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QUADRATIC FUNCTIONS IN GEOMETRY, TOPOLOGY, AND M-THEORY

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Contents

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1. Introduction

Quadratic refinements of the intersection pairing on a Riemann surface appear to have two mathematical origins: one in complex function theory dating back to Riemann in the 1870's, and one in topology stemming from the work of Pontryagin in the 1930's.

Pontryagin's ideas were taken up and generalized by Kervaire [**38**] in the late 1950's, who, among other things, used them to produce an example of a topological manifold of dimension 10 which does not admit a smooth structure. Analogous invariants for manifolds of other dimensions were investigated by many topologists, most notably Kervaire, Browder [**10**], Brown-Peterson [**16**, **15**] and Brown [**14**], and play an important role in the surgery classification of manifolds and in the homotopy groups of spheres.

Riemann's quadratic function occurred in his theory of ϑ-functions and while its topological aspects were clarified in 1971 by Atiyah [**8**, Proposition 4.1] and Mumford [**52**], until recently quadratic functions in higher dimensions have remained in the province of topology. Our purpose in this paper is to bring to quadratic functions in higher dimensions more of the geometry present in Riemann's work. There are two issues involved, one having to do with constructing quadratic functions, and the other having to do with the mathematical language in which to describe them. As we explain below, our motivation for doing this came from theoretical physics, and our theory owes much to the papers [**65**, **66**] of Witten.

In the case of Riemann surfaces, links between the topological approach of Pontryagin and the analytic approach of Riemann can be made using index theory. In [**8**] Atiyah interprets Riemann's quadratic function in terms of the mod 2 index of the Dirac operator. It is also possible to deduce Riemann's results from the theory of the determinant of the $\bar{\partial}$-operator. Though this point of view seems relatively modern, it is arguably the closest to Riemann's original analysis. Riemann's quadratic function occurred in the functional equation for his ϑ-function. The ϑ-function is (up to scale) the unique holomorphic section of the determinant of the $\bar{\partial}$-operator, and it's functional equation can studied from the symmetries of the determinant line. In §2.2 we will give a proof of Riemann's results along these lines. Our proof also works in the algebraic setting. While apparently new, it is related to that of Mumford, and gives another approach to his results [**52**].

While it doesn't seem possible to construct quadratic functions in higher dimensions using index theory alone, there is a lot to be learned from the example of determinant line bundles on Riemann surfaces. Rather than trying to use the index of an operator, our approach will be to generalize the index *formula*, i.e., the topological index. The index formula relates the determinant of the $\bar{\partial}$ operator in dimension 2 to the index of the Spin^c Dirac operator in dimension 4, and ultimately, quadratic functions in dimension 2 to the signature of 4-manifolds. Now on a 4-manifold M the relation between Spin^c-structures and quadratic refinements of the intersection pairing has a simple algebraic interpretation. The first Chern class λ of the Spin^c-structure is a characteristic element of the bilinear form on $H^2(M)$:

$$\int_M x \cup x \equiv \int_M x \cup \lambda \mod 2.$$

The expression

$$q(x) = \frac{1}{2}\int_M \left(x^2 - x\lambda\right) \tag{1.1}$$

is then a quadratic refinement of the intersection pairing. It is useful to compare this with the formula for the index of the Spin$^{\mathrm{c}}$ Dirac operator:

$$\kappa(\lambda) = \frac{1}{8}\int_M (\lambda^2 - L(M)), \tag{1.2}$$

where $L(M)$ is the characteristic class which gives the signature when integrated over M. Formula (1.1) gives the change of $\kappa(\lambda)$ resulting from the change of Spin$^{\mathrm{c}}$-structure $\lambda \mapsto \lambda - 2x$.

The fact that (1.2) is an integer also has an algebraic explanation: the square of the norm of a characteristic element of a non-degenerate symmetric bilinear form over $\mathbb{Z}$ is always congruent to the signature mod 8. This points the way to a generalization in higher dimensions. For manifolds of dimension $4k$, the characteristic elements for the intersection pairing in the middle dimension are the integer lifts λ of the Wu-class ν_{2k}. The expression (1.2) is then an integer, and its variation under to $\lambda \mapsto \lambda - 2x$ gives a quadratic refinement of the intersection pairing. This can almost be described in terms of index theory. A Spin$^{\mathrm{c}}$-structure on a manifold of dimension $4k$ determines an integral Wu-structure, and the integer $\kappa(\lambda)$ turns out to be the index of an operator. But we haven't found a good analytic way to understand the variation $\lambda \mapsto \lambda - 2x$. In dimension 4 this variation can be implemented by coupling the Spin$^{\mathrm{c}}$ Dirac operator to a $U(1)$-bundle with first Chern class $-x$. In higher dimensions one would need to couple the operator to something manufactured out of a cohomology class of degree $2k$.

In this paper we refine the expression (1.2) to a cobordism invariant for families of manifolds. The cobordism theory is the one built from families E/S of manifolds equipped with an integer cocycle $\lambda \in Z^{2k}(E;\mathbb{Z})$ whose mod 2 reduction represents the Wu class ν_{2k} of the relative normal bundle. If E/S has relative dimension $(4k-i)$, then our topological interpretation of (1.2) will produce an element

$$\kappa(\lambda) \in \tilde{I}^i(S)$$

of a certain generalized cohomology group. The cohomology theory $\tilde{I}$ is a generalized cohomology theory known as the Anderson dual of the sphere. It is the dualizing object in the category of cohomology theories (spectra). When $i = 2$, the group $\tilde{I}^2(S)$ classifies graded line bundles. By analogy with the case of Riemann surfaces, we think of $\kappa(\lambda)$ the determinant line of the Wu-structure λ on E/S. This "generalized determinant" $\kappa(\lambda)$ can be coupled to cocycles of degree $2k$, and can be used to construct quadratic functions.

The relationship between Wu-structures, quadratic functions, and the Kervaire invariant goes back to the early work on the Kervaire invariant in [**12**, **15**, **16**, **5**], and most notably to the paper of Browder [**10**]. It was further clarified by Brown [**14**]. The relationship between the signature in dimension $4k$ and the Kervaire invariant in dimension $(4k-$

2) was discovered by Milgram [**47**] and Morgan–Sullivan [**51**]. Our construction of $\kappa(\lambda)$ is derived from [**14**, **47**, **51**], though our situation is somewhat different. In [**14**, **47**, **51**] the emphasis is on surgery problems, and the class λ is necessarily 0. In this work it is the variation of λ that is important. Our main technical innovation involves a systematic exploitation of duality.

Even though $\kappa(\lambda)$ generalizes the determinant line, as described so far our cobordism approach produces objects which are essentially topological. To enrich these objects with more geometric content we introduce the language of *differential functions* and *differential cohomology theories*. Let X be a topological space, equipped with a real cocycle $\iota \in Z^n(X;\mathbb{R})$, and M a smooth manifold. A *differential function* from M to (X,ι) is a triple

$$(c, h, \omega)$$

with $c : M \to X$ a continuous function, $\omega \in \Omega^n(M)$ a closed n-form, and $h \in C^{n-1}(M;\mathbb{R})$ a cochain satisfying

$$\delta h = \omega - c^*\iota.$$

Using differential functions, we then revisit the basic constructions of algebraic topology and introduce differential cobordism groups and other differential cohomology theories. It works out, for instance, that the differential version of the group $\tilde{I}^1(S)$ is the group of smooth maps from S to $U(1)$. Using the differential version of $\tilde{I}^2(S)$ one can recover the category of $U(1)$-bundles with connection. In this way, by using *differential* rather than continuous functions, our topological construction refines to something richer in its geometric aspects.

The differential version of $H^k(M;\mathbb{Z})$ is the group of Cheeger-Simons differential characters $\check{H}^{k-1}(M)$ [**20**], and in some sense our theory of differential functions is a non-linear generalization. We began this project intending to work entirely with differential characters. But they turned out not robust enough for our purposes.

The bulk of this paper is devoted to working out the theory of differential function spaces. To make them into *spaces* at all we need to consider differential functions on the products $M \times \Delta^n$ of M with a varying simplex. This forces us at the outset to work with manifolds with corners (see Appendix C). Throughout this paper, the term *manifold* will mean *manifold with corners*. The term *manifold with boundary* will have its usual meaning, as will the term *closed manifold*.

Using the language of differential function spaces and differential cohomology theories the construction of quadratic functions in higher dimension can be made to arise very much in the way it did for Riemann in dimension 2. A differential integral Wu-structure is the analogue of the canonical bundle λ, a choice of $\lambda/2$ is the analogue of a ϑ-characteristic (or Spin-structure), and $\kappa(\lambda-2x)$ the analogue of the determinant of the

$\bar{\partial}$-operator coupled to a holomorphic line bundle. For a more detailed discussion, see §2.

Our interest in this project originated in a discussion with Witten. It turns out that quadratic functions in dimension 6 appear in the ring of "topological modular forms" [**36**] as a topologically defined mod 2 invariant. Modulo torsion, the invariants coming from topological modular forms, are accounted for by index theory on loop space. This suggested that it might be possible to generalize Atiyah's interpretation of Riemann's quadratic function to dimension 6 using some kind of mod 2 index on loop space. We asked Witten about this and he pointed out that he had used quadratic refinements of the intersection pairing on certain 6-manifolds in describing the fivebrane partition function in M-theory [**65**, **66**]. The fivebrane partition function is computed as the unique (up to scale) holomorphic section of a certain holomorphic line bundle, and the quadratic function is used to construct this line bundle. We then realized that an analogue of determinant lines could be used instead of mod 2 indices to generalize Riemann's quadratic functions to higher dimensions.

The organization of this paper is as follows. Section 2 is devoted to background material and the statement of our main result. More specifically, §2.1 recalls the results of Riemann and Pontryagin. In §2.2 we give a proof of Riemann's results using determinants. Sections 2.3 and 2.5 introduce differential cocycles, which play a role in higher dimensions analogous to the one played by line bundles in dimension 2. We state our main result (Theorem 2.17) in §2.6, and in §2.7 relate it to Witten's construction. In §3 we review Cheeger-Simons cohomology. In §4 we lay out the foundations of differential function complexes and differential cohomology theories. Section 5 contains the proof of Theorem 2.17, and Appendix E contains a construction of a stable exponential characteristic class for Spin bundles, taking values in cohomology with integer coefficients, whose mod 2 reduction is the total Wu-class.

We had originally included in this paper an expository discussion, primarily for physicists, describing the role of quadratic and differential functions in the construction of certain partition functions. In the end we felt that the subject matter deserved a separate treatment, which we hope to complete soon.

Our theory of differential function spaces provides a variation of algebraic topology more suited to the needs of mathematical physics. It has already proved useful in anomaly cancellation [**30**], and it appears to be a natural language for describing fields and their action functionals. We have many examples in mind, and hope to develop this point of view in a later paper.

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2. Determinants, differential cocycles and statement of results

2.1. Background. Let X be a Riemann surface of genus g. A *theta characteristic* of X is a square root of the line bundle ω of holomorphic 1-forms. There are 2^g theta characteristics of X, and they naturally form a principal homogeneous space for the group of square roots of the trivial line bundle. Riemann associated to each theta characteristic x a parity $q(x)$, defined to be the dimension mod 2 of the space of its holomorphic sections. He showed that $q(x)$ is invariant under holomorphic deformations and has remarkable algebraic properties—namely that

$$q(x \otimes a) - q(x) \tag{2.1}$$

is a quadratic function of a, and that the associated bilinear form

$$e(a, b) = q(x \otimes a \otimes b) - q(x \otimes a) - q(x \otimes b) + q(x)$$

is non-degenerate and independent of x. In these equations, a and b are square roots of the trivial bundle. They are classified by elements $H^1(X; \mathbb{Z}/2)$, and the form $e(a, b)$ corresponds to the cup product. One can check that the expression (2.1) depends only on the cohomology class underlying a, and so a choice of theta characteristic thus gives rise, via Riemann's parity, to a quadratic refinement of the intersection pairing. Riemann derived the algebraic properties of the function q using his ϑ-function and the Riemann singularities theorem. In the next section we will deduce these results from properties of determinant line bundles, in a way that can be generalized to higher dimensions.

Quadratic functions in topology arise from a famous error, an unwitting testimony to the depth of the invariants derived from quadratic functions. In the 1930's, Pontryagin introduced a geometric technique for investigating the homotopy groups of spheres [**56**, **55**]. His method was to study a map between spheres in terms of the geometry of the inverse image of a small disk centered on a regular value. It led eventually to a remarkable relationship between homotopy theory and differential topology, and one can find in these papers the beginnings of both bordism and surgery theories.

Pontryagin's first results concerned the homotopy groups $\pi_{n+1}S^n$ and $\pi_{n+2}S^n$, for $n \geq 2$ ([**56**, **55**]). Because of a subtle error, he was led to the conclusion[1]

$$\pi_{n+2}S^n = 0 \qquad n \geq 2.$$

[1] They are actually cyclic of order 2

The groups were later correctly determined by George Whitehead [**64**] and by Pontryagin himself [**57**] (see also [**58**] and [**37**]).

The argument went as follows. A map $f : S^{n+2} \to S^n$ is homotopic to a generic map. Choose a small open disk $D \subset S^n$ not containing any singular values, and let $a \in D$ be a point. The subspace $X = f^{-1}(a) \subset S^{n+2}$ is a Riemann surface, the neighborhood $f^{-1}D$ is diffeomorphic to the normal bundle to the embedding $X \subset S^{n+2}$, and the map f gives this normal bundle a framing. The homotopy class of the map f is determined by this data (via what is now known as the Pontryagin–Thom construction). If X has genus 0, then the map f can be shown to be null homotopic. Pontryagin sketched a geometric procedure for modifying f in such a way as to reduce the genus of X by 1. This involved choosing a simple closed curve C on X, finding a disk $D \subset S^{n+2}$ bounding C, whose interior is disjoint from X, and choosing a suitable coordinate system in a neighborhood of D. Pontryagin's procedure is the basic manipulation of framed surgery. It is not needed for the correct evaluation of the groups $\pi_{n+2}S^n$ and, except for the dimension 0 analogue, seems not to appear again until [**49**].

The choice of coordinate system is not automatic, and there is an obstruction

$$\phi : H_1(X; \mathbb{Z}/2) \to \mathbb{Z}/2$$

to its existence. As indicated, it takes its values in $\mathbb{Z}/2$ and depends only on the mod 2 homology class represented by C. As long as ϕ takes the value 0 on a non-zero homology class, the genus of X can be reduced by 1. Pontryagin's error concerned the algebraic nature of ϕ, and in [**55**] it was claimed to be linear. He later determined that ϕ is quadratic [**57**], and in fact

$$\phi(C_1 + C_2) = \phi(C_1) + \phi(C_2) + I(C_1, C_2),$$

where $I(C_1, C_2)$ is the intersection number of C_1 and C_2. Thus ϕ is a *quadratic refinement of the intersection pairing*. The Arf invariant of ϕ can be used to detect the non-trivial element of $\pi_{n+2}(S^n)$. This was the missing invariant.

Around 1970 Mumford called attention to Riemann's parity, and raised the question of finding a modern proof of its key properties. Both he [**52**] and Atiyah [**8**] provided answers. It is Atiyah's [**8**, Proposition 4.1] that relates the geometric and topological quadratic functions. Atiyah identifies the set of theta-characteristics with the set of Spin structures, and Riemann's parity with the mod 2 index of the Dirac operator. This gives immediately that Riemann's parity is a Spin-cobordism invariant, and that the association

$$(X, x) \mapsto q(x)$$

is a surjective homomorphism from the cobordism group MSpin_2 of Spin-manifolds of dimension 2 to $\mathbb{Z}/2$. Now the map from the cobordism group $\Omega_2^{\mathrm{fr}} = \pi_2 S^0$ of (stably) framed manifolds of dimension 2 to MSpin_2 is an isomorphism, and both groups are cyclic of order 2. It follows that Riemann's parity gives an *isomorphism* $\mathrm{MSpin}_2 \to \mathbb{Z}/2$, and that the restriction of this invariant to Ω_2^{fr} coincides with the invariant of Pontryagin. The key properties of q can be derived from this fact.

