} \langle \langle \tau_{k-1}(\mathcal{O}_\sigma)\mathcal{O}_\rho \rangle \rangle_0 \right\} \end{aligned}$$

$$\begin{aligned}
 & - \sum_{\sigma} (k + b_{\mu}) C_{\nu}^{\sigma} \left\{ \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_{l+1}(\mathcal{O}_{\sigma}) \rangle \rangle_0 \right. \\
 & \qquad - \langle \langle \tau_{k+2}(\mathcal{O}_{\mu}) \tau_{l-1}(\mathcal{O}_{\sigma}) \rangle \rangle_0 \\
 & \qquad - \delta_{l,0} \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_1(\mathcal{O}_{\sigma}) \rangle \rangle_0 \\
 & \qquad \left. - \delta_{l,-1} \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\sigma} \rangle \rangle_0 \right\} \\
 & - \sum_{\sigma, \rho} C_{\sigma}^{\rho} \langle \langle \mathcal{X} \tau_k(\mathcal{O}_{\mu}) \mathcal{O}^{\sigma} \rangle \rangle_0 \langle \langle \mathcal{O}_{\rho} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & + \sum_{\alpha, \sigma, \rho} C_{\sigma}^{\rho} C_{\mu}^{\alpha} \langle \langle \tau_{k-1}(\mathcal{O}_{\alpha}) \mathcal{O}^{\sigma} \rangle \rangle_0 \langle \langle \mathcal{O}_{\rho} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & + \sum_{\sigma, \rho} C_{\sigma}^{\rho} \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\rho} \rangle \rangle_0 \langle \langle \mathcal{O}^{\sigma} \mathcal{X} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & - \sum_{\alpha, \sigma, \rho} C_{\sigma}^{\rho} C_{\nu}^{\alpha} \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\rho} \rangle \rangle_0 \langle \langle \mathcal{O}^{\sigma} \tau_{l-1}(\mathcal{O}_{\alpha}) \rangle \rangle_0 \\
 & + \sum_{\sigma, \rho} (k + b_{\mu}) C_{\sigma}^{\rho} \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}^{\sigma} \rangle \rangle_0 \langle \langle \mathcal{O}_{\rho} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & + (l + b_{\nu} + 1) \left\{ \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{X} \tau_{l+2}(\mathcal{O}_{\nu}) \rangle \rangle_0 \right. \\
 & \qquad \left. - \delta_{l,-1} \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{X} \tau_1(\mathcal{O}_{\nu}) \rangle \rangle_0 \right\} \\
 & - \sum_{\sigma} (l + b_{\nu} + 1) C_{\mu}^{\sigma} \left\{ \langle \langle \tau_{k-1}(\mathcal{O}_{\sigma}) \tau_{l+2}(\mathcal{O}_{\nu}) \rangle \rangle_0 \right. \\
 & \qquad - \langle \langle \tau_{k+1}(\mathcal{O}_{\sigma}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & \qquad + \delta_{k,0} \langle \langle \tau_1(\mathcal{O}_{\sigma}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & \qquad \left. - \delta_{l,-1} \langle \langle \tau_{k-1}(\mathcal{O}_{\sigma}) \tau_1(\mathcal{O}_{\nu}) \rangle \rangle_0 \right\} \\
 & - (l + b_{\nu} + 1)(k + b_{\mu}) \left\{ \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_{l+2}(\mathcal{O}_{\nu}) \rangle \rangle_0 \right. \\
 & \qquad - \langle \langle \tau_{k+2}(\mathcal{O}_{\mu}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & \qquad \left. - \delta_{l,-1} \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_1(\mathcal{O}_{\nu}) \rangle \rangle_0 \right\} \\
 & - (l + b_{\nu} + 1) \sum_{\sigma, \rho} C_{\sigma}^{\rho} \langle \langle \tau_k(\mathcal{O}_{\mu}) \mathcal{O}_{\rho} \rangle \rangle_0 \langle \langle \mathcal{O}^{\sigma} \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & - \delta_{k,0} \sum_{\sigma} b_{\sigma} C_{\mu}^{\sigma} \langle \langle \tau_1(\mathcal{O}_{\sigma}) \tau_l(\mathcal{O}_{\nu}) \rangle \rangle_0 \\
 & + \delta_{l,0} \sum_{\sigma} (b_{\sigma} + 1) C_{\nu}^{\sigma} \langle \langle \tau_k(\mathcal{O}_{\mu}) \tau_1(\mathcal{O}_{\sigma}) \rangle \rangle_0.
 \end{aligned}$$

Using Lemma 1.4 to remove \mathcal{X} in this expression and simplifying, we

obtain the desired formula. q.e.d.

