J. DIFFERENTIAL GEOMETRY
**75** (2007) 109-141

# OZSVÁTH–SZABÓ INVARIANTS AND TIGHT CONTACT THREE–MANIFOLDS, II

Paolo Lisca & András I. Stipsicz

**Abstract**

Let $p$ and $n$ be positive integers with $p > 1$, and let $E_{p,n}$ be the oriented 3–manifold obtained by performing $p^2n - pn - 1$ surgery on a positive torus knot of type $(p, pn + 1)$. We prove that $E_{2,n}$ does not carry tight contact structures for any $n$, while $E_{p,n}$ carries tight contact structures for any $n$ and any odd $p$. In particular, we exhibit the first infinite family of closed, oriented, irreducible 3–manifolds which do not support tight contact structures. We obtain the nonexistence results via standard methods of contact topology, and the existence results by using a quite delicate computation of contact Ozsváth–Szabó invariants.

## 1. Introduction

Let $S^3_r(K)$, $r \in \mathbb{Q}$, be the oriented 3–manifold obtained by performing rational $r$–surgery along a knot $K \subset S^3$. In [**15**] we used the Ozsváth–Szabó invariants to study the existence of tight contact structures on $S^3_r(K)$. In particular, we proved that if $T_{p,q}$ is the positive $(p, q)$ torus knot, then $S^3_r(T_{p,q})$ carries positive, tight contact structures for every $r \neq pq - p - q$.

On the other hand, it was proved by Etnyre and Honda [**6**] that $S^3_1(T_{2,3})$ supports no positive tight contact structure. Therefore, the question whether the 3–manifolds $S^3_{pq-p-q}(T_{p,q})$ carry positive, tight contact structures seems to be particularly interesting.

Consider the oriented 3–manifold

$$E_{p,n} := S^3_{p^2n-pn-1}(T_{p,pn+1}).$$

The first main result of this paper is the following:

**Theorem 1.1.** *Let $p$, $n$ be positive integers with $p > 1$. Then, the number of isotopy classes of tight contact structures carried by $E_{p,n}$ is at most*

$$2\max\{p(p-1) - 4, 0\}.$$

An immediate corollary of Theorem 1.1 is:

**Corollary 1.2.** *Let $n$ be a positive integer. Then, the oriented 3–manifold $E_{2,n}$ admits no positive, tight contact structures.*

Notice that Corollary 1.2 generalizes the result of Etnyre and Honda [**6**]. Since the 3–manifolds $E_{2,n}$ are Seifert fibered with base $S^2$ and three exceptional fibers, by [**26**] they are irreducible. Therefore, Corollary 1.2 gives the first infinite family of closed, oriented, irreducible 3–manifolds not carrying positive, tight contact structures.

In the second part of the paper we prove the following:

**Theorem 1.3.** *Let $n$, $p$ be positive integers with $p > 1$ odd. Then, $E_{p,n}$ carries positive, tight contact structures.*

In order to motivate this result, we also prove that the oriented 3–manifolds $E_{p,n}$ do not support any fillable contact structures (Proposition 4.1). Therefore, one cannot prove the existence of tight contact structures by presenting the 3–manifolds $E_{p,n}$ as boundaries of symplectic fillings. In fact, we need to use the more sophisticated methods provided by Heegaard Floer theory.

The paper is organized as follows. In Section 2 we prove Theorem 1.1 and so verify Corollary 1.2. The proof uses convex surface theory along the lines of [**6**, **8**]. In the second part of the paper (Sections 3 to 6) we prove Theorem 1.3 using the Ozsváth–Szabó invariants. In Section 3 we recall the relevant facts of Heegaard Floer theory. In Section 4 we show that the 3–manifolds $E_{p,n}$ do not support symplectically fillable contact structures. In Section 5 we define suitable contact structures on the manifolds $E_{p,n}$ ($p > 1$ odd) and in Section 6 we verify their tightness. The techniques used in the first part of the paper (Section 2) are completely independent from the methods applied in the second part (Sections 3–6). However, the two approaches nicely complement each other, in the sense that using both of them on the same 3–manifold appears to be an effective way to attack the classification problem for tight contact structures.

**Acknowledgments.** The second author was partially supported by OTKA T034885. Part of this collaboration took place when the second author visited the University of Pisa. He wishes to thank the Geometry Group of the Mathematics Department for hospitality and support. Both authors were also partially supported by the EU Marie Curie TOK grant BudAlgGeo.

## 2. Proof of Theorem 1.1

We will follow the methods developed in [**6**] and implemented in [**8**]. We will assume that the reader is familiar with the theory of convex surfaces [**9**] as well as the references [**6**, **8**].

We now recall the notations used in [**6**, **8**]. Denote the Seifert fibered 3–manifold given by the surgery diagram of Figure 1 by $M(a, b, c)$ (with $a, b, c \in \mathbb{Q}$).

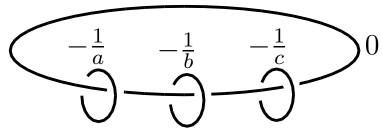


**Figure 1.** Surgery diagram for the Seifert fibered 3–manifold $M(a,b,c)$.

**Lemma 2.1.** *Let $p, n \in \mathbb{N}$ with $p \geq 2$ and $n \geq 1$. Then, there exists an orientation–preserving diffeomorphism*

$$S^3_{p^2n-pn-1}(T_{p,pn+1}) \cong M\left(-\frac{1}{p}, \frac{n}{pn+1}, \frac{1}{p(n+1)+1}\right).$$

*Proof.* An orientation–preserving diffeomorphism is given by the sequence of Kirby moves of Figure 2 for $r = p^2n - pn - 1$ (see e.g., [**11**] for an introduction to Kirby calculus). q.e.d.

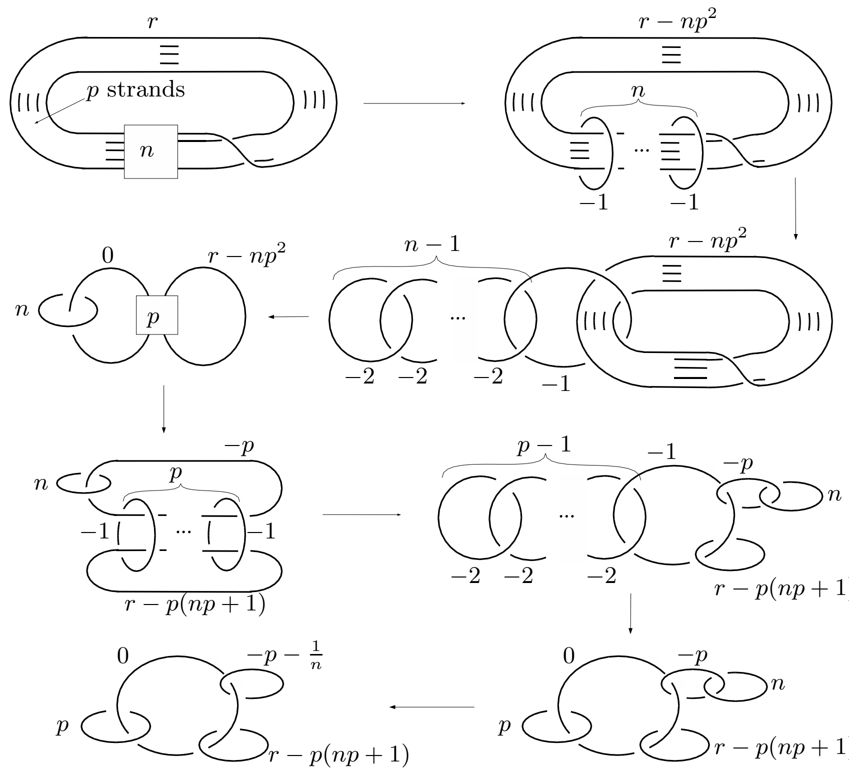


**Figure 2.** A diffeomorphism between $S^3_r(T_{p,pn+1})$ and $M(-\frac{1}{p}, \frac{n}{pn+1}, \frac{1}{p(np+1)-r})$.

Define

$$E_{p,n} := S^3_{p^2n-pn-1}(T_{p,pn+1}).$$

In view of Lemma 2.1 and following [**6**, **8**], we start by decomposing $E_{p,n}$ into $S^1 \times \Sigma_0$, where $\Sigma_0$ is $S^2$ minus three disks, and three copies of $S^1 \times D^2$ identified with neighbourhoods $V_i$ of the singular fibers $F_i$, $i = 1, 2, 3$. In order to recover $E_{p,n}$ from $S^1 \times \Sigma_0$ we need to glue these three copies of $S^1 \times D^2$ to its three boundary tori. We can prescribe the gluing maps by matrices once we fix identifications of the boundary tori with $\mathbb{R}^2/\mathbb{Z}^2$. To do that, for each boundary component of $\partial(S^1 \times \Sigma_0)$ we identify the intersection with a section $\{*\} \times \Sigma_0$ with the image of the line $\langle(1,0)\rangle$, and the fiber with the image of the line $\langle(0,1)\rangle$. For the boundaries of the solid tori $S^1 \times D^2$, the meridional direction is uniquely determined by the property of being homologically trivial in $S^1 \times D^2$. The longitude is unique only up to a $\mathbb{Z}$–action. This indeterminacy results in a certain degree of freedom in choosing the particular gluing matrices. We choose:

$$A_i \colon \partial(S^1 \times D^2) \to -\partial(E_{p,n} \setminus V_i), \quad i = 1, 2, 3,$$

$$A_1 = \begin{pmatrix} p & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} pn+1 & pn-p+1 \\ -n & 1-n \end{pmatrix}, \quad A_3 = \begin{pmatrix} p(n+1)+1 & 1 \\ -1 & 0 \end{pmatrix}.$$

The matrices $A_i$ have determinant one, and the ratios of the elements in their first columns equal the surgery coefficients appearing in the surgery diagram; therefore using the gluing maps $A_i$ recovers $E_{p,n}$. We shall denote by $F_i$ the singular fibers inside the glued-up tori, while each neighbourhood of $F_i$ (as a subspace of $E_{p,n}$) will be called $V_i$, $i = 1, 2, 3$. From the matrices $A_i$ it is immediate to compute that a regular fiber of the fibration has slope

$$v_1 = p, \quad v_2 = -\frac{pn+1}{pn-p+1} \quad \text{and} \quad v_3 = -(p(n+1)+1)$$

when viewed, respectively, in $\partial V_i$, $i = 1, 2, 3$, while the meridian of each $V_i$ has slope

$$c_1 = \frac{1}{p}, \quad c_2 = -\frac{n}{pn+1} \quad \text{and} \quad c_3 = -\frac{1}{p(n+1)+1}$$

when viewed in $-\partial(E_{p,n} \setminus V_i)$, $i = 1, 2, 3$. The numbers $v_1$, $v_2$ and $v_3$ are called the *vertical slopes*, while $c_1$, $c_2$ and $c_3$ are the *critical slopes*.

Recall that the *slope* of a convex torus in standard form identified with $\mathbb{R}^2/\mathbb{Z}^2$ is, by definition, the slope of any component of its dividing set.

**Remark.** If $T$ is a convex torus in standard form isotopic to $\partial V_i$ and the slope of $T$ with respect to the identification $-\partial(E_{p,n} \setminus V_i) \cong \mathbb{R}^2/\mathbb{Z}^2$ given above is equal to the critical slope of $F_i$, then the contact structure under consideration is overtwisted. In fact, any Legendrian divide on $T$ bounds an overtwisted disk in $V_i$.

Let $f \subset E_{p,n}$ be a Legendrian curve isotopic to a regular fiber of the fibration. There are two framings of $f$: the one coming from the fibration and the one induced by the contact structure. The difference between the fibration framing and the contact framing is, by definition, the *twisting number* of $f$.

Let $F_i$ be a Legendrian singular fiber with twisting number $m_i$ and standard neighbourhood $V_i$. Then, the slope of the torus $\partial V_i$ is $\frac{1}{m_i}$ with respect to the identification $\partial V_i \cong \mathbb{R}^2/\mathbb{Z}^2$ given above. The same slope is equal to, respectively,

$$b_1 = \frac{m_1}{pm_1 - 1}, \quad b_2 = -\frac{n(m_2+1)-1}{(pn+1)m_2 + p(n-1)+1}$$

and

$$b_3 = -\frac{m_3}{(p(n+1)+1)m_3 + 1}$$

when computed with respect to the chosen identification $-\partial(E_{p,n} \setminus V_i) \cong \mathbb{R}^2/\mathbb{Z}^2$. The numbers $b_1$, $b_2$ and $b_3$ are called the *boundary slopes.*

**Lemma 2.2.** *Let $\xi$ be a positive, tight contact structure on $E_{p,n}$. Then, the singular fibers $F_1$, $F_2$ and $F_3$ can be isotoped to Legendrian positions such that*

$$m_1 = 0 \quad \textit{and} \quad m_2 = m_3 = -1.$$

*Moreover, we can find (nonstandard) neighbourhoods $V_i' \supset V_i$ with convex boundaries such that the slopes of $-\partial(E_{p,n} \setminus V_i')$ are all infinite.*

*Proof.* The argument is a simple adaptation of the proof of [**6**, Lemma 7]. Notice that the statement of [**6**, Lemma 7] coincides with the statement we want to prove for $(n,p) = (1,2)$. Therefore, we will assume $(n,p) \neq (1,2)$.

