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THE YAMABE PROBLEM FOR HIGHER ORDER CURVATURES

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Abstract

Let $\mathcal{M}$ be a compact Riemannian manifold of dimension $n > 2$. The k-curvature, for $k = 1, 2, \ldots, n$, is defined as the k-th elementary symmetric polynomial of the eigenvalues of the Schouten tenser. The k-Yamabe problem is to prove the existence of a conformal metric whose k-curvature is a constant. When $k = 1$, it reduces to the well-known Yamabe problem. Under the assumption that the metric is admissible, the existence of solutions is known for the case $k = 2$, $n = 4$, for locally conformally flat manifolds and for the cases $k > n/2$. In this paper we prove the solvability of the k-Yamabe problem in the remaining cases $k \leq n/2$, under the hypothesis that the problem is variational. This includes all of the cases $k = 2$ as well as the locally conformally flat case.

1. Introduction

In recent years the Yamabe problem for the k-curvature of the Schouten tensor, or simply the k-Yamabe problem, has been extensively studied. Let $(\mathcal{M}, g_0)$ be a compact Riemannian manifold of dimension $n > 2$ and denote by 'Ric' and R respectively the Ricci tensor and the scalar curvature. The k-Yamabe problem is to prove the existence of a conformal metric $g = g_v = v^{\frac{4}{n-2}} g_0$ that solves the equation

$$\sigma_k(\lambda(A_g)) = 1 \quad \text{on } \mathcal{M}, \tag{1.1}$$

where $1 \leq k \leq n$ is an integer, and $\lambda = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of A_g with respect to the metric g. As usual, we denote by

$$A_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{2(n-1)} g\right) \tag{1.2}$$

the Schouten tensor, and by

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} \tag{1.3}$$

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the k-th elementary symmetric polynomial. When $k = 1$, we arrive at the well-known Yamabe problem.

When $k > 1$, the k-Yamabe problem is a fully nonlinear partial differential equation for the function v, which is elliptic if the eigenvalues $\lambda(A_g)$ lie in the convex cone Γ_k (or $-\Gamma_k$) [**CNS**], given by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0 \text{ for } j = 1, \dots, k\}. \tag{1.4}$$

Under the assumption $\lambda(A_{g_0}) \in \Gamma_k$, the k-Yamabe problem has been solved in the case $k = 2$, $n = 4$ by Chang, Gursky and Yang [**CGY1**, **CGY2**], for locally conformally flat manifolds by Li and Li [**LL1**] (see also [**GW2**]), and for $k > n/2$ by Gursky and Viaclovski [**GV2**]. In this paper we employ a variational method to treat the problem for the cases $2 \le k \le \frac{n}{2}$. We prove that equation (1.1) has a solution as long as it is variational, namely it is the Euler equation of a functional, which includes the cases when $k = 2$ and when $\mathcal{M}$ is locally conformally flat.

The progressive resolution of the Yamabe problem ($k = 1$) by the second author, Aubin and Schoen [**Ya**, **Tr**, **Au**, **S1**] was a milestone in differential geometry. Roughly speaking, the overall proof consists of two parts. The first one is to show that the Yamabe problem is solvable if the Yamabe constant Y_1 satisfies the condition

$$Y_1(\mathcal{M}) < Y_1(S^n), \tag{1.5}$$

and the second one is to verify the condition (1.5) for manifolds not conformally diffeomorphic to the unit sphere S^n with standard metric. When $\mathcal{M}$ is locally conformally flat, different proofs were found later [**SY1**, **Ye**].

For the k-Yamabe problem, $2 \le k \le \frac{n}{2}$, our variational approach basically comprises the same two steps. Namely, one first shows that (1.1) has a solution when the k-Yamabe constant Y_k satisfies

$$Y_k(\mathcal{M}) < Y_k(S^n), \tag{1.6}$$

and then verify the condition (1.6) for manifolds not conformal to the unit sphere S^n. For the first step, we cannot apply the variational method directly, as equation (1.1) is fully nonlinear and we need to restrict to a subset of the conformal class $[g_0] = \{g \mid g = v^{\frac{4}{n-2}} g_0,\ v > 0\}$, given by

$$[g_0]_k = \{g \mid g = v^{\frac{4}{n-2}} g_0, v > 0, \lambda(A_g) \in \Gamma_k\}. \tag{1.7}$$

Our approach is as follows. When $k < \frac{n}{2}$, a solution of (1.1) corresponds to a critical point of the functional

$$J(g) = \frac{n-2}{2(n-2k)} \int_{\mathcal{M}} \sigma_k(\lambda(g)) d\,\mathrm{vol}_g - \frac{n-2}{2n} \int_{\mathcal{M}} d\,\mathrm{vol}_g. \tag{1.8}$$

Through the functional (1.8), we introduce a descent gradient flow, establish appropriate a priori estimates, and prove the convergence of solutions to the flow under assumption (1.6). We need to choose a particular gradient flow to obtain the a priori estimates, locally in time. Our proof also leads to a Sobolev type inequality (see also [**GW3**]). That is for $2 \le k < \frac{n}{2}$, there exists a constant $C > 0$ independent of $g \in [g_0]_k$ such that the inequality

$$[Vol(\mathcal{M}_g)]^{\frac{n-2}{2n}} \le C\Big[\int_{\mathcal{M}} \sigma_k(\lambda(A_g))d\,\mathrm{vol}_g\Big]^{\frac{n-2}{2n-4k}}. \tag{1.9}$$

When $k = \frac{n}{2}$, we will prove that the integral

$$\mathcal{F}_k(g) =: \int_{\mathcal{M}} \sigma_k(\lambda(g))d\,\mathrm{vol}_g = \mathrm{const} \tag{1.10}$$

on $[g_0]_k$. From (1.6) and (1.10) we show that the set of solutions is compact. A crucial ingredient for the proof of (1.10) is Proposition 2.1 below, which shows that a partial differential equation is variational if and only if its linearized operator is self-adjoint.

For the second step, the verification of (1.6), we invoke a new idea of using the solution to the original problem when $k = 1$, so that (1.6) is deduced directly from (1.5).

This paper is arranged as follows. In Section 2, we state the main results, specifically in §2.1, while in §2.2 we outline the proof. In §2.3 we collect some related results on the k-Hessian equation. In §2.4 we give a necessary and sufficient condition for a partial differential equation to be variational. In Section 3, we introduce the gradient flow for the functional (1.8) and prove the necessary derivative estimates for solutions and the ensuing existence theorem (Theorem 3.2). We also provide a counterexample to regularity when the eigenvalues $\lambda(A_g)$ lie in the negative cone $(-\Gamma_k)$. In Section 4 we investigate the asymptotic behavior of a descent gradient flow and prove the convergence of the flow in the subcritical case. Using a standard blow up argument, we then infer the solvability of the Yamabe problem under condition (1.6). We then prove (1.6) for manifolds not conformal to S^n in Section 5. The final Section 6 contains some remarks.

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2. The main results

2.1. The main results. Let $(\mathcal{M}, g_0)$ be a Riemannian manifold. If $g = v^{\frac{4}{n-2}} g_0$ is a solution of (1.1), then the Schouten tensor is given by

$A_g = \frac{2}{(n-2)v} V$, and v satisfies the equation

$$L[v] := v^{(1-k)\frac{n+2}{n-2}} \sigma_k(\lambda(V)) = v^{\frac{n+2}{n-2}}, \tag{2.1}$$

where

$$V = -\nabla^2 v + \frac{n}{n-2} \frac{\nabla v \otimes \nabla v}{v} - \frac{1}{n-2} \frac{|\nabla v|^2}{v} g_0 + \frac{n-2}{2} v A_{g_0}. \tag{2.2}$$

Equation (2.1) is a fully nonlinear equation of similar type to the k-Hessian equations [**CNS**, **CW2**, **I**, **TW1**]. For the operator L to be elliptic, we need to restrict to metrics with eigenvalues $\lambda(A_g) \in \cup(\pm\Gamma_k)$. Therefore equation (2.1) has two elliptic branches: one is when the eigenvalues $\lambda \in \Gamma_k$ and the other one is when $\lambda \in (-\Gamma_k)$. In this paper we will mainly consider solutions with eigenvalues in Γ_k. Accordingly we say a metric g, or a function v, is *k-admissible* if $g \in [g_0]_k$, where $g = v^{\frac{4}{n-2}} g_0$ and $[g_0]_k$ was introduced in (1.7). In this paper we will always assume, unless otherwise indicated, that $2 \le k \le \frac{n}{2}$ and the following two conditions hold:

(C1) The set $[g_0]_k \neq \emptyset$;
(C2) The operator L is variational.

Note that condition (C1) may be replaced by $Y_j(\mathcal{M}) > 0$ for $j = 1, \dots, k$ [**S**], as in the case when $k = 2$ and $n = 4$ [**CGY1**, **GV1**]. The condition does not imply the metric $g_0 \in [g_0]_k$, but there is a conformal metric in $[g_0]_k$. Conditions (C1) (C2) are automatically satisfied when $k = 1$. Condition (C2) is satisfied when $k = 2$ or $\mathcal{M}$ is locally conformally flat.

As for the Yamabe problem, we introduce the k-Yamabe constant for $2 \le k \le \frac{n}{2}$,

$$Y_k(\mathcal{M}) = \inf\{\mathcal{F}_k(g) \mid g \in [g_0]_k, \mathrm{Vol}(\mathcal{M}_g) = 1\}, \tag{2.3}$$

where

$$\begin{aligned} \mathcal{F}_k(g) &= \int_{\mathcal{M}} \sigma_k(\lambda(A_g)) d\,vol_g \\ &= \int_{\mathcal{M}} v^{\frac{2n}{n-2} - k\frac{n+2}{n-2}} \sigma_k(\lambda(V))\, d\,vol_{g_0}. \end{aligned} \tag{2.4}$$

Note that we have ignored a coefficient $(\frac{2}{n-2})^k$ in the second equality. The main result of the paper is the following.

Theorem 2.1. *Assume $2 \le k \le \frac{n}{2}$ and the conditions* (C1) (C2) *hold. Then the k-Yamabe problem* (1.1) *is solvable.*

As indicated in the introduction, the proof of Theorem 2.1 is divided into two parts. The first part is the following lemma.

Lemma 2.1. *If the critical inequality* (1.6) *holds, then the k-Yamabe problem* (1.1) *is solvable.*

The second part provides the condition for (1.6).

Lemma 2.2. *The critical inequality* (1.6) *holds for any compact manifold which is not conformal to the unit sphere* S^n.

In particular when $k = \frac{n}{2}$, we have $\mathcal{F}_{n/2}(g) \equiv$const for any $g \in [g_0]_k$ (Lemma 4.6). Hence (1.6) implies that $\mathcal{F}_{n/2}(g) < Y_{n/2}(S^n)$ provided $\mathcal{M}$ is not conformal to the unit sphere.

2.2. Strategy of the proof. A solution of the k-Yamabe problem is a min-max type critical point of the corresponding functional. As we need to restrict ourselves to k-admissible functions, we cannot directly use variational theory (such as the Ekeland variational principle). Instead, we study a descent gradient flow of the functional and investigate its convergence. We need to choose a special gradient flow (similar to [**CW2**]) for which the necessary a priori estimates can be established.

As with the original Yamabe paper [**Ya**], we first study the approximating problems

$$L(v) = v^p, \tag{2.5}$$

where $1 < p \le \frac{n+2}{n-2}$. When $k < \frac{n}{2}$, equation (2.5) is the Euler equation of the functional

(2.6)

$$\begin{aligned} J_p(v) &= J_p(v; \mathcal{M}) \\ &= \frac{n-2}{2n-4k} \int_{(\mathcal{M},g_0)} v^{\frac{2n}{n-2} - k\frac{n+2}{n-2}} \sigma_k(\lambda(V)) - \frac{1}{p+1} \int_{(\mathcal{M},g_0)} v^{p+1}. \end{aligned}$$

Let $\varphi_1 = \varepsilon$ and $\varphi_2 = \varepsilon^{-1}$, where $\varepsilon > 0$ is a small constant. Then $J_p(\varphi_1) \to 0$ and $J_p(\varphi_2) \to -\infty$ as $\varepsilon \to 0$. Let P denote the set of paths in Φ_k connecting φ_1 and φ_2, namely

$$P = \{\gamma \in C([0,1], \Phi_k) \mid \gamma(0) = \varphi_1, \gamma(1) = \varphi_2\}, \tag{2.7}$$

where Φ_k denote the set of k-admissible functions. Denote

$$c_p[\mathcal{M}] = \inf_{\gamma \in P} \sup_{s \in [0,1]} J_p(\gamma(s); \mathcal{M}). \tag{2.8}$$

Then (1.6) is equivalent to

$$c_p[\mathcal{M}] < c_p[S^n] \tag{2.9}$$

with $p = \frac{n+2}{n-2}$. We will prove that J_p has a min-max critical point v_p with $J_p(v_p) = c_p[\mathcal{M}]$, in the sub-critical case $p < \frac{n+2}{n-2}$. By a blow-up argument, we prove furthermore that v_p converges to a solution of (2.1) under the assumption (2.9).

The descent gradient flow will be chosen so that appropriate a priori estimates can be established. To simplify the computations, we will also use the conformal transformations $g = u^{-2}g_0$ or $g = e^{-2w}g_0$. That is

$$u = e^w = v^{-\frac{2}{n-2}}. \tag{2.10}$$

We say u or w is k-admissible if v is.