Applying Lemma 4.4 to the last 7 terms of (28) and simplifying yield

$$\begin{aligned}
 & \sum_{\beta} \langle \langle \mathcal{L}_1 \tau_k(\mathcal{O}_\mu) \mathcal{O}_\beta \rangle \rangle_0 \langle \langle \mathcal{O}^\beta (\mathcal{L}_0 - 2\mathcal{D}) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &= (k + b_\mu)(k + b_\mu + 1)(k + b_\mu + 2) \langle \langle \tau_{k+2}(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & \quad + \sum_{\sigma} \{3(k + b_\mu)^2 + 6(k + b_\mu) + 2\} \mathcal{C}_\mu^\sigma \langle \langle \tau_{k+1}(\mathcal{O}_\sigma) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & \quad + \sum_{\sigma} 3(k + b_\mu + 1)(\mathcal{C}_\mu^2)^\sigma \langle \langle \tau_k(\mathcal{O}_\sigma) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & \quad + \sum_{\sigma} (\mathcal{C}_\mu^3)^\sigma \langle \langle \tau_{k-1}(\mathcal{O}_\sigma) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & \quad + \sum_{\beta} (1 - b_\beta) b_\beta (1 + b_\beta) \langle \langle \tau_k(\mathcal{O}_\mu) \tau_1(\mathcal{O}_\beta) \rangle \rangle_0 \langle \langle \mathcal{O}^\beta \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & \quad + \sum_{\alpha, \beta} b_\alpha (1 - b_\alpha) (b_\alpha + b_\beta) \langle \langle \mathcal{O}^\alpha \rangle \rangle_0 \\
 (31) \quad & \quad \cdot \langle \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle \rangle_0 \langle \langle \mathcal{O}^\beta \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & \quad + \sum_{\alpha, \beta} b_\alpha (1 - b_\alpha) \mathcal{C}_{\alpha\beta} \langle \langle \mathcal{O}^\alpha \rangle \rangle_0 \langle \langle \mathcal{O}^\beta \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & \quad + \sum_{\alpha} b_\alpha (1 - b_\alpha) (b_\alpha + 1) \langle \langle \tau_k(\mathcal{O}_\mu) \mathcal{O}^\alpha \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_\alpha) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & \quad - \sum_{\alpha, \beta} (3b_\alpha^2 - 1) \mathcal{C}_\alpha^\beta \langle \langle \tau_k(\mathcal{O}_\mu) \mathcal{O}_\beta \rangle \rangle_0 \langle \langle \mathcal{O}^\alpha \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & \quad + \delta_{k,0} \delta_{l,0} (\mathcal{C}^3)_{\mu\nu} \\
 & \quad + (l + b_\nu)(l + b_\nu + 1)(l + b_\nu + 2) \langle \langle \tau_k(\mathcal{O}_\mu) \tau_{l+2}(\mathcal{O}_\nu) \rangle \rangle_0 \\
 & \quad + \sum_{\sigma} \{3(l + b_\nu)^2 + 6(l + b_\nu) + 2\} \mathcal{C}_\nu^\sigma \langle \langle \tau_k(\mathcal{O}_\mu) \tau_{l+1}(\mathcal{O}_\sigma) \rangle \rangle_0 \\
 & \quad + \sum_{\sigma} 3(l + b_\nu + 1)(\mathcal{C}_\nu^2)^\sigma \langle \langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\sigma) \rangle \rangle_0 \\
 & \quad + \sum_{\sigma} (\mathcal{C}_\nu^3)^\sigma \langle \langle \tau_k(\mathcal{O}_\mu) \tau_{l-1}(\mathcal{O}_\sigma) \rangle \rangle_0.
 \end{aligned}$$

Now it is straightforward to check that the difference of the right-hand sides of (31) and (26) is $\frac{\partial}{\partial t_l} \frac{\partial}{\partial t_k} \Psi_{0,2}$, by using topological recursion relation and the relationships between b_α 's stated in Section 1.1. The generalized WDVV equation implies that the left-hand sides of (31)

and (26) are equal. Therefore $\frac{\partial}{\partial t_l^\nu} \frac{\partial}{\partial t_k^\mu} \Psi_{0,2} = 0$, for arbitrary (k, μ) and (l, ν) . As noted at the end of Section 2, this leads to

Proposition 4.5. *The genus-0 free energy function F_0 satisfies the L_2 constraint.*

5. \tilde{L}_n constraints for genus-0 free energy function

Besides L_n constraints, Eguchi, Hori, and Xiong ([8]) also conjectured the existence of another sequence of constraints for the free energy functions. We call them \tilde{L}_n constraints. The \tilde{L}_0 constraint is the dilaton equation. In this section we will prove the \tilde{L}_1 and \tilde{L}_2 constraints for the genus-0 free energy function F_0 .