Let $V_2$ and $V_3$ be standard neighbourhoods of $F_2$ and $F_3$ with vertical rulings on their boundaries. Up to stabilizing $F_2$ and $F_3$, we may assume $m_2, m_3 < -1$. Then, there are two possible cases.

*Case* I. Suppose there is a vertical annulus $A$ between $V_2$ and $V_3$ having no boundary parallel dividing curves. Then, by the Imbalance Principle [**12**, Proposition 3.17],

$$(pn+1)m_2 + p(n-1) + 1 = (p(n+1)+1)m_3 + 1, \tag{2.1}$$

that is,

$$m_3 = \frac{(pn+1)m_2 + p(n-1)}{p(n+1)+1} = m_2 + 1 - \frac{pm_2 + 2p + 1}{p(n+1)+1}.$$

Since $m_3 \in \mathbb{Z}$, this implies that $p(n+1)+1 \geq 7$ divides $pm_2 + 2p + 1 \neq 0$, therefore $m_2 < -2$ and we have

$$|pm_2 + 2p + 1| = p|m_2| - 2p - 1 \geq p(n+1) + 1.$$

This observation implies that Equation (2.1) can hold only if

$$|m_2| \geq n + 3 + \frac{2}{p},$$

i.e., if $m_2 \leq -(n+4)$. If we cut along $A$ and round corners, we get a torus $T$ of slope

$$-s_T = -\frac{n(m_2+1) + \frac{(pn+1)m_2+p(n-1)}{p(n+1)+1}}{(pn+1)m_2 + p(n-1)+1}. \tag{2.2}$$

surrounding the fibers $F_2$ and $F_3$. When viewed as minus the boundary of the complement of a neighbourhood of $F_1$, the slope of $T$ is $s_T$. We claim that

$$s_T > \frac{m_1}{pm_1 - 1}. \tag{2.3}$$

In fact, it is easy to check that $s_T$ is a strictly decreasing function of $m_2$, and takes the value $s_T = \frac{1}{p}$ for

$$m_2 = -1 - \frac{p^2 n}{p^2 n - p - 1}.$$

Moreover, an easy calculation shows that, since $(n,p) \neq (1,2)$,

$$-(n+4) < -1 - \frac{p^2 n}{p^2 n - p - 1}.$$

It follows that for $m_2 \leq -(n+4)$ we have $s_T > \frac{1}{p}$. Therefore, since

$$\frac{1}{p} > \frac{1}{p - \frac{1}{m_1}} = \frac{m_1}{pm_1 - 1},$$

the claim (2.3) is proved. This immediately implies the existence of a convex vertical torus $T'$ with slope $\infty$. Then, let $A_i$, $i = 1,2,3$, be vertical convex annuli between a Legendrian divide of $T'$ and a ruling of $\partial V_i$, $i = 1,2,3$. As long as $m_i < 0$, we can find bypasses on $A_i$ attached to $\partial V_i$ for each $i = 1,2,3$. By attaching those bypasses to $V_i$ we can find bigger standard neighbourhoods of the singular fibers $F_i$, which amounts to increasing the twisting numbers $m_i$ as long as the assumptions of the Twist Number Lemma [**6**, Lemma 6] hold, i.e., as long as

$$\frac{1}{p} \geq m_1 + 1, \quad -\frac{pn - p + 1}{pn+1} \geq m_2 + 1, \quad -\frac{1}{p(n+1)+1} \geq m_3 + 1.$$

Consequently, we can increase the $m_i$'s up to $m_1 = 0$ and $m_2 = m_3 = -1$. Moreover, the Legendrian divide of $T'$ allows us to attach further vertical bypasses to the standard neighbourhoods until we obtain the neighbourhoods $V_i'$ of the statement.

*Case* II. Suppose there is a vertical annulus $A$ between $V_2$ and $V_3$ with some boundary parallel dividing curve. Then, we can attach a vertical bypass to either $V_2$ or $V_3$ and increase either $m_2$ or $m_3$. Since under Case I we have proved the statement, we may assume that we fall again under Case II. Using Equation (2.1) it is easy to check that if $m_2 = -1$ we can always attach a vertical bypass to $V_3$ as long as $m_3 < -1$, while if $m_3 = -1$ we can attach a vertical bypass to $V_2$ as long as $m_2 < -1$. Therefore, we may assume to be able to increase $m_2$ and $m_3$ until $m_2 = m_3 = -1$. At this point the values of the boundary slopes $b_2$ and $b_3$ are

$$b_2 = -\frac{1}{p} \quad \text{and} \quad b_3 = -\frac{1}{p(n+1)}.$$

We can keep attaching vertical bypasses until the slopes of the resulting neighbourhoods are both $-\frac{1}{k}$, for some $0 \leq k \leq p$. Since for $k = 0$ this gives a vertical convex torus of infinity slope and the conclusion follows as in Case I, we may assume that at some point we can find an annulus $A$ between the two neighbourhoods with no boundary parallel curves. After cutting and rounding we get a torus of slope $-\frac{1}{k}$ surrounding $F_2$ and $F_3$, which can be viewed as a torus of slope $s = \frac{1}{k}$ around $V_1$. For $k = p$, $s$ is the critical slope of the first singular fiber, hence its existence contradicts the tightness of $\xi$. For $0 \leq k < p$ we have

$$b_1 = \frac{m_1}{pm_1 - 1} = \frac{1}{p - \frac{1}{m_1}} < \frac{1}{k}.$$

Therefore there is a torus of slope $\infty$ around $F_1$, and the conclusion follows as before. q.e.d.

Using Lemma 2.2, we can assume the boundary slopes to be

$$b_1 = 0, \quad b_2 = -\frac{1}{p} \quad \text{and} \quad b_3 = -\frac{1}{p(n+1)}.$$

Let $V_i'$ ($i = 1, 2, 3$) be the neighbourhoods given in the statement of Lemma 2.2. Each of the thickened tori $V_i' \setminus V_i$ has a decomposition into basic slices. Following the notation of [**8**], any tight contact structure on $\cup_i V_i'$ with infinity boundary slopes can be represented and is uniquely determined by a diagram as in Figure 3 for some choice of signs, where each sign denotes the corresponding type of basic slice.

Let $q_i$ denote the number of '+' signs in $V_i$. Then,

$$q_1 \in \{0, 1\}, \quad q_2 \in \{0, \ldots, p\} \quad \text{and} \quad q_3 \in \{0, \ldots, p(n+1)\}.$$

Let us denote by $\xi(q_1, q_2, q_3)$ the contact structure on $\cup_i V_i'$ corresponding to the vector $(q_1, q_2, q_3)$. We are going to use the following result in the proofs of Lemmas 2.4, 2.5, 2.6 and 2.7.

**Lemma 2.3** ([**8**], Lemma 4.13)**.** *Let $\Sigma$ be a pair of pants and $\xi$ a tight contact structure on $\Sigma \times S^1$. Suppose that the boundary $-\partial(\Sigma \times S^1) = T_0 \cup T_1 \cup T_2$ consists of tori in standard form with $\#\Gamma_{T_i} = 2$ for $i = 0, 1, 2$,*

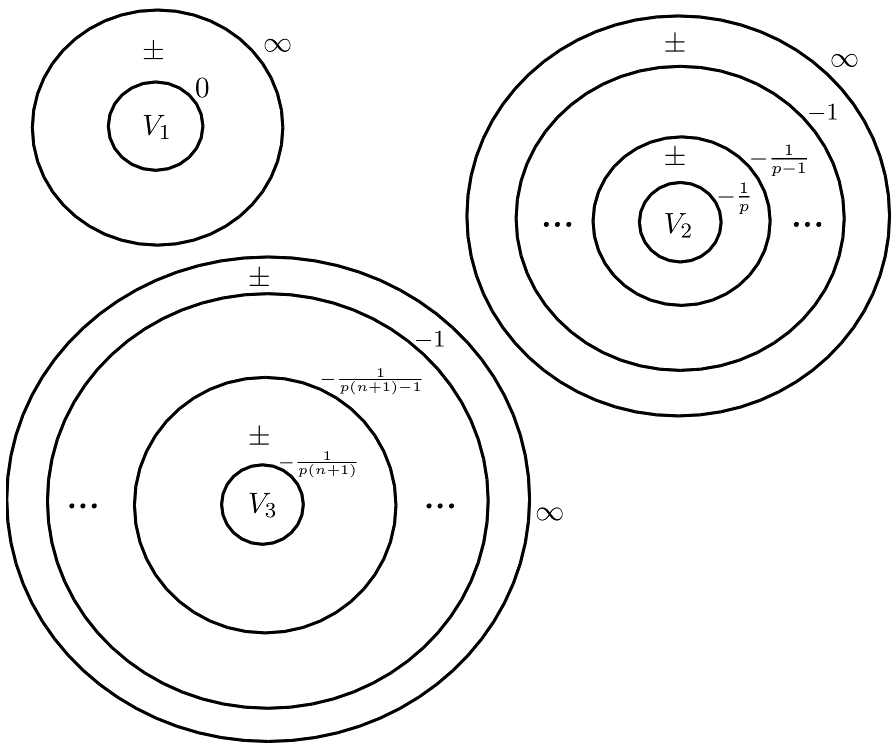


**Figure 3.** A tight contact structure with infinity boundary slopes on $\cup_i V_i'$.

*and slopes $s(T_0) = \frac{p_0}{q}$, $s(T_1) = \infty$, $s(T_2) = \frac{p_2}{q}$. Suppose also that there exists a pair of pants $\Sigma' \subset \Sigma$ such that $\Sigma \times S^1$ decomposes as $\Sigma \times S^1 = \Sigma' \times S^1 \cup (T_0 \times I) \cup (T_2 \times I)$, with $\xi|_{T_i \times I}$ minimally twisting for $i = 0, 2$ and where $\xi|_{\Sigma' \times S^1}$ is a tight contact structure with infinite boundary slopes such that for some $s \in S^1$ the surface $\Sigma' \times \{s\} \subset \Sigma' \times S^1$ is convex and its dividing set consists of arcs, each connecting two different boundary components. Suppose that one of the following holds:*

1) *$s(T_0) = s(T_2) = -\frac{1}{q}$ and $\xi|_{T_0 \times I}$ is isotopic to $\xi|_{T_2 \times I}$;*
2) *$s(T_2) < 0$ and $\xi|_{T_i \times I}$, for $i = 0, 2$, decomposes into basic slices of the same sign (i.e., with relative Euler class $\pm(q, p_i - 1)$).*

*Then there exists a convex annulus with Legendrian boundary consisting of vertical Legendrian rulings of $T_0$ and $T_2$ without boundary parallel dividing curves.*

Observe that in our situation $E_{p,n} \setminus \cup_i V_i' \cong \Sigma' \times S^1$, and a standard argument shows that the restriction to $E_{p,n} \setminus \cup_i V_i'$ of any tight contact structure on $E_{p,n}$ satisfies the assumptions on the dividing set of $\Sigma' \times \{s\}$ stated in Lemma 2.3 (cf. [**8**, Lemma 4.6]).

**Lemma 2.4.** *Let $\xi$ be a positive contact structure on $E_{p,n}$ such that*

$$\xi|_{\cup_i V_i'} = \xi(q_1, q_2, q_3).$$

*If $q_2 \leq q_3 \leq q_2 + pn$, then $\xi$ is overtwisted.*

*Proof.* By contradiction, suppose that $\xi$ is tight. The assumption is equivalent to

$$q_3 \geq q_2 \quad \text{and} \quad p(n+1) - q_3 \geq p - q_2. \tag{2.4}$$

Denote by $V_2''$ and $V_3''$ the neighborhoods of $F_2$ and $F_3$, respectively, bounded by vertical tori inside $V_2'$ and $V_3'$ with slope $-\frac{1}{p}$. Since by [**12**, Lemma 4.14] the basic slices of $V_i' \setminus V_i$ can be shuffled, by (2.4) we may assume that

$$\xi|_{V_2' \setminus V_2''} \quad \text{and} \quad \xi|_{V_3' \setminus V_3''}$$

are isotopic. By Lemma 2.3(1) there exists a vertical convex annulus $A$ with no boundary parallel dividing curve connecting two ruling curves of $\partial V_2''$ and $\partial V_3''$. Cutting along $A$ and rounding corners we get a convex vertical torus $T$ surrounding $F_2$ and $F_3$ with slope $-\frac{1}{p}$. When viewed as minus the boundary of the complement of a neighbourhood of $F_1$, the slope of $T$ becomes $\frac{1}{p}$, which is the critical slope $c_1$. This implies that $\xi$ is overtwisted, giving a contradiction. q.e.d.