Our gradient flow is given by

$$F[w] - w_t = \mu(f(x,w)), \tag{2.11}$$

where

$$F[w] := \mu(\sigma_k(\lambda(A_g))) \tag{2.12}$$

and $g = e^{-2w}g_0$. When $f(x,w) = e^{-2kw}$, a stationary solution of (2.11) is a solution to the k-Yamabe problem. The function μ is monotone increasing and satisfies

$$\lim_{t\to 0^+} \mu(t) = -\infty. \tag{2.13}$$

Condition (2.13) ensures the solution is k-admissible at any time t. For if $u(\cdot,t)$ is a smooth solution, then (2.13) implies $\sigma_k(\lambda) > 0$ at any time $t > 0$. A natural candidate for the choice of μ is the logarithm function $\mu(t) = \log t$ [**Ch**, **W1**, **TW2**]. However, for the flow (2.11), we need to choose a different μ to ensure appropriate a priori estimates.

In the case $k = \frac{n}{2}$, $\mathcal{F}_{n/2}$ is a constant less than $Y_{n/2}(S^n)$. Hence by the Liouville theorem in [**LL1**], it is easy to prove that the set of solutions of (2.1) is compact. Hence the existence of solutions can be obtained by a degree argument. When $2 \le k < \frac{n}{2}$, by the Liouville theorem in [**LL2**], one can also prove the set of solutions of (2.5) is compact when $p < \frac{n+2}{n-2}$. But to use the condition (1.6) in the blow-up argument, we need a solution v_p of (2.5) satisfying $J_p(v_p) = c_p$. This is the reason for us to employ the gradient flow.

For the verification of (1.6), let v_1 be the solution to the Yamabe problem ($k = 1$), and v be the k-admissible solution of the equation

$$\sigma_k(\lambda(V)) = v_1^{k\frac{n+2}{n-2}}. \tag{2.14}$$

We will verify (1.6) by using the solution v as the test function. Even when $k = 1$, an adaptation of this approach gives a new proof of the inequality $Y_1(\mathcal{M}) < Y_1(S^n)$, see [**W2**].

The idea of using a gradient flow was inspired by [**CW2**], where a similar problem to the k-Hessian problem (see (2.22) below) was studied. However, technically the argument in this paper is different. For example, for the k-Yamabe problem, the a priori estimates only allow us to get a local (in time) solution. The argument in this paper is also self-contained, except we will use the Liouville theorem in [**LL1**, **LL2**], proved by the moving plane method; see also [**CGY3**].

2.3. The k-Hessian equation. Equation (2.1) is closely related to the k-Hessian equation

$$\sigma_k(\lambda(D^2v)) = f(x) \quad x \in \Omega, \tag{2.15}$$

where $1 \leq k \leq n$, $\lambda = (\lambda_1, \dots, \lambda_n)$ denote the eigenvalues of the Hessian matrix $(D^2 v)$, and Ω is a bounded domain in the Euclidean n-space $\mathbb{R}^n$. For later applications, we collect here some elementary properties of the polynomial σ_k, and give a very brief summary of related results for the equation (2.15).

We write $\sigma_0(\lambda) = 1$, $\sigma_k(\lambda) = 0$ for $k > n$, and denote $\sigma_{k;i}(\lambda) = \sigma_k(\lambda)\big|_{\lambda_i=0}$.

Lemma 2.3. *Let $\lambda \in \Gamma_k$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Then*

$$
\begin{aligned}
&\text{(i)} && \lambda_k \geq 0,\\
&\text{(ii)} && \sigma_k(\lambda) = \sigma_{k;i}(\lambda) + \lambda_i \sigma_{k-1;i}(\lambda),\\
&\text{(iii)} && \Sigma_{i=1}^n \sigma_{k-1;i}(\lambda) = (n-k+1)\sigma_{k-1}(\lambda),\\
&\text{(iv)} && \sigma_{k-1;n}(\lambda) \geq \cdots \geq \sigma_{k-1;1}(\lambda) > 0,\\
&\text{(v)} && \sigma_{k-1;k}(\lambda) \geq C_{n,k} \Sigma_{i=1}^n \sigma_{k-1;i}(\lambda),\\
&\text{(vi)} && \sigma_{k-1}(\lambda) \geq \frac{k}{n-k+1} \tbinom{n}{k}^{1/k} [\sigma_k(\lambda)]^{(k-1)/k}.
\end{aligned}
$$

Moreover, the function $[\sigma_k]^{1/k}$ is concave on Γ_k.

We just listed a few basic formulae. There are many other useful ones, as given in for example [**CNS**, **LT**]. For our investigation of equation (2.1) and its parabolic counterpart, Lemma 2.3 will be sufficient. These formulae can be extended to $\sigma_k(\lambda(r))$, regarded as functions of $n \times n$ symmetric matrices r. In particular, $[\sigma_k(\lambda(r))]^{1/k}$ is concave in r [**CNS**].

We say a function $v \in C^2(\Omega)$ is k-admissible (relative to equation (2.15)) if the eigenvalues $\lambda(D^2 v) \in \Gamma_k$. Equation (2.15) is elliptic if v is k-admissible. The existence of k-admissible solutions to the Dirichlet problem for (2.15) was proved by Caffarelli-Nirenberg-Spruck [**CNS**], see also Ivochkina [**I**].

Relevant to the k-Yamabe problem is the variational property of the k-Hessian equation (2.15), investigated in [**CW2**, **TW2**, **W1**]. It is well known that the k-Hessian equation is the Euler equation of the functional

$$I_k(v) = \frac{1}{k+1} \int_\Omega (-v)\sigma_k(\lambda(D^2 v)). \tag{2.16}$$

The Sobolev-Poincaré type inequality, for k-admissible functions vanishing on the boundary,

$$I_l^{1/(l+1)}(v) \leq C I_k^{1/(k+1)}(v), \tag{2.17}$$

was established in [**W1**] for the case $l = 0$ and $k \geq 1$, and in [**TW2**] for the case $k > l \geq 1$, where $0 \leq l \leq k \leq n$,

$$I_0(v) = \left[\int_\Omega |v|^{k^*} dx\right]^{1/k^*}, \tag{2.18}$$

and

$$k^* \begin{cases} = n(k+1)/(n-2k) & \text{if } k < n/2, \\ < \infty & \text{if } k = n/2, \\ = \infty & \text{if } k > n/2. \end{cases} \tag{2.19}$$

The best constant in the inequality (2.17) is attained by

$$v(x) = (1+|x|^2)^{(2k-n)/2k} \tag{2.20}$$

when $l = 0$, $k < \frac{n}{2}$, and $\Omega = \mathbb{R}^n$; and by the unique solution of

$$\frac{\sigma_k}{\sigma_l}(\lambda(D^2 v)) = 1 \text{ in } \Omega \tag{2.21}$$

for $1 \le l < k \le n$.

From the inequalities (2.17) ($l = 0$), it was proved in [**CW2**] that the Dirichlet problem

$$\begin{aligned} \sigma_k(\lambda(D^2 v)) &= |v|^p + f(v) \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.22}$$

admits a nonzero k-admissible solution, where $1 \le k \le \frac{n}{2}$, $1 < p < k^* - 1$, f is a lower order term of $|v|^p$. The existence result was proved for the problem with a more general right hand side. When $p = k^* - 1$, the existence of solutions to (2.22) was also established in [**CW1**] by a blow-up argument. See also the recent survey article [**W4**].

2.4. A necessary and sufficient condition for an equation to be variational. The following proposition was communicated to the authors by Kaiseng Chou several years ago.

Proposition 2.1. *Let $\mathcal{M}$ be a compact manifold without boundary, $v \in C^4(\mathcal{M})$. An operator $F[v] = F[\nabla^2 v, \nabla v, v, x]$ is variational if and only if its linearized operator is self-adjoint. The functional is given by*

$$I[v] = \int G[v], \tag{2.23}$$

except when F is homogeneous of degree -1, where

$$G[v] = \int_0^1 vF[\lambda v]. \tag{2.24}$$

This proposition can be found in [**O**]. We give a proof of the "if" part, as we need some related formulae.

Proof. The linearized operator of $F[v]$ is given by

$$L(\varphi) = F^{ij}\varphi_{ij} + F_{p_j}\varphi_j + F_v\varphi. \tag{2.25}$$

We have

$$\begin{aligned}
\int_{\mathcal{M}} v\,L(\varphi) &= \int_{\mathcal{M}} [v\nabla_i(F^{ij}\nabla_j\varphi) + v\varphi F_v] - A \\
&= \int_M [-v_i\varphi_j F^{ij} + v\varphi F_v] - A \\
&= \int_{\mathcal{M}} \varphi[F^{ij}v_{ij} + F_{p_i}v_i + F_v v] - A + B \\
&= \int_{\mathcal{M}} \varphi\,L(v) - A + B,
\end{aligned}$$

where

$$\begin{aligned}
A &= \int_{\mathcal{M}} v\varphi_j(\nabla_i F^{ij} - F_{p_j}), \\
B &= \int_{\mathcal{M}} v_i\varphi(\nabla_j F^{ij} - F_{p_i}), \\
-A + B &= -\int_{\mathcal{M}} \left(\frac{\varphi}{v}\right)_i v^2(\nabla_j F^{ij} - F_{p_i}) \\
&= \int_{\mathcal{M}} \frac{\varphi}{v}\nabla_i[v^2(\nabla_j F^{ij} - F_{p_i})].
\end{aligned}$$

Hence, L is self-adjoint if and only if

$$\sum_{i,j=1}^{n} \nabla_i[v^2(\nabla_j F^{ij} - F_{p_i})] = 0. \tag{2.26}$$

If L is self-adjoint,

$$\begin{aligned}
\langle I'[v], \varphi\rangle &= \int_{\mathcal{M}} \varphi \int_0^1 F[\lambda v] + \int_0^1 \int_{\mathcal{M}} \lambda v[F^{ij}[\lambda v]\varphi_{ij} + F_{p_i}\varphi_i + F_v\varphi] \\
&= \int_{\mathcal{M}} \varphi \int_0^1 F[\lambda v] + \int_0^1 \int_{\mathcal{M}} \lambda\varphi[F^{ij}[\lambda v]v_{ij} + F_{p_i}v_i + F_v v] \\
&= \int_{\mathcal{M}} \varphi \int_0^1 F[\lambda v] + \int_{\mathcal{M}} \varphi \int_0^1 \lambda\frac{d}{d\lambda}F[\lambda v]d\lambda \\
&= \int_{\mathcal{M}} \varphi F[v].
\end{aligned}$$

Hence F is the Euler equation of the functional I. q.e.d.

Conversely, if the operator F is the Euler operator of the functional I, from the above argument we must have $-A + B = 0$, namely (2.26) holds. In other words, F is the Euler operator of I if and only if (2.26) holds. Observe that if

$$\sum_i \nabla_i F^{ij} = F_{p_j} \quad \forall\ j, \tag{2.27}$$

then (2.26) holds.

From Proposition (2.5) we can recover the results on the variational structure of (2.1) in [**V1**]. First, if locally $\mathcal{M}$ is Euclidean, one verifies directly that (2.26) holds, as it is a pointwise condition. The locally conformally flat case is equivalent to the Euclidean case by a conformal deformation to the Euclidean metric. Finally, if $k = 2$, we note that to verify (2.26) for arbitrary v with a fixed background metric g_0 is equivalent to verifying it for $v \equiv 1$ with respect to an arbitrary conformal metric $g = \hat{v}^{\frac{4}{n-2}} g_0$. However, when $v \equiv 1$, condition (2.26) becomes $\sum_{i,j=1}^{n} \nabla_i \nabla_j F^{ij} = 0$, where $F^{ij} = \frac{\partial}{\partial r_{ij}} \sigma_k(\lambda(r))$ at $r = A_g$. But we have

$$\nabla_i F^{ij} = \frac{1}{2(n-2)} (R_{,j} - 2R_{ij,i}) = 0 \tag{2.28}$$

by the second Bianchi identity.

By (2.23) we also see that (2.6) is the functional of (2.5). When $k = \frac{n}{2}$, the integral (2.24) may not exist. We may consider v as a composite function $v = \varphi(w)$ and write equation (2.5) in the form

$$F[w] =: \varphi'(w) L(\varphi(w)) = \varphi^p(w) \varphi'(w). \tag{2.29}$$

If the operator L in (2.1) satisfies (2.26) with respect to v, the operator F in (2.29) satisfies (2.26) with respect to w. Hence the corresponding functional is given by

$$\mathcal{E}_{n/2}(w) = \int_{(\mathcal{M}, g_0)} \int_0^1 w F[tw]. \tag{2.30}$$

In particular, if $v = e^{-\frac{n-2}{2} w}$, then we obtain the functional in [**BV**], see also [**CY**],

$$\mathcal{E}_{n/2}(w) = -\int_{(\mathcal{M}, g_0)} \int_0^1 w \sigma_{n/2}(\lambda(A_{g_t})), \tag{2.31}$$

where $g_t = e^{-2tw} g_0$.

3. The a priori estimates

In this section we study the regularity of k-admissible solutions ($2 \le k \le n$) to equation (2.1) and its parabolic counterpart (2.11). Global a priori estimates for the elliptic equation (2.1) (for solutions with eigenvalues in Γ_k) were established by Viaclovsky [**V2**], with interior estimates given by Guan and Wang [**GW1**]. We will provide a simpler proof for the elliptic equation (2.1), with essentially the same idea as in [**GW1**], and extend the estimates to the parabolic equation (2.11) on general manifolds, which is necessary for our proof of Theorem 2.1. Regularity has also been studied in many other papers [**CGY1**, **LL1**].

We will also present an example showing that the interior a priori estimates do not hold for solutions with eigenvalues in the negative cone $-\Gamma_k$.