5.1 \tilde{L}_1 constraint

The \tilde{L}_1 constraint predicts the vanishing of the following function:

$$\tilde{\Psi}_{0,1} := - \sum_{m,\alpha} \tilde{t}_m^\alpha \langle\langle \tau_{m+1}(\mathcal{O}_\alpha) \rangle\rangle_0 + \frac{1}{2} \sum_{\alpha} \langle\langle \mathcal{O}_\alpha \rangle\rangle_0 \langle\langle \mathcal{O}^\alpha \rangle\rangle_0.$$

We now prove that this prediction is true.

Proposition 5.1.

$$\tilde{\Psi}_{0,1} = 0.$$

Proof. As noted at the end of Section 2, we only need to show that all the second derivatives of $\tilde{\Psi}_{0,1}$ vanish. In fact, for any (k, μ) and (l, ν) ,

$$\begin{aligned} \frac{\partial^2}{\partial t_k^\mu \partial t_l^\nu} \tilde{\Psi}_{0,1} &= - \sum_{m,\alpha} \tilde{t}_m^\alpha \langle\langle \tau_{m+1}(\mathcal{O}_\alpha) \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle\rangle_0 \\ &\quad - \langle\langle \tau_{k+1}(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle\rangle_0 - \langle\langle \tau_k(\mathcal{O}_\mu) \tau_{l+1}(\mathcal{O}_\nu) \rangle\rangle_0 \\ &\quad + \sum_{\alpha} \langle\langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \mathcal{O}_\alpha \rangle\rangle_0 \langle\langle \mathcal{O}^\alpha \rangle\rangle_0 \\ &\quad + \sum_{\alpha} \langle\langle \tau_k(\mathcal{O}_\mu) \mathcal{O}_\alpha \rangle\rangle_0 \langle\langle \mathcal{O}^\alpha \tau_l(\mathcal{O}_\nu) \rangle\rangle_0. \end{aligned}$$

By formula (10), the second, third, and fifth terms cancelled with each other. Hence, applying the topological recursion relation to the first

term gives

$$\begin{aligned} \frac{\partial^2}{\partial t_k^\mu \partial t_l^\nu} \tilde{\Psi}_{0,1} &= - \sum_{m,\alpha,\beta} \tilde{t}_m^\alpha \langle \langle \tau_m(\mathcal{O}_\alpha) \mathcal{O}_\beta \rangle \rangle_0 \langle \langle \mathcal{O}^\beta \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\ &\quad + \sum_\alpha \langle \langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \mathcal{O}_\alpha \rangle \rangle_0 \langle \langle \mathcal{O}^\alpha \rangle \rangle_0 \\ &= \sum_\beta \langle \langle \mathcal{D} \mathcal{O}_\beta \rangle \rangle_0 \langle \langle \mathcal{O}^\beta \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\ &\quad + \sum_\alpha \langle \langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \mathcal{O}_\alpha \rangle \rangle_0 \langle \langle \mathcal{O}^\alpha \rangle \rangle_0. \end{aligned}$$

By Lemma 1.2 (2), we have

$$\frac{\partial^2}{\partial t_k^\mu \partial t_l^\nu} \tilde{\Psi}_{0,1} = 0.$$

q.e.d.

5.2 \tilde{L}_2 constraint

The \tilde{L}_2 constraint predicts the vanishing of the function

$$\begin{aligned} \tilde{\Psi}_{0,2} &:= \sum_{m,\alpha} (m + b_\alpha + 1) \tilde{t}_m^\alpha \langle \langle \tau_{m+2}(\mathcal{O}_\alpha) \rangle \rangle_0 \\ &\quad + \sum_{m,\alpha,\beta} c_\alpha^\beta \tilde{t}_m^\alpha \langle \langle \tau_{m+1}(\mathcal{O}_\beta) \rangle \rangle_0 \\ &\quad - \sum_\alpha b_\alpha \langle \langle \mathcal{O}^\alpha \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_\alpha) \rangle \rangle_0 - \frac{1}{2} \sum_{\alpha,\beta} c_\alpha^\beta \langle \langle \mathcal{O}^\alpha \rangle \rangle_0 \langle \langle \mathcal{O}_\beta \rangle \rangle_0. \end{aligned}$$

To prove this constraint, we need to study correlation functions involving the following vector field,

$$\tilde{\mathcal{L}}_1 := \sum_{m,\alpha} \tilde{t}_m^\alpha \frac{\partial}{\partial t_{m+1}^\alpha}.$$

The \tilde{L}_1 constraint can be reformulated as

$$(32) \quad \langle \langle \tilde{\mathcal{L}}_1 \rangle \rangle_0 = \frac{1}{2} \sum_\alpha \langle \langle \mathcal{O}_\alpha \rangle \rangle_0 \langle \langle \mathcal{O}^\alpha \rangle \rangle_0.$$

Using the fact that

$$[\tilde{\mathcal{L}}_1, \frac{\partial}{\partial t_m^\alpha}] = -\frac{\partial}{\partial t_{m+1}^\alpha},$$

we can show the following.