In [**52**], Mumford describes an algebraic proof of these results, and generalizes them to more general sheaves on non-singular algebraic curves. In [**32**] Harris extends Mumford's results to the case of singular curves.

2.2. Determinants and the Riemann parity. In this section we will indicate how the key properties of the Riemann's $q(x)$ can be deduced using determinants. Let E/S be a holomorphic family[2] of Riemann surfaces, and L a holomorphic line bundle over E. Denote by

$$\det L$$

the determinant line bundle of the $\bar\partial$-operator coupled to L. The fiber of $\det L$ over a point $s \in S$ can be identified with

$$\det H^0(L_s) \otimes \det H^1(L_s)^*.$$

If $K = K_{E/S}$ is the line bundle of relative Kahler differentials (holomorphic 1-forms along the fibers), then by Serre duality this equation can be re-written as

$$\det H^0(L_s) \otimes \det H^0(K \otimes L_s^{-1}). \tag{2.2}$$

This leads to an isomorphism

$$\det L \approx \det\left(K \otimes L^{-1}\right),$$

which, fiberwise, is given by switching the factors in (2.2). An isomorphism $L^2 \approx K$ (if one exists) then gives rise to automorphism

$$\det L \to \det L.$$

This automorphism squares to the identity, and so is given by a holomorphic map

$$q : S \to \{\pm 1\}.$$

To compute the value of q at a point $s \in S$, write

$$(\det L)_s = \Lambda^{2d}\left(H^0(L_s) \oplus (H^0(L_s)\right)$$

($d = \dim H^0(L_s)$). The sign encountered in switching the factors is

$$(-1)^d.$$

[2]In the language of algebraic geometry, E/S needs to be a smooth proper morphism of complex analytic spaces. The manifold S can have singularities. In this paper, we will not be dealing with holomorphic structures, and S will typically have corners. The notion "smooth morphism" will be replaced with notion of "neat map" (see C).

It follows that

$$q(s) = (-1)^{\dim H^0(L_s)}$$

and so we recover Riemann's parity in terms of the symmetry of $\det L$. This key point will guide us in higher dimensions. Note that our approach shows that *Riemann's* $q : S \to \{\pm 1\}$ is holomorphic, and so invariant under holomorphic deformations.

Riemann's algebraic properties are derived from the quadratic nature of det, i.e., the fact that the line bundle

$$B(L_1, L_2) = \frac{\det(1)\det(L_1 \otimes L_2)}{\det(L_1)\det(L_2)}$$

is bilinear in L_1 and L_2 with respect to tensor product. We will not give a proof of this property (see Deligne [**23**] in which the notation $\langle L_1, L_2\rangle$ is used), but note that it is suggested by the formula for the first Chern class of $\det L$

$$2\, c_1(\det L) = \int_{E/S} (x^2 - x c_1) \tag{2.3}$$

where x is the first Chern class of L, and c_1 is the first Chern class of the relative tangent bundle of E/S.

2.3. Differential cocycles. Now suppose E/S has relative dimension $2n$. In order for the quadratic term in (2.3) to contribute to the first Chern class of a line bundle over S, x must be an element of $H^{n+1}(E;\mathbb{Z})$. This motivates looking for mathematical objects classified by $H^{n+1}(E;\mathbb{Z})$ in the way complex line bundles are classified by $H^2(E;\mathbb{Z})$. In the discussion in §2.2 it was crucial to work with line bundles and the isomorphisms between them, rather than with isomorphism classes of line bundles; we need to construct a *category* whose isomorphism classes of objects are classified by $H^{n+1}(E;\mathbb{Z})$.

Definition 2.4. Let M be a manifold and $n \geq 0$ an integer. The category of *n-cocycles*, $\mathcal{H}^n(M)$ is the category whose objects are smooth n-cocycles

$$c \in Z^n(M;\mathbb{Z})$$

and in which a morphism from c_1 to c_2 is an element

$$b \in C^{n-1}(M;\mathbb{Z})/\delta\, C^{n-2}(M;\mathbb{Z})$$

such that

$$c_1 + \delta b = c_2$$

There is an important variation involving forms.

Definition 2.5. Let M be a manifold, and $n \geq 0$ an integer. The category of *differential n-cocycles*, $\check{\mathcal{H}}^n(M)$ is the category whose objects are triples

$$(c, h, \omega) \subset C^n(M;\mathbb{Z}) \times C^{n-1}(M;\mathbb{R}) \times \Omega^n(M),$$

satisfying

$$\delta c = 0 \tag{2.6}$$
$$d\omega = 0$$
$$\delta h = \omega - c.$$

A morphism from (c_1, h_1, ω_1) to (c_2, h_2, ω_2) is an equivalence class of pairs

$$(b, k) \in C^{n-1}(M; \mathbb{Z}) \times C^{n-2}(M; \mathbb{R})$$

satisfying

$$c_1 - \delta b = c_2$$
$$h_1 + \delta k + b = h_2.$$

The equivalence relation is generated by

$$(b, k) \sim (b - \delta a, k + \delta k' + a).$$

For later purposes it will be convenient to write

$$d(c, h, \omega) = (\delta c, \omega - c - \delta h, d\,\omega).$$

Note that $d^2 = 0$, and that condition (2.6) says $d(c, h, \omega) = 0$. We will refer to general triples

$$(c, h, \omega) \subset C^n(M; \mathbb{Z}) \times C^{n-1}(M; \mathbb{R}) \times \Omega^n(M),$$

as *differential cochains* (of degree n), and to those which are differential cocycles as *closed*. As will be explained in more detail in §3.2, these are the n-cochains and cocycles in the cochain complex $\check{C}(n)^*(M)$ with

$$\check{C}(n)^k(M)$$
$$= \{(c, h, \omega) \mid \omega = 0 \text{ if } k < n\} \subseteq C^k(M; \mathbb{Z}) \times C^{k-1}(M; \mathbb{R}) \times \Omega^n(M)$$

and differential given by

$$d(c, h, \omega) = (\delta c, \omega - c - \delta h, d\omega).$$

The k^{th} cohomology group of $\check{C}(n)^*(M)$ is denoted

$$\check{H}(n)^k(M),$$

and $\check{H}(n)^n(M)$, can be identified with the group of differential characters $\check{H}^{n-1}(M)$ of Cheeger–Simons [**20**].

The operations of addition of cochains and forms define abelian group structures on the categories $\check{\mathcal{H}}^n$. The set of isomorphism classes of objects in $\mathcal{H}^n(M)$ is the group $H^n(M; \mathbb{Z})$, and the automorphism group of the trivial object 0, is $H^{n-1}(M; \mathbb{Z})$. The set of isomorphism classes of objects in $\check{\mathcal{H}}^n(M)$ is the group $\check{H}(n)^n(M)$. The automorphism of the trivial object $(0, 0, 0)$ is naturally isomorphic to the group $H^{n-2}(M; \mathbb{R}/\mathbb{Z})$. There is a natural functor $\check{\mathcal{H}}^n(M) \to \mathcal{H}^n(M)$ which is compatible with the abelian group structures. On isomorphism classes of objects it corresponds to the natural map $\check{H}(n)^n(M) \to H^n(M; \mathbb{Z})$, and on the automorphism group of the trivial object it is the connecting homomorphism

$$H^{n-2}(M;\mathbb{R}/\mathbb{Z}) \to H^{n-1}(M;\mathbb{Z})$$

of the long exact sequence associated to

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0.$$

Example 2.7. The category $\check{\mathcal{H}}^2(M)$ is equivalent to the category of $U(1)$-bundles with connection, with the group structure of tensor product. One way to present a $U(1)$ bundle is to give for each open set U of M a principal homogeneous space $\Gamma(U)$ (possibly empty) for the group of smooth maps $U \to U(1)$. The points of $\Gamma(U)$ correspond to local sections of a principal bundle, and must restrict along inclusions and patch over intersections. To add a connection to such a bundle comes down to giving maps

$$\nabla : \Gamma(U) \to \Omega^1(U)$$

which are "equivariant" in the sense that

$$\nabla(g \cdot s) = \nabla(s) + g^{-1}dg. \tag{2.8}$$

An object $x = (c, h, \omega) \in \check{\mathcal{H}}^2(M)$ gives a $U(1)$-bundle with connection as follows: the space $\Gamma(U)$ is the quotient of the space of

$$s = (c^1, h^0, \theta^1) \in C^1(U;\mathbb{Z}) \times C^0(U;\mathbb{R}) \times \Omega^1(U)$$

satisfying

$$ds = x$$

by the equivalence relation

$$s \sim s + dt, \qquad t \in C^0(U;\mathbb{Z}) \times \{0\} \times \Omega^0(U).$$

Any two sections in $\Gamma(U)$ differ by an

$$\alpha \in C^1(U;\mathbb{Z}) \times C^0(U;\mathbb{R}) \times \Omega^1(U)$$

which is closed—in other words, an object in the category $\check{\mathcal{H}}^1(M)$. The equivalence relation among sections corresponds to the isomorphisms in $\check{\mathcal{H}}^1(M)$, and so the space $\Gamma(U)$ is a principal homogeneous space for the group of isomorphism classes in $\check{\mathcal{H}}^1(U)$ ie, the group of smooth maps from U to $\mathbb{R}/\mathbb{Z}$. The function ∇ associates to $s = (c^1, h^0, \theta^1)$ the 1-form θ^1. The equivariance condition 2.8 is obvious, once one identifies $\mathbb{R}/\mathbb{Z}$ with $U(1)$ and writes the action multiplicatively.

Quite a bit of useful terminology is derived by reference to the above example. Given a differential cocycle

$$x = (c, h, \omega) \in \check{\mathcal{H}}^n(M),$$

we will refer to $\omega = \omega(x)$ as the *curvature* of x , $c = c(x)$ as the *characteristic cocycle* (it is a cocycle representing the 1st Chern class when $n = 2$), and

$$e^{2\pi i\,h} = e^{2\pi i\,h(x)}$$

as the *monodromy*, regarded as a homomorphism from the group of $(n-1)$-chains into $U(1)$. The cohomology class $[x] \in H^n(M;\mathbb{Z})$ represented by c will be called the *characteristic class* of x. The set of differential cochains

$$s \in C^{n-1}(M;\mathbb{Z}) \times C^{n-2}(M;\mathbb{R}) \times \Omega^{n-1}(M)$$

satisfying

$$ds = x$$

will be called the space of *trivializations* of x. Note that any differential cochain s is a trivialization of ds.

The reduction of h modulo $\mathbb{Z}$ is also known as the *differential character* of x, so that

$$\text{monodromy} = e^{2\pi i\,(\text{differential character})}.$$

The curvature form, the differential character (equivalently the monodromy) and the characteristic class are invariants of the isomorphism class of x, while the characteristic cocycle is not. In fact the differential character (equivalently the monodromy) determines x up to isomorphism in $\check{\mathcal{H}}^n(M)$.

It is tempting to refer to the form component of a trivialization $s = (c, h, \theta)$ as the "curvature," but this does not reduce to standard terminology. By analogy with the case in which s has degree 1, we will refer to $\theta = \nabla(s)$ as the *connection form* associated to s.

It is useful to spell this out in a couple of other cases. The category $\check{\mathcal{H}}^1(M)$ is equivalent to the category whose objects are smooth maps from M to $\mathbb{R}/\mathbb{Z}$, and with morphisms, only the identity maps. The correspondence associates to (c^1, h^0, ω^1) its differential character. From the point of view of smooth maps to $\mathbb{R}/\mathbb{Z}$, the curvature is given by the derivative, and the characteristic cocycle is gotten by pulling pack a fixed choice of cocycle representing the generator of $H^1(\mathbb{R}/\mathbb{Z};\mathbb{Z})$. The characteristic class describes the effect of $f : M \to \mathbb{R}/\mathbb{Z}$ in cohomology, and determines f up to homotopy. A *trivialization* works out to be a lift of f to $\mathbb{R}$. If the trivialization is represented by $(c^0, 0, \theta^0)$, then the lift is simply given by θ^0—-the connection form of the trivialization. Finally, as the reader will easily check, $\check{\mathcal{H}}^0(M)$ is equivalent to the category whose objects are maps from M to $\mathbb{Z}$, and morphisms the identity maps.

2.4. Integration and $\check{H}$-orientations. Let M be a smooth compact manifold and $V \to M$ a (real) vector bundle over M of dimension k. A *differential Thom cocycle* on V is a (*compactly supported*) cocycle

$$U = (c, h, \omega) \in \check{Z}(k)^k_c(V)$$

with the property that for each $m \in M$,

$$\int_{V_m} \omega = \pm 1.$$

A choice of a differential Thom cocycle determines an orientation of each V_m by requiring that the sign of the above integral be $+1$.

The integral cohomology class underlying a differential Thom cocycle is a Thom class $[U]$ in $H^k_c(V;\mathbb{Z})$. There is a unique $[U]$ compatible with a fixed orientation, and so any two choices of U differ by a cocycle of the form δb, with

$$b = (b, k, \eta) \in \check{C}(k-1)^{k-1}.$$

Using the ideas of Mathai–Quillen [**59**, **43**], a differential Thom cocycle[3] can be associated to a metric and connection on V, up to addition of a term $d\,(b, k, 0)$.

Definition 2.9. A *$\check{H}$-orientation of* $p : E \to S$ consists of the following data

(1) A smooth embedding $E \subset S \times \mathbb{R}^N$ for some N;
(2) A tubular neighborhood[4]
$$W \subset S \times \mathbb{R}^N;$$
(3) A differential Thom cocycle U on W.

An *$\check{H}$-oriented map* is a map $p : E \to S$ together with a choice of $\check{H}$-orientation.

While every map of compact manifolds $p : E \to S$ factors through an embedding $E \subset S \times \mathbb{R}^N$, not every embedding $E \subset S \times \mathbb{R}^N$ admits a tubular neighborhood. A necessary and sufficient condition for the existence of a tubular neighborhood is that $p : E \to S$ be *neat*. A *neat map* of manifolds with corners is a map carrying corner points of codimension j to corner points of codimension j, and which is transverse (to the corner) at these points. A neat map $p : E \to S$ of compact manifolds factors through a neat embedding $E \subset S \times \mathbb{R}^N$, and a neat embedding has a normal bundle and admits a tubular neighborhood. Every smooth map of closed manifolds is neat, and a map $p : E \to S$ of manifolds with boundary is neat if $f \circ p$ is a defining function for the boundary of E whenever f is a defining function for the boundary of S. See Appendix C.

[3]The Thom form constructed by Mathai–Quillen is not compactly supported, but as they remark in [**43**, **59**] a minor modification makes it so.

[4]A *tubular neighborhood* of $p : E \hookrightarrow S$ is a vector bundle W over E, and an extension of p to a diffeomorphism of W with a neighborhood of $p(E)$. The derivative of the embedding $W \to M$ gives a vector bundle isomorphism $W \to \nu_{E/S}$ of W with the normal bundle to E in S.

Let E/S be an $\check{H}$-oriented map of relative dimension k. In §3.4 we will define natural, additive integration functors

$$\int_{E/S} : \mathcal{H}^n(E) \to \mathcal{H}^{n-k}(S)$$
$$\int_{E/S} : \check{\mathcal{H}}^n(E) \to \check{\mathcal{H}}^{n-k}(S),$$

compatible with the formation of the "connection form," and hence curvature:

$$\omega\left(\int_{E/S} x\right) = \int_{E/S} \omega\,(x)$$
$$\nabla\left(\int_{E/S} s\right) = \int_{E/S} \nabla\,(s)\,.$$

In the special case when E/S is a fibration over an open dense set, the integration functors arising from different choices of $\check{H}$-orientation are naturally isomorphic (up to the usual sign).