3.1. A priori estimates for equation (2.1). For the regularity of (2.1), we will use the conformal changes $g = u^{-2}g_0$. For the function u, equation (2.1) becomes

$$\sigma_k(\lambda(U)) = u^{-k}, \tag{3.1}$$

where

$$U = \nabla^2 u - \frac{|\nabla u|^2}{2u} g_0 + uA_{g_0}.$$

Lemma 3.1 ([**GW1**]). *Let $u \in C^3$ be a k-admissible positive solution of (3.1) in a geodesic ball $B_r(0) \subset \mathcal{M}$. Suppose $A_{g_0} = (a_{ij}) \in C^1(B_r(0))$. Then we have*

$$\frac{|\nabla u|}{u}(0) \leq C, \tag{3.2}$$

where C depends only on n, k, r, $\inf u$, and $\|A_{g_0}\|_{C^1}$, and ∇ denotes the covariant derivative with respect to the initial metric g_0.

Proof. Let μ be a smooth, monotone increasing function. Write equation (3.1) in the form

$$F[u] = \mu[f(x,u)], \tag{3.3}$$

where $F[u] = \mu[\sigma_k(\lambda(U))]$. We will prove (3.2) for the function $f = u^{-p}$ for some constant $p > 0$. The gradient estimate is indeed independent of μ. But we will need to choose proper μ for the second derivative estimate.

Let $z = |\nabla u|^2 \varphi^2(u)\rho^2$, where $\varphi(u) = \frac{1}{u}$, and $\rho(x) = (1 - \frac{|x|^2}{r^2})^+$ is a cut-off function; $|x|$ denotes the geodesic distance from 0. For any number a, we denote $a^+ = \max(0,a)$. Suppose z attains its maximum at $x_0 \in B_1(0)$, and $|\nabla u(x_0)| = u_1(x_0)$. Then at x_0, in an orthonormal frame,

$$\frac{1}{2}(\log z)_i = \frac{u_{1i}}{u_1} + \frac{\varphi'}{\varphi}u_i + \frac{\rho_i}{\rho} = 0, \tag{3.4}$$

$$\begin{aligned} \frac{1}{2}(\log z)_{ij} = {} & \frac{u_{1ij}}{u_1} + \sum_{\alpha>1} \frac{u_{\alpha i}u_{\alpha j}}{u_1^2} - \frac{u_{1i}u_{1j}}{u_1^2} \\ & + \frac{\varphi'}{\varphi}u_{ij} + \left(\frac{\varphi''}{\varphi} - \frac{\varphi'^2}{\varphi^2}\right)u_iu_j + \left(\frac{\rho_{ij}}{\rho} - \frac{\rho_i\rho_j}{\rho^2}\right). \end{aligned} \tag{3.5}$$

Differentiating equation (3.3), we get

$$F^{ij}[u_{ij1} + \left(\frac{u_1^3}{2u^2} - \frac{u_1u_{11}}{u}\right)\delta_{ij}] = \Delta, \tag{3.6}$$

where for a matrix $r = (r_{ij})$,

$$\begin{aligned} F^{ij}(r) &= \frac{\partial}{\partial r_{ij}}\mu[\sigma_k(\lambda(r))] = \mu' \frac{\partial}{\partial r_{ij}}\sigma_k(\lambda(r)), \\ \Delta &= \nabla_1\mu(f) - F^{ij}\nabla_1(a_{ij}u). \end{aligned}$$

By (3.4)–(3.6) we have, at x_0,

$$0 \geq \frac{1}{2} F^{ij} (\log z)_{ij} = \frac{1}{u_1} \left[\frac{u_1 u_{11}}{u} - \frac{u_1^3}{2u^2} \right] \mathcal{F} + \sum_{\alpha>1} F^{ij} \frac{u_{\alpha i} u_{\alpha j}}{u_1^2}$$
$$- F^{ij} \left(\frac{\varphi'}{\varphi} u_i + \frac{\rho_i}{\rho} \right) \left(\frac{\varphi'}{\varphi} u_j + \frac{\rho_j}{\rho} \right) + \frac{\varphi'}{\varphi} F^{ij} \left(u_{ij} - \frac{|\nabla u|^2}{2u} \delta_{ij} \right) + \frac{u_1^2}{2u} \frac{\varphi'}{\varphi} \mathcal{F}$$
$$+ \left(\frac{\varphi''}{\varphi} - \frac{\varphi'^2}{\varphi^2} \right) F^{11} u_1^2 + F^{ij} \left(\frac{\rho_{ij}}{\rho} - \frac{\rho_i \rho_j}{\rho^2} \right) + \frac{\Delta}{u_1} + \Delta',$$

where $\mathcal{F} = \sum F^{ii}$, and Δ' arises in the exchange of derivatives, with $|\Delta'| \leq C\mathcal{F}$. Note that

$$F^{ij}(u_{ij} - \frac{|\nabla u|^2}{2u} \delta_{ij}) = k\mu' \sigma_k(\lambda) - uF^{ij} a_{ij} \geq -C_a u \mathcal{F},$$

where $C_a = 0$ if $A_{g_0} = (a_{ij}) = 0$. By (3.4) and since $\varphi(u) = \frac{1}{u}$,

$$\left[\frac{u_{11}}{u} - \frac{u_1^2}{2u^2} \right] + \frac{u_1^2}{2u} \frac{\varphi'}{\varphi} = -\frac{u_1 \rho_1}{u\rho},$$

$$-F^{ij} \left(\frac{\varphi'}{\varphi} u_i + \frac{\rho_i}{\rho} \right) \left(\frac{\varphi'}{\varphi} u_j + \frac{\rho_j}{\rho} \right) + \left(\frac{\varphi''}{\varphi} - \frac{\varphi'^2}{\varphi^2} \right) F^{11} u_1^2$$
$$= -F^{ij} \left(\frac{2\varphi' u_i \rho_j}{\varphi\rho} + \frac{\rho_i \rho_j}{\rho^2} \right).$$

Hence, we obtain

$$0 \geq \sum_{\alpha>1} F^{ij} \frac{u_{\alpha i} u_{\alpha j}}{u_1^2} - C \left(\frac{1}{r^2 \rho^2} + \frac{u_1}{u} \frac{1}{r\rho} + C_a \right) \mathcal{F} + \frac{\Delta}{u_1} + \Delta'. \tag{3.7}$$

Denote $b = \frac{|\nabla u|^2}{2u}(x_0)$. We claim

$$\sum_{\alpha>1} F^{ij} u_{\alpha i} u_{\alpha j} \geq C b^2 \mathcal{F} - C' u^2 \mathcal{F} \tag{3.8}$$

for some positive constant C, C' ($C' = 0$ if $a_{ij} = 0$). Note that by Lemma 2.3 (iii) (vi), $\mathcal{F} \geq C_{n,k} \mu' \sigma_k^{(k-1)/k}$. From (3.8) we have

$$\frac{|\nabla u|}{u} \rho \leq \frac{C_1}{r} + C_2 \tag{3.9}$$

at x_0, where C_1 is independent of f and C_2 independent of r. Hence $z(0) \leq z(x_0) \leq C$, namely (3.2) holds.

Denote $\widetilde{u}_{ij} = u_{ij} + u a_{ij}$. For any two unit vectors ξ, η, we denote formally $\widetilde{u}_{\xi\eta} = \sum \xi_i \eta_j \widetilde{u}_{ij}$. Then to prove (3.8) it suffices to prove

$$A =: \sum_{\alpha>1} F^{ij} \widetilde{u}_{\alpha i} \widetilde{u}_{\alpha j} \geq C b^2 \mathcal{F}. \tag{3.10}$$

By a rotation of the coordinates we suppose $\{\widetilde{u}_{ij}\}$ is diagonal at x_0. Then

$$\lambda_1 = \widetilde{u}_{11} - b, \dots, \lambda_n = \widetilde{u}_{nn} - b$$

are the eigenvalues of the matrix $\{\widetilde{u}_{ij} - \frac{|Du|^2}{2u}\delta_{ij}\}$. Suppose $\lambda_1 \geq \cdots \geq \lambda_n$. At x_0 we have $|Du(x_0)| = u_\xi(x_0)$ for some unit vector ξ. In the new coordinates we have

$$A = \sum_i (F^{ii}\widetilde{u}_{ii}^2 - F^{ii}\widetilde{u}_{\xi i}^2).$$

If there exists a small $\delta_0 > 0$ such that $\langle e_i, \xi\rangle < 1 - \delta_0$ for all unit axial vectors e_i, then $A \geq \delta_0 F^{ii}\widetilde{u}_{ii}^2$. Since $\lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma_k$, we have $\lambda_k > 0$ and so $\widetilde{u}_{kk} > b$. Hence, by Lemma 2.3(v), $A \geq \delta_0 b^2 F^{kk} \geq \delta_1 b^2 \mathcal{F}$. We obtain (3.8).

So there is i^* such that $\langle e_{i^*}, \xi\rangle \geq 1 - \delta_0$ and $A \geq \frac{1}{2}\sum_{i\neq i^*} F^{ii}\widetilde{u}_{ii}^2$. If there exists $j \geq k$, $j \neq i^*$ such that $\widetilde{u}_{jj} \geq \alpha b$ for some $\alpha > 0$, then by Lemma 2.3(iv)(v), $A \geq \frac{1}{2}F^{jj}(\alpha b)^2 \geq \delta_2 b^2 \mathcal{F}$ and the claim holds. Otherwise we have $i^* = k$ since $\widetilde{u}_{kk} = \lambda_k + b \geq b$.

Case 1: $k \leq n-2$. Observing that $\frac{\partial}{\partial\lambda_1}\cdots\frac{\partial}{\partial\lambda_{k-1}}\sigma_k(\lambda) = \lambda_k + \cdots + \lambda_n \geq 0$, we have $\lambda_k \geq -(\lambda_{k+1} + \cdots + \lambda_n)$. Since $\widetilde{u}_{jj} \leq \alpha b$ for $j \geq k+1$, we have $\lambda_j \leq -(1-\alpha)b$. Hence $\lambda_k \geq (n-k)(1-\alpha)b \geq 2(1-\alpha)b$.

On the other hand, by (3.4), we may suppose that at x_0, $|\frac{\rho_\xi}{\rho}| \leq \alpha\frac{u_\xi}{u}$, for otherwise we have the required estimate (3.2). Hence $\widetilde{u}_{\xi\xi} \leq (2+\alpha)b$ for a different small $\alpha > 0$. By the relation $\widetilde{u}_{\xi\xi} = \sum_i \xi_i^2\widetilde{u}_{ii} \geq \sum_{i\leq k}\xi_i^2\widetilde{u}_{ii} - n\alpha b$ where $\xi = (\xi_1, \dots, \xi_n)$, we have $\widetilde{u}_{kk} \leq (1+\alpha)\widetilde{u}_{\xi\xi} \leq (2+2\alpha)b$. Hence $\lambda_k = \widetilde{u}_{kk} - b \leq (1+2\alpha)b$. We reach a contradiction when α is sufficiently small.

Case 2: $k = n-1$. We have

$$\frac{\partial\sigma_k}{\partial\lambda_{k-1}}\lambda_{k-1} = \sigma_k(\lambda) - \frac{\lambda_1\cdots\lambda_n}{\lambda_{k-1}} \geq -\frac{\lambda_1\cdots\lambda_n}{\lambda_{k-1}}.$$

Since $\lambda_n = \widetilde{u}_{nn} - b \leq -(1-\alpha)b$ and by $\frac{\partial}{\partial\lambda_1}\cdots\frac{\partial}{\partial\lambda_{n-2}}\sigma_k(\lambda) = \lambda_{n-1} + \lambda_n \geq 0$, we have $\lambda_{n-1} \geq (1-\alpha)b$ and so $\lambda_i \geq (1-\alpha)b$ for any $1 \leq i \leq n-1$. Hence $\frac{\partial\sigma_k}{\partial\lambda_{k-1}}\lambda_{k-1} \geq (1-\alpha)^2 b^2\lambda_1\cdots\lambda_{n-3}$. Note that $\mu'\frac{\partial\sigma_k}{\partial\lambda_i}(\lambda) = F^{ii}$. It follows that

$$\begin{aligned} A &\geq \tfrac{1}{2}\mu'\tfrac{\partial\sigma_k}{\partial\lambda_{k-1}}\widetilde{u}_{k-1\,k-1}^2 \geq \tfrac{1}{2}\mu'\tfrac{\partial\sigma_k}{\partial\lambda_{k-1}}\lambda_{k-1}^2 \\ &\geq \tfrac{1}{2}\mu'(1-\alpha)^2 b^2\lambda_1\cdots\lambda_{n-2} \geq Cb^2\mathcal{F}. \end{aligned} \tag{3.11}$$

Case 3: $k = n$. As in Case 1, we assume that $|\frac{\rho_\xi}{\rho}| \leq \alpha\frac{u_\xi}{u}$. Then by (3.4), $\widetilde{u}_{\xi\xi} \geq (2-\alpha)b$. Note that when $k = n$, $\lambda_i > 0$ for all i. Hence $\widetilde{u}_{ii} = \lambda_i + b > b$. Recall that when $k = n$, we have $i^* = n$. It follows that $\widetilde{u}_{nn} \geq (2-\alpha)b$ for a different small $\alpha > 0$. Hence $\lambda_n \geq (1-\alpha)b$

and

$$A \geq \frac{1}{2}\Sigma_{i\neq i^*}F^{ii}\tilde{u}_{ii}^2 \geq \frac{1}{2}F^{ii}\lambda_i^2 = \frac{1}{2}\lambda_i\lambda_n F^{nn} \geq Cb^2\mathcal{F}. \tag{3.12}$$

This completes the proof. q.e.d.

The above proof essentially belongs to Guan and Wang [**GW1**]; we included it here as it will also be needed for the parabolic equation (3.21) in §3.2 below. The main point is that the proof of (3.10) does not use the equation (3.3) and so it also applies to the corresponding parabolic equation. We also note that the gradient estimate is independent of the choice of μ. From Lemma 3.1, we obtain the following Liouville theorem.

Corollary 3.1. *Let $u \in C^3$ be an entire k-admissible positive solution of*

$$\sigma_k\left(\lambda\left(\nabla^2 u - \frac{|\nabla u|^2}{2u}I\right)\right) = 0 \quad in \quad \mathbb{R}^n. \tag{3.13}$$

Then $u \equiv$ constant.