Lemma 5.2.

- (1) $\left\langle \left\langle \tilde{\mathcal{L}}_1 \tau_m(\mathcal{O}_\alpha) \right\rangle \right\rangle_0 = -\langle \tau_{m+1}(\mathcal{O}_\alpha) \rangle_0 + \sum_{\sigma} \langle \tau_m(\mathcal{O}_\alpha) \mathcal{O}_\sigma \rangle_0 \langle \mathcal{O}^\sigma \rangle_0,$
- (2) $\left\langle \left\langle \tilde{\mathcal{L}}_1 \tau_m(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \right\rangle \right\rangle_0 = \sum_{\sigma} \langle \tau_m(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \mathcal{O}_\sigma \rangle_0 \langle \mathcal{O}^\sigma \rangle_0.$

Proof. The first equation follows directly from (32) and the fact that

$$\left\langle \left\langle \tilde{\mathcal{L}}_1 \tau_m(\mathcal{O}_\alpha) \right\rangle \right\rangle_0 = \tilde{\mathcal{L}}_1 \frac{\partial}{\partial t_m^\alpha} F_0 = \frac{\partial}{\partial t_m^\alpha} (\tilde{\mathcal{L}}_1 F_0) - \frac{\partial}{\partial t_{m+1}^\alpha} F_0.$$

Now we prove the second equation.

$$\begin{aligned} \left\langle \left\langle \tilde{\mathcal{L}}_1 \tau_m(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \right\rangle \right\rangle_0 &= \tilde{\mathcal{L}}_1 \frac{\partial}{\partial t_m^\alpha} \frac{\partial}{\partial t_n^\beta} F_0 \\ &= \left\{ \frac{\partial}{\partial t_m^\alpha} \frac{\partial}{\partial t_n^\beta} \tilde{\mathcal{L}}_1 - \frac{\partial}{\partial t_{m+1}^\alpha} \frac{\partial}{\partial t_n^\beta} - \frac{\partial}{\partial t_m^\alpha} \frac{\partial}{\partial t_{n+1}^\beta} \right\} F_0. \end{aligned}$$

By (32), we have

$$\begin{aligned} \left\langle \left\langle \tilde{\mathcal{L}}_1 \tau_m(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \right\rangle \right\rangle_0 &= \sum_{\sigma} \langle \tau_m(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \mathcal{O}_\sigma \rangle_0 \langle \mathcal{O}^\sigma \rangle_0 \\ &\quad + \sum_{\sigma} \langle \tau_m(\mathcal{O}_\alpha) \mathcal{O}_\sigma \rangle_0 \langle \mathcal{O}^\sigma \tau_n(\mathcal{O}_\beta) \rangle_0 \\ &\quad - \langle \tau_{m+1}(\mathcal{O}_\alpha) \tau_n(\mathcal{O}_\beta) \rangle_0 \\ &\quad - \langle \tau_m(\mathcal{O}_\alpha) \tau_{n+1}(\mathcal{O}_\beta) \rangle_0. \end{aligned}$$

By formula (10), the last three terms are canceled with each other. This proves the second equation. q.e.d.

We also need

Lemma 5.3.

$$\begin{aligned}
& \sum_{\sigma} b_{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \mathcal{O}^{\sigma} \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_{\sigma}) \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
& + \sum_{\sigma} b_{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \tau_1(\mathcal{O}_{\sigma}) \rangle \rangle_0 \langle \langle \mathcal{O}^{\sigma} \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
& = (m + b_{\alpha} + 1) \langle \langle \tau_{m+2}(\mathcal{O}_{\alpha}) \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
& + \sum_{\sigma} C_{\alpha}^{\sigma} \langle \langle \tau_{m+1}(\mathcal{O}_{\sigma}) \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
& + (n + b_{\beta} + 1) \langle \langle \tau_m(\mathcal{O}_{\alpha}) \tau_{n+2}(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
& + \sum_{\sigma} C_{\beta}^{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \tau_{n+1}(\mathcal{O}_{\sigma}) \rangle \rangle_0 \\
& - \sum_{\sigma, \rho} C_{\sigma}^{\rho} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \mathcal{O}^{\sigma} \rangle \rangle_0 \langle \langle \mathcal{O}_{\rho} \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0.
\end{aligned}$$