**Lemma 2.5.** *Let $\xi$ be a positive contact structure on $E_{p,n}$ such that*

$$\xi|_{\cup_i V_i'} = \xi(q_1, q_2, q_3).$$

*If $q_1 = 0$ and $q_3 \leq p-1$, or $q_1 = 1$ and $q_3 \geq pn+1$, then $\xi$ is overtwisted.*

*Proof.* We consider the case $q_1 = 0$ only, because the case $q_1 = 1$ follows by a symmetric argument. Assume by contradiction that $\xi$ is tight. Stabilize $F_1$ $n$ times by adding zig-zags to it in such a way that the newly created basic slices all have negative signs. The new Legendrian singular fiber has a standard neighbourhood $V_1'' \subset V_1$ such that the boundary slope of $-\partial(E_{p,n} \setminus V_1'')$ is

$$\frac{n}{pn+1}.$$

Inside $V_3$ there is a convex neighbourhood $V_3''$ of $F_3$ such that $-\partial(E_{p,n} \setminus V_3'')$ has boundary slope

$$-\frac{1}{pn+1}.$$

Moreover, since we can shuffle the basic slices of $V_3' \setminus V_3$, by the assumption $q_3 \leq p-1$ we may assume that

$$\xi|_{V_1' \setminus V_1''} \quad \text{and} \quad \xi|_{V_3' \setminus V_3''}$$

decompose into basic slices of the same sign. Therefore, by Lemma 2.3(2) there exists a convex vertical annulus $A$ between $V_1''$ and $V_3''$ with no boundary parallel dividing curves. Cutting along $A$ and rounding corners we get a vertical convex torus which, when viewed as minus the boundary of the complement of a neighbourhood of $F_2$ has slope $-\frac{n}{pn+1}$, which is exactly the critical slope $c_2$. This implies that $\xi$ is overtwisted, giving a contradiction. q.e.d.

**Lemma 2.6.** *Let $\xi$ be a positive contact structure on $E_{p,n}$ such that*

$$\xi|_{\cup_i V_i'} = \xi(q_1, q_2, q_3).$$

*If $(q_1, q_2) \in \{(0,0),(1,p)\}$, then for any $q_3 \in \{0,\dots,p(n+1)\}$ the structure $\xi$ is overtwisted .*

*Proof.* Suppose by contradiction that $\xi$ is tight. Stabilize $F_1$ $(n+1)$ times and $F_2$ once, and denote by $V_1''$ and $V_2''$ standard neighbourhoods of the new Legendrian curves. The slopes of $-\partial(E_{p,n}\setminus V_1'')$ and $-\partial(E_{p,n}\setminus V_2'')$ are, respectively,

$$\frac{n+1}{p(n+1)+1} \quad \text{and} \quad -\frac{n+1}{p(n+1)+1}.$$

Since $(q_1, q_2) \in \{(0,0),(1,p)\}$, the stabilizations can be chosen so that

$$\xi|_{V_1'\setminus V_1''} \quad \text{and} \quad \xi|_{V_2'\setminus V_2''}$$

decompose into basic slices of the same sign. Therefore, by Lemma 2.3(2) we can find a convex vertical annulus $A$ between $V_1''$ and $V_2''$ with no boundary parallel dividing curves. Cutting and rounding provides a torus with slope $\frac{1}{p(n+1)+1}$, which turns into the critical slope $c_3$ when viewed as minus the boundary of the complement of a neighbourhood of $F_3$. Therefore, $\xi$ is overtwisted and we have a contradiction. q.e.d.

**Lemma 2.7.** *Let $\xi$ be a positive contact structure on $E_{p,n}$ such that*

$$\xi|_{\cup_i V_i'} = \xi(q_1, q_2, q_3).$$

*Suppose that*

$$(q_1, q_2, q_3) \in \{(0,1,pn+2),(0,p-1,pn+p),(1,1,0),(1,p-1,p-2)\}.$$

*Then, $\xi$ is overtwisted.*

*Proof.* By contradiction, suppose that $\xi$ is tight. Since the basic slices of $V_i' \setminus V_i$, $i = 2,3$ can be shuffled, the assumption on $(q_1, q_2, q_3)$ guarantees that we can find convex neighbourhoods $V_2''$ and $V_3''$ with boundary slope $-\frac{1}{p-1}$ such that $V_i \subset V_i'' \subset V_i'$, $i = 2,3$, and such that

$$\xi|_{V_2'\setminus V_2''} \quad \text{and} \quad \xi|_{V_3'\setminus V_3''}$$

are isotopic. Then, by Lemma 2.3(1), we can find a convex vertical annulus between $V_2''$ and $V_3''$ with no boundary parallel dividing curves. Cutting and rounding gives a convex vertical torus $T$ which, when viewed as minus the boundary of the complement of a neighbourhood of $F_1$ has slope $\frac{1}{p-1}$.

Now we follow the line of the argument given in the last paragraph of the proof of [**8**, Theorem 4.14]. By substituting $m_1 = 1$ into the formula for the boundary slope $b_1$, we get exactly $\frac{1}{p-1}$. This shows that $F_1$ can be destabilized to a Legendrian curve $F_1'$, and $T$ can be viewed as the boundary of a standard neighbourhood of $F_1'$. If now we stabilize $F_1'$, we get a new singular fiber $F_1$ and a new standard neighbourhood $V_1$ inside $V_1'$. But there is a degree of freedom in the choice of the stabilization of $F_1'$, which corresponds to the choice of "zig–zag" to be added to it. By choosing the appropriate stabilization, we can arrange a different sign for the basic slice $\xi|_{V_1' \setminus V_1}$.

The above argument shows that there is an isotopy between $\xi$ and a contact structure which restricts to $\cup_i V_i'$ as $\xi(1-q_1, q_2', q_3')$, for some $q_2'$ and $q_3'$ which are a priori different from $q_2$ and $q_3$. In fact, when we create the torus $T$ we do not touch $V_2''$ and $V_3''$, but we destroy $V_2' \setminus V_2''$ and $V_3' \setminus V_3''$. Using $-\partial(E_{p,n} \setminus V_1')$, which has slope infinity, we can find new convex neighbourhoods $V_i' \supset V_i''$ with infinity boundary slope, but we loose control on the signs in the basic slice decompositions of $V_2' \setminus V_2''$ and $V_3' \setminus V_3''$. Since $V_3''$ has been preserved, an easy computation shows that $q_3' \geq pn + 1$ if $q_1 = 0$, and $q_3' \leq p - 1$ if $q_1 = 1$. By Lemma 2.5, any contact structure which restricts to $\cup_i V_i'$ as $\xi(1-q_1, q_2', q_3')$ is overtwisted in these cases and we get a contradiction. q.e.d.

*Proof of Theorem* 1.1. Let $V_i'$ $(i = 1, 2, 3)$ be the neighborhoods given in the statement of Lemma 2.2. By [**6**, Lemmas 10, 11], there are exactly two positive, tight contact structures on $E_{p,n} \setminus \cup_i V_i'$ with convex boundary and boundary slopes $(\infty, \infty, \infty)$. The statement is now an immediate consequence of Lemmas 2.4, 2.5, 2.6 and 2.7. q.e.d.

**Remark 2.8.** Shortly after the first version of the present paper was circulated, Paolo Ghiggini pointed out to the authors that the upper bound given in Theorem 1.1 is not sharp for $p > 2$.

## 3. Generalities in Heegaard Floer theory

In the second part of the paper we will apply Heegaard Floer theory in proving tightness of certain contact structures (specified by contact surgery diagrams later) on the oriented 3–manifolds

$$E_{p,n} = S^3_{p^2n-pn-1}(T_{p,pn+1})$$

for $p > 1$ and odd. As it was indicated earlier, the methods used in the subsequent sections are completely different from the ones used earlier. For the sake of completeness we begin our discussion by shortly reviewing the basics of Heegaard Floer theory and contact surgery.

**Ozsváth–Szabó homologies.** In a remarkable series of papers [**19**, **20**, **21**, **24**] Ozsváth and Szabó defined new invariants of many low–dimensional objects, including contact structures on closed 3–manifolds. Heegaard Floer theory associates a finetely generated abelian group $\widehat{HF}(Y,\mathbf{t})$ (the *Ozsváth–Szabó homology group*) to a closed, oriented spin$^c$ 3–manifold $(Y,\mathbf{t})$, and a homomorphism

$$F_{W,\mathbf{s}}\colon \widehat{HF}(Y_1,\mathbf{t}_1)\to \widehat{HF}(Y_2,\mathbf{t}_2)$$

to an oriented spin$^c$ cobordism $(W,\mathbf{s})$ between two spin$^c$ 3–manifolds $(Y_1,\mathbf{t}_1)$ and $(Y_2,\mathbf{t}_2)$.

Throughout this paper we shall assume that $\mathbb{Z}/2\mathbb{Z}$ coefficients are being used in the complexes defining the $\widehat{HF}$–groups. With this assumption, the groups are actually $\mathbb{Z}/2\mathbb{Z}$–vector spaces. The group $\widehat{HF}(Y)$ will denote the sum of $\widehat{HF}(Y,\mathbf{t})$ for all spin$^c$ structures. A fundamental property of these groups is that there are only finitely many spin$^c$ structures on any 3–manifold with nontrivial Ozsváth–Szabó homology groups; hence $\widehat{HF}(Y)$ is also finitely generated. For a rational homology sphere $Y$ the Ozsváth–Szabó homology group $\widehat{HF}(Y,\mathbf{t})$ is nontrivial for any spin$^c$ structure $\mathbf{t}\in Spin^c(Y)$, see [**20**, Proposition 5.1]. In particular, for a rational homology 3–sphere $Y$ we have

$$\dim \widehat{HF}(Y)\geq |H_1(Y;\mathbb{Z})|.$$

A rational homology 3–sphere $Y$ is called an *L–space* if

$$\dim \widehat{HF}(Y)= |H_1(Y;\mathbb{Z})|.$$

In the light of the above nonvanishing result, this property is equivalent to

$$\widehat{HF}(Y,\mathbf{t})=\mathbb{Z}/2\mathbb{Z}$$

for all $\mathbf{t}\in \mathrm{Spin}^c(Y)$.

Let $Y$ be a closed, oriented 3–manifold and let $K\subset Y$ be a framed knot with framing $f$. Let $Y(K)$ denote the 3–manifold given by surgery along $K\subset Y$ with respect to the framing $f$. The surgery can be viewed at the 4–manifold level as a 2–handle addition. The resulting cobordism $X$ induces a homomorphism

$$F_X:=\sum_{\mathbf{t}\in\mathrm{Spin}^c(X)} F_{X,\mathbf{t}}\colon \widehat{HF}(Y)\to\widehat{HF}(Y(K))$$

obtained by summing over all spin$^c$ structures on $X$. Similarly, there is a cobordism $U$ defined by adding a 2–handle to $Y(K)$ along a normal circle $N$ to $K$ with framing $-1$ with respect to a normal disk to $K$. The boundary components of $U$ are $Y(K)$ and the 3–manifold $Y'(K)$ obtained from $Y$ by a surgery along $K$ with framing $f+1$. As before, $U$ induces a homomorphism

$$F_U\colon \widehat{HF}(Y(K))\to\widehat{HF}(Y'(K)).$$

Finally, by attaching a 4–dimensional 2–handle to $Y'(K)$ along a normal circle $D$ to $N$ with framing $-1$ with respect to the normal disk to $N$, we obtain a cobordism $V$. As it is shown in [**15**], the 4–manifold $V$ is a cobordism from $Y'(K)$ to $Y$. As above, $F_V$ denotes the induced homomorphism

$$F_V \colon \widehat{HF}(Y'(K)) \to \widehat{HF}(Y).$$

**Theorem 3.1** (Surgery exact triangle; [**20**], Theorem 9.16)**.** *The homomorphisms $F_X, F_U$ and $F_V$ fit into an exact triangle*

$$\begin{tikzcd} \widehat{HF}(Y) \arrow[rr, "F_X"] & & \widehat{HF}(Y(K)) \arrow[dl, "F_U"] \\ & \widehat{HF}(Y'(K)) \arrow[ul, "F_V"] & \end{tikzcd}$$

It was proved in [**19**, **22**] that the Ozsváth–Szabó homology groups $\widehat{HF}(Y)$ split as

$$\widehat{HF}(Y) = \oplus_{(d,\mathbf{t})\in\mathcal{H}} \widehat{HF}_d(Y, \mathbf{t}),$$

where $\mathcal{H}$ denotes the set of homotopy types of oriented 2–plane fields on $Y$. The set $\mathcal{H}$ can be identified with $[Y, S^2]$, which is isomorphic to the set of framed 1–manifolds via the Pontrjagin–Thom construction. The 1–manifold determines a spin$^c$ structure $\mathbf{t} \in \mathrm{Spin}^c(Y)$, while the framing corresponds to the degree $d$. This invariant of the oriented 2–plane field $\xi$ is naturally an element of $\mathbb{Z}/\mathrm{div}\,(\xi)\mathbb{Z}$, where $\mathrm{div}\,(\xi)$ is the divisibility of $c_1(\xi)$ in $H^2(Y;\mathbb{Z})$. If $c_1(\xi)$ is torsion then $\mathrm{div}\,(\xi) = 0$. Therefore if $\mathbf{t} \in \mathrm{Spin}^c(Y)$ is torsion, that is, $c_1(\mathbf{t}) \in H^2(Y;\mathbb{Z})$ is a torsion element, then the Ozsváth–Szabó homology group $\widehat{HF}(Y, \mathbf{t})$ comes with a natural relative $\mathbb{Z}$–grading. As it was shown in [**22**], this relative $\mathbb{Z}$–grading admits a natural lift to an absolute $\mathbb{Q}$–grading. In conclusion, for a torsion spin$^c$ structure $\mathbf{t}$ the Ozsváth–Szabó homology group $\widehat{HF}(Y, \mathbf{t})$ splits as

$$\widehat{HF}(Y, \mathbf{t}) = \oplus_{d\in\mathbb{Q}} \widehat{HF}_d(Y, \mathbf{t}),$$

where the degree $d$ is determined mod 1 by $\mathbf{t}$. When $\mathbf{t} \in \mathrm{Spin}^c(Y)$ has torsion first Chern class, there is an isomorphism between the homology groups $\widehat{HF}_d(Y, \mathbf{t})$ and $\widehat{HF}_{-d}(-Y, \mathbf{t})$.