When E is a manifold with boundary and S is a closed manifold, a map $p : E \to S$ cannot be neat, and so cannot admit an $\check{H}$-orientation, even if E/S is orientable in the usual sense. Integration can be constructed in this case by choosing a defining function $f : E \to [0,1]$ for the boundary of E. An orientation of E/S *can* be extended to an $\check{H}$-orientation of

$$E \to [0,1] \times S.$$

Integration along this map is then defined, and gives a functor

$$\int_{E/[0,1]\times S} : \check{\mathcal{H}}^n(E) \to \check{\mathcal{H}}^{n-k+1}([0,1]\times S), \qquad (k = \dim E/S),$$

which commutes with the differential d, and is compatible with restriction to the boundary:

$$\begin{array}{ccc}
\check{\mathcal{H}}^n(E) & \xrightarrow{\int_{E/S}} & \check{\mathcal{H}}^{n-k}([0,1]\times S) \\
{\scriptstyle \text{res.}}\big\downarrow & & \big\downarrow{\scriptstyle \text{res.}} \\
\check{\mathcal{H}}^n(\partial E) & \xrightarrow[\int_{\partial E/S}]{} & \check{\mathcal{H}}^{n-k}(\{0,1\}\times S).
\end{array}$$

The usual notion of integration over manifolds with boundary is the iterated integral

$$\int_{E/S} x = \int_0^1 \int_{E/[0,1]\times S} x.$$

The second integral does not commute with d (and hence does not give rise to a functor), but rather satisfies Stokes theorem:

$$d\int_0^1 s = \int_0^1 ds - (-1)^{|s|}\left(s|_{\{1\}\times S} - s|_{\{0\}\times S}\right). \tag{2.10}$$

So, for example, if x is closed,

$$d\int_0^1 \int_{E/[0,1]\times S} x = (-1)^{n-k}\int_{\partial E/S} x; \tag{2.11}$$

in other words

$$(-1)^{n-k}\int_0^1 \int_{E/[0,1]\times S} x$$

is a trivialization of

$$\int_{\partial E/S} x.$$

In low dimensions these integration functors work out to be familiar constructions. If $E \to S$ is an oriented family of 1-manifolds without boundary, and $x \in \check{\mathcal{H}}^2(E)$ corresponds to a line bundle then

$$\int_{E/S} x$$

represents the function sending $s \in S$ to the monodromy of x computed around the fiber E_s.

The cup product of cocycles, and the wedge product of forms combine to give bilinear functors

$$\check{\mathcal{H}}^n(M) \times \check{\mathcal{H}}^m(M) \to \check{\mathcal{H}}^{n+m}(M)$$

which are compatible with formation of the connection form (hence curvature) and characteristic cocycle. See §3.2.

2.5. Integral Wu-structures. As we remarked in the introduction, the role of the canonical bundle is played in higher dimensions by an integral Wu-structure.

Definition 2.12. Let $p : E \to S$ be a smooth map, and fix a cocycle $\nu \in Z^{2k}(E; \mathbb{Z}/2)$ representing the Wu-class ν_{2k} of the relative normal bundle. A *differential integral Wu-structure of degree* $2k$ on E/S is a differential cocycle

$$\lambda = (c, h, \omega) \in \check{Z}(2k)^{2k}(E)$$

with the property that $c \equiv \nu \mod 2$.

We will usually refer to a differential integral Wu-structure of degree $2k$ as simply an *integral Wu-structure*. If λ and λ' are integral Wu-structures, then there is a unique

$$\eta \in \check{C}(2k)^{2k}(E)$$

with the property that

$$\lambda' = \lambda + 2\eta.$$

We will tend to overuse the symbol λ when referencing integral Wu-structures. At times it will refer to a differential cocycle, and at times merely the underlying topological cocycle (the c component). In all cases it should be clear from context which meaning we have in mind.

For a cocycle $\nu \in Z^{2k}(E;\mathbb{Z}/2)$, let $\check{\mathcal{H}}^{2k}_\nu(E)$ denote the category whose objects are differential cocycles $x = (c,h,\omega)$ with $c \equiv \nu \mod 2$, and in which a morphism from x to x' is an equivalence class of differential cochains

$$\tau = (b,k) \in \check{C}(2k)^{2k-1}(E)/d\left(\check{C}(2k)^{2k-2}(E)\right)$$

for which $x' = x + 2d\tau$. The set of isomorphism classes in $\check{\mathcal{H}}^{2k}_\nu(E)$ is a torsor (principal homogeneous space) for the group $\check{H}(2k)^{2k}(M)$, and the automorphism group of any object is $H^{2k-2}(E;\mathbb{R}/\mathbb{Z})$. We will write the action of $\mu \in \check{H}(2k)^{2k}(M)$ on $x \in \check{\mathcal{H}}^{2k}_\nu(E)$ as

$$x \mapsto x + (2)\mu,$$

with the parentheses serving as a reminder that the multiplication by 2 is formal; even if $\mu \in \check{H}(2k)^{2k}(M)$ has order 2, the object $x + (2)\mu$ is not necessarily isomorphic to x.[5] A differential cochain

$$y = (c', h', \omega') \in \check{C}(2k)^{2k}$$

gives a functor

$$\begin{gathered}\check{\mathcal{H}}^{2k}_\nu(E) \to \check{\mathcal{H}}^{2k}_{\nu+d\bar{y}}(E)\\ x \mapsto x + dy\end{gathered}$$

in which $\bar{y}$ denotes the mod 2 reduction of c'. Up to natural equivalence, this functor depends only on the value of $\bar{y}$. In this sense the category $\check{\mathcal{H}}^{2k}_\nu(E)$ depends only on the cohomology class of ν.

With this terminology, a differential integral Wu-structures is an object of the category $\check{\mathcal{H}}^{2k}_\nu(E)$, with ν a cocycle representing the Wu-class ν_{2k}. An *isomorphism* of integral Wu-structures is an isomorphism in $\check{\mathcal{H}}^{2k}_\nu(E)$.

We show in Appendix E that it is possible to associate an integral Wu-structure $\bar{\nu}_{2k}(E/S)$ to a Spin structure on the relative normal bundle of E/S. Furthermore, as we describe in §3.3, a connection on $\nu_{E/S}$ gives a refinement of $\bar{\nu}_{2k}(E/S)$ to a differential integral Wu-structure $\check{\nu}_{2k}(E/S) \in \check{H}(2k)^{2k}_\nu(E)$. We'll write $\lambda(s,\nabla)$ (or just $\lambda(s)$) for the integral Wu-structure associated to a Spin-structure s and connection ∇. By Proposition E.9, if the Spin-structure is changed by an element

[5]It could happen that 2μ can be written as $d(b',k')$, but not as $2d(b,k)$.

$\alpha \in H^1(E;\mathbb{Z}/2)$ then, up to isomorphism, the integral Wu-structure changes according to the rule

$$\lambda(s+\alpha) \equiv \lambda(s) + (2)\beta \sum_{\ell \geq 0} \alpha^{2^\ell - 1} \nu_{2k-2^\ell} \in \check{\mathcal{H}}^{2k}_\nu(E), \tag{2.13}$$

where ν_j is the j^{th} Wu class of the relative normal bundle, and β denotes the map

$$H^{2k-1}(E;\mathbb{Z}/2) \to H^{2k-1}(E;\mathbb{R}/\mathbb{Z}) \hookrightarrow \check{H}(2k)^{2k}(E)$$

described in §3.2.

We now reformulate the above in terms of "twisted differential characters." These appear in the physics literature [**67**] as Chern-Simons terms associated to characteristic classes which do not necessarily take integer values on closed manifolds. We will not need this material in the rest of the paper.

Definition 2.14. Let M be a manifold,

$$\nu \in Z^{2k}(M;\mathbb{R}/\mathbb{Z})$$

a smooth cocycle, and $2k \geq 0$ an integer. The category of *ν-twisted differential $2k$-cocycles*, $\check{\mathcal{H}}^{2k}_\nu(M)$, is the category whose objects are triples

$$(c,h,\omega) \subset C^{2k}(M;\mathbb{R}) \times C^{2k-1}(M;\mathbb{R}) \times \Omega^{2k}(M),$$

satisfying

$$\begin{aligned} c &\equiv \nu \mod \mathbb{Z} \\ \delta c &= 0 \\ d\omega &= 0 \\ \delta h &= \omega - c \end{aligned}$$

(we do not distinguish in notation between a form and the cochain represented by integration of the form over chains). A morphism from (c_1,h_1,ω_1) to (c_2,h_2,ω_2) is an equivalence class of pairs

$$(b,k) \in C^{2k-1}(M;\mathbb{Z}) \times C^{2k-2}(M;\mathbb{R})$$

satisfying

$$\begin{aligned} c_1 - \delta b &= c_2 \\ h_1 + \delta k + b &= h_2. \end{aligned}$$

The equivalence relation is generated by

$$(b,k) \sim (b - \delta a, k + \delta k' + a).$$

The category $\check{\mathcal{H}}^{2k}_\nu(M)$ is a torsor for the category $\check{\mathcal{H}}^{2k}(M)$. We will write the translation of $v \in \check{\mathcal{H}}^{2k}_\nu(M)$ by $x \in \check{\mathcal{H}}^{2k}(M)$ as $v + x$.

Remark 2.15. We have given two meanings to the symbol $\check{\mathcal{H}}^{2k}_{\nu}(M)$; one when ν is a cocycle with values in $\mathbb{Z}/2$, and one when ν is a cocycle taking values in $\mathbb{R}/\mathbb{Z}$. For a cocycle $\nu \in Z^{2k}(M;\mathbb{Z}/2)$ there is an *isomorphism* of categories

$$\check{\mathcal{H}}^{2k}_{\nu}(M) \to \check{\mathcal{H}}^{2k}_{\frac{1}{2}\nu}(M)$$
$$(c,h,\omega) \mapsto \frac{1}{2}(c,h,\omega),$$

in which $\frac{1}{2}\nu \in Z^{2k}(M;\mathbb{R}/\mathbb{Z})$ is the composite of ν with the inclusion

$$\mathbb{Z}/2 \hookrightarrow \mathbb{R}/\mathbb{Z}$$
$$t \mapsto \frac{1}{2}t.$$

The isomorphism class of a ν-twisted differential cocycle (c,h,ω) is determined by ω and the value of h modulo $\mathbb{Z}$.

Definition 2.16. Let M be a smooth manifold, and

$$\nu \in Z^{2k}(M;\mathbb{R}/\mathbb{Z})$$

a smooth cocycle. A *ν-twisted differential character* is a pair

$$(\chi,\omega)$$

consisting of a character

$$\chi : Z_{2k-1}(M) \to \mathbb{R}/\mathbb{Z}$$

of the group of smooth $(2k-1)$-cycles, and a $2k$-form ω with the property that for every smooth $2k$-chain B,

$$\int_B \omega - \chi(\partial B) \equiv \nu(B) \mod \mathbb{Z}.$$

The set of ν-twisted differential characters will be denoted

$$\check{H}^{2k-1}_{\nu}(M).$$

It is torsor for the group $\check{H}^{2k-1}(M)$.

2.6. The main theorem.

Theorem 2.17. *Let E/S be an $\check{H}$-oriented map of manifolds of relative dimension $4k-i$, with $i \le 2$. Fix a differential cocycle L_{4k} refining the degree $4k$ component of the Hirzebruch L polynomial, and fix a cocycle $\nu \in Z^{2k}(E;\mathbb{Z}/2)$ representing the Wu-class ν_{2k}. There is a functor*

$$\kappa_{E/S} : \check{\mathcal{H}}^{2k}_{\nu}(E) \to \check{\mathcal{H}}^{i}(S)$$

with the following properties:

i) (*Normalization*) *modulo torsion,*

$$\kappa(\lambda) \approx \frac{1}{8}\int_{E/S} \lambda \cup \lambda - \check{L}_{4k};$$

ii) (*Symmetry*) *there is an isomorphism*

$$\tau(\lambda) : \kappa(-\lambda) \xrightarrow{\approx} \kappa(\lambda)$$

satisfying

$$\tau(\lambda) \circ \tau(-\lambda) = \textit{identity map of } \kappa(\lambda);$$

iii) (*Base change*) *Suppose that* $S' \subset S$ *is a closed submanifold and* $p : E \to S$ *is transverse to* S'. *Then the* $\check{H}$*-orientationof* E/S *induces an* $\check{H}$*-orientationof*

$$E' = p^{-1}(S') \xrightarrow{p'} S'$$

and, with $\tilde{f}$ *denoting the map* $E' \to E$, *there is an isomorphism*

$$\tilde{f}^* \kappa_{E/S}(\lambda) \approx \kappa_{E'/S'}(f^*\lambda);$$

iv) (*Transitivity*) *let*

$$E \to B \to S$$

be a composition of $\check{H}$*-oriented maps, and suppose given a framing of the stable normal bundle of* B/S *which is compatible with its* $\check{H}$*-orientation. The* $\check{H}$*-orientations of* E/B *and* B/S *combine to give an* $\check{H}$*-orientation of* E/S, *the differential cocycle* $\check{L}_{4k}$ *represents the Hirzebruch L-class for* E/S, *and a differential integral Wu–structure* λ *on* E/B *can be regarded as a differential integral Wu–structure on* E/S. *In this situation there is an isomorphism*

$$\kappa_{E/S}(\lambda) \approx \int_{B/S} \kappa_{E/B}(\lambda).$$

We use κ and a fixed differential integral Wu-structure λ to construct a quadratic functor

$$q_{E/S} = q^{\lambda}_{E/S} : \check{\mathcal{H}}^{2k}(E) \to \check{\mathcal{H}}^{i}(S)$$

by

$$q^{\lambda}_{E/S}(x) = \kappa(\lambda - 2x).$$

Corollary 2.18.

i) *The functor* $q_{E/S}$ *is a quadratic refinement of*

$$B(x,y) = \int_{E/S} x \cup y$$

i.e., there is an isomorphism

$$t(x,y) : q_{E/S}(x+y) - q_{E/S}(x) - q_{E/S}(y) + q_{E/S}(0) \xrightarrow{\approx} B(x,y);$$

ii) (*Symmetry*) *There is an isomorphism*

$$s^{\lambda}(x) : q^{\lambda}_{E/S}(\lambda - x) \approx q^{\lambda}_{E/S}(x)$$

satisfying

$$s^{\lambda}(x) \circ s^{\lambda}(\lambda - x) = \textit{identity map};$$

iii) (*Base change*) *The functor q satisfies the base change property* iii) *of Theorem* 2.17*:*

$$q_{E'/S'}(\tilde{f}^* x) \approx f^* q_{E/S}(x);$$

iv) (*Transitivity*) *The functor q satisfies the transitivity property* iv) *of Theorem* 2.17*:*

$$q_{E/S}(x) \approx \int_{B/S} q_{E/B}(x).$$

Remark 2.19. All the isomorphisms in the above are natural isomorphisms, ie they are isomorphisms of *functors*.

Remark 2.20. Suppose that E/S is a fibration of relative dimension n, with Spin manifolds M_s as fibers, and $E \subset \mathbb{R}^N \times S$ is an embedding compatible with the metric along the fibers. Associated to this data is an $\check{H}$-orientation, differential integral Wu-structure, and a differential L-cocycle. To describe these we need the results of §3.3 where we show that a connection on a principal G-bundle and a classifying map determines differential cocycle representatives for characteristic classes, and a differential Thom cocycle for vector bundles associated to oriented orthogonal representations of G.

Let W be the normal bundle to the embedding $E \subset S \times \mathbb{R}^N$. The fiber at $x \in M_s \subset E$ of the normal bundle W of $E \subset S \times \mathbb{R}^N$ can be identified with the orthogonal complement in $\mathbb{R}^N$ of the tangent space to M_s at x. Because of this, W comes equipped with a metric, a connection, a Spin-structure, and a classifying map

$$\begin{array}{ccc} W & \longrightarrow & \xi^{\mathrm{Spin}}_{N-n} \\ \big\downarrow & & \big\downarrow \\ E & \longrightarrow & Gr^{\mathrm{Spin}}_{N-n}\left(\mathbb{R}^N\right), \end{array}$$

where $Gr^{\mathrm{Spin}}_{N-n}\left(\mathbb{R}^N\right)$ is defined by the homotopy pullback square

$$\begin{array}{ccc} Gr^{\mathrm{Spin}}_{N-n}\left(\mathbb{R}^N\right) & \longrightarrow & \mathrm{BSpin}(N-n) \\ \big\downarrow & & \big\downarrow \\ \widetilde{Gr}_{N-n}\left(\mathbb{R}^N\right) & \longrightarrow & BSO(N-n). \end{array}$$

In the above, $\xi^{\mathrm{Spin}}_{N-n}$ is the universal $(N-n)$-plane bundle, and $\widetilde{Gr}_{N-n}\left(\mathbb{R}^N\right)$ refers to the oriented Grassmannian. The metric on W can be used to construct a tubular neighborhood

$$W \subset S \times \mathbb{R}^N,$$

and the differential Thom cocycle and L-cocycle are the ones constructed in §3.3.