Proof. By Lemma 1.4 (3), we have

$$\begin{aligned}
& \sum_{\sigma} b_{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \mathcal{O}^{\sigma} \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_{\sigma}) \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
& = \sum_{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \mathcal{O}^{\sigma} \rangle \rangle_0 (n + b_{\beta} + b_{\sigma} + 1) \langle \langle \tau_1(\mathcal{O}_{\sigma}) \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
& - (n + b_{\beta} + 1) \sum_{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \mathcal{O}^{\sigma} \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_{\sigma}) \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
& = \sum_{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \mathcal{O}^{\sigma} \rangle \rangle_0 \left\{ \langle \langle \mathcal{X} \tau_1(\mathcal{O}_{\sigma}) \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \right. \\
(33) \quad & \quad \left. - \sum_{\rho} C_{\sigma}^{\rho} \langle \langle \mathcal{O}_{\rho} \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \right. \\
& \quad \left. - \sum_{\rho} C_{\beta}^{\rho} \langle \langle \tau_1(\mathcal{O}_{\sigma}) \tau_{n-1}(\mathcal{O}_{\rho}) \rangle \rangle_0 \right\} \\
& - (n + b_{\beta} + 1) \sum_{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \mathcal{O}^{\sigma} \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_{\sigma}) \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \sum_{\sigma} b_{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \tau_1(\mathcal{O}_{\sigma}) \rangle \rangle_0 \langle \langle \mathcal{O}^{\sigma} \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
 &= - \sum_{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \tau_1(\mathcal{O}_{\sigma}) \rangle \rangle_0 (n + b_{\beta} + 1 - b_{\sigma}) \langle \langle \mathcal{O}^{\sigma} \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
 & \quad + (n + b_{\beta} + 1) \sum_{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \tau_1(\mathcal{O}_{\sigma}) \rangle \rangle_0 \langle \langle \mathcal{O}^{\sigma} \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
 (34) \quad &= - \sum_{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \tau_1(\mathcal{O}_{\sigma}) \rangle \rangle_0 \left\{ \langle \langle \mathcal{X} \mathcal{O}^{\sigma} \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 - \delta_{n,0} \mathcal{C}_{\beta}^{\sigma} \right. \\
 & \quad \left. - \sum_{\rho} \mathcal{C}_{\beta}^{\rho} \langle \langle \mathcal{O}^{\sigma} \tau_{n-1}(\mathcal{O}_{\rho}) \rangle \rangle_0 \right\} \\
 & \quad + (n + b_{\beta} + 1) \sum_{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \tau_1(\mathcal{O}_{\sigma}) \rangle \rangle_0 \langle \langle \mathcal{O}^{\sigma} \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0.
 \end{aligned}$$

Summing up (33) and (34) together, then using Lemma 4.3 (ii) and (iii) to simplify it, we obtain

$$\begin{aligned}
 & \sum_{\sigma} b_{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \mathcal{O}^{\sigma} \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_{\sigma}) \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
 & \quad + \sum_{\sigma} b_{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \tau_1(\mathcal{O}_{\sigma}) \rangle \rangle_0 \langle \langle \mathcal{O}^{\sigma} \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
 &= \langle \langle \tau_{m+2}(\mathcal{O}_{\alpha}) \mathcal{X} \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
 & \quad - \sum_{\sigma, \rho} \mathcal{C}_{\beta}^{\rho} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \mathcal{O}^{\sigma} \rangle \rangle_0 \langle \langle \mathcal{O}_{\rho} \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 \\
 & \quad - \sum_{\sigma} \mathcal{C}_{\beta}^{\sigma} \left\{ \langle \langle \tau_{m+2}(\mathcal{O}_{\alpha}) \tau_{n-1}(\mathcal{O}_{\sigma}) \rangle \rangle_0 - \langle \langle \tau_m(\mathcal{O}_{\alpha}) \tau_{n+1}(\mathcal{O}_{\sigma}) \rangle \rangle_0 \right. \\
 & \quad \quad \left. + \delta_{n,0} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \tau_1(\mathcal{O}_{\sigma}) \rangle \rangle_0 \right\} \\
 & \quad + \delta_{n,0} \sum_{\sigma} \mathcal{C}_{\beta}^{\sigma} \langle \langle \tau_m(\mathcal{O}_{\alpha}) \tau_1(\mathcal{O}_{\sigma}) \rangle \rangle_0 \\
 & \quad - (n + b_{\beta} + 1) \left\{ \langle \langle \tau_{m+2}(\mathcal{O}_{\alpha}) \tau_n(\mathcal{O}_{\beta}) \rangle \rangle_0 - \langle \langle \tau_m(\mathcal{O}_{\alpha}) \tau_{n+2}(\mathcal{O}_{\beta}) \rangle \rangle_0 \right\}.
 \end{aligned}$$

Using Lemma 1.4 (3) to remove \mathcal{X} in the first term and simplifying, we arrive at the desired equation. q.e.d.