Next we describe the relation between degrees and the maps induced by 4–dimensional cobordisms. Let $(W, \mathbf{s})$ be a spin$^c$ cobordism between two spin$^c$ manifolds $(Y_1, \mathbf{t}_1)$ and $(Y_2, \mathbf{t}_2)$. If the spin$^c$ structures $\mathbf{t}_i$ are both torsion and $x \in \widehat{HF}(Y_1, \mathbf{t}_1)$ is a nonzero homogeneous element of degree $d(x)$, then either $F_{W,\mathbf{s}}(x) \in \widehat{HF}(Y_2, \mathbf{t}_2)$ is zero or it is homogeneous of degree

$$d(x) + \frac{1}{4}(c_1^2(\mathbf{s}) - 3\sigma(W) - 2\chi(W)).$$

Notice that $F_W$ (being equal to the sum $\sum_{\mathbf{s}\in Spin^c(W)} F_{W,\mathbf{s}}$) might map a homogeneous element $x \in \widehat{HF}_d(Y_1,\mathbf{t})$ into a nonhomogeneous element $F_W(x) \in \widehat{HF}(Y_2)$.

We need one more piece of information. Recall that the set of spin$^c$ structures comes equipped with a natural involution, usually denoted by $\mathbf{t} \mapsto \overline{\mathbf{t}}$. The spin$^c$ structure $\overline{\mathbf{t}}$, called the *conjugate* of $\mathbf{t}$, is defined as follows: If one thinks of a spin$^c$ structure as a suitable equivalence class of nowhere zero vector fields (cf. [**19**]), then the above involution is the map induced by multiplying a representative vector field by $(-1)$. Equivalently, viewing a spin$^c$ structure as an equivalence class of oriented 2–plane fields, the conjugate action is induced by reversing the orientation of the planes in the oriented 2–plane field.

**Theorem 3.2** ([**20**], Theorem 2.4)**.** *The groups* $\widehat{HF}(Y,\mathbf{t})$ *and* $\widehat{HF}(Y,\overline{\mathbf{t}})$ *are canonically isomorphic.*

A spin$^c$ structure $\mathbf{t} \in \mathrm{Spin}^c(Y)$ is induced by a *spin* structure exactly when $c_1(\mathbf{t}) = 0$, or equivalently when $\mathbf{t} = \overline{\mathbf{t}}$. Let $\mathcal{J}_Y$ denote the isomorphism of Theorem 3.2 between $\widehat{HF}(Y,\mathbf{t})$ and $\widehat{HF}(Y,\overline{\mathbf{t}})$. Then, according to [**21**, Theorem 3.6], given a spin$^c$ cobordism $(W,\mathbf{s})$ we have

$$F_{W,\mathbf{s}} = \mathcal{J}_{Y'} \circ F_{W,\overline{\mathbf{s}}} \circ \mathcal{J}_Y, \tag{3.1}$$

where $\overline{\mathbf{s}}$ is the spin$^c$ structure on the 4–manifold $W$ conjugate to $\mathbf{s}$. (If we think of $\mathbf{s} \in \mathrm{Spin}^c(W)$ as a suitable equivalence class of almost–complex structures defined on $W - \{\text{finitely many points}\}$, then $\overline{\mathbf{s}}$ corresponds to the conjugate of the almost–complex structure defining $\mathbf{s}$.) As an easy corollary of (3.1), we get that $F_{W,\mathbf{s}}$ is nontrivial if and only if $F_{W,\overline{\mathbf{s}}}$ is nontrivial. Viewing $\widehat{HF}(Y)$ with the conjugate actions as a $\mathbb{Z}/2\mathbb{Z}$–representation, the above identity (3.1) simply says that the induced map $F_W$ for the cobordism $W$ is $\mathbb{Z}/2\mathbb{Z}$–equivariant.

The special relation between spin structures and maps induced by cobordisms is demonstrated by the following simple observation. Suppose that $Y$ is a rational homology sphere which is an $L$–space. We identify the nontrivial element in each group $\widehat{HF}(Y,\mathbf{t}) = \mathbb{Z}/2\mathbb{Z}$ with $\mathbf{t} \in \mathrm{Spin}^c(Y)$. With this convention, the set of spin$^c$ structures provides a basis for $\widehat{HF}(Y)$. Let $V$ be a cobordism between the rational homology spheres $Y_1$ and $Y_2$, $Y_i$ be $L$–spaces and $\mathbf{t}_i$ be *spin* structures on $Y_i$ $(i = 1,2)$. Let

$$\mathcal{S} = \{\mathbf{s} \in \mathrm{Spin}^c(V) \mid \mathbf{s}|_{Y_i} = \mathbf{t}_i \quad i = 1,2\}.$$

The set $\mathcal{S}$ decomposes as the collection $\mathcal{S}_1$ of spin$^c$ structures which are *not* spin structures and the set of spin structures $\mathcal{S}_2$ among the elements of $\mathcal{S}$. As always, let $F_V$ denote the map induced by the cobordism $V$, that is, $F_V = \sum_{\mathbf{s}\in\mathrm{Spin}^c(V)} F_{V,\mathbf{s}}$.

**Lemma 3.3.** *Suppose that $V$ and $\mathbf{t}_i$ $(i=1,2)$ are given as above. If $\mathcal{S}_2=\emptyset$ then the $\mathbf{t}_2$–component of $F_V(\mathbf{t}_1)$ is zero.*

*Proof.* Notice that the $\mathbf{t}_2$–component of $F(\mathbf{t}_1)$ is computed by considering the sum $\sum_{\mathbf{s}\in\mathcal{S}} F_{V,\mathbf{s}}(\mathbf{t}_1)$. By assumption, this sum is equal to $\sum_{\mathbf{s}\in\mathcal{S}_1} F_{V,\mathbf{s}}(\mathbf{t}_1)$. Since $\mathcal{S}_1=\{\mathbf{s}_1,\overline{\mathbf{s}_1},\ldots,\mathbf{s}_k,\overline{\mathbf{s}_k}\}$, $\mathbf{t}_1=\overline{\mathbf{t}_1}$ by assumption and $F_{V,\mathbf{s}_i}(\mathbf{t}_1)+F_{V,\overline{\mathbf{s}_i}}(\mathbf{t}_1)=0$, the lemma follows. q.e.d.

**Contact $(\pm1)$–surgery.** Suppose that $L\subset(Y,\xi)$ is a Legendrian knot in a contact 3–manifold. Let $Y_L^{\pm}$ denote the 3–manifold we get by doing $(\pm1)$–surgery along $L$, where the surgery coefficient is measured with respect to the contact framing of $L$. According to the classification of tight contact structures on a solid torus [**12**], the contact structure $\xi|_{Y-\nu L}$ extends uniquely (up to isotopy) to the surgered manifolds $Y_L^+$ and $Y_L^-$ as a tight structure on the glued–up torus. Therefore, the knot $L$ with a $(+1)$ or $(-1)$ on it uniquely specifies a contact 3–manifold $(Y_L^+,\xi_L^+)$ or $(Y_L^-,\xi_L^-)$. (For more about contact surgery see [**1**, **2**, **3**].) In particular, a Legendrian link $\mathbb{L}\subset(S^3,\xi_{st})$ in the standard contact 3–sphere (which can be represented by its front projection) defines a contact structure once the surgery coefficients $(+1)$ and $(-1)$ are specified on its components. In order to keep diagrams as simple as possible, we will follow the convention that when in a diagram a Legendrian knot has no coefficient, then contact $(-1)$–surgery is performed on it. Contact $(-1)$–surgery is also frequently called *Legendrian surgery* in the literature.

**Contact Ozsváth–Szabó invariants.** In [**24**] Ozsváth and Szabó define an invariant

$$c(Y,\xi)\in\widehat{HF}(-Y,\mathbf{t}_\xi)$$

assigned to a positive, cooriented contact structure $\xi$ on $Y$.[1] In fact, $\xi$ (as an oriented 2–plane field) determines an element $(d(\xi),\mathbf{t}_\xi)\in\mathcal{H}$ and according to [**24**] the contact invariant $c(Y,\xi)$ is an element of $\widehat{HF}_{-d(\xi)}(-Y,\mathbf{t}_\xi)$. Moreover, if $c_1(\xi)\in H^2(Y;\mathbb{Z})$ is torsion then

$$d(\xi)=\frac{1}{4}(c_1^2(X,J)-3\sigma(X)-2\chi(X)+2),$$

where $X$ is a compact almost–complex 4–manifold with $\partial X=Y$, and $\xi$ is homotopic to the distribution of complex tangencies on $\partial X$.

The main properties of the contact Ozsváth–Szabó invariant are summarized in the following two theorems.

**Theorem 3.4** ([**24**]). *If $(Y,\xi)$ is overtwisted, then $c(Y,\xi)=0$. If $(Y,\xi)$ is Stein fillable then $c(Y,\xi)\neq0$. In particular, for the standard contact structure $(S^3,\xi_{st})$ the invariant $c(S^3,\xi_{st})\in\widehat{HF}(S^3)=\mathbb{Z}/2\mathbb{Z}$ is nonzero.*

[1]When $\mathbb{Z}$–coefficients are used, the invariant $c(Y,\xi)$ is only defined up to sign, but since we are using $\mathbb{Z}/2\mathbb{Z}$–coefficients we do not need to worry about such ambiguity.

**Theorem 3.5** ([**14**, **24**]). *Suppose that $(Y_2, \xi_2)$ is obtained from $(Y_1, \xi_1)$ by a contact $(+1)$–surgery. Then*

$$F_{-W}(c(Y_1, \xi_1)) = c(Y_2, \xi_2),$$

*where $-W$ is the cobordism induced by the surgery with reversed orientation and $F_{-W}$ is the sum $\sum_{\mathbf{s}} F_{-W,\mathbf{s}}$ over all spin$^c$ structures on $W$. In particular, if $c(Y_2, \xi_2) \neq 0$ then $(Y_1, \xi_1)$ is tight.*

Since by [**1**, Proposition 8] contact $(-1)$–surgery along a Legendrian push–off inverts contact $(+1)$–surgery, the above theorem implies

**Corollary 3.6.** *If $(Y_2, \xi_2)$ is given as Legendrian surgery along a Legendrian knot in $(Y_1, \xi_1)$ and $c(Y_1, \xi_1) \neq 0$ then $c(Y_2, \xi_2) \neq 0$; in particular, $(Y_2, \xi_2)$ is tight.*

An easy application of the surgery exact triangle and Theorem 3.5 provides

**Lemma 3.7** ([**14**], Lemma 2.5). *The contact structure $\eta_1$ on $S^1 \times S^2$ given as contact $(+1)$–surgery on a Legendrian unknot with Thurston–Bennequin number $-1$ has nonvanishing contact Ozsváth–Szabó invariant $c(S^1 \times S^2, \eta_1) \in \widehat{HF}(S^1 \times S^2)$.*

## 4. Symplectic fillings

In this section we show, assuming $n \geq 1$ and $p > 1$, that the 3–manifold $E_{p,n}$ does not support fillable contact structures, thus justifying our use of Heegaard Floer theory in the proof of tightness of the contact structures described below.