Proof. For equation (3.13), the constant C_2 in (3.9) vanishes. Letting $r \to \infty$, by (3.9), we see that either $\frac{|\nabla u|}{u} \equiv 0$, or $\mathcal{F} = 0$. In the former case, u is a constant. In the latter case, u satisfies $\sigma_{k-1}(\lambda) = 0$ and so it is also a constant by induction. q.e.d.

By approximation, as in [**MTW**], one can show that Corollary 3.1 holds for continuous positive viscosity solutions. The proof of the interior gradient estimate (3.2) can be simplified if one allows the estimate to depend on both $\inf_{B(0,r)} u$ and $\sup_{B(0,r)} u$. Indeed, let $\varphi(u) = \frac{1}{u-\delta}$ in the auxiliary function z, where $\delta = \frac{1}{2}\inf_{B(0,r)} u$. Then one obtains the extra good term $\frac{\delta u_1^2}{(u-\delta)u^2}\mathcal{F}$ on the right hand side of (3.7). The proof after (3.8) is not needed.

For the k-Yamabe problem, $f(u) = u^{-k}$. The constant C in (3.2) is independent of $\sup u$. Therefore we have the Harnack type inequality [**GW1**].

Corollary 3.2. *Let $u \in C^3$ be a positive solution of* (3.1)*. If* $\inf u \geq C_0 > 0$*, then* $\sup u \leq C_1$.

Next we prove the second order derivative estimate.

Lemma 3.2. *Let $u \in C^4$ be a k-admissible positive solution of* (3.1) *in a geodesic ball $B_r(0) \subset \mathcal{M}$. Suppose $A \in C^2(B_r(0))$. Then we have*

$$|\nabla^2 u|(0) \leq C, \tag{3.14}$$

where C depends only on n, k, r, $\inf u$, $\sup u$, and $\|A_{g_0}\|_{C^2}$.

Proof. Again we will consider the more general equation (3.3). Choose $\mu(t) = t^{1/k}$ such that equation (3.3) is concave in U_{ij}. Differentiating (3.3), we get

$$F^{ij}U_{ij,kk} = -\frac{\partial^2\mu(\sigma_k(\lambda(U)))}{\partial U_{ij}\partial U_{rs}}U_{ij,k}U_{rs,k} + \nabla_k^2\mu(f) \geq \nabla_k^2\mu(f), \tag{3.15}$$

where $U_{ij,k} = \nabla_k U_{ij}$. As above denote $\widetilde{u}_{ij} = u_{ij}+ua_{ij}$. Let T denote the unit tangent bundle of $B_r(0)$ with respect to g_0. Assume the auxiliary function z on T, $z(x, e_p) = \rho^2\nabla^2\widetilde{u}(e_p, e_p)$, attains its maximum at x_0 and in direction $e_1 = (1, 0, \dots, 0)$, where $\rho(x) = (1 - \frac{|x|^2}{r^2})^+$. In an orthonormal frame at x_0, we may assume by a rotation of axes that $\{U_{ij}\}$ is diagonal at x_0. Then at x_0, F^{ij} is diagonal and

$$0 = (\log z)_i = \frac{2\rho_i}{\rho} + \frac{\widetilde{u}_{11,i}}{\widetilde{u}_{11}}, \tag{3.16}$$

$$0 \geq (\log z)_{ii} = \left(\frac{2\rho_{ii}}{\rho} - \frac{6\rho_i^2}{\rho^2}\right) + \frac{\widetilde{u}_{11,ii}}{\widetilde{u}_{11}}. \tag{3.17}$$

By (3.16), the gradient estimate, and the Ricci identities,

$$\begin{aligned} U_{ij,11} &= u_{ij11} - \frac{u_{k1}^2}{u}\delta_{ij} + O\left(\frac{1+u_{11}}{\rho}\right) \\ &= u_{11ij} - \frac{u_{k1}^2}{u}\delta_{ij} + O\left(\frac{1+u_{11}}{\rho}\right). \end{aligned} \tag{3.18}$$

Hence we obtain

$$\begin{aligned} 0 \geq \sum_i F^{ii}(\log z)_{ii} &\geq -\frac{C}{\rho^2}\mathcal{F} + F^{ii}\frac{\widetilde{u}_{11,ii}}{\widetilde{u}_{11}} \\ &\geq -\frac{C}{\rho^2}\mathcal{F} + \frac{u_{11}^2}{2u\widetilde{u}_{11}}\mathcal{F} + \frac{1}{\widetilde{u}_{11}}\nabla_k^2\mu(f). \end{aligned}$$

Since $\mu(t) = t^{1/k}$, we have $\mathcal{F} \geq C > 0$. Hence (3.14) holds. q.e.d.

The second order derivative estimate (3.14) was established in [**GW1**]. As the proof is straightforward, we included it here for completeness. The estimate is also similar to that in [**GW4**] for the equation

$$\det\left(\nabla^2 u - \frac{|\nabla u|^2}{2u}I + \frac{u}{2}I\right) = f(x, u, \nabla u) \quad \text{in} \ \ \Omega \subset S^n, \tag{3.19}$$

which arises in the design of a reflector antenna, where I is the unit matrix.

By Lemma 3.2, equation (3.1) becomes a uniformly elliptic equation. By the Evans-Krylov estimates and linear theory [**GT**], we have the following interior estimates.

Theorem 3.1. *Let $u \in C^{3,1}$ be a positive solution of* (3.1) *in a geodesic ball $B_r(0) \subset \mathcal{M}$. Suppose $f > 0, \in C^{1,1}$. Then for any $\alpha \in (0,1)$,*

$$\|u\|_{C^{3,\alpha}(B_{r/2}(0))} \le C, \tag{3.20}$$

where C depends only on n, k, r, $\inf_{\mathcal{M}} u$, and g_0.

Theorem 3.1 also holds for equation (3.3) with $f = \kappa u^{-p}$ for a constant $p > 0$ and a smooth, positive function κ.

3.2. The parabolic equation. It is more convenient to study the parabolic equation for the function $w = \log u$. In this section we will extend the a priori estimates in §3.1 to the parabolic equation

$$F[w] - w_t = \mu(f), \tag{3.21}$$

where $F[w] = \mu[\sigma_k(\lambda(W))]$, and

$$W = \nabla^2 w + \nabla w \otimes \nabla w - \frac{1}{2}|\nabla w|^2 g_0 + A_{g_0}.$$

When $f = e^{-2kw}$, a stationary solution of (3.21) satisfies the equation

$$\sigma_k(\lambda(W)) = e^{-2kw},$$

which is equivalent to (3.1).

We choose a monotone increasing function μ such that F is concave in $D^2 w$ and

$$\mu(t) = \begin{cases} t^{1/k} & t \ge 10, \\ \log t & t \in (0, \frac{1}{10}), \end{cases}$$

and furthermore

$$(t-s)(\mu(t) - \mu(s)) \ge c_0 (t-s)(t^{1/k} - s^{1/k}) \tag{3.22}$$

for some constant $c_0 > 0$ independent of t. Condition (3.22) will be used in the next section.

We say w is k-admissible if for any fixed t, w is k-admissible as a function of x. Denote $Q_r = B_r(0) \times (t_0, t_0 + r^2]$ for some $t_0 \ge 0$. In the following lemmas we establish interior (in both time and spatial variables) a priori estimates for w. For brevity, we take $t_0 = 0$.

Lemma 3.3. *Let w be a k-admissible solution of* (3.21) *on Q_r. Then we have the estimates*

$$|\nabla_x w(0, r^2)| \le C, \tag{3.23}$$

where C is independent of $\sup w$, if $f = e^{-pw}$ for some constant $p > 0$.

Proof. The proof is similar to that of Lemma 3.1. Let $u = e^w$ so that u satisfies the equation

$$\widetilde{F}[u] - \frac{u_t}{u} = \mu(f), \tag{3.24}$$

where

$$\widetilde{F}[u] = \mu\left[\frac{1}{u^k}\sigma_k\left(\lambda\left(\nabla^2 u - \frac{|\nabla u|^2}{2u}g_0 + uA_{g_0}\right)\right)\right].$$

Let $z = \left(\frac{|\nabla u|}{u}\right)^2\rho^2$ be the auxiliary function as in the proof of Lemma 3.1. Here we choose

$$\rho(x,t) = \frac{t}{r^2}\left(1 - \frac{|x|^2}{r^2}\right)^+.$$

Suppose z attains its maximum at (x_0, t_0). Then $t_0 > 0$. By a rotation of axes we assume $|\nabla u| = u_1$. Then at (x_0,t_0), $z_i = 0$, $\{z_{ij}\} \le 0$, and $z_t \ge 0$. Hence we have (3.4), (3.5) and

$$\frac{u_{1t}}{u} - \frac{u_1 u_t}{u^2} + \frac{u_1\rho_t}{u\rho} \ge 0. \tag{3.25}$$

Differentiating equation (3.24), we obtain (3.6) with F^{ij} and Δ replaced by

$$\begin{aligned}\widetilde{F}^{ij}(r) &= \frac{\partial}{\partial r_{ij}}\mu\left[\frac{1}{u^k}\sigma_k(\lambda(r))\right]\\ &= \frac{\mu'}{u^k}\frac{\partial}{\partial r_{ij}}\sigma_k(\lambda(r)),\\ \Delta &= \left[\frac{u_{1t}}{u} - \frac{u_1u_t}{u^2}\right] + \frac{ku_1\mu'}{u^{k+1}}\sigma_k(\lambda) + [\nabla_1\mu(f) - \widetilde{F}^{ij}\nabla_1(a_{ij}u)]\\ &\ge -\frac{u_1\rho_t}{u\rho} + \nabla_1\mu(f) - \widetilde{F}^{ij}\nabla_1(a_{ij}u).\end{aligned}$$

Hence we obtain (3.7) in the form

$$0 \ge \sum_{\alpha>1}\widetilde{F}^{ij}\frac{u_{\alpha i}u_{\alpha j}}{u_1^2} - C\left(\frac{1}{r^2\rho^2} + \frac{u_1}{u}\frac{1}{r\rho} + C_a\right)\widetilde{\mathcal{F}} + \frac{\Delta}{u_1} + \Delta',$$

where Δ' arises in the exchange of derivatives, and satisfies $|\Delta'| \le C\widetilde{\mathcal{F}}$. Recall that (3.8) holds in the present case as well (with F^{ij} and $\mathcal{F}$ replaced by $\widetilde{F}^{ij}$ and $\widetilde{\mathcal{F}}$), as the proof of (3.10) does not use equation (3.3). Therefore

$$0 \ge \widetilde{\mathcal{F}}\left(\frac{u_1}{u}\right)^2 - C\left(\frac{1}{r^2\rho^2} + \frac{u_1}{u}\frac{1}{r\rho} + C_a\right)\widetilde{\mathcal{F}} + \frac{\Delta}{u_1} - C\widetilde{\mathcal{F}}. \tag{3.26}$$

By Lemma 2.3 and our choice of μ, $\widetilde{\mathcal{F}} = \sum \widetilde{F}^{ii}$ has a positive lower bound,

$$\widetilde{\mathcal{F}} \ge \frac{\mu'(u^{-k}\sigma_k(\lambda))}{u^k}\sigma_k^{(k-1)/k}(\lambda(U)) \ge \frac{C}{u} \tag{3.27}$$

for some $C > 0$ depending only on n, k. Note that at the minimum point of z,

$$\frac{\Delta}{u_1} \ge -\frac{1}{t_0 u} + \frac{1}{u_1}\nabla_1\mu(f) - C\widetilde{\mathcal{F}}$$

and if $f(w) = e^{-pw} = u^{-p}$,

$$\frac{1}{u_1}\nabla_1\mu(f) = -pu^{-p-1}\mu' \geq -Cu^{-1}(1+u^{-p/k}).$$

Therefore we obtain

$$t_0\left(\frac{u_1}{u}\right)^2 \leq C_1 + C_2(1+u^{-p/k}). \tag{3.28}$$

This completes the proof. q.e.d.

Remark 3.1. For the fixed μ as above, the a priori estimate (3.23) holds for the equation

$$\frac{1}{a}\mu(a^k\sigma(\lambda(W)) - w_t = \frac{1}{a}\mu(a^k f), \tag{3.29}$$

where $a > 0$ is a constant, and the constant C in (3.23) is independent of $a \geq 1$. Note that when $a = 1$, (3.29) reduces to equation (3.21) or (3.24).

Indeed, for equation (3.29), we have instead of (3.26),

$$\widetilde{\mathcal{F}} \geq \frac{1}{u}\left(\frac{a^k}{u^k}\sigma_k(\lambda(U))\right)^{(k-1)/k}\mu'\left(\frac{a^k}{u^k}\sigma_k(\lambda(U))\right) \geq \frac{C}{u}.$$

Next we have

$$\nabla_1\left(\frac{1}{a}\mu(a^k f)\right) = a^{k-1}\mu'(a^k f)\partial_1 f = -pa^{k-1}u^{-p-1}\mu'(a^k f).$$

One can easily verify that

$$\nabla_1\left(\frac{1}{a}\mu(a^k f)\right) = \frac{1}{k}u^{-\frac{p}{k}-1} \ \text{ if } \ a^k u^{-p} > 10,$$
$$\nabla_1\left(\frac{1}{a}\mu(a^k f)\right) = \frac{1}{au} \ \text{ if } \ a^k u^{-p} < \frac{1}{10}.$$

We may choose μ concave such that $\mu'(s) > \mu'(10)$ for $s \in (\frac{1}{10}, 10)$. Again we obtain (3.28). Hence (3.23) holds uniformly for $a \geq 1$.