Now we are ready to prove the \tilde{L}_2 constraint.

Proposition 5.4.

$$\tilde{\Psi}_{0,2} = 0.$$

Proof. As in the proof of \tilde{L}_1 constraint, we only need to show that all the second derivatives of $\tilde{\Psi}_{0,2}$ are equal to zero. For arbitrary (μ, k) and (ν, l) , we have

$$\begin{aligned} \frac{\partial^2}{\partial t_k^\mu \partial t_l^\nu} \tilde{\Psi}_{0,2} &= \sum_{m,\alpha} (m + b_\alpha + 1) \tilde{t}_m^\alpha \langle \langle \tau_{m+2}(\mathcal{O}_\alpha) \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\ &+ \sum_{m,\alpha,\beta} C_\alpha^\beta \tilde{t}_m^\alpha \langle \langle \tau_{m+1}(\mathcal{O}_\beta) \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\ &+ (k + b_\mu + 1) \langle \langle \tau_{k+2}(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\ &+ \sum_\sigma C_\mu^\sigma \langle \langle \tau_{k+1}(\mathcal{O}_\sigma) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\ &+ (l + b_\nu + 1) \langle \langle \tau_k(\mathcal{O}_\mu) \tau_{l+2}(\mathcal{O}_\nu) \rangle \rangle_0 \\ &+ \sum_\sigma C_\nu^\sigma \langle \langle \tau_k(\mathcal{O}_\mu) \tau_{l+1}(\mathcal{O}_\sigma) \rangle \rangle_0 \\ &- \sum_\sigma b_\sigma \langle \langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \mathcal{O}^\sigma \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_\sigma) \rangle \rangle_0 \\ &- \sum_\sigma b_\sigma \langle \langle \tau_k(\mathcal{O}_\mu) \mathcal{O}^\sigma \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_\sigma) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\ &- \sum_\sigma b_\sigma \langle \langle \tau_k(\mathcal{O}_\mu) \tau_1(\mathcal{O}_\sigma) \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\ &- \sum_\sigma b_\sigma \langle \langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \tau_1(\mathcal{O}_\sigma) \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \rangle \rangle_0 \\ &- \sum_{\sigma,\rho} C_\sigma^\rho \langle \langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \mathcal{O}^\sigma \rangle \rangle_0 \langle \langle \mathcal{O}_\rho \rangle \rangle_0 \\ &- \sum_{\sigma,\rho} C_\sigma^\rho \langle \langle \tau_k(\mathcal{O}_\mu) \mathcal{O}^\sigma \rangle \rangle_0 \langle \langle \mathcal{O}_\rho \tau_l(\mathcal{O}_\nu) \rangle \rangle_0. \end{aligned}$$

Applying the topological recursion relation to the first two terms, and Lemma 5.3 to the eighth and ninth terms yields

$$\begin{aligned}
 \frac{\partial^2}{\partial t_k^\mu \partial t_l^\nu} \tilde{\Psi}_{0,2} &= \sum_{m,\alpha,\sigma} (m + b_\alpha + 1) \tilde{t}_m^\alpha \langle \langle \tau_{m+1}(\mathcal{O}_\alpha) \mathcal{O}_\sigma \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &\quad + \sum_{m,\alpha,\beta,\sigma} c_\alpha^\beta \tilde{t}_m^\alpha \langle \langle \tau_m(\mathcal{O}_\beta) \mathcal{O}_\sigma \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 (35) \quad &\quad - \sum_\sigma b_\sigma \langle \langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \mathcal{O}^\sigma \rangle \rangle_0 \langle \langle \tau_1(\mathcal{O}_\sigma) \rangle \rangle_0 \\
 &\quad - \sum_\sigma b_\sigma \langle \langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \tau_1(\mathcal{O}_\sigma) \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \rangle \rangle_0 \\
 &\quad - \sum_{\sigma,\rho} c_\sigma^\rho \langle \langle \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \mathcal{O}^\sigma \rangle \rangle_0 \langle \langle \mathcal{O}_\rho \rangle \rangle_0.
 \end{aligned}$$