Recall that a compact symplectic 4–manifold $(X, \omega)$ is a *symplectic filling* of the closed contact 3–manifold $(Y, \xi)$ if $\partial X = Y$ and $\omega|_\xi \neq 0$ along the boundary $\partial X$.

**Proposition 4.1.** *For each $p > 1$ and $n \geq 1$ the oriented 3–manifold $E_{p,n} = S^3_{p^2n-pn-1}(T_{p,pn+1})$ is an $L$–space and supports no positive, fillable contact structure.*

*Proof.* Arguing by contradiction, suppose that $E_{p,n}$ supports a fillable contact structure. Recall that the slice genus of the $(p, q)$–torus knot $T_{p,q}$ is equal to $\frac{1}{2}(p-1)(q-1)$. Since $(pq-1)$–surgery on the torus knot $T_{p,q}$ is a lens space [**16**], by [**15**, Proposition 4.1] $E_{p,n}$ is an $L$-space. By [**25**, Theorem 1.4] this implies that if $(X, \omega)$ is a symplectic filling of $E_{p,n}$, then $b_2^+(X) = 0$. On the other hand, Figure 4 shows that $-E_{p,n}$ is the boundary of a negative definite plumbing 4–manifold $W_{p,n}$.

Therefore the closed 4–manifold $Z = X \cup_{E_{p,n}} W_{p,n}$ is negative definite, and by Donaldson's celebrated result [**4**, **5**] $Z$ has a diagonal intersection form. This implies that any intersection lattice contained in $Q_{W_{p,n}}$ embeds into the diagonal intersection form $Q_Z$. But the argument of

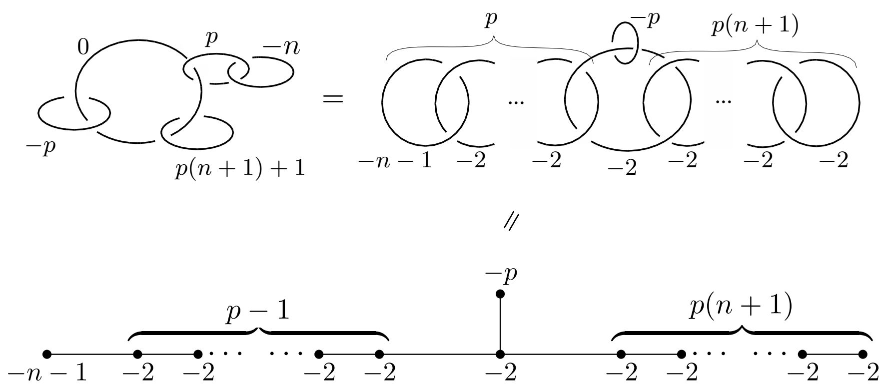


**Figure 4.** Presentation of $-E_{p,n}$ as the boundary of a plumbing.

[**13**, Lemma 4.3] with the minor modification given in [**15**, Theorem 4.2] (due to the presence of the framing $-n-1$ instead of $-2$ at the end of one long leg) shows that $Q_{W_{p,n}}$ contains an intersection lattice which does not embed into any diagonal intersection form, yielding a contradiction. q.e.d.

## 5. Tight contact structures on $E_{p,n}$

Now we outline our approach to the proof of Theorem 1.3. The strategy is the following: in this section we specify a contact structure $\xi_{p,n}$ on a certain 3–manifold $S_{p,n}$ so that the contact invariant $c(S_{p,n}, \xi_{p,n})$ is nonzero. Since $S_{p,n}$ turns out to be an $L$-space, we can identify the invariant $c(S_{p,n}, \xi_{p,n}) \in \widehat{HF}(-S_{p,n})$ by determining the spin$^c$ structure induced by $\xi_{p,n}$. By specifying an appropriate Legendrian knot in $\xi_{p,n}$ and doing contact $(+1)$–surgery along it, we define a contact structure $\zeta_{p,n}$ on $E_{p,n}$ and a cobordism $X$ from $S_{p,n}$ to $E_{p,n}$. In the next section we show that $c(S_{p,n}, \xi_{p,n})$ is not in $\ker F_{-X}$, which implies that $c(E_{p,n}, \zeta_{p,n}) = F_{-X}(c(S_{p,n}, \xi_{p,n}))$ is nonzero, hence that the contact structure $\zeta_{p,n}$ on $E_{p,n}$ is tight, concluding the argument. Throughout the rest of the paper we assume that $p > 1$ is odd. The contact structure $\xi_{p,n}$ is defined by the contact surgery diagram of Figure 5. The numbers different from $+1$ next to the vertical braces denote the number of left cusps immediately to their right. Moreover (as noted earlier) we adopt the convention that when in a diagram a Legendrian knot has no surgery coefficient, then contact $(-1)$–surgery is performed on it.

Notice that the diagram also specifies the underlying oriented 3–manifold $S_{p,n}$.

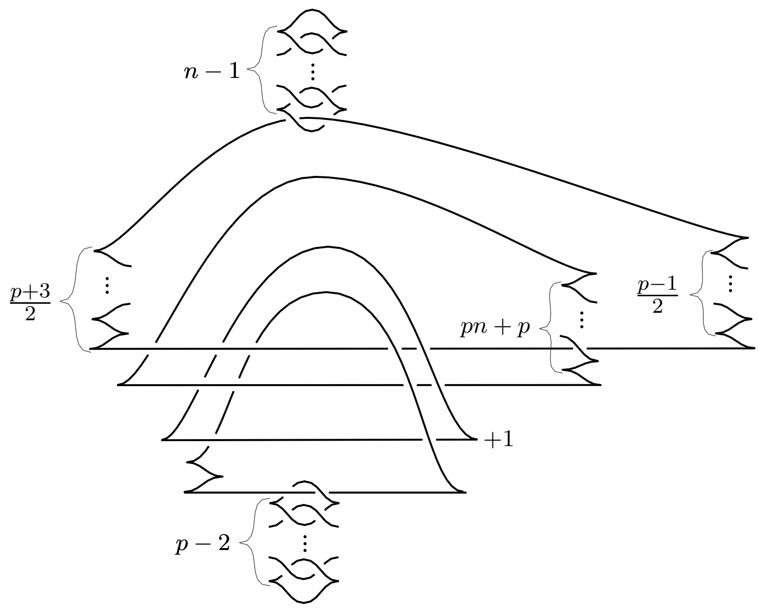


**Figure 5.** The tight contact structure $\xi_{p,n}$ on $S_{p,n}$ with $p > 1$ odd.

**Proposition 5.1.** *The* 3*–manifold* $S_{p,n}$ *defined by the contact surgery diagram of Figure* 5 *is an* $L$*–space, and the invariant* $c(S_{p,n}, \xi_{p,n})$ *is nonzero.*

*Proof.* The first statement can be proved in two steps. First observe, by converting contact surgery coefficients into smooth ones, that $S_{p,n}$ is orientation preserving diffeomorphic to $S^3_r(T_{p,pn+1})$, with

$$r = p(np+1) - \frac{p(n+1)+1}{p(n+1)+2}.$$

For the Kirby moves see Figure 6 and compare the result with Figure 2.

Since the above $r$ is greater than $2g_s(T_{p,pn+1}) - 1 = p^2n - pn - 1$, by [**15**, Proposition 4.1] the 3–manifold $S_{p,n}$ is an $L$–space.

The second statement follows from the fact that the structure $\xi_{p,n}$ is given as Legendrian surgery on the contact structure $\eta_1$ of Lemma 3.7. Therefore, Lemma 3.7 and Corollary 3.6 imply that the invariant of $\xi_{p,n}$ is nonzero. q.e.d.

**Remark.** In fact, the contact structure $\xi_{p,n}$ can be proved to be Stein fillable. We will not make use of this fact in our further arguments.

Next, we want to identify the spin$^c$ structure induced by $\xi_{p,n}$. In order to do this, we need a little preparation.

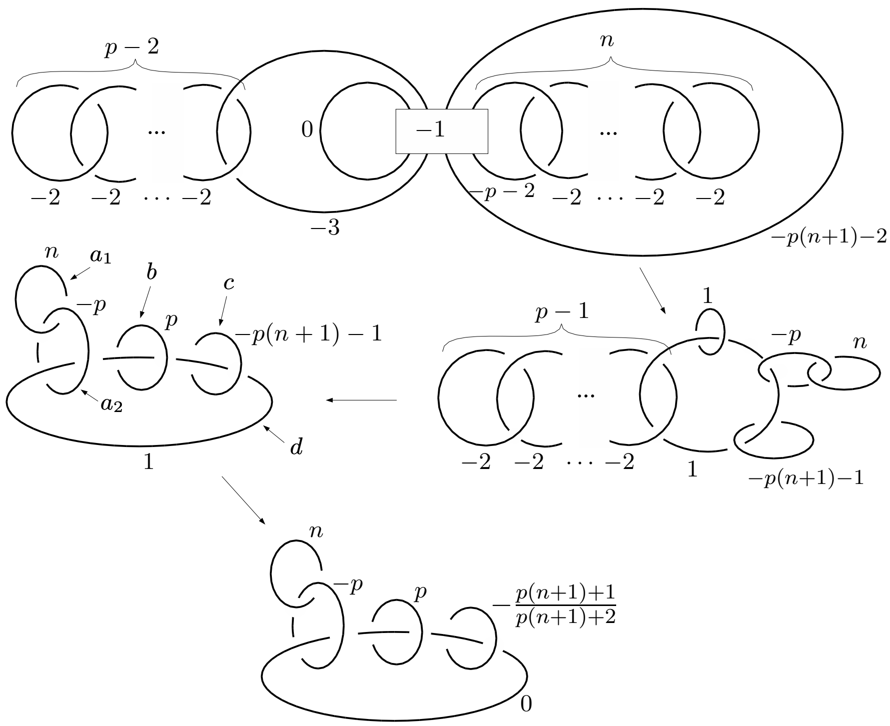


**Figure 6.** Surgery diagrams for $S_{p,n}$.

It follows from Figure 6 that the homology group $H_1(S_{p,n};\mathbb{Z})$ has order

$$h_S := |H_1(S_{p,n};\mathbb{Z})| = p(pn+1)(p(n+1)+2) - p(n+1) - 1. \tag{5.1}$$

Moreover, $H_1(S_{p,n};\mathbb{Z})$ is generated by the classes $\mu_{a_1}$, $\mu_{a_2}$, $\mu_b$, $\mu_c$, $\mu_d$ of suitably oriented meridional circles to the knots $a_1$, $a_2$, $b$, $c$, $d$ given in Figure 6. These elements are subject to the relations:

$$n\mu_{a_1} + \mu_{a_2} = 0, \quad -p\mu_{a_2} + \mu_{a_1} + \mu_d = 0, \quad p\mu_b + \mu_d = 0,$$

$$(-p(n+1)-1)\mu_c + \mu_d = 0, \quad \mu_{a_2} + \mu_b + \mu_c + \mu_d = 0.$$

The relations above imply that $\mu_d$ generates the homology group, since $\mu_{a_1}$, $\mu_{a_2}$, $\mu_b$ and $\mu_c$ can be expressed in terms of $\mu_d$ as

- $\mu_{a_1} = [n(n+1)p^2 + 2np - 1 - n]\mu_d, \qquad \mu_{a_2} = -n\mu_{a_1}$,
- $\mu_b = [(-n^2-n)p^2 + (-1-3n)p - 1 + n]\mu_d$,
- $\mu_c = [(n^2+2n+1)np^2 + p(2n^2+3n+1) - (n+2)n]\mu_d$.

Notice that the order of $H_1(S_{p,n};\mathbb{Z})$ is always odd. Therefore, there is no 2–torsion in the second cohomology of $S_{p,n}$, and the spin$^c$ structures on $S_{p,n}$ are determined by their first Chern classes.

**Lemma 5.2.** *Let $\mathbf{t}_{p,n}$ be the* spin$^c$ *structure induced by $\xi_{p,n}$. Then, if $p$ is odd we have $c_1(\mathbf{t}_{p,n}) = PD(\mu_d)$.*

*Proof.* Consider the 4–manifold $X$ determined by the surgery diagram of Figure 5. Since $X$ is simply connected, a spin$^c$ structure on $X$ is determined by its first Chern class. Let $\alpha \in H^2(X;\mathbb{Z})$ be the unique cohomology class which evaluates on each 2–homology class corresponding to an oriented knot $K$ of the diagram as the rotation number of $K$. Then, the spin$^c$ structure corresponding to $\alpha$ restricts to the spin$^c$ structure of $\xi_{p,n}$ (see e.g., [**3**] for details).