Lemma 3.4. *Let w be a k-admissible solution of* (3.21) *on Q_r. Then we have the estimate*

$$|\nabla_x^2 w(0, r^2)| \leq C, \tag{3.30}$$

where C depends only on n, k, r, μ, $\inf w$, $\sup w$, *and* $\|A_{g_0}\|_{C^2}$.

Proof. Differentiating equation (3.21) twice, we get

$$\begin{aligned} F^{ij}W_{ij,k} &= w_{tk} + \nabla_k\mu(f), \\ F^{ij}W_{ij,kk} &= -F^{ij,rs}W_{ij,k}W_{rs,k} + w_{tkk} + \nabla_k^2\mu(f) \\ &\geq w_{tkk} + \nabla_k^2\mu(f), \end{aligned}$$

where $F^{ij} = \frac{\partial F}{\partial W_{ij}}$ (note that F^{ij} here is different from $\widetilde{F}^{ij}$ in the proof of Lemma 3.3), $W_{ij,k} = \nabla_k W_{ij}$, and $F^{ij,rs} = \frac{\partial^2 \mu(\sigma_k(\lambda(W)))}{\partial W_{ij}\partial W_{rs}}$. Denote $\widetilde{w}_{ij} = w_{ij} + a_{ij}$, $a_{ij} = (A_{g_0})_{ij}$. Let T denote the unit tangent bundle of $\mathcal{M}$ with respect to g_0. Consider the auxiliary function z defined on $T \times [0, r^2]$, given by $z = \rho^2\big(\nabla^2\widetilde{w} + (\nabla w)^2\big)(e_p, e_p)$, where ρ is the cut-off function in the proof of Lemma 3.3. Assume that z attains its maximum at (x_0, t_0) and in direction $e_1 = (1, 0, \dots, 0)$. We choose an orthonormal frame at (x_0, t_0), such that after a rotation of axes, $\{W_{ij}\}$ is diagonal. Then F^{ij} is diagonal and at (x_0, t_0),

$$
\begin{aligned}
0 &= (\log z)_i = \frac{2\rho_i}{\rho} + \frac{\widetilde{w}_{11,i} + 2w_1 w_{1i}}{\widetilde{w}_{11} + w_1^2}, \\
0 &\le (\log z)_t = \frac{2\rho_t}{\rho} + \frac{w_{11t} + 2w_1 w_{1t}}{\widetilde{w}_{11} + w_1^2}, \\
0 &\ge (\log z)_{ii} = \left(\frac{2\rho_{ii}}{\rho} - \frac{6\rho_i^2}{\rho^2}\right) + \frac{\widetilde{w}_{11,ii} + 2w_1 w_{1ii} + 2w_{1i}^2}{\widetilde{w}_{11} + w_1^2}.
\end{aligned}
\tag{3.31}
$$

We have, by (3.31) and the Ricci identities,

$$
\begin{aligned}
W_{ij,11} &= w_{ij11} + w_{i11}w_j + w_{j11}w_i + 2w_{i1}w_{j1} \\
&\quad - w_{k1}^2\delta_{ij} + O\left(\frac{1}{\rho}(\widetilde{w}_{11} + w_1^2)\right) \\
&= w_{11ij} + 2w_{i1}w_{j1} - w_{k1}^2\delta_{ij} + O(\frac{1}{\rho}(\widetilde{w}_{11} + w_1^2)).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
0 &\ge \sum_i F^{ii}(\log z)_{ii} - (\log z)_t \\
&\ge -\frac{C}{\rho^2}\mathcal{F} + \frac{1}{\widetilde{w}_{11} + w_1^2}F^{ii}(\widetilde{w}_{11,ii} + 2w_1 w_{ii1} + 2w_{i1}^2) \\
&\quad - \frac{w_{11t} + 2w_1 w_{1t}}{\widetilde{w}_{11} + w_1^2} - \frac{2\rho_t}{\rho} \\
&\ge -\frac{C}{\rho^2}\mathcal{F} + \frac{1}{\widetilde{w}_{11} + w_1^2}F^{ii}[(W_{ii,11} + w_{k1}^2) + 2w_1 w_{1ii}] \\
&\quad - \frac{w_{11t} + 2w_1 w_{1t}}{\widetilde{w}_{11} + w_1^2} - \frac{2\rho_t}{\rho} \\
&\ge -\frac{C}{\rho^2}\mathcal{F} + \frac{1}{\widetilde{w}_{11} + w_1^2}(F^{ii}W_{ii,11} - w_{11t}) + w_{11}\mathcal{F} \\
&\quad + \frac{2w_1}{\widetilde{w}_{11} + w_1^2}(F^{ii}w_{ii1} - w_{t1}) - \frac{2\rho_t}{\rho} \\
&\ge -\frac{C}{\rho^2}\mathcal{F} + \frac{1}{\widetilde{w}_{11} + w_1^2}\nabla_1^2\mu(f) + w_{11}\mathcal{F}
\end{aligned}
\tag{3.32}
$$

$$+\frac{2w_1}{\widetilde{w}_{11}+w_1^2}\nabla_1\mu(f)-\frac{2\rho_t}{\rho}.$$

By our choice of μ, $\mathcal{F}\geq C$ for some C depending only on n,k. We obtain $tw_{11}\rho^2\leq C$ at (x_0,t_0). Whence $z(0,r^2)\leq z(x_0,t_0)\leq C$. q.e.d.

Remark 3.2. The a priori estimate (3.30) also holds for the equation (3.29) and the constant C in (3.30) is independent of $a\geq 1$.

Indeed, by Lemma 2.3, we have

$$\begin{aligned}\mathcal{F} &= \sum_i \frac{\partial}{\partial W_{ii}}\left[\frac{1}{a}\mu(a^k\sigma_k(\lambda(W))\right]\\ &= (n-k+1)a^{k-1}\sigma_{k-1}(\lambda(W))\mu'(a^k\sigma_k(\lambda))\\ &\geq Ca^{k-1}\sigma_k^{(k-1)/k}\mu'(a^k\sigma_k(\lambda))\\ &\geq C\inf_{t>0}t^{(k-1)/k}\mu'(t).\end{aligned}$$

By our choice of μ, $\inf_{t>0}t^{(k-1)/k}\mu'(t)\geq C>0$. Hence $\mathcal{F}>C>0$.

Therefore by (3.32) it suffices to show that

$$|\nabla_1 g|+|\nabla_1^2 g|\leq C \tag{3.33}$$

for some $C>0$ independent of $a\geq 1$, where $g=\frac{1}{a}\mu(a^kf)$. By our choice of μ,

$$g=\begin{cases}\mu(f) & \text{if } a^kf>10\\ \frac{1}{a}(k\log a+\log f) & \text{if } a^kf<\frac{1}{10}.\end{cases}$$

Hence $\sup(|\nabla_1 g|+|\nabla_1^2 g|)$ is independent of $a\geq 1$ if $a^kf>10$ or $a^kf<\frac{1}{10}$. When $a^kf\in(\frac{1}{10},10)$, we also have (3.33) as μ is monotone and concave.

Lemma 3.5. *Let w be a k-admissible solution of* (3.21) *on Q_r. Then we have the estimates*

$$|w_t(0,r^2)|\leq C, \tag{3.34}$$

where C depends only on $n,k,\ r,\mu,\ \inf w,\ \sup w$, and $\|A_{g_0}\|_{C^2}$.

Proof. From the equation (3.21) and by the estimate (3.30) we have an upper bound for w_t. It suffices to show that w_t is bounded from below. Let $z=\frac{w_t}{(M-w)^\alpha}\rho^\beta$, where $M=2\sup_{Q_r}|w|$, and ρ is the cut-off function as above. Suppose $\min_{Q_r}z$ attains its minimum at (x_0,t_0), $t_0>0$. Then at the point we have $z_t\leq 0$, $z_i=0$ and the matrix

$\{z_{ij}\} \geq 0$, namely

$$(3.35)\quad \frac{w_{tt}}{w_t} + \alpha\frac{w_t}{M-w} + \beta\frac{\rho_t}{\rho} \geq 0,$$

$$(3.36)\quad \frac{w_{ti}}{w_t} + \alpha\frac{w_i}{M-w} + \beta\frac{\rho_i}{\rho} = 0 \quad i = 1, \dots, n,$$

$$(3.37)\quad \left\{\frac{w_{ijt}}{w_t} - \frac{w_{it}w_{jt}}{w_t^2} + \alpha\frac{w_{ij}}{M-w} + \alpha\frac{w_iw_j}{(M-w)^2} + \beta\frac{\rho_{ij}}{\rho} - \beta\frac{\rho_i\rho_i}{\rho^2}\right\} \leq 0,$$

where we have changed the direction of the inequalities as we assume that $w_t < 0$. Differentiating equation (3.21) gives

$$(3.38)\qquad F^{ij}W_{ijt} - w_{tt} = \frac{\partial}{\partial t}\mu(f).$$

Hence, by (3.35),

$$(3.39)\qquad \begin{aligned} \alpha\frac{w_t}{M-w} &\geq -\frac{w_{tt}}{w_t} - \beta\frac{\rho_t}{\rho} \\ &= \frac{-1}{w_t}F^{ij}W_{ijt} + \frac{1}{w_t}\frac{\partial}{\partial t}\mu(f) - \beta\frac{\rho_t}{\rho}. \end{aligned}$$

By (3.36), the matrix in (3.37) is equal to

$$\left\{\frac{w_{ijt}}{w_t} + \frac{\alpha w_{ij}}{M-w} + \frac{\alpha(1-\alpha)w_iw_j}{(M-w)^2} - \frac{2\alpha\beta w_i\rho_j}{(M-w)\rho} + \beta\frac{\rho_{ij}}{\rho} - \beta(1+\beta)\frac{\rho_i\rho_i}{\rho^2}\right\} \leq 0.$$

We have

$$\begin{aligned} \frac{-1}{w_t}F^{ij}W_{ijt} &= \frac{-1}{w_t}F^{ij}(w_{ijt} + w_{it}w_j + w_{jt}w_i - w_kw_{kt}\delta_{ij}) \\ &\geq F^{ij}\left(\frac{\alpha w_{ij}}{M-w} + \frac{\alpha(1-\alpha)w_iw_j}{(M-w)^2} - \frac{2\alpha\beta w_i\rho_j}{(M-w)\rho}\right) \\ &\quad + F^{ij}\left(\beta\frac{\rho_{ij}}{\rho} - \beta(1+\beta)\frac{\rho_i\rho_i}{\rho^2}\right) \\ &\quad + F^{ij}\left(2\alpha\frac{w_iw_j}{M-w} + 2\beta\frac{\rho_iw_j}{\rho} - \alpha\frac{|\nabla w|^2}{M-w}\delta_{ij} - \beta\frac{w_k\rho_k}{\rho}\delta_{ij}\right) \\ &\geq \frac{\alpha}{M-w}F^{ij}(w_{ij} + 2w_iw_j - |\nabla w|^2\delta_{ij}) \\ &\quad + \frac{\alpha}{M-w}F^{ij}\left(\frac{(1-\alpha)w_iw_j}{M-w} - 2\beta\frac{w_i\rho_j}{\rho}\right) - \frac{C}{\rho^2}, \end{aligned}$$

where the constant C depends on the gradient estimate (3.23) and the second derivative estimate (3.30). Choose $\alpha = \frac{1}{2}$. By the Holder inequality,

$$F^{ij}\left(\frac{w_iw_j}{2(M-w)} - 2\beta\frac{w_i\rho_j}{\rho}\right) \geq -\frac{C}{\rho^2}.$$

By the k-admissibility, $F^{ij}W_{ij} \geq 0$. Hence we obtain

$$\begin{aligned}\frac{-1}{w_t}F^{ij}W_{ijt} &\geq \frac{\alpha}{M-w}F^{ij}(W_{ij}+w_iw_j-\frac{1}{2}|\nabla w|^2\delta_{ij}-a_{ij})-\frac{C}{\rho^2}\\ &\geq -\frac{C}{\rho^2}.\end{aligned}$$

It follows that

$$\alpha\frac{w_t}{M-w} \geq -\frac{C}{\rho^2}+\frac{1}{w_t}\frac{\partial}{\partial t}\mu(f)-\beta\frac{\rho_t}{\rho}. \tag{3.40}$$

Now we choose $\beta = 2$. Then we obtain

$$z(x_0,t_0)=\frac{w_t}{(M-w)^{1/2}}\rho^2(x_0,t_0) \geq -C.$$

It follows that $z(0,r^2) \geq z(x_0,t_0) \geq -C$. Hence w_t is bounded from below. q.e.d.

Theorem 3.2. *For any k-admissible function w_0, there is a k-admissible solution $w \in C^{3,2}(\mathcal{M}\times[0,T))$ of* (3.21) *with $w(\cdot,0)=w_0$ on a maximal time interval $[0,T)$. If $T<\infty$, we have $\inf_{\mathcal{M}} w(\cdot,t)\to-\infty$ as $t\nearrow T$.*

Proof. First we point out that a k-admissible solution of (3.21) is locally bounded. Indeed, at the minimum point of w, by equation (3.21) we have

$$w_t = F[w]-\mu(f) \geq \mu\big(\sigma_k(\lambda(A_{g_0}))\big)-\mu(f).$$

Hence, locally in time, the solution is bounded from below. By the interior gradient estimate (3.23), the solution is also bounded from above. Therefore by Lemmas 3.3-3.5, equation (3.21) is uniformly parabolic. By Krylov's regularity theory, we obtain the $C^{3,2}$ a priori estimate for (3.21), and so the local existence follows. Let $[0,T)$ be the maximal time interval for the solution. If $T<\infty$, we must have $\inf_{\mathcal{M}} w(\cdot,t)\to-\infty$ as $t\nearrow T$. q.e.d.