Fix once and for all a cocycle

$$\tilde{c} \in Z^{2k}\left(Gr^{\mathrm{Spin}}_{N-n}\left(\mathbb{R}^N\right);\mathbb{Z}\right)$$

representing $\bar{\nu}_{2k}$. The the mod 2 reduction of $\tilde{c}$ represents the universal Wu-class ν_{2k}. We set

$$\begin{aligned} c &= f^*(\tilde{c}) \\ \nu &= c \mod 2 \in Z^{2k}(E;\mathbb{Z}/2). \end{aligned}$$

We take λ to be the differential cocycle associated to characteristic class $\bar{\nu}_{2k}$ and the Spin-connection on the normal bundle to $E \subset \mathbb{R}^N \times S$.

Remark 2.21. To be completely explicit about the $\check{H}$-orientation constructed in Remark 2.20 a convention would need to be chosen for associating a tubular neighborhood to a (neat) embedding equipped with a metric on the relative normal bundle. It seems best to let the geometry of a particular situation dictate this choice. Different choices can easily be compared by introducing auxiliary parameters.

Remark 2.22. Another important situation arises when $K \to S$ is a map from a compact manifold with boundary to a compact manifold without boundary. As it stands, Theorem 2.17 doesn't apply since $K \to S$ is not a neat map. To handle this case choose a defining function f for $\partial K = E$, and suppose we are given an orientation and integral Wu–structure λ of $K/[0,1] \times S$. Then by (2.10), for each differential cocycle $y \in \check{\mathcal{H}}^{2k}(K)$

(2.23)

$$(-1)^{i+1}\, d \int_0^1 q^{\lambda}_{K/[0,1]\times S}(y) = q^{\lambda}_{K/[0,1]\times S}(y)|_{S\times\{1\}} - q^{\lambda}_{K/[0,1]\times S}(y)|_{S\times\{0\}}.$$

Since

$$\begin{array}{ccc} E & \longrightarrow & K \\ \downarrow & & \downarrow \\ S \times \{0,1\} & \longrightarrow & [0,1] \times S \end{array}$$

is a transverse pullback square, the base change property of q gives a canonical isomorphism

$$q^{\lambda}_{K/[0,1]\times S}(y)|_{S\times\{1\}} - q^{\lambda}_{K/[0,1]\times S}(y)|_{S\times\{0\}} \approx q^{\lambda}_{E/S}(y|_E). \tag{2.24}$$

Combining (2.23) and (2.24), and writing $x = y|_E$, gives

$$(-1)^{i+1}\, d \int_0^1 q^{\lambda}_{K/[0,1]\times S}(y) = q^{\lambda}_{E/S}(x). \tag{2.25}$$

In other words, *writing the pair $(E/S, x)$ as a boundary gives a trivialization of $q^{\lambda}_{E/S}(x)$.*

Theorem 2.17 remains true when $i > 2$, but with the modification that $q^{\lambda}_{E/S}$ is not a differential cocycle, but a differential function to $(\tilde{I}_i; \iota)$. See §5 for more details.

The properties of $q^{\lambda}_{E/S}(x)$ lead to an explicit formula for $2q^{\lambda}_{E/S}(x)$. First of all, note that

$$q^{\lambda}_{E/S}(\lambda) = \kappa(-\lambda) = \kappa(\lambda).$$

It follows that

$$\begin{aligned} q^{\lambda}_{E/S}(0) &= q^{\lambda}_{E/S}(\lambda) = q^{\lambda}_{E/S}(x + (\lambda - x)) \\ &= q^{\lambda}_{E/S}(x) + q^{\lambda}_{E/S}(\lambda - x) - q^{\lambda}_{E/S}(0) + B(x, \lambda - x). \end{aligned}$$

Using the symmetry $q^{\lambda}_{E/S}(\lambda - x) = q^{\lambda}_{E/S}(x)$, and collecting terms gives

Corollary 2.26. *Let E/S be as in Theorem* 2.17. *There is an isomorphism*

$$2q^{\lambda}_{E/S}(x) \approx 2\kappa_{E/S}(\lambda) + \int_{E/S} x \cup x - x \cup \lambda.$$

We now exhibit the effect of q on automorphisms in the first nontrivial case, $i = 2$. We start with

$$x = 0 \in \check{\mathcal{H}}^{2k}(E).$$

The automorphism group of x is $H^{2k-2}(E; \mathbb{R}/\mathbb{Z})$, and the automorphism group of $q(x) = \kappa(\lambda) \in \check{\mathcal{H}}^2(S)$ is $H^0(S; \mathbb{R}/\mathbb{Z})$. The map

$$q : H^{2k-2}(E; \mathbb{R}/\mathbb{Z}) \to H^0(S; \mathbb{R}/\mathbb{Z})$$

is determined by its effect on fibers

$$q : H^{2k-2}(E_s; \mathbb{R}/\mathbb{Z}) \to H^0(\{s\}; \mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z} \qquad s \in S.$$

By Poincaré duality, such a homomorphism is given by

$$q(\alpha) = \int_{E_s} \alpha \cup \mu \qquad \alpha \in \mathrm{Aut}(0),$$

for some $\mu \in H^{2k}(E_s)$. After some work (involving either chasing through the definitions, or replacing automorphisms with their mapping tori in the usual way), the quadratic nature of q implies that for general $x \in \check{\mathcal{H}}^{2k}(E_s)$ one has

$$q(\alpha) = \int_{E_s} \alpha \cup ([x] + \mu) \qquad \alpha \in \mathrm{Aut}(x). \tag{2.27}$$

Again, with a little work, Corollary 2.26 gives the identity

$$2\mu = -[\lambda].$$

This is a somewhat surprising formula. The theory of Wu-classes shows that for each $2n$-manifold M, the Wu class ν_{2k} of the normal bundle vanishes, and hence the cohomology class $[\lambda]$ is divisible by 2.

Our theory shows that the choice of integral Wu-structure actually leads to a canonical way of dividing $[\lambda]$ by 2, namely $-\mu$.

The cohomology class μ arises in another context. Suppose that E/S is the product family $E = M \times S$, and the integral Wu-structure is gotten by change of base from an integral Wu-structure on $M \to \{\mathrm{pt}\}$. If $x \in \check{\mathcal{H}}^{2k}(E)$ is arbitrary, and $y \in \check{\mathcal{H}}^{2k}(E)$ is of the form $(0, h, 0)$, then

$$q(x+y) \approx q(x) + \int_{E/S} y \cup (x + \mu \otimes 1), \tag{2.28}$$

where $\mu \in H^{2k}(M;\mathbb{Z})$ is the class described above.

2.7. The fivebrane partition function. We will now show how Corollary 2.18 can be used to construct the holomorphic line bundle described by Witten in [**65**]. We will need to use some of the results and notation of §3.2, especially the exact sequences (3.3).

Suppose that M is a Riemannian Spin manifold of dimension 6, and let J be the torus

$$J = H^3(M) \otimes \mathbb{R}/\mathbb{Z}.$$

The Hodge $*$-operator on 3-forms determines a complex structure on $H^3(M;\mathbb{R})$, making J into a complex torus. In fact J is a polarized abelian variety—the $(1,1)$-form ω is given by the cup product, and the Riemann positivity conditions follow easily from Poincaré duality. These observations are made by Witten in [**65**], and he raises the question of finding a symmetric, holomorphic line bundle $\mathcal{L}$ on J with first Chern form ω. There is, up to scale, a unique holomorphic section of such an $\mathcal{L}$ which is the main ingredient in forming the fivebrane partition function in M-theory.

Witten outlines a construction of the line bundle $\mathcal{L}$ in case $H^4(M;\mathbb{Z})$ has no torsion. We construct the line bundle $\mathcal{L}$ in general. As was explained in the introduction this was one motivation for our work.

The group J is naturally a subgroup of the group

$$\check{H}(4)^4(M) = \check{H}^{4-1}(M)$$

of differential characters (§3.2) of degree 3 on M. In principle, the fiber of $\mathcal{L}$ over a point $z \in J$ is simply the line

$$q^{\lambda}_{M/\mathrm{pt}}(x) = q^{\lambda}(x),$$

where x is a differential cocycle representing z, and λ is the integral Wu-structure corresponding to the Spin-structure on M. The key properties of $\mathcal{L}$ are a consequence of Theorem 2.17. One subtlety is that the line $q^{\lambda}(x)$ need not be independent of the choice of representing cocycle. Another is that the symmetry is not a symmetry about 0.

To investigate the dependence on the choice of cocycle representing z, note that any two representatives x and x' are isomorphic, and so the lines $q^{\lambda}(x)$ and $q^{\lambda}(x')$ are also isomorphic. Any two isomorphisms of x and x' differ by an automorphism $\alpha \in \mathrm{Aut}(x)$, and by (2.27) the corresponding automorphism $q^\lambda(\alpha)$ of $q^\lambda(x)$ is given by

$$q^\lambda(\alpha) = \int_M \alpha \cup ([z] + \mu).$$

This means that as long as $[z] = -\mu$ the isomorphism

$$q^\lambda(x) \approx q^\lambda(x')$$

is independent of the choice of isomorphism $x \approx x'$, and the lines $q^\lambda(x)$ and $q^\lambda(x')$ can be canonically identified. The desired $\mathcal{L}$ is therefore naturally found over the "shifted" torus J_t, where $t \in A^4(M)$ is any element whose underlying cohomology class is $-\mu$, and

$$J_t \subset \check{H}(4)^4(M)$$

denotes the inverse image of t under the second map of the sequence

$$H^3(M) \otimes \mathbb{R}/\mathbb{Z} \to \check{H}(4)^4(M) \to A^4(M).$$

The sequence shows that each space J_t is a principal homogeneous space for J.

To actually construct the line bundle $\mathcal{L}$ over J_t, we need to choose a universal $Q \in \check{H}(4)^4(M \times J_t)$ (an analogue of a choice of Poincaré line bundle). We require Q to have the property that for each $p \in J_t$, the restriction map

$$(1 \times \{p\})^* : \check{H}(4)^4(M \times J_t) \to \check{H}(4)^4(M \times \{p\})$$

satisfies

$$(1 \times \{p\})^* Q = p. \tag{2.29}$$

The line bundle $\mathcal{L}$ will then turn out to be

$$q(x) = q^\lambda_{M \times J_t / J_t}(x)$$

where $x = (c, h, G)$ is a choice of differential cocycle representing Q. We first construct

$$Q_0 \in \check{H}(4)^4(M \times J)$$

and then use a point of J_t to translate it to

$$Q \in \check{H}(4)^4(M \times J_t).$$

Write

$$V = H^3(M; \mathbb{R})$$
$$L = H^3(M; \mathbb{Z})$$

so that V is the tangent space to J at the origin, and there is a canonical isomorphism

$$V/L = J.$$

Let

$$\theta \in \Omega^3(M \times V)$$

be the unique 3-form which is vertical with respect to the projection

$$M \times V \to V$$

and whose fiber over $v \in V$ is harm(v)—the unique harmonic form whose deRham cohomology class is v. The form θ gives rise to an element

$$\bar{\theta} \in \check{H}(4)^4(M \times V)$$

via the embedding

$$\Omega^3/\Omega^3_0 \to \check{H}(4)^4(M \times V).$$

The restriction map

$$(1 \times \{v\})^* : \check{H}(4)^4(M \times V) \to \check{H}(4)^4(M \times \{v\})$$

has the property that for each $v \in V$,

$$(1 \times \{v\})^* \bar{\theta} = v. \tag{2.30}$$

This guarantees that (2.29) will hold for the class Q we are constructing.

We will now show that there is a class $Q_0 \in \check{H}(4)^4(M \times J)$ whose image in $\check{H}(4)^4(M \times V)$ is $\bar{\theta}$. Note that the curvature $(d\bar{\theta})$ of $\bar{\theta}$ is a translation invariant, and is the pullback of a translation invariant four form

$$\omega(Q_0) \in \Omega^4_0(M \times J)$$

with integral periods. Consider the following diagram in which the rows are short exact:

$$\begin{array}{ccccc}
H^3(M \times V; \mathbb{R}/\mathbb{Z}) & \longrightarrow & \check{H}(4)^4(M \times V) & \longrightarrow & \Omega^4_0(M \times V) \\
\uparrow & & \uparrow & & \uparrow \\
H^3(M \times J; \mathbb{R}/\mathbb{Z}) & \longrightarrow & \check{H}(4)^4(M \times J) & \longrightarrow & \Omega^4_0(M \times J).
\end{array}$$

The leftmost vertical map is surjective. We've already seen that the curvature of $\bar{\theta}$ descends to $M \times J$. An easy diagram chase gives the existence of the desired $Q_0 \in \check{H}(4)^4(M \times J)$. Such a class Q_0 is unique up to the addition of an element

$$a \in \ker H^3(M \times J; \mathbb{R}/Z) \to H^3(M \times V; \mathbb{R}/\mathbb{Z}) = H^3(M; \mathbb{R}/\mathbb{Z}).$$

When $t \in A^4(M \times J)$ is an element whose underlying cohomology class is $-\mu \otimes 1$, the line bundle $q(x)$ is independent of both the choice of Q and the choice of x representing Q. The independence from the choice of x amounts, as described above, to checking that q sends automorphisms of x to the identity map of $q(x)$. If $\alpha \in H^2(M \times J)$ is an automorphism of x, then by (2.27)

$$q(\alpha) = \int_{M \times J_t/J_t} \alpha \cup ([x] + \mu \otimes 1).$$

The integral vanishes since the Künneth component of $([x] + \mu \otimes 1)$ in $H^4(M) \otimes H^0(J_t)$ is zero. If $Q' = Q + Y$ is another choice of "Poincaré bundle" then, by (2.28)

$$q(Q') = q(Q) + \int_{M \times J/J} Y \cup (Q + \mu \otimes 1).$$

Once again, one can check that the integral vanishes by looking at Künneth components.

Witten constructs $\mathcal{L}$ (up to isomorphism) by giving a formula for monodromy around loops in J. We now show that the monodromy of $q(x)$ can be computed by the same formula.

Let $\gamma : S^1 \to J$ be a loop, and consider the following diagram:

$$\begin{tikzcd} M \times S^1 \arrow[r, "\tilde{\gamma}"] \arrow[d] & M \times J \arrow[d] \\ S^1 \arrow[r, "\gamma"'] \arrow[d] & J \\ \mathrm{pt} & . \end{tikzcd}$$

The monodromy of $q(x)$ around γ given by

$$\exp\left(2\pi i \int_{S^1} \gamma^* q(x)\right).$$

By the base change property of q (Corollary 2.18, iii)), the integral above can be computed as

$$\int_{S^1} q_{M \times S^1/S^1}(\tilde{\gamma}^* x).$$

Choose any framing of the stable normal bundle of S^1. By transitivity (Corollary 2.18, iv)), this, in turn, is given by

$$q_{M \times S^1/\mathrm{pt}}(\tilde{\gamma}^*(x)).$$

We can compute this by finding a section, which we do by writing $M \times S^1$ as a boundary, extending $\tilde{\gamma}^*(x)$), and using (2.25).

The 7-dimensional Spin-manifold $M \times S^1$, together with the cohomology class represented by $\tilde{\gamma}^* c$ defines an element of

$$M\,\mathrm{Spin}_7\, K(\mathbb{Z}, 4).$$

Since this group is zero, there is a Spin-manifold N with $\partial N = M \times S^1$ and a cocycle $\tilde{c}$ on N extending $\tilde{\gamma}^* c$. We can then find

$$y \in \check{\mathcal{H}}^4(N)$$

whose restriction to $M \times S^1$ is $\tilde{\gamma}^*(x)$. By (2.25)

$$q_{M \times S^1/\mathrm{pt}}(\tilde{\gamma}^*(x)) = -d\, q_N(y).$$

This means that $q_N(y)$ is a real number whose reduction modulo $\mathbb{Z}$ is $\frac{1}{2\pi i}$ times the log of the monodromy. Note, by the same reasoning, that $q_N(0)$ must be an integer since it gives the monodromy of the constant line bundle $q^\lambda(0)$. So the monodromy of $q^\lambda(x)$ (divided by $2\pi i$) is also given by

$$q_N(y) - q_N(0) \mod \mathbb{Z}.$$

Using Corollary 2.26 we find

$$q_N(y) - q_N(0) = \frac{-1}{2}\int_N G\wedge G - G\wedge F,$$

where G is the 4-form $\omega(y)$, and F is $\frac{1}{2}$ the first Pontryagin form of N—that is $\omega(\lambda(N))$. This is (up to sign) Witten's formula.