Therefore, after choosing a suitable orientation of the curves in Figure 5, we have

$$PD(c_1(\mathbf{t}_{p,n})) = \sum_K \mathrm{rot}(K)\mu_K, \tag{5.2}$$

where the sum is over all surgery curves, rot$(K)$ denotes the rotation number of the oriented Legendrian knot $K$, and $\mu_K$ denotes the first homology class induced by its meridian. Recall that according to [**10**, **11**] the front projection determines the rotation number of the corresponding Legendrian knot as

$$\mathrm{rot}(K) = \frac{1}{2}(c_d - c_u), \tag{5.3}$$

where $c_u$ and $c_d$ denote the number of up and down cusps in the projection. Using Formulas (5.2) and (5.3), and following the Kirby moves of Figure 6, one can easily check that

$$PD(c_1(\mathbf{t}_{p,n})) = -\mu_{a_2} - \mu_b + p(n+1)\mu_c - \mu_d.$$

Replacing each of $\mu_{a_2}$, $\mu_b$ and $\mu_c$ by the corresponding multiple of $\mu_d$ yields, after a somewhat tedious calculation, $PD(c_1(\mathbf{t}_{p,n})) = \mu_d$. q.e.d.

**Definition 5.3.** Let $\zeta_{p,n}$ be the contact structure defined by the upper–left contact surgery picture of Figure 7.

**Proposition 5.4.** *The contact structure $\zeta_{p,n}$ is supported by $E_{p,n}$.*

*Proof.* The proof requires only a minor modification of the Kirby calculus of Figure 6. This modification is shown in Figure 7. q.e.d.

## 6. Maps between the Ozsváth–Szabó homologies

In this section we show that the contact Ozsváth–Szabó invariant of the contact 3–manifold $(E_{p,n}, \zeta_{p,n})$ is nonzero. This proves Theorem 1.3. Note that $\zeta_{p,n}$ is obtained by contact $(+1)$–surgery on $\xi_{p,n}$ along the Legendrian knot $L$ shown in Figure 7. There is a cobordism naturally associated to the surgery which we denote by $X$. By the properties of the contact Ozsváth–Szabó invariants we know that $c(E_{p,n}, \zeta_{p,n}) = F_{-X}(c(S_{p,n}, \xi_{p,n}))$. This section is devoted to collecting

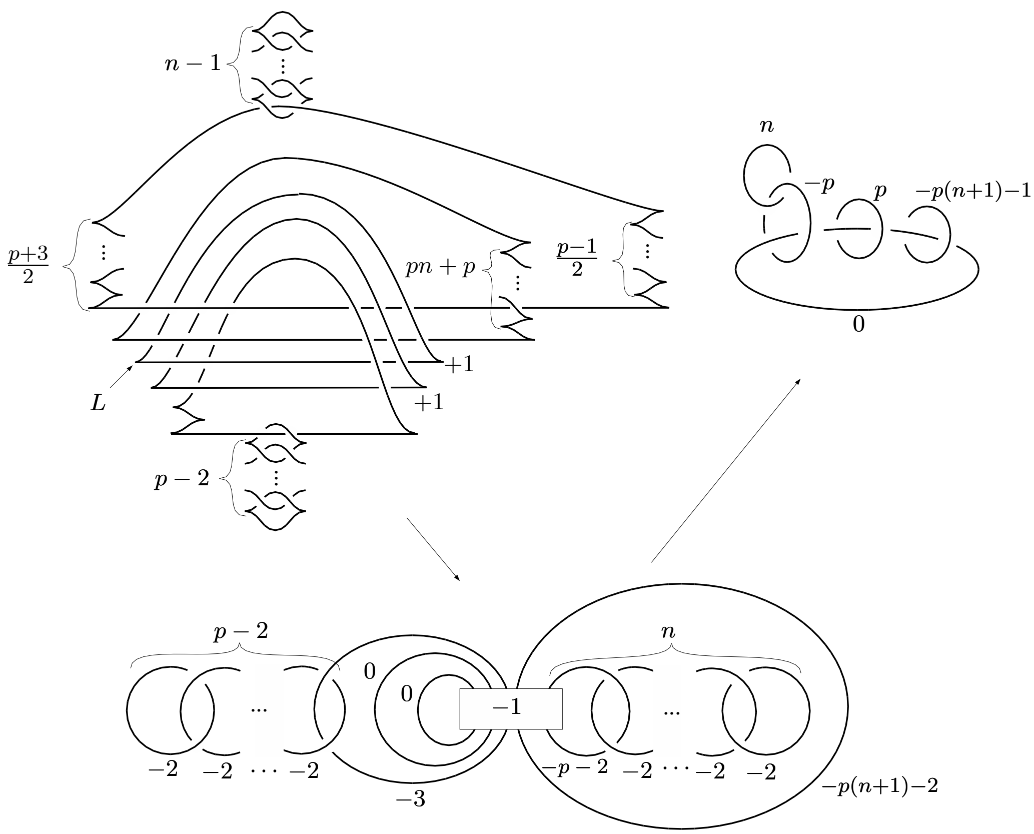


**Figure 7.** The contact structure $\zeta_{p,n}$ on $E_{p,n}$.

partial information about the map $F_{-X}$. In particular, we show that $c(S_{p,n}, \xi_{p,n})$ is not in $\ker F_{-X}$. Recall that we have assumed that $p > 1$ is odd. The cobordism $-X$ induced by the surgery on the knot $L$ of Figure 7 (after reversing its orientation) fits into the triangle given by Figure 8.

In the remaining figures of the paper we adopt the convention of denoting the 3–manifold under examination by solid framed links, while dashed curves denote the 2–handles of the cobordism built on the given 3–manifold. We shall use the corresponding exact triangle involving the Ozsváth–Szabó homology groups to study the map

$$F := F_{-X} \colon \widehat{HF}(-S_{p,n}) \to \widehat{HF}(-E_{p,n}).$$

The strategy to show that the contact invariant

$$c(E_{p,n}, \zeta_{p,n}) = F_{-X}(c(S_{p,n}, \xi_{p,n}))$$

is nonzero will be the following. Let $G_V$ be the map induced by the cobordism $V$. First we show that there exists an element of $\widehat{HF}(-L_{p,n})$

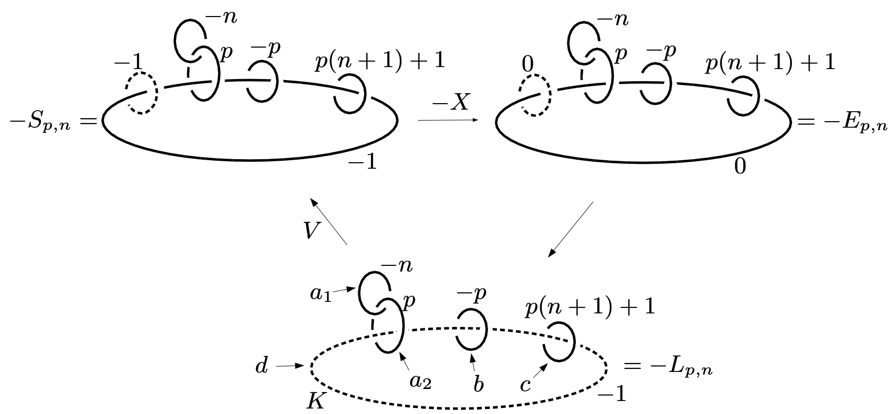


**Figure 8.** Manifolds and cobordisms in the main surgery triangle.

corresponding to a spin structure on $-L_{p,n}$ with the property that its $G_V$–image is equal to $a + \overline{a}$ for some $a \in \widehat{HF}(-S_{p,n})$. (Recall that $\overline{a}$ denotes the image of $a \in \widehat{HF}(-S_{p,n})$ under the $\mathcal{J}$–action induced by conjugation on spin$^c$ structures.) Next we consider the decomposition of this element $a$ into a sum of homogeneous terms, and we find a homogeneous component $a_1 \in \widehat{HF}(-S_{p,n}, \mathbf{t})$ which maps to a nonzero element under $F_{-X}$. In the final step of the proof we determine the spin$^c$ structure $\mathbf{t}$ corresponding to the above element $a_1$ and show that it is equal to the spin$^c$ structure induced by the contact structure $\xi_{p,n}$. Since $S_{p,n}$ was proved to be an $L$–space, the nonzero elements $a_1$ and $c(S_{p,n}, \xi_{p,n})$ inducing the same spin$^c$ structure must be equal. In particular, $F_{-X}(c(S_{p,n}, \xi_{p,n})) \neq 0$, concluding the proof. In identifying the spin$^c$ structure of the element $a_1$ we appeal to a computation which determines the degree difference between two spin structures on $-L_{p,n}$ and $-E_{p,n}$; this computation relies on the study of a related exact triangle and is given in a separate subsection. Notice that all the 3–manifolds in the triangle of Figure 8 are $L$–spaces: this property was verified for $E_{p,n}$ and $S_{p,n}$ in Propositions 4.1 and 5.1, while $L_{p,n}$ is the connected sum of three lens spaces, hence the $L$–space property trivially follows. (Recall that $\widehat{HF}(Y)$ is isomorphic to $\widehat{HF}(-Y)$, hence $Y$ is an $L$–space if and only if $-Y$ is an $L$–space.) To set up notation, consider the surgery exact triangle defined by the cobordisms of Figure 8:

$$\begin{tikzcd} \widehat{HF}(-S_{p,n}) \arrow[rr, "F = F_{-X}"] & & \widehat{HF}(-E_{p,n}) \arrow[dl, "H"] \\ & \widehat{HF}(-L_{p,n}) \arrow[ul, "G_V"] & \end{tikzcd} \tag{6.1}$$

Using the surgery descriptions it follows that

$$h_E := |H_1(E_{p,n};\mathbb{Z})| = p^2n - pn - 1, \quad \text{and} \tag{6.2}$$

$$h_L := |H_1(L_{p,n})| = p(pn+1)(p(n+1)+1). \tag{6.3}$$

**Proposition 6.1.** *The map $H$ is equal to* 0, *therefore $F$ is surjective and $G_V$ is injective.*

*Proof.* Since the three 3–manifolds are all $L$–spaces, their Ozsváth–Szabó homology groups can be determined from their first homologies. Now a simple computation using Equations (5.1), (6.2) and (6.3) shows that $h_E + h_L = h_S$, hence the statement of the lemma follows from the exactness of the triangle and elementary algebra (cf. also the concluding remark of [**15**, Section 2]). q.e.d.

**Lemma 6.2.** *The manifolds $S_{p,n}$ and $E_{p,n}$ admit a unique spin structure, while $L_{p,n}$ supports exactly two spin structures.*

*Proof.* Recall that any orientable 3–manifold $Y$ admits a spin structure, and the number of inequivalent spin structures is $|H^1(Y;\mathbb{Z}/2\mathbb{Z})|$. Using Equations (5.1), (6.2) and (6.3) it is easy to check that $S_{p,n}$ and $E_{p,n}$ have first homology groups of odd order, while for $L_{p,n}$ (as the connected sum of the three lens spaces of Figure 8) we have

$$H^1(L_{p,n};\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

q.e.d.

**Lemma 6.3.** *Let $V$ and $W$ be the cobordisms defined, respectively, in Figure* 8 *and Figure* 10. *Then, each spin structure on $-L_{p,n}$ extends as a spin structure to one of the cobordisms $V$ and $W$, but not to the other.*

*Proof.* Recall that we are assuming that $p$ is odd. In the proof we will distinguish two cases according to the parity of $n$. We would like to present $-L_{p,n}$ as the boundary of two spin 4–manifolds. Consider the bottom pictures of Figures 8 and 10. Suppose first that $n$ is even. By anti-blowups we can transform the $(-n)$–framed unknot linking the $p$–framed unknot into a chain of $(+2)$'s. During this operation we change the framing $p$ into $p+1$. Do the same operation with the $(-p)$–framed circle. Notice that after the above blow ups and blow downs the parity of the framing of the knot $K$ shown by the figures has changed. Since $n$ is even, $p(n+1)+1$ is also even. Therefore the diagram defines a simply connected spin 4–manifold with a unique spin structure, and we define $\mathbf{t}_V \in \mathrm{Spin}(L_{p,n})$ as the restriction of this unique spin structure to the boundary. Since the framing of $K$ when defining $V$ is even, $\mathbf{t}_V$ extends to $V$ as a spin structure but does not extend to $W$ (as a spin structure), since it would give a spin 4–manifold with a homology class of odd square, hence with nontrivial second Stiefel–Whitney class.

To find the other spin 4–manifold, we turn the $(p(n+1)+1)$–framed circle into a chain of $(-2)$'s by blowing up and down. This operation changes the parity of the framing of $K$ again. We define $\mathbf{t}_W$ as the restriction of the unique spin structure of the resulting simply connected spin 4–manifold. Since the parity of the framing of $K$ is now different than in the previous case, the spin structure $\mathbf{t}_W$ extends to the cobordism $W$ as a spin structure but does not extend to $V$ as a spin structure. Clearly $\mathbf{t}_V \neq \mathbf{t}_W$, and when $n$ is even we are done.