3.3. Counterexamples. Theorem 3.1 applies to solutions of (3.1) with eigenvalues in the positive cone Γ_k. The a priori estimate (3.14) relies critically on the negative sign of the term $\frac{|\nabla u|^2}{2u}$, which yields the dominating term u_{k1}^2 in (3.18). Equation (3.1) has another elliptic branch, namely when the eigenvalues λ lie in the negative cone $-\Gamma_k$. An open problem is whether the a priori estimate (3.14) holds for solutions with eigenvalues in the negative cone $-\Gamma_k$. This is also an open problem for equations from optimal transportation [**MTW**], in particular the reflector antenna design problem (3.19). Here we give a counter example to the regularity. Our example is a modification of the Heinz-Levy counterexample in [**Sc**].

We will consider the two dimensional case. By making the change $u \to -u$, we consider equation

$$\det(u_{ij} + |\nabla u|^2 I + a_{ij}) = f \tag{3.41}$$

with positive sign before the term $|\nabla u|^2$, where f is a $C^{1,1}$ positive function to be determined. We want to show that equation (3.41) has no interior a priori estimates for solutions with eigenvalues in the positive cone.

Set

$$u(x) = \frac{b}{2}x_2^2 + \varphi(x_1), \tag{3.42}$$

where b is constant, φ is an even function. Let

$$a_{11} = -b^2x_2^2, \quad a_{12} = 0, \quad a_{22} = -b - b^2x_2^2. \tag{3.43}$$

Then equation (3.41) becomes

$$(\varphi'' + \varphi'^2)\varphi'^2 = f. \tag{3.44}$$

Let $\psi = (\varphi')^3$. Then ψ satisfies the equation

$$\frac{1}{3}\psi' + \psi^{4/3} = f. \tag{3.45}$$

Let

$$\psi(x_1) = x_1 - \frac{9}{7}x_1^{7/3}. \tag{3.46}$$

Then

$$f(x) = \frac{1}{3} - x_1^{4/3} + \left(x_1 - \frac{9}{7}x_1^{7/3}\right)^{4/3} \tag{3.47}$$

is a positive C^2 function, but the solution $u \notin C^2$.

If, instead of (3.43), we choose

$$a_{11} = c_0 - b^2x_2^2, \quad a_{12} = 0, \quad a_{22} = \varepsilon - b - b^2x_2^2, \tag{3.48}$$

where c_0, ε are constants, $\varepsilon > 0$ small, then we have the equation

$$(\varphi'' + \varphi'^2 + c_0)(\varepsilon + \varphi'^2) = f. \tag{3.49}$$

Let $f \equiv 1$ and denote $g = \varphi'$. Then $g(0) = 0$ and g satisfies

$$g' = \frac{1}{\varepsilon + g^2} - g^2 - c_0. \tag{3.50}$$

This equation has a unique solution g_ε. Obviously the gradient of g_ε is not uniformly bounded. Hence there is no interior $C^{1,1}$ a priori estimate for equation (3.42). Note that the matrix $A = (a_{ij})$ can either be in the positive cone or in the negative cone by choosing proper constants b, c_0.

Write equation (3.1) in the form

$$\sigma_k\left(\lambda\left(\nabla^2 w - \nabla w \otimes \nabla w + \frac{1}{2}|\nabla w|^2 I + A\right)\right) = f. \tag{3.51}$$

Then in a way similar to above, we can construct a sequence of functions satisfying equation (3.51) with $f = 1$ whose second derivatives are not uniformly bounded.

Remark 3.3. In many situations [**MTW**] there arise equations of the form

$$\sigma_k(\lambda(D^2u + A(x, u, Du)) = f, \tag{3.52}$$

where A is a matrix. From the discussions in this section, we see that the interior a priori estimates hold in general when A is negative definite with respect to Du, and do not hold if A is positive definite. When $A = 0$, there is no interior regularity in general, but if the solution vanishes on the boundary, interior a priori estimates have been established in [**CW2**].

4. Proof of Lemma 2.1

4.1. Existence of solutions in the sub-critical growth case. In this subsection we first study the existence of k-admissible solutions, for $2 \le k < \frac{n}{2}$, to equation (2.5) in the subcritical growth case $1 < p < \frac{n+2}{n-2}$. We then extend the existence result to the critical case $p = \frac{n+2}{n-2}$ in §4.2 by the blow-up argument. In §4.3 we consider the case $k = \frac{n}{2}$.

Theorem 4.1. *Suppose $2 \le k < \frac{n}{2}$. Then for any given $1 < p < \frac{n+2}{n-2}$, there is a solution v_p of* (2.5) *with $J_p(v_p) = c_p > 0$, where J_p, c_p are defined respectively in* (2.6) *and* (2.8). *Moreover, the set of solutions of* (2.5) *is compact.*

A solution of (2.5) is a critical point of the functional $J = J_p$. To study the critical points of the functional J, we will employ the parabolic equation (3.21), which can also be written in terms of v as (ignoring a coefficient $\frac{2}{n-2}$ before v_t)

$$F[v] + \frac{v_t}{v} = \mu(f(v)), \tag{4.1}$$

where $f(v) = v^{\frac{4k}{n-2} - \varepsilon}$, $F[v] = \mu(\sigma_k(\lambda(\frac{V}{v})))$, μ is the function in (3.21), and

$$\varepsilon = \frac{n+2}{n-2} - p. \tag{4.2}$$

Write functional (2.6) in the form

$$J(v) = \frac{n-2}{2n-4k} \int_{(\mathcal{M}, g_0)} v^{\frac{2n-4k}{n-2}} \sigma_k \left(\lambda \left(\frac{V}{v} \right) \right) - \frac{1}{p+1} \int_{(\mathcal{M}, g_0)} v^{\frac{2n-4k}{n-2}} v^{\frac{4k}{n-2} - \varepsilon}. \tag{4.3}$$

Equation (4.1) is a descent gradient flow of the functional J,

$$\begin{aligned}(4.4)\qquad \frac{d}{dt}J(v) &= \int_{(\mathcal{M},g_0)} v^{\frac{2n-4k}{n-2}-1}\left[\sigma_k\left(\lambda\left(\frac{V}{v}\right)\right) - v^{\frac{4k}{n-2}-\varepsilon}\right] v_t \\ &= -\int_{(\mathcal{M},g_0)} v^{\frac{2n-4k}{n-2}}\left[\sigma_k\left(\lambda\left(\frac{V}{v}\right)\right) - v^{\frac{4k}{n-2}-\varepsilon}\right] \\ &\qquad \cdot\left[\mu\left(\sigma_k\left(\lambda\left(\frac{V}{v}\right)\right)\right) - \mu\left(v^{\frac{4k}{n-2}-\varepsilon}\right)\right] \\ &\le 0.\end{aligned}$$

Given an initial k-admissible function v_0, by Theorem 3.2, the flow (4.1) has a unique smooth positive solution v on a maximal time interval $[0,T)$, where $T \le \infty$.

Lemma 4.1. *Suppose $J(v(\cdot,t))$ is bounded from below for all $t \in (0,T)$. If $v(\cdot,t)$ is uniformly bounded, then either $v(\cdot,t) \to 0$ or there is a sequence $t_j \to \infty$ such that $v(\cdot,t_j)$ converges to a solution of* (2.5).

Proof. By the assumption that $v(\cdot,t)$ is uniformly bounded, we have $T=\infty$. At the maximum point of $v(\cdot,t)$, by equation (4.1) we have

$$(4.5)\qquad v_t \le v[\mu(f(v)) - \mu(\sigma_k(\lambda(A_{g_0})))].$$

Hence if $\sup v(\cdot,t_0)$ is sufficiently small at some t_0, by the assumptions $g_0 \in [g_0]_k$ and $v>0$, we have $v(\cdot,t)\to 0$ uniformly. Therefore if v does not converges to zero uniformly, by the gradient estimate (3.23), we have $v \ge c$ for some constant $c>0$. In the latter case, by Theorem 3.2 and the assumption that v is uniformly bounded, we have $\|v(\cdot,t)\|_{C^3(\mathcal{M})} \le C$ for any $t \ge 0$.

Choose a sequence $t_j \to \infty$ such that

$$(4.6)\qquad \frac{d}{dt}J(v(\cdot,t_j)) \to 0.$$

By the above C^3 a priori estimate, we may abstract a subsequence, still denoted as t_j, such that $v(\cdot,t_j)$ converges in $C^{2,\alpha}$. By (4.4) we conclude that $v(x,t_j)$ converges as $j\to\infty$ to a solution of (2.5). q.e.d.

Lemma 4.2. *Suppose $J(v(\cdot,t))$ is bounded from below for all $t \in (0,T)$. Then $T=\infty$ and $v(\cdot,t)$ is uniformly bounded.*

Proof. Suppose to the contrary that there exists a sequence $t_j \nearrow T$ such that $m_j = \sup v(\cdot,t_j) \to \infty$. Assume the maximum is attained at $z_j \in \mathcal{M}$. By choosing a normal coordinate centered at z_j, we may identify a neighbourhood of z_j in $\mathcal{M}$ with the unit ball in $\mathbb{R}^n$ such that

z_j becomes the origin. We make the local transformation

$$\begin{aligned}(4.7)\qquad v_j(y,s) &= m_j^{-1} v(x,t),\\ y &= m_j^{\frac{2}{n-2}-\frac{\varepsilon}{2k}} x,\\ s &= m_j^{\frac{4}{n-2}-\frac{\varepsilon}{k}}(t-t_j).\end{aligned}$$

For the transformation $x \to y$, more precisely it should be understood as a dilation of $\mathcal{M}$, regarded as a submanifold in $\mathbb{R}^N$ for some $N > n$ with induced metric. Denote $\mathcal{M}_j = \{Y = m_j^{\frac{2}{n-2}-\frac{\varepsilon}{2k}} X \mid X \in \mathcal{M} \subset \mathbb{R}^N\}$, with induced metric from $\mathbb{R}^N$. Then we have $0 < v_j(y,0) \le m_j^{-1} v(0,t_j) = 1$, and v_j is defined for $y \in \mathcal{M}_j$ and $s \le s_0$, where by (4.5), $s_0 > 0$ is a positive constant independent of j. Moreover, v_j satisfies the equation

$$\begin{aligned}(4.8)\qquad m_j^{-\frac{4}{n-2}+\frac{\varepsilon}{k}} \mu\left[m_j^{\frac{4k}{n-2}-\varepsilon} \sigma_k\left(\lambda\left(\frac{V_j}{v_j}\right)\right)\right] - \frac{(v_j)_s}{v_j}\\ = m_j^{-\frac{4}{n-2}+\frac{\varepsilon}{k}} \mu(m_j^{\frac{4k}{n-2}-\varepsilon} f(v_j)).\end{aligned}$$

By direct computation,

(4.9)

$$\begin{aligned}\int_{\mathcal{M}_j} v_j^{\frac{2n-4k}{n-2}} \sigma_k\left(\lambda\left(\frac{V_j}{v_j}\right)\right) dy &= m_j^{\varepsilon(1-\frac{n}{2k})} \int_{\mathcal{M}} v^{\frac{2n-4k}{n-2}} \sigma_k\left(\lambda\left(\frac{V}{v}\right)\right) dx\\ \int_{\mathcal{M}_j} v_j^{\frac{2n}{n-2}-\varepsilon} dy &= m_j^{\varepsilon(1-\frac{n}{2k})} \int_{\mathcal{M}} v^{\frac{2n}{n-2}-\varepsilon} dx.\end{aligned}$$

Hence,

$$\begin{aligned}(4.10)\qquad J(v_j,\mathcal{M}_j) =: \frac{n-2}{2n-4k} &\int_{\mathcal{M}_j} v_j^{\frac{2n-4k}{n-2}} \sigma_k\left(\lambda\left(\frac{V_j}{v_j}\right)\right) dy\\ &- \frac{1}{p+1}\int_{\mathcal{M}_j} v_j^{\frac{2n}{n-2}-\varepsilon} dy\\ &= m_j^{\varepsilon(1-\frac{n}{2k})} J(v,\mathcal{M}) \le C.\end{aligned}$$

We may choose $s_j \in (0, \frac{1}{2}s_0)$ such that

$$(4.11)\qquad \frac{d}{ds} J(v_j(\cdot, s_j)) \to 0.$$

By (4.4), (4.11) is equivalent to

$$\begin{aligned}&\int_{\mathcal{M}_j} v_j^{\frac{2n-4k}{n-2}} \left\{\sigma_k\Big(\lambda\Big(\frac{V_j}{v_j}\Big)\Big) - v_j^{\frac{4k}{n-2}-\varepsilon}\right\}\cdot\\ &\left\{m_j^{-\frac{4}{n-2}+\frac{\varepsilon}{k}}\left[\mu\Big(m_j^{\frac{4k}{n-2}-\varepsilon}\sigma_k\Big(\lambda\Big(\frac{V_j}{v_j}\Big)\Big)\Big) - \mu\Big(m_j^{\frac{4k}{n-2}-\varepsilon} v_j^{\frac{4k}{n-2}-\varepsilon}\Big)\right]\right\} \to 0.\end{aligned}$$

By (3.22), we obtain
(4.12)
$$\int_{\mathcal{M}_j} v_j^{\frac{2n-4k}{n-2}} \left\{\sigma_k(\lambda(\frac{V_j}{v_j})) - v_j^{\frac{4k}{n-2}-\varepsilon}\right\} \cdot \left\{\left(\sigma_k(\lambda(\frac{V_j}{v_j}))\right)^{1/k} - \left(v_j^{\frac{4k}{n-2}-\varepsilon}\right)^{1/k}\right\} \to 0.$$

By the gradient estimate (3.23), $v_j + \frac{1}{v_j}$ (at $s = s_j$) is locally uniformly bounded. Hence

$$\sigma_k\Big(\lambda\Big(\frac{V_j}{v_j}\Big)\Big) - v_j^{\frac{4k}{n-2}-\varepsilon} \to 0 \quad \text{in} \quad L^{(k+1)/k}. \tag{4.13}$$

From equation (4.8) and by Remarks 3.1 and 3.2, we see that v_j are locally uniformly bounded in $C^{1,1}$ and the convergence in (4.13) is locally uniform.