Several comments are in order.

(1) Our construction of the line bundle $\mathcal{L}$ works for any Spin manifold M of dimension $(4k-2)$. The computation of monodromy we have given only applies when the reduced bordism group

$$\mathrm{MSpin}_{4k-1}\, K(\mathbb{Z}, 2k)$$

vanishes. This group can be identified with

$$H_{2k-1}(\mathrm{BSpin}; \mathbb{Z})$$

and does not, in general, vanish.

(2) In the presence of torsion, as the manifold M moves through bordisms, the different components of the shifted Jacobians can come together, suggesting that the "correct" shifted Jacobian to use is not connected. We have constructed a line bundle over the entire group $\check{H}(4)^4(M)$, but it is independent of the choices only on the component indexed by $-\mu$. This suggests that it is better to work with a line bundle over the *category* $\check{\mathcal{H}}^4(M)$. Of course these remarks also apply in dimension $4k-2$.

(3) In our formulation the quadratic function arises from a symmetry of the bundle. Its value on the points of order two in the connected component of the identity of J can be computed from the monodromy of the connection, but not in general. This suggests that it is important to remember automorphisms of objects in $\check{\mathcal{H}}^4(M)$, and once again places priority on $\check{\mathcal{H}}^4(M)$ over its set of isomorphism classes.

(4) Formula (2.13) gives the effect of a change of Spin–structure on $\mathcal{L}$, and the resulting quadratic function.

3. Cheeger–Simons cohomology

3.1. Introduction. We wish to define a cohomology theory of smooth manifolds M, which encodes the notion of closed q-forms with integral periods. More specifically, we are looking for a theory to put in the corner of the square

$$\begin{array}{ccc} ? & \longrightarrow & \Omega^q_{\text{closed}}(M) \\ \big\downarrow & & \big\downarrow \\ H^q(M;\mathbb{Z}) & \longrightarrow & H^q(M;\mathbb{R}), \end{array} \tag{3.1}$$

which will fit into a long exact Mayer-Vietoris type of sequence. Here $\Omega^q_{\text{closed}}(M)$ is the space of closed q-forms on M. There are two approaches to doing this, and they lead to equivalent results. The first is via the complex of smooth cochains and is due to Cheeger and Simons. The second is via complexes of sheaves, and is due to Deligne and Beilinson. In this paper we follow the approach of Cheeger–Simons, and ultimately generalize it to the theory of differential function spaces (§4).[6]

3.2. Differential Characters. To construct the cohomology theory sketched in the previous section, we refine (3.1) to a diagram of cochain complexes, and define a complex $\check{C}(q)^*(M)$ by the homotopy cartesian square

$$\begin{array}{ccc} \check{C}(q)^*(M) & \longrightarrow & \Omega^{*\geq q}(M) \\ \big\downarrow & & \big\downarrow \\ C^*(M;\mathbb{Z}) & \longrightarrow & C^*(M;\mathbb{R}). \end{array} \tag{3.2}$$

More explicitly, the complex $\check{C}(q)^*(M)$ is given by

$$\check{C}(q)^n(M) = \begin{cases} C^n(M;\mathbb{Z}) \times C^{n-1}(M;\mathbb{R}) \times \Omega^n(M) & n \geq q \\ C^n(M;\mathbb{Z}) \times C^{n-1}(M;\mathbb{R}) & n < q, \end{cases}$$

with differential

$$d(c,h,\omega) = (\delta c, \omega - c - \delta h, d\omega)$$

$$d(c,h) = \begin{cases} (\delta c, -c - \delta h, 0) & (c,h) \in \check{C}(q)^{q-1} \\ (\delta c, -c - \delta h) & \text{otherwise.} \end{cases}$$

It will be convenient to write

$$\check{C}(0)^* = \check{C}^*$$

and regard

$$\check{C}(q)^* \subset \check{C}^*$$

as the subcomplex consisting of triples (c,h,ω) for which $\omega = 0$ if $\deg \omega < q$.

[6]Since writing this paper we have learned of the work of Harvey and Lawson (see for example [**33**]) which contains another treatment of Cheeger–Simons cohomology.

The Mayer-Vietoris sequence associated to (3.2)

$$\to \dots \check{H}(q)^n(M) \to H^n(M;\mathbb{Z}) \times H^n(\Omega^{*\geq q}) \\ \to H^n(M;\mathbb{R}) \to \check{H}(q)^{2k}(M) \to \dots$$

leads to natural isomorphisms

$$\check{H}(q)^n(M) = \begin{cases} H^n(M;\mathbb{Z}) & n > q \\ H^{n-1}(M;\mathbb{R}/\mathbb{Z}) & n < q, \end{cases}$$

and a short exact sequence

$$0 \mapsto H^{q-1}(M) \otimes \mathbb{R}/\mathbb{Z} \to \check{H}(q)^q(M) \to A^q(M) \to 0.$$

Here $A^q(M)$ is defined by the pullback square

$$\begin{array}{ccc} A^k(M) & \longrightarrow & \Omega^k_{\mathrm{cl}} \\ \big\downarrow & & \big\downarrow \\ H^k(M;\mathbb{Z}) & \longrightarrow & H^k(M;\mathbb{R}), \end{array}$$

and is thus the subgroup of $H^q(M;\mathbb{Z}) \times \Omega^q_{\mathrm{cl}}$, consisting of pairs for which the (closed) form is a representative of the image of the cohomology class in deRham cohomology. This sequence can be arranged in three ways

$$\begin{gathered} 0 \to H^{q-1}(M;\mathbb{R}/\mathbb{Z}) \to \check{H}(q)^q(M) \to \Omega^q_0 \to 0 \\ 0 \to H^{q-1} \otimes \mathbb{R}/\mathbb{Z} \to \check{H}(q)^q(M) \to A^q(M) \to 0 \\ 0 \to \Omega^{q-1}/\Omega^{q-1}_0 \to \check{H}(q)^q(M) \to H^q(M;\mathbb{Z}) \to 0. \end{gathered} \tag{3.3}$$

In this, Ω^j_0 denotes the space of closed j-forms with integral periods.

A cocycle for the group $\check{H}(q)^q$ is a triple

$$(c^q, h^{q-1}, \omega),$$

for which

$$\delta h^{q-1} = \omega - c^q.$$

The equivalence class of

$$(c^q, h^{q-1}, \omega)$$

in $\check{H}(q)^q(M)$ determines, and is determined by the pair

$$(\chi, \omega),$$

where

$$\chi : Z_{q-1} \to \mathbb{R}/\mathbb{Z}$$

is given by

$$\chi z = h^{q-1}(z) \mod \mathbb{Z}.$$

Definition 3.4. Let M be a smooth manifold. A *differential character* of M of degree $(k-1)$ is a pair (χ, ω) consisting of a character

$$\chi : \mathbb{Z}_{k-1} \to \mathbb{R}/\mathbb{Z}$$

of the group of smooth k-cycles, and a k-form ω with the property that for every smooth k-chain B,

$$\chi(\partial B) = \int_B \omega. \tag{3.5}$$

The group of differential characters was introduced by Cheeger-Simons [**20**] and is denoted

$$\check{H}^{k-1}(M).$$

We will refer to $\check{H}^{k-1}(M)$ as the $(k-1)^{st}$ *Cheeger-Simons cohomology group.* As indicated, the map

$$(c^q, h^{q-1}, \omega) \mapsto (\chi, \omega)$$

gives an isomorphism

$$\check{H}(q)^q(M) \approx \check{H}^{q-1}(M).$$

The cup product in cohomology and the wedge product of forms lead to pairings

$$\check{C}(k)^*(M) \otimes \check{C}(l)^*(M) \to \check{C}(k+l)^* \tag{3.6}$$

$$\check{H}(k)^*(M) \otimes \check{H}(l)^*(M) \to \check{H}(k+l)^* \tag{3.7}$$

making $\check{H}(*)^*$ into a graded commutative ring. As Cheeger and Simons point out, the formula for these pairings is complicated by the fact that the map from forms to cochains does not map the wedge product to the cup product. For

$$\omega \in \Omega^r, \qquad \eta \in \Omega^s,$$

let

$$\omega \cup \eta \in C^{r+s}$$

be the cup product of the cochains represented by ω and η (using, for example, the Alexander–Whitney chain approximation to the diagonal). Let

$$B(\omega, \eta) \in C^{r+s-1} \tag{3.8}$$

be any natural chain homotopy between $\wedge$ and $\cup$:

$$\delta B(\omega, \eta) + B(d\,\omega, \eta) + (-1)^{|\omega|} B(\omega, d\,\eta) = \omega \wedge \eta - \omega \cup \eta$$

(any two choices of B are naturally chain homotopic). The product of cochains (c_1, h_1, ω_1) and (c_2, h_2, ω_2) is given by the formula

$$(c_1 \cup c_2, (-1)^{|c_1|} c_1 \cup h_2 + h_1 \cup \omega_2 + B(\omega_1, \omega_2), \omega_1 \wedge \omega_2).$$

As described in section 2.3 the group

$$\check{H}(2)^2(M)$$

can be identified with the group of isomorphism classes of $U(1)$ bundles with connection on M, and the complex $\check{C}(2)^*(M)$ can be used to give a more refined description of the whole category of $U(1)$-bundles with connection on M. We refer the reader back to §2.3 for an interpretation of the groups $\check{H}(k)^k(M)$ and of the complex $\check{C}(k)^*(M)$ for $k < 2$.

3.3. Characteristic classes. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and let $W(\mathfrak{g})$ be the Weyl algebra of polynomial functions on $\mathfrak{g}$ invariant under the adjoint action. We will regard $W(\mathfrak{g})$ as a differential graded algebra, with differential 0, and graded in such a way that $W(\mathfrak{g})^{2n}$ consists of the polynomials of degree n, and $W(\mathfrak{g})^{\mathrm{odd}} = 0$. Using [**53**, **54**] choose a system of smooth n-classifying spaces $B^{(n)}(G)$ and compatible connections on the universal bundles. Write $BG = \varinjlim B^{(n)}G$,

$$C^*(BG;\mathbb{Z}) = \varprojlim C^*(B^{(n)}G;\mathbb{Z})$$
$$C^*(BG;\mathbb{R}) = \varprojlim C^*(B^{(n)}G;\mathbb{R}),$$

∇_{univ} for the universal connection and Ω_{univ} for its curvature. The Chern–Weil homomorphism (with respect to the universal connection) is a co-chain homotopy equivalence

$$W(\mathfrak{g}) \xrightarrow{\approx} C^*(BG;\mathbb{R}).$$

Define $\check{C}^*(BG)$ by the homotopy Cartesian square

$$\begin{array}{ccc} \check{C}^*(BG) & \longrightarrow & W(\mathfrak{g}) \\ \big\downarrow & & \big\downarrow{\scriptstyle \text{Chern–Weil}} \\ C^*(BG;\mathbb{Z}) & \longrightarrow & C^*(BG;\mathbb{R}). \end{array}$$

The map $\check{C}^*(BG) \to C^*(BG)$ is then a cochain homotopy equivalence, and we can take $\check{C}^*(BG)$ as a complex of integer cochains on BG. Specifically, an element $\check{C}^n(BG)$ is a triple

$$(c,h,w) \in C^n(BG;\mathbb{Z}) \times C^{n-1}(BG;\mathbb{R}) \times W(\mathfrak{g})^n$$

and the differential is given by

$$d(c,h,w) = (\delta c, w - c - \delta h, dw) = (\delta c, w - c - \delta h, 0).$$

Suppose that M is a manifold, and $f : M \to BG$ is a map classifying a principal G-bundle $P \to M$. The map f is determined up to homotopy by P. A *characteristic class for principal G-bundles* is a cohomology class $\gamma \in H^k(BG)$. Because of the homotopy invariance of cohomology, the class $f^*\gamma \in H^kM$ depends only on $P \to M$ and not on the map f which classifies P.

Now suppose we have, in addition to the above data, a connection ∇ on P (with curvature Ω), and that the map f is smooth. Choose a cocycle $(c, h, w) \in \check{C}^k(BG)$ representing γ. As in [**21**] consider the connection

$$\nabla_t = (1-t) f^* \nabla_{\text{univ}} + t\nabla$$

on $P \times I$, with curvature Ω_t. Then the form

$$\eta = \int_0^1 w(\Omega_t)$$

satisfies

$$\begin{aligned} d\eta &= w(\Omega_1) - w(\Omega_0) \\ &= w(\Omega) - f^* w(\Omega_{\text{univ}}). \end{aligned}$$

We can associate to this data the *characteristic differential cocycle*

$$(f^*c, f^*h + \eta, w(\Omega)) \in \check{Z}(k)^k(M). \tag{3.9}$$

The characteristic differential cocycle depends on a connection on P, a (universal) choice of cocycle representing γ, and a smooth map f classifying P (but not the connection). Since the differential in $W(\mathfrak{g})$ is zero, varying (c, h, w) by a coboundary changes (3.9) by the coboundary of a class in $\check{C}(k)^{k-1}(M)$. Varying f by a smooth homotopy also results in a change of (3.9) by the coboundary of a class in $\check{C}(k)^{k-1}(M)$. It follows that the underlying cohomology class $\check{\gamma}$ of (3.9) depends only on the principal bundle P and the connection ∇. In this way we recover the result of Cheeger-Simons which, in the presence of a connection, refines an integer characteristic class to a differential cocycle.

To summarize, given a principal G-bundle $P \to M$ and a cohomology class $\gamma \in H^k(BG; Z)$ one has a characteristic class

$$\gamma(P) \in H^k(M; \mathbb{Z}).$$

A choice of connection ∇ on P gives a refinement of $\gamma(P)$ to a cohomology class

$$\check{\gamma}(P, \nabla) \in \check{H}(k)^k(M; \mathbb{Z}).$$

A choice of cocycle (in $\check{C}^*(BG)$) representing γ, and a map $M \to BG$ classifying P gives a cocycle representative of $\check{\gamma}(P, \nabla)$.

Now suppose that V is an oriented orthogonal representation of G of dimension n, and $P \to M$ is a principal G-bundle. Associated to the orientation is a Thom class $U \in H^n(\bar{V}; \mathbb{Z})$. By the results of Mathai-Quillen [**43**, §6], and the methods above, a choice of connection ∇ on P gives a differential Thom cohomology class $[\check{U}] \in \check{H}(n)^n(\bar{V})$, and the additional choice of a map $M \to BG$ classifying P gives a differential Thom cocycle $\check{U} \in \check{Z}(n)^n(\bar{V})$.

3.4. Integration. We begin with the *integration map*

$$\int_{S\times\bar{\mathbb{R}}^N/S} : \check{C}(p+N)_c^{q+N}(S\times\mathbb{R}^N)\to\check{C}(p)^q(S).$$

Choose a fundamental cycle

$$Z_N\in C_N(\mathbb{R}^N;\mathbb{Z})$$

(for example by choosing Z_1 and then taking Z_N to be the N-fold product), and map

$$(c,h,\omega)\mapsto\left(c/Z_N,h/Z_N,\int_{S\times\bar{\mathbb{R}}^N/S}\omega\right), \tag{3.10}$$

where a/b denotes the slant product, as described in [**62**]. One checks that the cochain ω/Z_N coincides with the form $\int_{S\times\bar{\mathbb{R}}^N/S}\omega$ (regarded as a cochain), so the above expression could be written

$$(c/Z_N,h/Z_N,\omega/Z_N).$$

It will be convenient to write this expression as

$$x/Z_N$$

with $x=(c,h,\omega)$. It follows immediately that (3.10) is a map of complexes, since "slant product" with a closed chain is a map of complexes.