Finally we address the case of odd $n$. In this case both $-p$ and $p(n+1)+1$ are odd, so first we turn these surgeries into chains of $(+2)$ (and $(-2)$, resp.) surgeries. Each one of these transformations changes the framing of the knot $K$ by $+1$ (and $-1$ resp.), so the net change of the framing of $K$ is zero. Now we have a choice for the remaining two odd framed surgery curves defining $-L_{p,n}$. If we turn the $(-n)$–framed unknot into a chain of $(+2)$'s, we change the framing $p$ into $p+1$, but we do not change the framing of $K$. Hence the resulting 4–manifold admits a spin structure $\mathbf{s}_W$ which extends to $W$ as a spin structure, but not to $V$. We denote the restriction of $\mathbf{s}_W$ to the boundary $-L_{p,n}$ by $\mathbf{t}_W$. On the other hand, the corresponding operation on the $p$–framed circle changes the framing of the $(-n)$–framed circle to $(-n-1)$ and also changes the parity of the framing of $K$. Therefore the spin structure of the resulting simply connected spin 4–manifold will extend to $V$ as a spin structure but not to $W$. The restriction of this spin structure to the boundary $-L_{p,n}$ will be called $\mathbf{t}_V$. Clearly $\mathbf{t}_W \neq \mathbf{t}_V$, and the proof is finished. q.e.d.

**Notation.** We denote the unique spin structures on $-S_{p,n}$ and $-E_{p,n}$, respectively, by $\mathbf{t}_S$ and $\mathbf{t}_E$. As in the proof of Lemma 6.3, we denote by $\mathbf{t}_V$ the spin structure on $-L_{p,n}$ which extends as a spin structure to $V$ but not to $W$ and by $\mathbf{t}_W$ the spin structure which extends (as a spin structure) to $W$ but not to $V$.

**Computations.** Now we return to the analysis of Triangle (6.1). Recall that when $Y$ is a rational homology sphere which is an $L$–space, we have identified the nontrivial element in each group $\widehat{HF}(Y,\mathbf{t})$ with $\mathbf{t} \in \mathrm{Spin}^c(Y)$. If $H_1(Y;\mathbb{Z})$ is of odd rank, then $Y$ admits a unique spin structure, which will be denoted by $\mathbf{t}_Y$. Using the conjugate action encountered in Section 3 (cf. Theorem 3.2), and denoting $\mathcal{J}(\mathbf{t})$ by $\overline{\mathbf{t}}$, in this case the vector space $\widehat{HF}(Y)$ has a basis of the form

$$\{\mathbf{t}_1, \overline{\mathbf{t}_1}, \mathbf{t}_2, \overline{\mathbf{t}_2}, \ldots, \mathbf{t}_k, \overline{\mathbf{t}_k}, \mathbf{t}_Y\}. \tag{6.4}$$

Let

$$C := \langle \mathbf{t}_1, \ldots, \mathbf{t}_k \rangle \subset \widehat{HF}(Y).$$

Then, we have

$$\widehat{HF}(Y) = \langle \mathbf{t}_Y \rangle \oplus C \oplus \overline{C}. \tag{6.5}$$

Notice that the subspace $C \subset \widehat{HF}(Y)$ depends on a choice of basis as in (6.4), and therefore the above splitting is not canonical. In analogy to Equation (6.5), there are direct sum decompositions

$$\begin{aligned} \widehat{HF}(-S_{p,n}) &= \langle \mathbf{t}_S \rangle \oplus A \oplus \overline{A}, \\ \widehat{HF}(-L_{p,n}) &= \langle \mathbf{t}_V \rangle \oplus \langle \mathbf{t}_W \rangle \oplus C \oplus \overline{C} \\ \widehat{HF}(-E_{p,n}) &= \langle \mathbf{t}_E \rangle \oplus T \oplus \overline{T}. \end{aligned} \tag{6.6}$$

Since by its definition $\mathbf{t}_W$ does not extend as a spin structure to $V$, Lemma 3.3 implies that

$$G_V(\mathbf{t}_W) \in A \oplus \overline{A}.$$

Since $\mathbf{t}_W$ is fixed under conjugation, so is $G_V(\mathbf{t}_W)$; therefore there is an element $a \in A$ such that $G_V(\mathbf{t}_W) = a + \overline{a}$. Notice that $F(a) = F(\overline{a}) = \overline{F(a)}$, because

$$F(a) + F(\overline{a}) = F(a + \overline{a}) = F(G_V(\mathbf{t}_W)) = 0,$$

and we work with $\mathbb{Z}/2\mathbb{Z}$–coefficients.

**Lemma 6.4.** *We have $F(a) \neq 0$.*

*Proof.* If $F(a) = 0$ then, by exactness, we have $a = G_V(c)$ for some element $c \in \widehat{HF}(-L_{p,n})$. Therefore

$$G_V(\mathbf{t}_W + c + \overline{c}) = 0.$$

Since $c + \overline{c} \in C \oplus \overline{C}$, the injectivity of $G_V$ would imply $\mathbf{t}_W \in C \oplus \overline{C}$, which is impossible by (6.6). q.e.d.

**Lemma 6.5.** *Suppose that $F(a) = \epsilon \mathbf{t}_E + t + \overline{t}$ for some $t \in \widehat{HF}(-E_{p,n})$. Then, $\epsilon \neq 0$.*

*Proof.* By contradiction, suppose that $\epsilon = 0$. By the surjectivity of $F$, there is $b \in \widehat{HF}(-S_{p,n})$ with $F(b) = t$, implying also $F(\overline{b}) = \overline{t}$. Now consider $x = a + b + \overline{b}$. Then, $F(x) = 0$, and so $F(\overline{x}) = 0$. By exactness this means that there is $u \in \widehat{HF}(-L_{p,n})$ satisfying $G_V(u) = x$, and so $G_V(\overline{u}) = \overline{x}$. This implies that $G_V(u + \overline{u} + \mathbf{t}_W) = 0$. By the the injectivity of $G_V$, this would imply

$$\mathbf{t}_W = u + \overline{u} \in C \oplus \overline{C},$$

which is impossible by (6.6). q.e.d.

In order to apply the degree–shift formula for the cobordisms $X$ and $V$, we need some understanding of their algebraic topology.

**Lemma 6.6.** *We have*

$$H_2(V; \mathbb{Z}) \cong H_2(-X; \mathbb{Z}) \cong \mathbb{Z}$$

*and*

$$\sigma(V) = \sigma(-X) = -1,$$

*where $\sigma$ denotes the signature.*

*Proof.* The cobordism $V$ is obtained by attaching a 2–handle to the rational homology sphere $-L_{p,n}$. Therefore, $H_2(V, -L_{p,n}; \mathbb{Z}) \cong \mathbb{Z}$, and the exactness of the sequence

$$0 \longrightarrow H_2(V;\mathbb{Z}) \longrightarrow H_2(V, -L_{p,n};\mathbb{Z}) \longrightarrow H_1(-L_{p,n};\mathbb{Z})$$

implies $H_2(V;\mathbb{Z}) \cong \mathbb{Z}$. A similar argument shows $H_2(-X;\mathbb{Z}) \cong \mathbb{Z}$.

It is easy to deduce from Figure 8 that

$$V \cup -X \cong Q \# \overline{\mathbb{CP}^2},$$

where the cobordism $Q\#\overline{\mathbb{CP}^2}$ is given by Figure 9, obtained by applying two Rolfsen twists to the bottom picture of Figure 8.

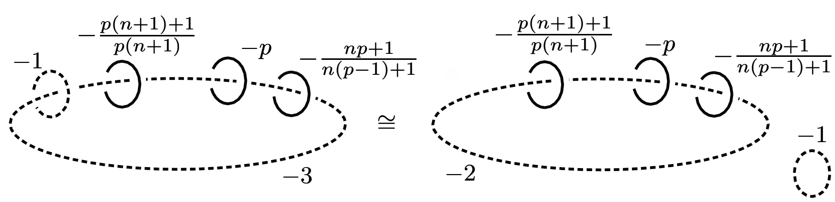


**Figure 9.** The cobordism $Q\#\overline{\mathbb{CP}^2}$.

Similarly to the proof of Lemma 6.3, we can replace the two unknots with nonintegral surgery coefficients by two chains of unknots with integral coefficients, with each coefficient less than or equal to $-2$. The resulting picture expresses $Q$ as a 4–dimensional 2–handle attached to the boundary of a 4–dimensional plumbing $P$ with $\partial P = -L_{p,n}$. Moreover, the union $P \cup Q$ is still a plumbing and we claim that it is negative definite. In fact, according to [**18**, Theorem 5.2], to see this it is enough to check that

$$-2 + \frac{n(p-1)+1}{np+1} + \frac{1}{p} + \frac{pn+1}{p(n+1)+1} < 0$$

for any $n \geq 1$ and $p \geq 2$. This implies that $Q$ is negative definite and concludes the proof. q.e.d.

Recall that $h_S, h_E$ and $h_L$ denote the cardinality of the homology groups $H_1(S_{p,n};\mathbb{Z})$, $H_1(E_{p,n};\mathbb{Z})$ and $H_1(L_{p,n};\mathbb{Z})$, respectively.

**Lemma 6.7.** *Let $g \in H_2(V;Z)$ and $g' \in H_2(-X;\mathbb{Z})$ be generators. Then,*

$$g \cdot g = -h_L h_S, \quad \textit{and} \quad g' \cdot g' = -h_S h_E.$$

*Proof.* We give the argument for $V$, the one for $-X$ being essentially the same. From Figure 8 we see that $V$ is obtained by attaching a 4–dimensional 2–handle along a circle which represents a generator of $H_1(-L_{p,n};\mathbb{Z})$. Therefore, $H_1(V;\mathbb{Z}) = 0$. Since by Lemma 6.6 we have that $H_2(V;\mathbb{Z}) \cong \mathbb{Z}$, the Theorem of Universal Coefficients gives

$$H_2(V,\partial V;\mathbb{Z}) \cong H^2(V;\mathbb{Z}) \cong \mathbb{Z}.$$

Consider the exact sequence

$$(6.7)\quad 0 \to H_2(V;\mathbb{Z}) \stackrel{i_*}{\to} H_2(V,\partial V;\mathbb{Z}) \to H_1(\partial V; Z) \cong \mathbb{Z}/h_L\mathbb{Z} \oplus \mathbb{Z}/h_S\mathbb{Z} \to 0.$$

It is easy to check that $h_L$ and $h_S$ are coprime, thus

$$\mathbb{Z}/h_L\mathbb{Z} \oplus \mathbb{Z}/h_S\mathbb{Z} \cong \mathbb{Z}/(h_L h_S)\mathbb{Z},$$

and $i_*(g)$ must be equal to $h_L h_S$ times a generator of $H_2(V,\partial V;\mathbb{Z})$. Therefore, since by Lemma 6.6 the cobordism $V$ has negative definite intersection form,

$$g \cdot g = \langle \mathrm{PD}(i_*(g)), g\rangle = -h_L h_S.$$

q.e.d.

**Lemma 6.8.** *Let $\mathbf{s} \in \mathrm{Spin}^c(V)$, and let $C \subset V$ be the cocore of the 2–handle defining $V$. If $\mathbf{s}|_{-L_{p,n}} = \mathbf{t}_W$, then*

$$\mathrm{PD}(c_1(\mathbf{s})) = k[C] \in H_2(V,\partial V;\mathbb{Z})$$

*for some odd integer $k$. Moreover,*

$$c_1(\mathbf{s}) \cdot c_1(\mathbf{s}) = -\frac{k^2 h_L}{h_S}.$$

*Proof.* According to the proof of Lemma 6.3, $\mathbf{t}_W$ is the restriction to $-L_{p,n}$ of a spin structure $\mathbf{u}$ on a spin 4–manifold $Z$ with $\partial Z = -L_{p,n}$. Moreover, $Z$ is obtained by attaching 4–dimensional 2–handles to the 4–ball $B^4$, and $V$ by attaching a last 2–handle $H$ to $\partial Z$. Recall that the framing of the attaching circle of $H$ is odd, because $\mathbf{t}_W$ does not extend over $V$ as a spin structure. Thus, if $\mathbf{s}|_{-L_{p,n}} = \mathbf{t}_W$, then $\mathbf{s}$ extends $\mathbf{u}$ to $W := Z \cup V$ as a spin$^c$ structure. Denote by $\tilde{\mathbf{s}}$ the extended spin$^c$ structure $\mathbf{u} \cup \mathbf{s}$. Thinking of $H$ as attached to $S^3 = \partial B^4$, let $F$ denote the surface obtained by capping off the core $D$ of $H$ by a Seifert surface with interior pushed in $B^4$. Since $c_1(\tilde{\mathbf{s}})$ is characteristic and $F$ has odd square, we have

$$\langle c_1(\tilde{\mathbf{s}}), [F]\rangle = k$$

for some odd integer $k$. Therefore, since $W$ is simply connected, $\mathrm{PD}(c_1(\tilde{\mathbf{s}})) = k[C]$. The first part of the statement follows because $\tilde{\mathbf{s}}$ restricts to $\mathbf{s}$ on $V$ and $C \subset V$.