We now use Lemma 1.4 (3) to compute the first two terms. Let

$$\begin{aligned}
 f &:= \sum_{m,\alpha,\sigma} (m + b_\alpha + 1) \tilde{t}_m^\alpha \langle \langle \tau_{m+1}(\mathcal{O}_\alpha) \mathcal{O}_\sigma \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &\quad + \sum_{m,\alpha,\beta,\sigma} c_\alpha^\beta \tilde{t}_m^\alpha \langle \langle \tau_m(\mathcal{O}_\beta) \mathcal{O}_\sigma \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &= \sum_{m,\alpha,\sigma} \tilde{t}_m^\alpha \langle \langle \mathcal{X} \tau_{m+1}(\mathcal{O}_\alpha) \mathcal{O}_\sigma \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &\quad - \sum_{m,\alpha,\sigma} b_\sigma \tilde{t}_m^\alpha \langle \langle \tau_{m+1}(\mathcal{O}_\alpha) \mathcal{O}_\sigma \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &= \sum_\sigma \langle \langle \mathcal{X} \tilde{\mathcal{L}}_1 \mathcal{O}_\sigma \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &\quad - \sum_\sigma b_\sigma \langle \langle \tilde{\mathcal{L}}_1 \mathcal{O}_\sigma \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0.
 \end{aligned}$$

By the generalized WDVV equation and Lemma 5.2, we get

$$\begin{aligned}
 f &= \sum_\sigma \langle \langle \mathcal{X} \tau_k(\mathcal{O}_\mu) \mathcal{O}^\sigma \rangle \rangle_0 \langle \langle \mathcal{O}_\sigma \tilde{\mathcal{L}}_1 \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &\quad - \sum_\sigma b_\sigma \langle \langle \tilde{\mathcal{L}}_1 \mathcal{O}_\sigma \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &= \sum_{\sigma,\rho} \langle \langle \mathcal{X} \tau_k(\mathcal{O}_\mu) \mathcal{O}^\sigma \rangle \rangle_0 \langle \langle \mathcal{O}_\sigma \tau_l(\mathcal{O}_\nu) \mathcal{O}_\rho \rangle \rangle_0 \langle \langle \mathcal{O}^\rho \rangle \rangle_0 \\
 &\quad - \sum_\sigma b_\sigma \left\{ - \langle \langle \tau_1(\mathcal{O}_\sigma) \rangle \rangle_0 + \sum_\rho \langle \langle \mathcal{O}_\sigma \mathcal{O}_\rho \rangle \rangle_0 \langle \langle \mathcal{O}^\rho \rangle \rangle_0 \right\} \\
 &\quad \cdot \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0.
 \end{aligned}$$

Using the generalized WDVV first and then applying Lemma 1.4 (3), we obtain

$$\begin{aligned}
 f &= \sum_{\sigma, \rho} \langle \langle \mathcal{X} \mathcal{O}_\rho \mathcal{O}_\sigma \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_l(\mathcal{O}_\nu) \tau_k(\mathcal{O}_\mu) \rangle \rangle_0 \langle \langle \mathcal{O}^\rho \rangle \rangle_0 \\
 &\quad + \sum_{\sigma} b_\sigma \langle \langle \tau_1(\mathcal{O}_\sigma) \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &\quad - \sum_{\sigma, \rho} b_\sigma \langle \langle \mathcal{O}_\sigma \mathcal{O}_\rho \rangle \rangle_0 \langle \langle \mathcal{O}^\rho \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &= \sum_{\sigma, \rho} \{ (b_\rho + b_\sigma) \langle \langle \mathcal{O}_\rho \mathcal{O}_\sigma \rangle \rangle_0 + \mathcal{C}_{\sigma\rho} \} \langle \langle \mathcal{O}^\sigma \tau_l(\mathcal{O}_\nu) \tau_k(\mathcal{O}_\mu) \rangle \rangle_0 \langle \langle \mathcal{O}^\rho \rangle \rangle_0 \\
 &\quad + \sum_{\sigma} b_\sigma \langle \langle \tau_1(\mathcal{O}_\sigma) \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &\quad - \sum_{\sigma} b_\sigma \sum_{\rho} \langle \langle \mathcal{O}_\sigma \mathcal{O}_\rho \rangle \rangle_0 \langle \langle \mathcal{O}^\rho \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &= \sum_{\sigma, \rho} \mathcal{C}_{\sigma\rho} \langle \langle \mathcal{O}^\sigma \tau_l(\mathcal{O}_\nu) \tau_k(\mathcal{O}_\mu) \rangle \rangle_0 \langle \langle \mathcal{O}^\rho \rangle \rangle_0 \\
 &\quad + \sum_{\sigma} b_\sigma \langle \langle \tau_1(\mathcal{O}_\sigma) \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0 \\
 &\quad + \sum_{\sigma} b_\rho \sum_{\rho} \langle \langle \mathcal{O}_\sigma \mathcal{O}_\rho \rangle \rangle_0 \langle \langle \mathcal{O}^\rho \rangle \rangle_0 \langle \langle \mathcal{O}^\sigma \tau_k(\mathcal{O}_\mu) \tau_l(\mathcal{O}_\nu) \rangle \rangle_0.
 \end{aligned}$$

Plugging this formula into (35) and using the topological recursion relation again lead to

$$\frac{\partial^2}{\partial t_k^\mu \partial t_l^\nu} \tilde{\Psi}_{0,2} = 0.$$

This proves the proposition. q.e.d.