By extracting a subsequence we can assume that $v_j(\cdot, s_j)$ converges to a function $v_0 \in C^{1,1}(\mathbb{R}^n)$ with $v_0(0) = 1$. We claim that v_0 is a smooth solution of the equation

$$F_0[v] := \sigma_k^{1/k}(\lambda(V)) = v^p \tag{4.14}$$

in $\mathbb{R}^n$. On the other hand, by the Liouville Theorem in [**LL2**], there is no entire positive solution to (4.14) when $\varepsilon > 0$. This is a contradiction. Hence Lemma 4.2 holds.

To prove that v_0 is a smooth solution of (4.14), we draw on an old trick of Evans [**E**]. Since $v_0 \in C^{1,1}(\mathbb{R}^n)$, v_0 is twice differentiable almost everywhere. Suppose now that $F_0[v_0] > v_0^p$ at some point x_0 where v_0 is twice differentiable. Without loss of generality we assume that $x = 0$. Let

$$\varphi(x) = v_0(0) + Dv_0(0)x + \frac{1}{2}D_{ij}v_0(0)x_ix_j + \frac{\varepsilon}{2}|x|^2 - \delta,$$

where ε, δ are positive constants. By choosing δ sufficiently small, we have

$$\varphi > v_0 \quad \text{on} \quad \partial B_r(0) \quad \text{and} \quad \varphi(0) < v_0(0).$$

Since $v_j \to v_0$ uniformly, we have $\varphi > v_j$ on $\partial B_r(0)$ and $\varphi(0) < v_j(0)$ when j is sufficiently large. Since v_0 is locally uniformly bounded in $C^{1,1}$, by the inequality $F_0[v_0] > v_0^p$ we have $\lambda(V_{v_0} - \varepsilon I) \in \Gamma_k$ and

$$F_j[\varphi] := \sigma_k^{1/k}[\lambda(\hat{V})] \geq v_j^p$$

when $\varepsilon > 0$ is sufficiently small, where V_{v_0} is the matrix relative to v_0, given in (2.2) and

$$\hat{V} = -\nabla^2\varphi + \frac{n}{n-2}\frac{\nabla v_j \times \nabla v_j}{v_j} - \frac{1}{n-2}\frac{|\nabla v_j|^2}{v_j}g_0 + \frac{n-2}{2}v_jA_{g_0}.$$

Hence, by the concavity of $\sigma_k^{1/k}$,

$$F^{ab}[v_j]D_{ab}(\varphi - v_j) \leq F(v_j) - v_j^p \to 0 \tag{4.15}$$

in $L^{\widetilde{p}}(\Omega)$ for any $\widetilde{p} < \infty$, where $F^{ab}[v_j] = \frac{\partial}{\partial r_{ab}}\sigma_k^{1/k}(\lambda(r))$ at $r = V_{v_j}$ $(a, b = 1, \dots, n)$, which satisfy $\det F^{ab} \geq C > 0$ for some $C > 0$ depending only on n, k. Applying the Aleksandrov-Bakelman maximum principle [**GT**] to (4.15) in $\{\varphi < v_j\}$ and sending $j \to \infty$, we conclude that $\varphi \geq v_0$ near 0, which is a contradiction so that $F_0[v_0] \leq v_0^p$ at x_0. By a similar argument, we obtain the reverse inequality and hence we conclude (4.14) for v_0. Since the limit equation (4.14) is locally uniformly elliptic with respect to v_0, we then conclude further regularity by the Evans-Krylov estimates and linear theory [**GT**]. In particular we obtain $v_0 \in C^\infty$. q.e.d.

Lemma 4.3. *There exists a k-admissible function v_0 such that the solution v of* (4.1) *satisfies $J(v(\cdot, t)) \geq -C$ and $\sup v(\cdot, t) \geq c_0 > 0$ for all $t \geq 0$.*

Proof. Let P be the set of paths introduced in §2.2. For $\gamma \in P$, let v_s $(s \in [0, 1])$ be the solution of (4.1) with initial condition $v_s(\cdot, 0) = \gamma(s)$. Then by (4.5) and the comparison principle, there is an $s_0 > 0$ such that $v_s(\cdot, t) \to 0$ uniformly for $s \leq s_0$. Denote by I_γ the set of $s \in [0, 1]$ such that $J(v_s(\cdot, t)) \geq 0$ for all $t > 0$. Then $(0, s_0) \subset I_\gamma$. Let $s^* = \sup\{s \mid s \in I_\gamma\}$.

Obviously $s^* \in I_\gamma$. For if there exists t such that $J(v_{s^*}(\cdot, t)) < 0$, then $J(v_s(\cdot, t)) < 0$ for $s < s^*$ sufficiently close to s^*, which implies $s^* \neq \sup\{s \mid s \in I_\gamma\}$. It is also easy to see that $v_{s^*}(\cdot, t)$ does not converges to zero uniformly, for otherwise $v_s(\cdot, t) \to 0$ uniformly for $s > s^*$ and near s^*. Finally, by our definition of the set P, we have $1 \notin I_\gamma$, namely $s^* < 1$. Hence $v_0 = \gamma(s^*)$ satisfies Lemma 4.3. q.e.d.

From the above three lemmas, one sees that there is a sequence $t_j \to \infty$ such that $v_{s^*}(\cdot, t_j)$ converges to a solution of (2.5) for $1 < p < \frac{n+2}{n-2}$. Next we prove

Lemma 4.4. *For any given $1 < p < \frac{n+2}{n-2}$, the set of admissible solutions of* (2.5) *is compact.*

Proof. By the a priori estimates, it suffices to show that the set of solutions is uniformly bounded. If, on the contrary, that there is a sequence of k-admissible solutions v_j such that $\sup v_j \to \infty$, denote $m_j = \sup v_j$ and assume that the sup is attained at z_j. Similar to (4.7), we make a translation and a dilation of coordinates and a scaling for solution, namely

$$\begin{aligned} \widetilde{v}_j(y) &= m_j^{-1} v_j(x), \\ y &= R_j x \quad R_j = m_j^{\frac{2}{n-2} - \frac{\varepsilon}{2k}}. \end{aligned}$$

Then $0 < \widetilde{v}_j \leq 1$, and $\widetilde{v}_j$ satisfies

$$\sigma_k(\lambda(\widetilde{V})) = \widetilde{v}^{k\frac{n+2}{n-2} - \varepsilon}.$$

By the a priori estimates in §3.1, $\widetilde{v}$ is locally uniformly bounded in C^3. Hence $\widetilde{v}_j$ converges by a subsequence to a positive solution $\widetilde{v}$ of

$$\sigma_k(\lambda(V)) = v^{k\frac{n+2}{n-2}-\varepsilon} \text{ in } \mathbb{R}^n. \tag{4.16}$$

By the Liouville Theorem [**LL2**], there is no nonzero solution to the above equation. We reach a contradiction. q.e.d.

Let v be a k-admissible solution of (2.5). Then we have

$$\int_{(\mathcal{M},g_0)} v^{\frac{2n-4k}{n-2}} \sigma_k\Big(\lambda\Big(\frac{V}{v}\Big)\Big) - \int_{(\mathcal{M},g_0)} v^{\frac{2n-4k}{n-2}} v^{\frac{4k}{n-2}-\varepsilon} = 0.$$

Hence

$$J(v) = \sup_{t>0} J(tv).$$

By (4.3), we have

$$\begin{aligned} J(v) &= \Big(\frac{n-2}{2n-4k} - \frac{1}{p+1}\Big) \int_{(\mathcal{M},g_0)} v^{\frac{2n-4k}{n-2}} v^{\frac{4k}{n-2}-\varepsilon} \\ &\geq C > 0. \end{aligned} \tag{4.17}$$

By the compactness in Lemma 4.4, the constant C is bounded away from zero.

Lemma 4.5. *There exists a solution v_p of* (2.5) *such that $J(v_p) = c_p$.*

Proof. For any given constant $\delta > 0$, choose a path $\gamma \in P$ such that $\sup_{s\in(0,1)} J(\gamma(s)) \leq c_p + \delta$. By the proof of Lemma 4.3, there exists $s^* \in (0,1)$ such that the solution of (4.1) with initial condition $v(\cdot,t) = \gamma_{s^*}$ converges to a solution v^*_δ of (2.5). Since (4.1) is a descent gradient flow, we have $J(v^*_\delta) < c_p+\delta$. Letting $\delta \to 0$, by the compactness in Lemma 4.4, v^*_δ converges along a subsequence to a solution v of (2.5) with $J(v) \leq c_p$. Note that $J(v) = \sup_{s>0} J(sv) \geq c_p$. Hence $J(v) = c_p$. q.e.d.

From (4.17) we also have

$$c_p \geq C > 0. \tag{4.18}$$

We have thus proved Theorem 4.1.

4.2. Proof of Lemma 2.1. In this subsection we prove Lemma 2.1 for $2 \leq k < \frac{n}{2}$. Let v_p be a solution of (2.5) with $J_p(v_p) = c_p$. If there is a sequence $p_j \nearrow \frac{n+2}{n-2}$ such that $\sup v_{p_j}$ is uniformly bounded, by the a priori estimate in §3.1, v_{p_j} sub-converges to a solution of (2.1) and Lemma 2.1 is proved.

If $\sup v_p \to \infty$ as $p \nearrow \frac{n+2}{n-2}$, noting that $c_p \le \sup_{s>0} J(sv_0)$ for any given admissible function v_0, we see that c_p is uniformly bounded from above for $p \in [1, \frac{n+2}{n-2}]$. By (4.17),

$$\int_{(\mathcal{M},g_0)} v_p^{p+1} \le C, \tag{4.19}$$

where C is independent of $p \le \frac{n+2}{n-2}$. Denote $m_p = \sup v_p$ and assume that the sup is attained at $z_p = 0$. As before, we make a dilation of coordinates and a scaling for solution, namely

$$\widetilde{v}_p(y) = m_p^{-1} v_p(x),$$
$$y = R_p x, \quad R_p = m_p^{\frac{2}{n-2} - \frac{\varepsilon}{2k}}.$$

Then $0 < \widetilde{v}_p \le 1$, and $\widetilde{v}_p$ satisfies

$$\sigma_k(\lambda(\widetilde{V})) = \widetilde{v}^{k\frac{n+2}{n-2} - \varepsilon}$$

in B_{cR_p} for some constant $c > 0$ independent of p. Note that in the present case, $\varepsilon = \frac{n+2}{n-2} - p \to 0$. By the a priori estimates in §3.1, $\widetilde{v}$ is locally uniformly bounded in C^3. Hence $\widetilde{v}_p$ converges by a subsequence to a positive solution $\widetilde{v}$ of

$$\sigma_k(\lambda(V) = v^{k\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n.$$

By the Liouville Theorem [**LL1**],

$$\widetilde{v}(y) = \overline{c}(1 + |y|^2)^{\frac{2-n}{2}}, \tag{4.20}$$

where $\overline{c} = [n(n-2)]^{(n-2)/4}$. Moreover

$$\sup_{s>0} J_{p^*}(s\widetilde{v}; \mathbb{R}^n) = c_{p^*}[S^n], \tag{4.21}$$

with $p^* = \frac{n+2}{n-2}$, where c_p was defined in (2.9).

The above argument implies that the metric $g = v_p^{\frac{4}{n-2}} g_0$ is a bubble near the maximum point z_p. By (4.20), v_p has the asymptotical behavior

$$v_p(x) = \overline{c}\left(\frac{\delta}{\delta^2 + r^2}\right)^{\frac{n-2}{2}} (1 + o(1)) \qquad \delta = m_p^{-\frac{2}{n-2} + \frac{\varepsilon}{2k}}. \tag{4.22}$$

For a sufficiently small $\theta > 0$, let $\Omega_p = \{x \in \mathcal{M} \mid v_p(x) > \theta m_p\}$, and let

$$\hat{v}_p(x) = \begin{cases} v_p(x) & x \in \mathcal{M} - \Omega_p, \\ \theta m_p & x \in \Omega_p. \end{cases}$$

Note that by assumption (1.6),

$$\sup_{s>0} J(sv_p) = c_p < c_{p^*}[S^n] \tag{4.23}$$

when $p < \frac{n+2}{n-2}$ and is close to $\frac{n+2}{n-2}$.

Combining (4.21), (4.22), and (4.23), we see that

$$\int_{(\mathcal{M},g_0)} \hat{v}_p^{p+1} \geq C > 0$$

for some C independent of θ, provided θ is sufficiently small and m_p is sufficiently large, and

$$\sup_{s>0} J(s\hat{v}_p) < \sup_{s>0} J(sv_p).$$

Namely $\sup_{s>0} J(s\hat{v}_p) < c_p$, which is in contradiction of our definition of c_p. Note that $\hat{v}_p$ is not smooth, but can be approximated by smooth, k-admissible functions. This completes the proof of Lemma 2.1. q.e.d.