Now observe that the boundary of $h_L$ parallel copies of $D$ is homologically trivial in $-L_{p,n}$. Thus, we can define $S \subset V$ to be the surface obtained by capping off $h_L D$ in $-L_{p,n}$ with a bounding surface. Moreover, since $C$ is disjoint from $-L_{p,n}$, by Exact Sequence (6.7) the relative homology class $[C]$ must be a multiple of $h_L$ times a generator $g'$ of $H_2(V, \partial V; \mathbb{Z})$. But the equality $[C] \cdot [S] = h_L$ implies at once that $[S]$ is a generator $g$ of $H_2(V; \mathbb{Z})$, and $[C]$ is $h_L g'$. Now recall that in the proof of Lemma 6.7 we showed that the image of $g$ under the map $i_*$ of Exact Sequence (6.7) is equal to $\pm h_L h_S g'$. Therefore,

$$h_S \,\mathrm{PD}(c_1(\mathbf{s})) = k h_S [C] = k h_S h_L g' = \pm k i_*(g)$$

which implies, by Lemma 6.7, that

$$c_1(\mathbf{s}) \cdot c_1(\mathbf{s}) = k^2 \frac{g \cdot g}{h_S^2} = -k^2 \frac{h_L}{h_S}.$$

q.e.d.

**Lemma 6.9.** *Let* $\mathbf{s} \in \mathrm{Spin}^c(-X)$*, and let* $D \subset -X$ *be the core of the* 2*–handle defining* $-X$*. If* $\mathbf{s}|_{-E_{p,n}} = \mathbf{t}_E$*, then*

$$\mathrm{PD}(c_1(\mathbf{s})) = l[D] \in H_2(-X, \partial(-X); \mathbb{Z})$$

*for some odd integer* $l$*. Moreover,*

$$c_1(\mathbf{s}) \cdot c_1(\mathbf{s}) = -l^2 \frac{h_E}{h_S}.$$

*Proof.* Observe that the spin structure $\mathbf{t}_E$ does not extend to $-X$ as a spin structure simply because $-X$ does not carry spin structures. This follows immediately from Lemma 6.7, since both $h_S$ and $h_E$ are odd numbers. Thus, the proof of this lemma is similar to the proof of Lemma 6.8, and we omit it. q.e.d.

We wish to find a relation between the degrees of $\mathbf{t}_W$ and $\mathbf{t}_E$. This can be done with a (quite tedious) direct computation: the gradings of generators of $\widehat{HF}(Y)$ for a lens space $Y$ are given in [**22**], and since $-L_{p,n}$ is a connected sum of three lens spaces and the degrees are additive under connected sums, the computation of the degree of $\mathbf{t}_W$ is a fairly easy exercise. The degree of an element in the Ozsváth–Szabó homology of a Seifert fibered 3–manifold can be computed using formulae from [**17**, **23**]. In particular, in [**17**] there is an explicit formula in terms of a vector with some special properties in the cohomology of a certain negative definite plumbing with boundary $Y$. This direct computation, however, is quite delicate, so we prefer to choose a theoretically more involved, less computational way of relating the degrees of $\mathbf{t}_W$ and $\mathbf{t}_E$. In particular, we will get the desired conclusion by studying a related triangle of manifolds.

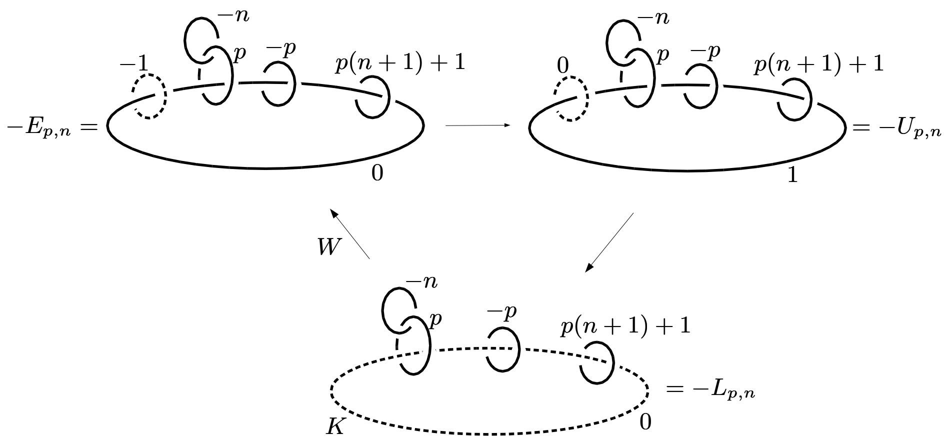


**Figure 10.** Manifolds and cobordisms in a related surgery triangle.

**Digression: study of a related triangle.** Let us consider the triangle of 3–manifolds and cobordisms given by Figure 10.

**Proposition 6.10.** *The* 3*–manifold* $-U_{p,n}$ *is an* $L$*–space.*

*Proof.* Kirby calculus, as in the proof of Proposition 5.1, shows that $U_{p,n}$ is diffeomorphic to $S^3_r(T_{p,pn+1})$, with

$$r = p^2n + p + 1 + \frac{1}{p(n+1)}.$$

Since the above $r$ is greater than $2g_s(T_{p,pn+1}) - 1 = p^2n - pn - 1$, by [**15**, Proposition 4.1] $U_{p,n}$ is an $L$–space. q.e.d.

The exact triangle on Ozsváth–Szabó homologies induced by the surgery triangle of Figure 10 has the following shape:

$$\begin{tikzcd}
\widehat{HF}(-E_{p,n}) \arrow[rr, "F'"] & & \widehat{HF}(-U_{p,n}) \arrow[dl, "H'"] \\
& \widehat{HF}(-L_{p,n}) \arrow[ul, "G_W"] &
\end{tikzcd}$$

Simple computation shows that

$$h_U := |H_1(U_{p,n}; \mathbb{Z})| = p^3n(n+1) + p(p+1)(n+1) + 1.$$

Since $h_U$ is odd, the 3–manifold $-U_{p,n}$ supports a unique spin structure, which will be denoted by $\mathbf{t}_U$. In analogy to Equation (6.5), there is a direct sum decomposition

$$\widehat{HF}(-U_{p,n}) = \langle \mathbf{t}_U \rangle \oplus S \oplus \overline{S}. \tag{6.8}$$

**Corollary 6.11.** *The map $F'$ in the above triangle is* 0*. Therefore $H'$ is injective and $G_W$ is surjective.*

*Proof.* Since all the manifolds involved are $L$–spaces, the argument boils down to the simple observation that $h_L = h_E + h_U$, cf. also the proof of Proposition 6.1. q.e.d.

**Lemma 6.12.** *The $\mathbf{t}_E$–component of the element*

$$G_W(\mathbf{t}_W) \in \widehat{HF}(-E_{p,n})$$

*is nonzero.*

*Proof.* Notice first that, since $\mathbf{t}_V$ does not extend to $W$ as a spin structure, by Lemma 3.3 the $\mathbf{t}_E$–component of $G_W(\mathbf{t}_V)$ is zero. Arguing by contradiction, suppose now that the $\mathbf{t}_E$–component of $G_W(\mathbf{t}_W)$ is also zero. Suppose that $G_W(\mathbf{t}_V) = x_V + \overline{x_V}$ and $G_W(\mathbf{t}_W) = x_W + \overline{x_W}$ with $x_V, x_W \in T$.

Since $G_W$ is onto, there exist elements $l_V, l_W \in \widehat{HF}(-L_{p,n})$ such that

$$G_W(l_V) = x_V \quad \text{and} \quad G_W(l_W) = x_W.$$

Therefore,

$$G_W(\mathbf{t}_V + l_V + \overline{l_V}) = 0 \quad \text{and} \quad G_W(\mathbf{t}_W + l_W + \overline{l_W}) = 0.$$

By exactness, this implies the existence of $u_V, u_W \in \widehat{HF}(-U_{p,n})$ such that

$$H'(u_V) = \mathbf{t}_V + l_V + \overline{l_V} \quad \text{and} \quad H'(u_W) = \mathbf{t}_W + l_W + \overline{l_W}.$$

Since $H'$ is injective, we have that $u_V$ and $u_W$ are both fixed under conjugation. Then, one of $u_V$, $u_W$ or $u_V + u_W$ belongs to $S \oplus \overline{S}$ and is therefore of the form $s + \overline{s}$ for some $s \in S$. But for any $s \in S$ we have $H'(s + \overline{s}) \in C \oplus \overline{C}$, so one of $t_V + l_V + \overline{l_V}$, $t_W + l_W + \overline{l_W}$ or their sum belongs to $C \oplus \overline{C}$, which is clearly impossible. This contradiction proves the lemma. q.e.d.

The following is the most important result of this subsection:

**Proposition 6.13.** *We have*

$$\deg(\mathbf{t}_E) = \deg(\mathbf{t}_W) + \frac{1}{4}.$$

*Proof.* By Lemma 6.12 the $\mathbf{t}_E$–component of the element $G_W(\mathbf{t}_W)$ is nontrivial, therefore there are spin$^c$ structures $\mathbf{s}_i$ on $W$ such that $G_{W,\mathbf{s}_i}(\mathbf{t}_W) = \mathbf{t}_E$. By the conjugation invariance we have $G_{W,\mathbf{s}_i}(\mathbf{t}_W) = G_{W,\overline{\mathbf{s}_i}}(\mathbf{t}_W)$. Since we use mod 2 coefficients, this shows that there are an odd number of $\mathbf{s}_i$'s with the above property, and therefore there exists a spin structure $\mathbf{s}$ on $W$ with the property that $G_{W,\mathbf{s}}(\mathbf{t}_W) = \mathbf{t}_E$. An argument similar to the one given in Lemma 6.6 shows that $W$ is negative definite. Since for a spin structure $c_1(\mathbf{s}) = 0$, the degree shift formula implies the result. q.e.d.

*Proof of Theorem* 1.3. Recall that there is an element $a \in \widehat{HF}(-S_{p,n})$ satisfying the equation $G_V(\mathbf{t}_W) = a + \overline{a}$. Express $a$ as a sum of homogeneous elements. Since by Lemma 6.5 the $\mathbf{t}_E$–component of $F(a)$ is nonzero, $a$ has a homogeneous component $a_1$ with the same property. By the degree–shift formula, Lemmas 6.8 and 6.9 immediately imply (with $|k| = |l| = 1$) that

$$\deg(\mathbf{t}_E) - \frac{1}{4}(-\frac{h_E}{h_S} + 1) \leq \deg(a_1) \leq \deg(\mathbf{t}_W) + \frac{1}{4}(-\frac{h_L}{h_S} + 1). \tag{6.9}$$

But since $h_S = h_L + h_E$, by Proposition 6.13 the inequalities of Equation (6.9) must in fact be equalities. This shows that the spin$^c$ structure corresponding to $a_1$ is the restriction of a spin$^c$ structure $\mathbf{s}$ as in Lemma 6.8 with $k = \pm 1$. Consequently, $a_1 \in \widehat{HF}(-S_{p,n}, \mathbf{t})$ with $c_1(\mathbf{t}) = \pm PD(\mu_d)$ in the basis of homologies given by Figure 6. According to Lemma 5.2, either $a_1$ or $\overline{a_1}$ belongs to the same summand $\widehat{HF}(-S_{p,n}, \mathbf{t})$ as $c(S_{p,n}, \xi_{p,n})$. Therefore, since $-S_{p,n}$ is an $L$–space, $c(S_{p,n}, \xi_{p,n})$ is equal to either $a_1$ or $\overline{a_1}$. But $F(\overline{a_1}) = \overline{F(a_1)}$. Therefore,

$$c(E_{p,n}, \zeta_{p,n}) = F_{-X}(c(S_{p,n}, \xi_{p,n}))$$

has nonzero $\mathbf{t}_E$–component, and therefore it coincides with $\mathbf{t}_E$. This fact implies that $\zeta_{p,n}$ is a tight, positive contact structure on $E_{p,n}$, concluding the proof. q.e.d.

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Dipartimento di Matematica
Università di Pisa
Largo Bruno Pontecorvo, 5
I-56127 Pisa, Italy
*E-mail address*: lisca@dm.unipi.it

Rényi Institute of Mathematics
Hungarian Academy of Sciences
H-1053 Budapest
Reáltanoda utca 13–15, Hungary
and
Institute for Advanced Study
Princeton, New Jersey
*E-mail address*: stipsicz@math-inst.hu