Definition 3.11. Suppose that $p:E\to S$ is an $\check{H}$-oriented map of smooth manifolds with boundary of relative dimension k. The "integration" map

$$\int_{E/S} : \check{C}(p+k)^{q+k}(E)\to\check{C}(p)^q(S)$$

is defined to be the composite

$$\check{C}(p+k)^{q+k}(E)\xrightarrow{\cup U}\check{C}(p+N)_c^{q+N}(S\times\mathbb{R}^N)\xrightarrow{\int_{\mathbb{R}^N}}\check{C}(p)_c^q(S).$$

Remark 3.12. In the terminology of §2.3, when E/S is a fibration over an open dense subspace of S, the map $\int_{E/S}$ commutes with the formation of the "connection form"; ie

$$\text{connection form}\left(\int_{E/S}(c,h,\omega)\right)=\int_{E/S}\omega$$

in which the right hand integral indicates ordinary integration over the fibers. In particular, the connection form of $\int_{E/S}(c,h,\omega)$ depends only on the orientation of the relative normal bundle, and not the other choices that go into the $\check{H}$-orientation of E/S. As we will see at the end of this section, up to natural isomorphism the *integration functor* depends only on the orientation of the relative normal bundle.

Suppose that

$$\begin{array}{ccc} E_1 & \xrightarrow{f_E} & E_2 \\ \Big\downarrow & & \Big\downarrow \\ S_1 & \xrightarrow[f_S]{} & S_2 \end{array}$$

is a transverse pullback square in which f_S is a closed embedding. An $\check{H}$-orientation of E_2/S_2 induces an $\check{H}$-orientation of E_1/S_1, and the resulting integration functors are compatible with base change in the sense that

$$f_S^*\left(\int_{E_2/S_2} x\right) = \int_{E_1/S_1} f_E^* x.$$

We now turn to our version of Stokes Theorem as described in §2.3. Let $p : E \to S$ be an orientable map in which E is a manifold with boundary, and S is closed. Choose a defining function

$$f : E \to [0,1]$$

for the boundary of E. Then

$$f \times p : E \to [0,1] \times S$$

is a neat map of manifolds with boundary, and

$$\begin{array}{ccc} \partial E & \longrightarrow & E \\ \Big\downarrow & & \Big\downarrow \\ \partial[0,1] \times S & \xrightarrow[\iota]{} & [0,1] \times S \end{array}$$

is a transverse pullback square. Choose an $\check{H}$-orientation of $f \times p$, and let Z_I be the fundamental chain of $I = [0,1]$. The expression

$$\int_0^1 \int_{E/[0,1]\times S} x := \left(\int_{E/[0,1]\times S} x\right) \Big/ Z_I$$

satisfies Stokes theorem:

(3.13)

$$\delta \int_0^1 \int_{E/[0,1]\times S} x = \int_0^1 \int_{E/[0,1]\times S} \delta x - (-1)^{|x|} \left(\int_{E_1/S} x - \int_{E_0/S} x\right)$$

where $E_i = f^{-1}(i)$. This follows easily from the formula

$$\delta(a/b) = (\delta a)/b + (-1)^{|a|+|b|} a/\partial b$$

and naturality. For a discussion of this sign, and the slant product in general, see §3.5.

When x is closed, (3.13) can be re-written as

$$\delta \int_0^1 \int_{E/[0,1]\times S} x = -\int_{\partial E/S} x.$$

Put more prosaically, this says that up to sign

$$\int_0^1 \int_{E/[0,1]\times S} x$$

is a trivialization of

$$\int_{\partial E/S} x.$$

Returning to the situation of Remark 3.12, suppose given two $\check{H}$-orientations μ_0 and μ_1 of E/S refining the same orientation of the relative normal bundle. To distinguish the two integration functors we will write them as

$$\int_{E/S} (\,-\,)\, d\mu_0 \quad \text{and} \quad \int_{E/S} (\,-\,)\, d\mu_1.$$

Choose an $\check{H}$-orientation μ of $E \times \Delta^1 \to S \times \Delta^1$ restricting to μ_i at $E \times \{i\}$. For a differential cocycle $\alpha = (c, h, \omega) \in \check{Z}(p)^q(E)$ set

$$\beta = \int_0^1 \int_{E\times\Delta^1/S\times\Delta^1} p_2^*\alpha \, d\mu,$$

with $p_2 : E \times \Delta^1 \to E$ the projection. By the above

$$d\beta = \int_{E/S} \alpha \, d\mu_1 - \int_{E/S} \alpha \, d\mu_0.$$

Now with no assumption, β is a cochain in $\check{C}(p-n-1)^{q-n-1}$. But since E/S is a fibration over an open dense set, and integration commutes with the formation of the curvature form, the curvature form of β is zero. It follows that

$$\beta \in \check{C}(p-n-1)^{q-n}.$$

This construction can then be regarded as giving a natural isomorphism between

$$\int_{E/S} (\,-\,)\, d\mu_0 \quad \text{and} \quad \int_{E/S} (\,-\,)\, d\mu_1.$$

3.5. Slant products. In this section we summarize what is needed to extend the definition of the slant product from singular cochains to differential cochains. The main thing to check is that the slant product of a form along a smooth chain is again a form (Lemma 3.15 below).

Suppose that M and N are smooth manifolds, and R a ring. The *slant product* is the map of complexes

$$C^{p+q}(M \times N; R) \otimes C_q(N) \to C^p(M; R) \tag{3.14}$$

adjoint to the contraction

$$\begin{aligned} &C^{p+q}(M \times N; R) \otimes C_p(M) \otimes C_q(N) \\ &\qquad\qquad \to C^{p+q}(M \times N; R) \otimes C_{p+q}(M \times N) \to R. \end{aligned}$$

There is a sign here, which is made troublesome by the fact that the usual convention for the differential in the cochain complex does not make it the dual of the chain complex (i.e. the evaluation map is not a map of complexes). To clarify, the map (3.14) is a map of complexes, provided the differential in $C_*(M)$ is modified to be

$$b \mapsto (-1)^{|b|}\partial b.$$

Thus the relationship between the (co-)boundary and the slant product is given by

$$\delta(a/b) = (\delta a)/b + (-1)^{|a|+|b|} a/\partial b.$$

To extend the slant product to the complex of differential cochains

$$\check{C}*(M \times N) \otimes C_*(N) \to \check{C}^*(M)$$

amounts to producing maps

$$\Omega^*(M \times N) \otimes C_*(N) \xrightarrow{/} \Omega^*(M)$$

$$C^*(M \times N; \mathbb{R}) \otimes C_*(N) \xrightarrow{/} C^*(M; \mathbb{R})$$

$$C^*(M \times N; \mathbb{Z}) \otimes C_*(N) \xrightarrow{/} C^*(M; \mathbb{R})$$

compatible with the inclusions

$$C^*(\,-\,;\mathbb{Z}) \hookrightarrow C^*(\,-\,;\mathbb{R}) \hookleftarrow \Omega^*(\,-\,;\mathbb{R}).$$

This reduces to checking that the slant product of a form along a smooth chain is again a form.

Lemma 3.15. *Suppose that ω is a $(p+q)$-form on $M \times N$, regarded as a cochain, and that Z_p and Z_q are p and q-chains on M and N respectively. Then the value of ω/Z_q on Z_p is*

$$\int_{Z_p \times Z_q} \omega.$$

In other words, the cochain ω/Z_q is represented by the form

$$\int_{M \times Z_q/M} \omega.$$

Proof. By naturality we are reduced to the case in which

$$M = \Delta^p$$
$$N = \Delta^q$$

and Z_p and Z_q are the identity maps respectively. The value of ω/Z_q on Z_p is

$$\int_{Z_p \otimes Z_q} \omega.$$

But $Z_p \otimes Z_q$ is by definition the sum of all of the non-degenerate $(p+q)$-simplices of $\Delta^p \times \Delta^q$, with orientation derived from that of $\Delta^p \times \Delta^q$. In other words, $Z_p \otimes Z_q$ is the fundamental chain of of $\Delta^p \times \Delta^q$. It follows that

$$\int_{Z_p \otimes Z_q} \omega = \int_{Z_p \times Z_q} \omega.$$

q.e.d.

4. Generalized differential cohomology

4.1. Differential function spaces. In §3 we introduced the cohomology groups $\check{H}(q)^n(S)$ by combining differential forms and ordinary cohomology. These groups are formed from triples (c, h, ω) with c a cocycle, h a cochain and ω a form. For practical purposes an n-cocycle can be regarded[7] as a map to the Eilenberg-MacLane space $K(\mathbb{Z}, n)$. We take this as our point of departure, and in this section shift the emphasis from cocycles to maps. Consider a topological space X (with no particular smooth structure), a cocycle $\iota \in Z^n(X; \mathbb{R})$, and a smooth manifold S.

Definition 4.1. A *differential function* $t : S \to (X; \iota)$ is a triple (c, h, ω)

$$c : S \to X, \quad h \in C^{n-1}(S; \mathbb{R}), \quad \omega \in \Omega^n(S)$$

satisfying

$$\delta\, h = \omega - c^* \iota.$$

Example 4.2. Suppose X is the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, which we can take to be CP^∞, though we don't make use of its smooth structure. Choose a two-cocycle ι_Z representing the first Chern class of the universal line bundle L, and let ι be its image in $\mathbb{Z}^2(X; \mathbb{R})$. As described in Example 2.7, to refine a map c to a differential function amounts to putting a connection on the line bundle $c^* L$. The $U(1)$-bundle with connection is the one associated to

$$(c^* \iota_Z, h, \omega) \in Z(2)^2(S).$$

To form a *space* of differential functions we use the *singular complex* of space X^S—a combinatorial object that retains the homotopy type of the function space. This requires the language of simplicial sets. We have outlined what is needed in Appendix A. For further details see [**31**, **45**, **22**]. Rest assured that we recover the complex for differential cohomology. (See Example 4.10, and Appendix D.)

[7]It is well-known [**61**] that for a CW complex S, the set $[S, K(\mathbb{Z}, n)]$ can be identified with the cohomology group $H^n(S; \mathbb{Z})$. In particular, any cocycle is cohomologous to one which is pulled back from a map to an Eilenberg-MacLane space. A more refined statement is that the space of maps from S to $K(\mathbb{Z}, n)$ has the homotopy type of the "space of n-cocycles" on S. For a more precise discussion see Appendix A

Let

$$\Delta^n = \{(t_0, \dots, t_n) \mid 0 \le t_i \le 1, \sum t_i = 1\}$$

be the standard n-simplex. The *singular complex* of a space M is the simplicial set $\operatorname{sing} M$ with n-simplices

$$\operatorname{sing} M_n = \{z : \Delta^n \to M\}\,.$$

The evaluation map (A.5)

$$|\operatorname{sing} M| \to M$$

induces an isomorphism of homotopy groups and of singular homology groups. The simplicial set $\operatorname{sing} M$ always satisfies the Kan extension condition (A.8), so the maps

$$\pi_n^{\mathrm{simp}} \operatorname{sing} M \to \pi_n |\operatorname{sing} M| \to \pi_n(M)$$

are all isomorphisms.

The singular complex of the function space X^S is the simplicial set whose k-simplices are maps

$$S \times \Delta^k \to X.$$

Definition 4.3. Suppose X is a space, $\iota \in Z^n(X; \mathbb{R})$ a cocycle, and S a smooth manifold. The *differential function complex*

$$\check{X}^S = (X; \iota)^S$$

is the simplicial set whose k-simplices are differential functions

$$S \times \Delta^k \to (X; \iota).$$

Remark 4.4. Phrased slightly differently, a k-simplex of $(X;\iota)^S$ is a k-simplex

$$c : \Delta^k \times S \to X$$

of X^S, together with a refinement of $c^*\iota$ to a differential cocycle. We will refer to the differential cocycle refining $c^*\iota$ as the *underlying differential cocycle.*

The differential function complex has an important filtration. Define a filtration of the deRham complex of Δ^k by

$$\operatorname{filt}_s \Omega^*(\Delta^k) = \left(\Omega^0(\Delta^k) \to \cdots \to \Omega^s(\Delta^k)\right),$$

and let $\operatorname{filt}_s \Omega^*(S \times \Delta^k)$ be the subspace generated by

$$\Omega^*(S) \otimes \operatorname{filt}_s \Omega^*(\Delta^k).$$

In other words, a form $\omega \in \Omega^*(S \times \Delta^k)$ lies in filtration $\le s$ if it vanishes on all sequences of vectors containing $(s+1)$ vectors tangent to Δ^k. Note that exterior differentiation shifts this filtration:

$$d : \operatorname{filt}_s \to \operatorname{filt}_{s+1}\,.$$

Definition 4.5. With the notation of Definition 4.3, a k-simplex (c, h, ω) of the differential function complex $(X; \iota)^S$ has *weight filtration* $\leq s$ if the form ω satisfies

$$\omega \in \operatorname{filt}_s \Omega^n(S \times \Delta^k)_{\mathrm{cl}}.$$

We will use the notation

$$\operatorname{filt}_s(X; \iota)^S$$

to denote the sub-simplicial set of $(X; \iota)^S$ consisting of elements of weight filtration $\leq s$.

Example 4.6. Continuing with Example 4.2, a 1-simplex of $(CP^\infty, \iota)^S$ gives rise to a $U(1)$-bundle with connection L over $S \times [0, 1]$, which, in turn, leads to an isomorphism of $U(1)$-bundles

$$L|_{S\times\{0\}} \to L|_{S\times\{1\}} \tag{4.7}$$

by parallel transport along the paths

$$t \mapsto (x, t).$$

Note that this isomorphism need *not* preserve the connections.

Example 4.8. A 1-simplex of $\operatorname{filt}_0(CP^\infty; \iota)^S$ also gives rise to a $U(1)$-bundle with connection L over $S \times [0, 1]$. But this time the curvature form must be pulled back[8] from a form on S. In this case the isomorphism (4.7) is *horizontal* in the sense that it *does* preserve the connections.

Remark 4.9. We will show later (Lemma D.2 and Proposition D.5) that this construction leads to a simplicial homotopy equivalence between $\operatorname{filt}_s(CP^\infty; \iota)^S$ and the simplicial abelian group associated to the chain complex

$$Z(2)^2(S) \leftarrow C(2)^1(S) \leftarrow C(0)^0(S).$$

Thus the fundamental groupoid[9] of $\operatorname{filt}_0(CP^\infty; \iota)^S$ is equivalent to the groupoid of $U(1)$-bundles with connection over S. The fundamental groupoid of $\operatorname{filt}_1(CP^\infty; \iota)^S$ is equivalent to the groupoid of $U(1)$-bundles over S and isomorphisms, and the fundamental groupoid of $\operatorname{filt}_2(CP^\infty; \iota)^S$ is equivalent to the fundamental groupoid of $(CP^\infty)^S$

[8]By definition it must be of the form $g(t)\omega$, where ω is independent of t. But it is also *closed*, so g must be constant.

[9]A more precise description of the relationship between the category of line bundles and the simplicial set $(CP^\infty; \iota)^S$ can be formulated in terms of the *fundamental groupoid*. Recall that the fundamental groupoid [**60**] of a space E is the groupoid $\pi_{\leq 1}E$ whose objects are the points of E, and in which a morphism from x to y is an equivalence of paths starting at x and ending at y. The equivalence relation is that of homotopy relative to the endpoints, and composition of maps is formed by concatenation of paths. The fundamental groupoid of a simplicial set is defined analogously.

which is equivalent to the category of principal $U(1)$-bundles over S, and homotopy classes of isomorphisms.