4.3. The case $k = \frac{n}{2}$. In this case, the corresponding functional is

$$J(v) = \mathcal{E}_{n/2}(v) - \frac{1}{p+1}\int_{(\mathcal{M},g_0)} v^{\frac{2n}{n-2}-\varepsilon},$$

where $\mathcal{E}_{n/2}$ is given in (2.31). The proof of Lemma 4.3 does not apply, due to that $J(v) \to -\infty$ as $v \to 0$, and also we don't know if $\mathcal{E}_{n/2}(v)$ is bounded from below for any admissible function v with $\mathrm{Vol}\mathcal{M}_{g_v} = 1$. However, when $k = \frac{n}{2}$, we have the following

Lemma 4.6. *Assume that equation* (2.1) *is variational. Then* $\mathcal{F}_{n/2}(v)$ *is a constant.*

Proof. When $k = \frac{n}{2}$, we write the equation (2.1) and the functional $\mathcal{F}_{n/2}$ in the form

$$\sigma_{n/2}(\lambda(W)) = e^{-nw}, \tag{4.24}$$

$$\mathcal{F}_{n/2}(w) = \int_{(\mathcal{M},g_0)} \sigma_{n/2}(\lambda(W)).$$

To prove that $\mathcal{F}_{n/2}$ is equal to a constant, we have

$$\begin{aligned}\mathcal{F}_{n/2}(w) - \mathcal{F}_{n/2}(w_0) &= \int_{(\mathcal{M},g_0)} \int_0^1 \frac{d}{dt}\sigma_{n/2}(\lambda(W_t)) \\ &= \int_0^1 \int_{(\mathcal{M},g_0)} L_{w_t}(w)\end{aligned}$$

where $w_t = tw$, $w_0 = 0$, and L_{w_t} is the linearized operator of $\sigma_{n/2}(\lambda(W))$ at w_t. By the assumption that equation (2.1) is variational, we have (see §2.4)

$$\int_{(\mathcal{M},g_0)} L_{w_t}(w) = \int_{(\mathcal{M},g_0)} wL_{w_t}(1) = 0.$$

This completes the proof of Lemma 4.6. q.e.d.

By assumption (1.6), we have

$$\mathcal{F}_{n/2}(v) = c_0 < Y_{n/2}(S^n) \tag{4.25}$$

for some constant c_0 depending on $(\mathcal{M}, g_0)$. Lemma 4.6 enables us to prove the following

Lemma 4.7. *For $1 < p \le \frac{n+2}{n-2}$, the set of solutions of* (2.5) *is compact.*

Proof. When $1 < p < \frac{n+2}{n-2}$, the proof is the same as that of Lemma 4.4.

When $p = \frac{n+2}{n-2}$, we use the same argument of Lemma 4.4. Instead of (4.16), we have the equation

$$\sigma_{n/2}(\lambda(V)) = v^{\frac{n}{2}\frac{n+2}{n-2}} \quad \text{in} \quad \mathbb{R}^n. \tag{4.26}$$

By the Liouville theorem [**LL1**], v must be the function given in (4.20). Hence we have

$$\int_{\mathbb{R}^n} \sigma_{n/2}(\lambda(V)) = Y_{n/2}(S^n).$$

By (4.9), we obtain that

$$\underline{\lim}_{j\to\infty} \mathcal{F}_{n/2}(v_j) \ge Y_{n/2}(S^n).$$

This is in contradiction with (4.25). q.e.d.

By Lemma 4.7, we can prove the existence of solutions of (2.1) by a degree argument, see [**CGY2**, **LL1**]. We omit the details here.

4.4. A Sobolev type inequality. As a consequence of our argument above, we have the following Sobolev type inequality.

Corollary 4.1. *Let $2 \le k < \frac{n}{2}$. Then there exists a constant $C > 0$ such that the inequality*

$$\left[\operatorname{Vol}(\mathcal{M}_g)\right]^{\frac{n-2}{2n}} \le C\left[\int_{\mathcal{M}} \sigma_k(\lambda(A_g))d\operatorname{vol}_g\right]^{\frac{n-2}{2n-4k}} \tag{4.27}$$

holds for any conformal metric $g \in [g_0]_k$.

Proof. Note that (4.27) is equivalent to

$$\left[\int_{(\mathcal{M},g_0)} v^{\frac{2n}{n-2}}\right]^{\frac{n-2}{2n}} \le C\left[\int_{(\mathcal{M},g_0)} v^{\frac{2n-4k}{n-2}}\sigma_k\Big(\lambda\Big(\frac{V}{v}\Big)\Big)\right]^{\frac{n-2}{2n-4k}} \tag{4.28}$$

for any k-admissible function v, which is equivalent to (4.18). q.e.d.

5. Verification of the critical inequality

We let v_ε be the function given by

$$v_\varepsilon(x) = \left(\frac{\varepsilon}{\varepsilon^2 + r^2}\right)^{\frac{n-2}{2}}, \tag{5.1}$$

where $r = |x|$, $x \in \mathbb{R}^n$, and $\varepsilon > 0$ is a small constant. Let V_ε be the matrix relative to v_ε, see (2.2). Then we have

$$\frac{V_\varepsilon}{v_\varepsilon} = (n-2)v_\varepsilon^{\frac{4}{n-2}} I.$$

Hence v_ε is k-admissible on $\mathbb{R}^n$ and

$$\sigma_k\Big(\lambda\Big(\frac{V_\varepsilon}{v_\varepsilon}\Big)\Big) = C_{n,k} v_\varepsilon^{\frac{4k}{n-2}},$$

where $C_{n,k} = \frac{n!(n-2)^k}{k!(n-k)!}$. It follows that

$$\int_{\mathbb{R}^n} v^{\frac{2n-4k}{n-2}} \sigma_k\Big(\lambda\Big(\frac{V}{v}\Big)\Big) = C_{n,k} \int_{R^n} v^{\frac{2n}{n-2}}.$$

So we have

$$Y_k(S^n) = \frac{\int_{\mathbb{R}^n} v^{\frac{2n-4k}{n-2}} \sigma_k(\lambda(\frac{V}{v}))}{\big[\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}}\big]^{(n-2k)/n}} = C_{n,k}\Big[\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}}\Big]^{2k/n}. \tag{5.2}$$

In particular, we have

$$Y_k(S^n) = \frac{C_{n,k}}{(n(n-2))^k}[Y_1(S^n)]^k. \tag{5.3}$$

In the above, $v = v_\varepsilon$ and the integrations are independent of ε.

To verify (1.6), it would be natural to use the function (5.1) as a test function, as in the case $k = 1$ [**Au**, **S1**]. However, to realize this idea it would involve complicated computations. We shall deduce (1.6) directly from (1.5). First note that by assumption, there exists a function $v > 0$ such that $\widetilde{g} = v^{4/(n-2)} g \in [g_0]_k$. Hence $\sigma_1(\lambda(A_{\widetilde{g}})) > 0$. That is, the scalar curvature of $(\mathcal{M}, \widetilde{g})$ is positive. Hence the comparison principle for the operator $\sigma_1(\lambda(A_g))$ holds on $\mathcal{M}$.

Let v_1 be a solution of the Yamabe problem (with $k = 1$) such that $Q_1(v_1) < Y_1(S^n)$, where

$$Q_1(v) = \frac{\int_{\mathcal{M}} v\sigma_1(\lambda(V))}{[\int_{\mathcal{M}} v^{2n/(n-2)}]^{(n-2)/n}}.$$

Let v_k be the solution of

$$\sigma_k(\lambda(V)) = C_{n,k} v_1^{k\frac{n+2}{n-2}} \quad \text{in} \quad \mathcal{M}. \tag{5.4}$$

By Lemma 2.3(vi), we have

$$-\Delta v_k + \frac{n-2}{4(n-1)} R v_k = \sigma_1(\lambda(V_k)) \geq n(n-2) v_1^{\frac{n+2}{n-2}}.$$

Since v_1 satisfies

$$-\Delta v + \frac{n-2}{4(n-1)} R v = n(n-2) v_1^{\frac{n+2}{n-2}},$$

by the comparison principle,

$$v_k \geq v_1. \tag{5.5}$$

Now writing

$$Q_k(v) = \frac{\int_{\mathcal{M}} v^{\frac{2n}{n-2} - k\frac{n+2}{n-2}} \sigma_k(\lambda(V))}{\left[\int_{\mathcal{M}} v^{2n/(n-2)}\, d\mathrm{vol}_g\right]^{(n-2k)/n}},$$

we claim that

$$Q_k(v_k) < Y_k(S^n), \tag{5.6}$$

namely (1.6) holds. Indeed, when $k \geq 2$, we have $\frac{2n}{n-2} - k\frac{n+2}{n-2} < 0$. Hence by (5.5),

$$v_1^{\frac{2n}{n-2} - k\frac{n+2}{n-2}} \geq v_k^{\frac{2n}{n-2} - k\frac{n+2}{n-2}}.$$

Hence

$$\begin{aligned}\int_{\mathcal{M}} v_k^{\frac{2n}{n-2} - k\frac{n+2}{n-2}} \sigma_k(\lambda(V_k))\, d\mathrm{vol}_g &\leq C_{n,k} \int_{B_\rho} v_1^{\frac{2n}{n-2} - k\frac{n+2}{n-2}} v_1^{k\frac{n+2}{n-2}}\, d\mathrm{vol}_g \\ &\leq C_{n,k} \int_{B_\rho} v_1^{\frac{2n}{n-2}}\, d\mathrm{vol}_g\end{aligned}$$

and

$$\int_{\mathcal{M}} v_k^{\frac{2n}{n-2}}\, d\mathrm{vol}_g \geq \int_{B_\rho} v_1^{\frac{2n}{n-2}}\, d\mathrm{vol}_g.$$

Therefore we obtain

$$Q_k(v_k) \leq C_{n,k} \left[\int_{\mathcal{M}} v_1^{\frac{2n}{n-2}}\, d\mathrm{vol}_g\right]^{2k/n}, \tag{5.7}$$

so that (5.6) follows from (5.3).

6. Supplementary remarks

The following remarks also take account of related development since this paper was originally written and submitted for publication.

6.1. Compactness of the solution set. For the Yamabe problem ($k = 1$), Schoen [**S2**] has shown that the set of solutions is compact if the manifold is locally conformally flat and not conformally equivalent to the sphere. Schoen also established the compactness for general manifolds in dimension 3. His result was improved to dimensions $n \leq 5$ in [**D**], $n \leq 7$ in [**LZ1**, **M**], and $n \leq 11$ in [**LZ2**], assuming the positive mass theorem. Most recently, the compactness was established for dimensions $n \leq 24$

in [**KMS**], and counterexamples were found for dimensions $n \geq 25$ in [**B**, **BM**].

When $k > \frac{n}{2}$, the compactness of solutions has been established in [**GV2**] for the more general equation

$$\sigma_k(\lambda(A_{g_v})) = f\, v^{\frac{n+2}{n-2}}, \tag{6.1}$$

where f is any positive, smooth function. Recently, in [**TW3**], it was proved that not only the set of solutions, but the set of admissible metrics $[g_0]_k$ is compact under the restriction $\text{Vol}(\mathcal{M}, g) = 1$. The compactness of solutions also extends to the cases $k = \frac{n}{2}$ and more general symmetric functions. The proofs in [**GV2**] and [**TW3**] rely crucially on the fact that the Ricci curvature $\text{Ric}_g \geq 0$ if $g \in [g_0]_k$, which is not true when $2 \leq k < \frac{n}{2}$.

6.2. Conditions (C1) and (C2). As indicated earlier, we impose condition (C1) so that equation (1.1) is elliptic. If a fully nonlinear partial differential equation is not elliptic, little is known about the existence and regularity of solutions. For example, it is unknown whether there is a local solution to the Monge-Ampere equation $\det D^2 u = f$ when the right hand side f changes sign, even in dimension two. But condition (C1) can be replaced by the positivity of the Yamabe constant $Y_k(\mathcal{M})$ [**S**], as in the case $k = 2$, $n = 4$ [**CGY1**, **GV1**].

As for the condition (C2), the variational approach to the k-Yamabe problem is natural for $k \leq \frac{n}{2}$, as in the case $k = 1$. The variational structure is also crucial in recent works on the compactness of solutions to the original Yamabe problem [**LZ2**, **M**]. Recently Branson and Gover [**BG**] have proved that for $k \geq 3$, the variational structure is equivalent to $\mathcal{M}$ being locally conformally flat.

6.3. The full k-Yamabe problem [La]. We bring to the attention of the readers the full k-Yamabe problem. On a Riemannian manifold (M^n, g), one can define a series of scalar curvatures

$$s_k = s_k(\text{Riem}) = s_k(W + A \odot g), \tag{6.2}$$

for $k = 1, 2, \ldots, [\frac{n}{2}]$, where Riem, W, A are introduced at the beginning in the introduction. The k-scalar curvature can also be expressed simply as

$$s_k = \frac{1}{(2k)!} c^{2k} \text{Riem}^k, \tag{6.3}$$

where c is the standard contraction operator, and Riem^k denotes the exterior product.

When $k = 1$, s_1 is the usual scalar curvature. When n is even, $s_{n/2}$ is the Lipschitz-Killing curvature. Furthermore, if $\mathcal{M}$ is a hypersurface, the k-scalar curvature s_k is the $2k^{th}$ mean curvature H_{2k}, which is equal to the $2k^{th}$ elementary symmetric polynomial of the principal curvatures of the hypersurface (which is an intrinsic quantity). When $\mathcal{M}$ is locally conformally flat, then the Weyl curvature in (6.2) vanishes, and s_k turns out to be the k-curvature given in (1.1), for $k = 1, 2, \ldots, [\frac{n}{2}]$.

The full k-Yamabe problem concerns the existence of a conformal metric such that the k-scalar curvature is a constant. This problem coincides with the k-Yamabe problem for locally conformally flat manifolds. The corresponding equation of the k-Yamabe problem is always variational, as in the case $k = 1$ [**La**].

6.4. We also mention here related but independent work of Ge and Wang [**GeW**], which appears to have been carried out at the same time. They considered the k-Yamabe problem in the cases $k = 2$, $n > 8$, for non-conformally locally flat manifolds, using a test function construction from [**GW3**] that parallels the original approach of Aubin [**Au**].

6.5. Finally, we mention that S. Chen [**Cn**] has found a different proof of the interior gradient estimate for solutions to the elliptic equations in Section 3. See [**Li**, **W3**] for another proof.

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