References

- [1] K. Behrend, *Gromov-Witten invariants in algebraic geometry*, Invent. Math. **127** (1997) 601–617.
- [2] P. Belorousski & R. Pandharipande, *A descendent relation in genus 2*, (math. AG/9803072).
- [3] L. Borisov, *On betti numbers and Chern classes of varieties with trivial odd cohomology groups*, (alg-geom/9703023).

- [4] R. Dijkgraaf, H. Verlinde & E. Verlinde, *Loop equations and Virasoro constraints in nonperturbative two-dimensional quantum gravity*, Nuclear Phys. **B 348** (1991) No. 3, 435–456.
- [5] R. Dijkgraaf & E. Witten, *Mean field theory, topological field theory, and multi-matrix models*, Nuclear Phys. **B 342** (1990) 486–522.
- [6] B. Dubrovin, *Geometry of 2D topological field theories*, Integrable systems and quantum groups, Lecture Notes in Math. Vol. 1620, Springer, Berlin, 1996, 120–348.
- [7] T. Eguchi, K. Hori & C. Xiong, *Gravitational quantum cohomology*, Internat. J. Modern Phys. **A 12** (1997) 1743–1782, (hep-th/9605225).
- [8] ———, *Quantum Cohomology and Virasoro Algebra*, Phys. Lett. **B 402** (1997) 71–80, (hep-th/9703086).
- [9] T. Eguchi, M. Jinzjiji & C. Xiong, *Quantum cohomology and free field representation*, Nuclear Phys. **B 510** (1998) 608–622, (hep-th/9709152).
- [10] T. Eguchi & C. Xiong, *Quantum cohomology at higher genus: topological recursion relations and Virasoro conditions*, To appear in Adv. Theoret. Math. Phys., (hep-th/9801010).
- [11] M. Fukuma, H. Kawai & R. Nakayama, *Continuum Schwinger-Dyson equations and universal structures in two-dimensional quantum gravity*, Internat. J. Modern Phys. **A6** (1991) No. 8, 1385–1406.
- [12] E. Getzler, *Intersection theory on $\bar{M}_{1,4}$ and elliptic Gromov-Witten Invariants*, J. Amer. Math. Soc. **10** (1997) 973–998 (alg-geom/9612004).
- [13] ———, *Topological recursion relations in genus 2*, (math.AG/9801003).
- [14] K. Hori, *Constraints for topological strings in $D \geq 1$* , Nuclear Phys. **B 439** (1995) 395, (hep-th/9411135).
- [15] V. Kac & A. Schwarz, *geometric interpretation of the partition function of 2D gravity*, Phys. Lett. **B 257** (1991) No. 3-4, 329–334.
- [16] S. Katz, unpublished, March, 1997.
- [17] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix airy function*, Comm. Math. Phys. **147** (1992).
- [18] M. Kontsevich & Y. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164** (1994) 525–562.
- [19] ———, *Relations between the correlators of the topological sigma-model coupled to gravity*, (alg-geom/9708024).
- [20] J. Li & G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. Soc. **11** (1998) 119–174. (alg-geom/9602007).

- [21] ———, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, (alg-geom/9608032).
- [22] D. McDuff & D. Salamon, *J-holomorphic curves and quantum cohomology*, Univ. Lect. Ser. Vol. 6 (AMS, Providence, RI, 1994).
- [23] Y. Ruan, & G. Tian, , *A mathematical theory of quantum cohomology* J. Differential Geom. **42** (1995) 259–367.
- [24] ———, *Higher genus symplectic invariants and sigma models coupled with gravity*, Invent. Math. **130** (1997) 455–516. (alg-geom/9601005)
- [25] G. Tian, *Quantum cohomology and its associativity*, Current Develop. Math. (Cambridge, MA.), Internat. Press. 1995, 360–401.
- [26] E. Witten, *Topological sigma models*, Comm. Math. Phys. **118** (1988).
- [27] ———, *Two dimensional gravity and intersection theory on Moduli space*, Surveys Differential Geom. **1** (1991) 243–310.
- [28] ———, *On the Kontsevich model and other models of two-dimensional gravity*, Proc. 20 Internat. Conf. Differential Geom. Methods in Theoret. Phys., World Sci. River Edge, NJ. 1992, 176–216.

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