Example 4.10. Take X to be the Eilenberg-MacLane space $K(\mathbb{Z}, n)$, and let $\iota_{\mathbb{R}} \in Z^n(X; \mathbb{R})$ be a fundamental cocycle: a cocycle whose underlying cohomology class corresponds to the inclusion $\mathbb{Z} \subset \mathbb{R}$ under the isomorphism

$$H^n(K(Z, n); \mathbb{R}) \approx \hom(\mathbb{Z}, \mathbb{R}).$$

We show in Appendix D that the simplicial set

$$\operatorname{filt}_s(X; \iota)^S$$

is a homotopy equivalent to the simplicial abelian group associated with the chain complex (§3.2)

$$\check{Z}(n-s)^n(S) \leftarrow \check{C}(n-s)^{n-1}(S) \ldots \leftarrow \check{C}(n-s)^0(S).$$

The equivalence is given by slant product of the underlying differential cocycle with the fundamental class of the variable simplex. It follows (Proposition A.11) that the homotopy groups of $\operatorname{filt}_s \check{X}^S$ are given by

$$\pi_i \operatorname{filt}_s \check{X}^S = \check{H}(n-s)^{n-i}(S),$$

and we recover the differential cohomology of S. In this way the homotopy groups of differential function spaces generalize the Cheeger-Simons cohomology groups.

Remark 4.11. We will also be interested in the situation in which we have several cocycles ι of varying degrees. These can be regarded as a single cocycle with values in a graded vector space. We will use the convention

$$\mathcal{V}_j = \mathcal{V}^{-j},$$

and so grade cochains and forms with values in a graded vector space $\mathcal{V}$ in such a way that the $C^i(X; \mathcal{V}_j)$ and $\Omega^i(X; \mathcal{V}_j)$ have total degree $(i-j)$. We will write

$$C^*(X; \mathcal{V})^n = \bigoplus_{i+j=n} C^i(X; \mathcal{V}^j)$$

$$\Omega^*(S; \mathcal{V})^n = \bigoplus_{i+j=n} \Omega^i(X; \mathcal{V}^j)$$

$$Z^*(X; \mathcal{V})^n = \bigoplus_{i+j=n} Z^i(X; \mathcal{V}^j),$$

and

$$H^n(X; \mathcal{V}) = \bigoplus_{i+j=n} H^i(X; \mathcal{V}^j).$$

We define $\operatorname{filt}_s \Omega^*(S \times \Delta^k; \mathcal{V})$ to be the subspace generated by

$$\Omega^*(S; \mathcal{V}) \otimes \operatorname{filt}_s \Omega^*(\Delta^k).$$

As before, a form $\omega \in \Omega^*(S \times \Delta^k; \mathcal{V})$ lies in filtration $\leq s$ if it vanishes on all sequences of vectors containing $(s+1)$ vectors tangent to Δ^k.

We now turn to the analogue of the square (3.2) and the second of the fundamental exact sequences (3.3). Using the equivalence between simplicial abelian groups and chain complexes (see §A.3), we can fit the differential function complex $(X;\iota)^S$ into a homotopy Cartesian square

$$\begin{array}{ccc} \operatorname{filt}_s(X;\iota)^S & \longrightarrow & \operatorname{filt}_s \Omega^*(S \times \Delta^\bullet; \mathcal{V})^n_{\mathrm{cl}} \\ \Big\downarrow & & \Big\downarrow \\ \operatorname{sing} X^S & \longrightarrow & Z^*(S \times \Delta^\bullet; \mathcal{V})^n. \end{array} \tag{4.12}$$

By Corollary D.15

$$\pi_m Z(S \times \Delta^\bullet; \mathcal{V})^n = H^{n-m}(S;\mathcal{V})$$

$$\pi_m \operatorname{filt}_s \Omega^*(S \times \Delta^\bullet; \mathcal{V})^n_{\mathrm{cl}} = \begin{cases} H^{n-m}_{\mathrm{DR}}(S;\mathcal{V}) & m < s \\ \Omega^*(S;\mathcal{V})^{n-s}_{\mathrm{cl}} & m = s \\ 0 & m > s. \end{cases}$$

This gives isomorphisms

$$\pi_k \operatorname{filt}_s(X;\iota)^S \xrightarrow{\approx} \pi_k X^S \qquad k < s$$

$$\pi_k \operatorname{filt}_s(X;\iota)^S \xrightarrow{\approx} H^{n-k-1}(S;\mathcal{V})/\pi_{k+1} X^S \qquad k > s,$$

and a short exact sequence

$$H^{n-s-1}(S;\mathcal{V})/\pi_{s+1}X^S \rightarrowtail \pi_s \operatorname{filt}_s(X;\iota)^S \twoheadrightarrow A^{n-s}(S;X,\iota),$$

where $A^{n-s}(S;X,\iota)$ is defined by the pullback square

$$\begin{array}{ccc} A^{n-s}(S;X,\iota) & \longrightarrow & \Omega(S;\mathcal{V})^{n-s}_{\mathrm{cl}} \\ \Big\downarrow & & \Big\downarrow \\ \pi_s X^S & \longrightarrow & H^{n-s}(S;\mathcal{V}). \end{array}$$

4.2. Naturality and homotopy. We now describe how the the differential function complex $(X;\iota)^S$ depends on X, S, and ι. First of all, a smooth map

$$g : S \to T$$

gives a map

$$\operatorname{filt}_s(X;\iota)^T \to \operatorname{filt}_s(X;\iota)^S$$

sending (c,h,ω) to $(c,h,\omega) \circ g = (c \circ g, c^*h, c^*\omega)$. We will refer to this map as *composition with g*.

Given a map $f : X \to Y$ and a cocycle

$$\iota \in Z^n(Y;\mathcal{V}),$$

composition with f gives a map

$$\check{f} : \operatorname{filt}_s(X; f^*\iota)^S \to \operatorname{filt}_s(Y;\iota)^S$$

sending (c,h,ω) to $(c\circ f,h,\omega)$.

Proposition 4.13. *Suppose that $f : X \to Y$ is a (weak) homotopy equivalence, and $\iota \in Z^n(Y;\mathcal{V})$ is a cocycle. Then for each manifold S, the map*

$$\check{f} : \operatorname{filt}_s(X; f^*\iota)^S \to \operatorname{filt}_s(Y;\iota)^S$$

is a (weak) homotopy equivalence.

Proof. When f is a homotopy equivalence, the vertical maps in the following diagram are homotopy equivalences (two of them are the identity map).

$$\begin{tikzcd}
\operatorname{sing} X^S \arrow[r] \arrow[d] & Z^*(S\times\Delta^\bullet;\mathcal{V})^n \arrow[d] & \operatorname{filt}_s \Omega^*(S\times\Delta^\bullet;\mathcal{V})^n_{\mathrm{cl}} \arrow[l] \arrow[d] \\
\operatorname{sing} Y^S \arrow[r] & Z^*(S\times\Delta^\bullet;\mathcal{V})^n & \operatorname{filt}_s \Omega^*(S\times\Delta^\bullet;\mathcal{V})^n_{\mathrm{cl}} \arrow[l]
\end{tikzcd}$$

It follows that the map of homotopy pullbacks (see (4.12))

$$\check{f} : \operatorname{filt}_s(X; f^*\iota)^S \to \operatorname{filt}_s(Y;\iota)^S \tag{4.14}$$

is a homotopy equivalence. If the map f is merely a weak equivalence, one needs to use the fact that a manifold with corners has the homotopy type of a CW complex to conclude that the left vertical map is a weak equivalence. Since the formation of homotopy pullbacks preserves weak equivalences the claim again follows. q.e.d.

Remark 4.15. Suppose that we are given a homotopy

$$H : X\times[0,1]\to Y,$$

with $H(x,0)=f(x)$ and $H(x,1)=g(x)$. We then have a diagram of differential function spaces

$$\begin{tikzcd}
\operatorname{filt}_s(X,f^*\iota)^S \arrow[r] \arrow[dr, "\check{f}"'] & \operatorname{filt}_s(X\times I;H^*\iota)^S \arrow[d] & \operatorname{filt}_s(X;g^*\iota)^S \arrow[l] \arrow[dl, "\check{g}"] \\
& \operatorname{filt}_s(Y,\iota)^S . &
\end{tikzcd}$$

By Proposition 4.13, the horizontal maps are homotopy equivalences, and so the construction can be regarded as giving a homotopy between $\operatorname{filt}_s \check{f}$ and $\operatorname{filt}_s \check{g}$.

Given two cocycles $\iota_1,\iota_2\in Z^*(X;\mathcal{V})^n$, and a cochain $b\in C^*(X;\mathcal{V})^{n-1}$ with $\delta b=\iota_1-\iota_2$, we get a map

$$\begin{aligned}(X;\iota_1)^S &\to (X;\iota_2)^S\\ (c,h,\omega)&\mapsto(c,h+c^*b,\omega).\end{aligned} \tag{4.16}$$

This map is an isomorphism, with inverse $-b$. In particular, the group $Z^*(X;\mathcal{V})^{n-1}$ acts on the differential function complex $(X;\iota)^S$.

Finally, suppose given a map $t : \mathcal{V} \to \mathcal{W}$ of graded vector spaces, and a cocycle $\iota \in Z^n(X; \mathcal{V})$. Composition with t defines a cocycle

$$t \circ \iota \in Z^n(X; W),$$

and a map of differential function complexes

$$(X; \iota)^S \to (X; t \circ \iota)^S .$$

Combining these, we see that given maps

$$f : X \to Y \qquad t : \mathcal{V} \to \mathcal{W},$$

cocycles

$$\iota_X \in Z^n(X; \mathcal{V}) \qquad \iota_Y \in Z^n(Y; W),$$

and a cochain $b \in C^{n-1}(X; W)$ with $\delta b = t \circ \iota_X - f^* \iota_Y$ we get a map of differential function complexes

$$\begin{aligned} (X; \iota_X)^S &\to (Y; \iota_Y)^S \\ (c, h, \omega) &\mapsto (f \circ c, t \circ h + c^* b, t \circ \omega) \end{aligned}$$

which is a weak equivalence when f is a weak equivalence and t is an isomorphism.

Remark 4.17. All of this means that the homotopy type of

$$\mathrm{filt}_s(X, \iota)^S$$

depends only on the cohomology class of ι and and the homotopy type of X. For example, suppose f and g are homotopic maps $X \to Y$, and that $\alpha, \beta \in Z^n(X)$ are cocycles in the cohomology class of $f^*\iota$ and $g^*\iota$ respectively. A choice of cochains

$$\begin{aligned} b_1, b_2 &\in C^{n-1}(X, \mathbb{R}) \\ \delta b_1 &= \alpha - f^* \iota \\ \delta b_2 &= \beta - g^* \iota \end{aligned}$$

gives isomorphisms

$$\begin{aligned} \mathrm{filt}_s(X; \alpha)^S &\approx \mathrm{filt}_s(X; f^* \iota)^S \\ \mathrm{filt}_s(X; \beta)^S &\approx \mathrm{filt}_s(X; g^* \iota)^S, \end{aligned}$$

and defines maps

$$\begin{aligned} \mathrm{filt}_s \check{f} &: \mathrm{filt}_s(X; \alpha)^S \to \mathrm{filt}_s(Y; \iota)^S \\ \mathrm{filt}_s \check{g} &: \mathrm{filt}_s(X; \beta)^S \to \mathrm{filt}_s(Y; \iota)^S. \end{aligned}$$

A homotopy $H : X \times [0, 1] \to Y$ from f to g, then leads to the diagram of Remark 4.15, and hence a homotopy between $\mathrm{filt}_s \check{f}$ and $\mathrm{filt}_s \check{g}$.

4.3. Thom complexes. In this section we describe Thom complexes and the Pontryagin-Thom construction in the context of differential function complexes.

Recall that the *Thom complex* of a vector bundle V over a compact space M is the 1-point compactification of the total space. We will use the notation $\mathrm{Thom}(M;V)$, or $\bar{V}$ to denote the Thom complex. (We are avoiding the more traditional M^V because of the conflict with the notation X^S for function spaces). When M is not compact, one sets

$$\mathrm{Thom}(M;V) = \bar{V} = \bigcup_{\substack{K_\alpha \subset M \\ \text{compact}}} \mathrm{Thom}(K_\alpha; V_\alpha) \qquad V_\alpha = V|_{K_\alpha}.$$

Suppose V is a vector bundle over a manifold M. We will call a map $g : S \to \bar{V}$ *smooth* if its restriction to

$$g^{-1}(V) \to V$$

is smooth. We define the deRham complex $\Omega^*(\bar{V})$ to be the sub-complex of $\Omega^*(V)$ consisting of forms which are fiber-wise compactly supported. With these definitions, the differential function complex

$$\mathrm{filt}_s(X;\iota)^{\bar{V}}$$

is defined, and a smooth map $S \to \bar{V}$ gives a map of differential function complexes

$$\mathrm{filt}_s(X;\iota)^{\bar{V}} \to \mathrm{filt}_s(X;\iota)^S$$

Definition 4.18. Let

$$W_1 \to X \quad \text{and} \quad W_2 \to Y$$

be vector bundles, and $\iota \in Z^k\left(\bar{W}_2, \{\infty\}; \mathcal{W}\right)$ a cocycle. A *(vector) bundle map* $W_1 \to W_2$ is a pullback square

$$\begin{tikzcd} W_1 \arrow[r, "t"] \arrow[d] & W_2 \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

for which the induced isomorphism $W_1 \to f^*W_2$ is an isomorphism of vector bundles. A *differential bundle map* is a differential function

$$\check{t} = (c, h, \omega) : \bar{W}_1 \to \left(\bar{W}_2; \iota\right).$$

which is a bundle map. The *complex of differential bundle maps*

$$\mathrm{filt}_s\left(W_2;\iota\right)^{W_1} \subset \mathrm{filt}_s\left(\bar{W}_2;\iota\right)^{\bar{W}_1}$$

is the subcomplex consisting of vector bundle maps.

In case W_2 is the universal bundle over some kind of classifying space, then a vector bundle map $W_1 \to W_2$ is a classifying map for W_1.

The set of bundle maps is topologized as a closed subspace of the space of all maps from $\mathrm{Thom}(X; W_1) \to \mathrm{Thom}(Y; W_2)$ (which, in turn is a closed subspace of the space of all maps $W_1 \to W_2$).

Suppose that B is a topological space, equipped with a vector bundle V, and $\iota \in Z^k\left(\bar{V}, \{\infty\}\right)$ is a cocycle.

Definition 4.19. A *B-oriented embedding* is a neat embedding $p : E \to S$, a tubular neighborhood[10] $W \hookrightarrow S$ of $p : E \to S$, and a vector bundle map $W \to V$ classifying W. A *differential B-oriented embedding* is a neat embedding $p : E \to S$, a tubular neighborhood $W \hookrightarrow S$ of $p : E \to S$, and a differential vector bundle map

$$\check{t} : W \to (V; \iota)$$

classifying W.

Let $p : E \hookrightarrow S$ be a differential oriented embedding. The construction of Pontryagin-Thom gives a smooth map

$$S \to \mathrm{Thom}(E, W). \tag{4.20}$$

Composition with (4.20) defines the *push-forward*

$$p_! : \mathrm{filt}_s(V; \iota)^W \subset \mathrm{filt}_s\,(X; \iota)^{\mathrm{Thom}(E,W)} \to \mathrm{filt}_s(X; \iota)^S.$$

4.4. Interlude: differential K-theory. Before turning to the case of a general cohomology theory, we apply the ideas of the previous section to the case of K-theory. The resulting *differential K-theory* originally came up in anomaly cancellation problems for D-branes in M-theory [**30**, **29**]. The actual anomaly cancellation requires a refinement of the families index theorem, an ongoing joint work of the authors and Dan Freed.

Let $\mathcal{F}$ be the space of Fredholm operators. We remind the reader that the space $\mathcal{F}$ is a classifying space for K-theory and in particular that any vector bundle can be obtained as the index bundle of a map into $\mathcal{F}$. Let

$$\iota = (\iota_n) \in \prod Z^{2n}(\mathcal{F}; \mathbb{R}) = Z^0(\mathcal{F}; \mathcal{V})$$

be a choice of cocycle representatives for the universal Chern character, so that if $f : S \to \mathcal{F}$ classifies a vector bundle V, then the characteristic class $\mathrm{ch}_n(V) \in H^{2n}(S; \mathbb{R})$ is represented by $f^*\iota_n$.

Definition 4.21. The differential K-group $\check{K}^0(S)$ is the group

$$\pi_0\,\mathrm{filt}_0\,(\mathcal{F}; \iota)^S$$

[10]Recall from footnote 4 that a *tubular neighborhood* of $p : E \hookrightarrow S$ is a vector bundle W over E, and an extension of p to a diffeomorphism of W with a neighborhood of $p(E)$.

Remark 4.22. In other words, an element of $\check{K}^0(S)$ is represented by a triple (c, h, ω) where $c : S \to \mathcal{F}$ is a map, $\omega = (\omega_n)$ is a sequence of $2n$-forms, and $h = (h_n)$ is a sequence of $(2n-1)$ cochains satisfying

$$\delta h = \omega - c^*\iota.$$

Two triples (c^0, h^0, ω^0) and (c^1, h^1, ω^1) are equivalent if there is a (c, h, ω) on $S \times I$, with ω constant in the I direction, and with

$$(c, h, \omega)|_{\{0\}} = (c^0, h^0, \omega^0)$$
$$(c, h, \omega)|_{\{1\}} = (c^1, h^1, \omega^1).$$

We will use the symbol $\check{ch}(\check{V})$ to denote the differential cocycle underlying a differential function $\check{V} : S \to (\mathcal{F}; \iota)$.

The space $\Omega^i\mathcal{F}$ is a classifying space for K^{-i}. Let

$$\iota^{-i} = (\iota^{-i}_{2n-i}) \in \prod Z^{2n-i}(\Omega^i\mathcal{F}; \mathbb{R})$$

be the cocycle obtained by pulling ι back along the evaluation map

$$S^i \times \Omega^i\mathcal{F} \to \mathcal{F}$$

and integrating along S^i (taking the slant product with the fundamental cycle of S^i). The cohomology classes of the ι^{-i}_{2n-i} are the universal even (odd) Chern character classes, when i is even (odd).

Definition 4.23. The differential K-group $\check{K}^{-i}(S)$ is the group

$$\pi_0 \operatorname{filt}_0 \left(\Omega^i\mathcal{F}; \iota^{-i}\right)^S$$

Remark 4.24. As above, an element of $\check{K}^{-i}(S)$ is represented by a triple (c, h, ω) where $c : S \to \Omega^i\mathcal{F}$ is a map, $\omega = (\omega_n)$ is a sequence of $2n - i$-forms, and $h = (h_n)$ is a sequence of $(2n - i - 1)$ cochains satisfying

$$\delta h = \omega - c^*\iota^{-i}.$$

Two triples (c^0, h^0, ω^0) and (c^1, h^1, ω^1) are equivalent if there is a (c, h, ω) on $S \times I$, with ω constant in the I direction, and with

$$(c, h, \omega)|_{\{0\}} = (c^0, h^0, \omega^0)$$
$$(c, h, \omega)|_{\{1\}} = (c^1, h^1, \omega^1).$$

These differential K-groups lie in short exact sequences

$$0 \to K^{-1}(S; \mathbb{R}/\mathbb{Z}) \to \check{K}^0(S) \to \prod \Omega^{2n}(S) \to 0$$
$$0 \to K^{-1}(S) \otimes \mathbb{R}/\mathbb{Z} \to \check{K}^0(S) \to A^0_K(S) \to 0$$

and

$$0 \to K^{-i-1}(S; \mathbb{R}/\mathbb{Z}) \to \check{K}^{-i}(S) \to \prod \Omega^{2n-i}(S) \to 0$$
$$0 \to K^{-i-1}(S) \otimes \mathbb{R}/\mathbb{Z} \to \check{K}^{-i}(S) \to A^{-i}_K(S) \to 0,$$

where A_K^{-i} is defined by the pullback square

$$\begin{array}{ccc} A_K^{-i}(S) & \longrightarrow & \prod \Omega_{\mathrm{cl}}^{2n-i}(S) \\ \big\downarrow & & \big\downarrow \\ K^{2n-i}(S) & \xrightarrow[\mathrm{ch}]{} & \prod H^{2n-i}(S;\mathbb{R}). \end{array}$$

So an element of $A_K^{-i}(S)$ consists of an element $x \in K^{2n-i}(S)$ and a sequence of closed $2n-i$-forms representing the Chern character of x.

Bott periodicity provides a homotopy equivalence

$$\Omega^n \mathcal{F} \approx \Omega^{n+2}\mathcal{F}$$

under which ι_n corresponds to ι_{n+2}. This gives an equivalence of differential function spaces

$$\mathrm{filt}_t \left(\Omega^n \mathcal{F}; \iota_n\right)^S \approx \mathrm{filt}_t \left(\Omega^{n+2}\mathcal{F}; \iota_{n+2}\right)^S$$

and in particular, of differential K-groups

$$\check{K}^{-n}(S) \approx \check{K}^{-n-2}(S).$$

This allows one to define differential K-groups $\check{K}^n$ for $n > 0$, by

$$\check{K}^n(S) = \check{K}^{n-2N}(S) \qquad n - 2N < 0.$$

In [**42**] Lott defines the group $K^{-1}(S;\mathbb{R}/\mathbb{Z})$ in geometric terms, and proves an index theorem, generalizing the index theorem for flat bundles in [**7**]. Lott takes as generators of $K^{-1}(S;\mathbb{R}/\mathbb{Z})$ pairs (V, h) consisting of a (graded) vector bundle V with a connection, and sequence h of odd forms satisfying

$$dh = \text{Chern character forms of } V.$$

Our definition is close in spirit to Lott's, with c corresponding to V, $c^*\iota$ to the Chern character forms of the connection, h to h and $\omega = 0$. In a recent preprint [**41**], Lott constructs the abelian gerbe with connection whose curvature is the 3-form part of the Chern character of the index of a family of self-adjoint Dirac-like operators. In our terminology he constructs the degree 3 part of the differential Chern character going from differential K-theory to differential cohomology.

4.5. Differential cohomology theories. In this section we build on our theory of differential functions and define differential cohomology theories. We explained in the introduction why we wish to do so.

Let E be a cohomology theory. The spaces representing E cohomology groups ($\Omega^n\mathcal{F}$ in the case of K-theory, Eilenberg-MacLane spaces for ordinary cohomology) fit together to form a *spectrum*.

Definition 4.25 (see [**40**, **35**, **27**, **28**, **1**])**.** A *spectrum* E consists of a sequence of pointed spaces E_n, $n = 0, 1, 2, \dots$ together with maps

$$s_n^E : \Sigma E_n \to E_{n+1} \tag{4.26}$$

whose adjoints

$$t_n^E : E_n \to \Omega E_{n+1} \tag{4.27}$$

are homeomorphisms.

Remark 4.28. By setting, for $n > 0$

$$E_{-n} = \Omega^n E_0 = \Omega^{n+k} E_k,$$

a spectrum determines a sequence of spaces E_n, $n \in \mathbb{Z}$ together with homeomorphisms

$$t_n^E : E_n \to \Omega E_{n+1}.$$

If X is a pointed space and E is a spectrum, then E-cohomology groups of X are given by

$$E^k(X) = [X, E_k] = [\Sigma^N X, E_{N+k}]$$

and the E-homology groups are given by

$$E_k(X) = \lim_{N\to\infty} \pi_{N+k} E_N \wedge X.$$

Example 4.29. Let A be an abelian group, and $K(A,n)$ an Eilenberg-MacLane space. Then $[S, K(A,n)] = H^n(S; A)$. To assemble these into a *spectrum* we need to construct homeomorphisms

$$K(A,n) \to \Omega K(A, n+1). \tag{4.30}$$

By standard algebraic topology methods, one can choose the spaces $K(A,n)$ so that (4.30) is a closed inclusion. Replacing $K(A,n)$ with

$$\varinjlim \Omega^N K(A, N+n)$$

then makes (4.30) a homeomorphism. This is the *Eilenberg-MacLane spectrum*, and is denoted HA. By construction

$$HA^n(S) = H^n(S; A)$$
$$HA_n(S) = H_n(S; A).$$

Example 4.31. Now take $E_{2n} = \mathcal{F}$, and $E_{2n-1} = \Omega\mathcal{F}$. We have a homeomorphism $E_{2n-1} = \Omega E_{2n}$ by definition, and a homotopy equivalence $E_{2n} \to \Omega E_{2n+1}$ by Bott periodicity:

$$E_{2n} \to \Omega^2 E_{4k} = \Omega E_{2n+1}.$$

As in Example 4.29, the spaces E_n can be modified so that the maps $E_n \to \Omega E_{n+1}$ form a spectrum, the K-theory spectrum.

Multiplication with the fundamental cycle Z_{S^1} of S^1 gives a map of singular chain complexes

$$C_* E_n \to C_{*+1} \Sigma E_n \to C_{*+1} E_{n+1}.$$

Definition 4.32. Let E be a spectrum. The *singular chain complex* of E is the complex

$$C_*(E) = \varinjlim C_{*+n} E_n,$$

and the *singular cochain complex* of E (with coefficients in an abelian group A) is the cochain complex

$$C^*(E; A) = \hom(C_* E, A) = \varprojlim (C^{*+n} E_n) \subset \prod C^{*+n} E_n.$$

The *homology* and *cohomology* groups of E are the homology groups of the complexes $C_*(E)$ and $C^* E$. Note that these groups can be non-zero even when k is negative.

Now fix a cocycle $\iota \in Z^p(E; \mathcal{V})$. By definition, this means that there are cocycles $\iota_n \in Z^{p+n}(E_n; \mathcal{V})$, $n \in \mathbb{Z}$ which are compatible in the sense that

$$\iota_n = (s_n^* \, \iota_{n+1}) \, / Z_{S^1}.$$

Once in a while it will be convenient to denote the pair $(E_n; \iota_n)$ as

$$(E_n; \iota_n) = (E; \iota)_n. \tag{4.33}$$

Definition 4.34. Let S be a manifold. The *differential E-cohomology group*

$$E(n-s)^n(S; \iota)$$

is the homotopy group

$$\pi_0 \operatorname{filt}_s (E_n; \iota_n)^S = \pi_0 \operatorname{filt}_s (E; \iota)_n^S.$$

Remark 4.35. We will see in the next section that the spaces

$$(E_n; \iota_n)^S$$

come equipped with homotopy equivalences

$$\operatorname{filt}_{s+n}(E_n; \iota_n)^S \to \Omega \operatorname{filt}_{s+n+1}(E_{n+1}; \iota_{n+1})^S. \tag{4.36}$$

Thus there exists a spectrum

$$\operatorname{filt}_s(E; \iota)^S$$

with

$$\left(\operatorname{filt}_s(E; \iota)^S\right)_n = \operatorname{filt}_{s+n}(E; \iota)_n^S = \operatorname{filt}_{s+n}(E_n; \iota_n)^S,$$

and the higher homotopy group

$$\pi_t \operatorname{filt}_s (E_n; \iota_n)^S$$

is isomorphic to

$$\pi_0 \operatorname{filt}_{s-t} (E_{n-t}; \iota_{n-t})^S = E(n-s)^{n-t}(S; \iota).$$

Example 4.37. We will show in Appendix D that when E is the Eilenberg-MacLane spectrum $H\mathbb{Z}$, one has

$$H\mathbb{Z}(n)^k(S) = \check{H}(n)^k(S).$$

4.6. Differential function spectra. In this section we will establish the homotopy equivalence (4.36). As a consequence there exists a *differential function spectrum* $\operatorname{filt}_s(E;\iota)^S$ whose n^{th} space has the homotopy type of $\operatorname{filt}_{s+n}(E_n;\iota_n)^S$.

There is one point here on which we have been deliberately vague, and which needs to be clarified. Differential function complexes are not spaces, but simplicial sets. Of course they can be made into spaces by forming their geometric realizations, and, since they are Kan complexes, no homotopy theoretic information is gained or lost by doing so. But this means that the notation $\Omega\operatorname{filt}_{s+n+1}(E_{n+1};\iota_{n+1})^S$ is misleading, and that the object we need to work with is the space of *simplicial loops.*

For any simplicial set X let $\Omega^{\text{simp}}X$ be the *simplicial loop space of* X; the simplicial set whose k-simplices are the maps of simplicial sets $h:\Delta^k_\bullet\times\Delta^1_\bullet\to X_\bullet$ for which $h(x,0)=h(x,1)=*$. Using the standard simplicial decomposition of $\Delta^k_\bullet\times\Delta^{1\bullet}$, a k-simplex of $\Omega^{\text{simp}}X$ can be described as a sequence

$$h_0,\dots,h_k\in X_{k+1}$$

of $(k+1)$-simplices of X satisfying

$$\begin{aligned}\partial_i h_i &= \partial_i h_{i-1}\\ \partial_0 h_0 &= \partial_{k+1}h_k = *.\end{aligned}\tag{4.38}$$

There is a canonical map

$$\left|\Omega^{\text{simp}}X\right|\to\Omega\left|X\right|$$

which is a homotopy equivalence if X satisfies the Kan extension condition (A.8). The simplicial set $\operatorname{filt}_{s+n+1}\left(\check{E}^S_{n+1};\iota_{n+1}\right)$ satisfies the Kan extension condition, and in this section we will actually produce a simplicial homotopy equivalence

$$\operatorname{filt}_{s+n}(E_n;\iota_n)^S\to\Omega^{\text{simp}}\operatorname{filt}_{s+n+1}(E_{n+1};\iota_{n+1})^S.\tag{4.39}$$

Let

$$E^{S\times\mathbb{R}}_{n+1,c}$$

be the space of "compactly supported" functions, i.e., the fiber of the map

$$E^{S\times\bar{\mathbb{R}}}_{n+1}\to E^{S\times\{\infty\}}_{n+1},$$

in which $\bar{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$ is the one point compactification of $\mathbb{R}$. Of course, $E^{S\times\mathbb{R}}_{n+1,c}$ is simply the loop space of E^S_{n+1}, and is homeomorphic to E^S_n. Define a simplicial set $\operatorname{filt}_s(E_{n+1};\iota_{n+1})^{S\times\mathbb{R}}_c$ by the homotopy Cartesian

square
(4.40)

$$\begin{array}{ccc} \operatorname{filt}_{n+s+1}(E_{n+1};\iota_{n+1})_c^{S\times\mathbb{R}} & \longrightarrow & \operatorname{filt}_{n+s+1}\Omega_c^*\left(S\times\mathbb{R}\times\Delta^\bullet;\mathcal{V}\right)_{\mathrm{cl}}^{n+p+1} \\ \downarrow & & \downarrow \\ \operatorname{sing} E_{n+1,c}^{S\times\mathbb{R}} & \longrightarrow & Z_c^*\left(S\times\mathbb{R}\times\Delta^\bullet;\mathcal{V}\right)^{n+p+1} \end{array}$$

The vector space $\Omega_c^*\left(S\times\mathbb{R}\times\Delta^k;\mathcal{V}\right)$ is the space of forms which are compactly supported (along $\mathbb{R}$), and the subspace

$$\operatorname{filt}_t\Omega_c^*\left(S\times\mathbb{R}\times\Delta^k;\mathcal{V}\right)$$

consists of those whose Kunneth components along $\Delta^k\times\mathbb{R}$ of degree greater than t vanish.

We will construct a diagram of simplicial homotopy equivalences

$$\begin{array}{ccc} \operatorname{filt}_{n+s+1}(E_{n+1};\iota_{n+1})_c^{S\times\mathbb{R}} & \longrightarrow & \Omega^{\mathrm{simp}}\operatorname{filt}_{n+s+1}(E_{n+1};\iota_{n+1})^S \\ \downarrow & & \\ \operatorname{filt}_{s+n}(E_n;\iota_n)^S. & & \end{array} \tag{4.41}$$

Choosing a functorial section of the leftmost map gives (4.39).

For the left map of (4.41), note that the homeomorphism $E_{n+1,c}^{S\times\mathbb{R}}\approx E_n^S$ prolongs to an isomorphism of simplicial sets $\operatorname{sing}E_{n+1,c}^{S\times\mathbb{R}}\approx\operatorname{sing}E_n^S$ which is compatible with the maps

$$\operatorname{filt}_{n+s+1}\Omega_c^*\left(S\times\mathbb{R}\times\Delta^\bullet;\mathcal{V}\right)_{\mathrm{cl}}^{n+p+1}\to\operatorname{filt}_{n+s}\Omega^*\left(S\times\Delta^\bullet;\mathcal{V}\right)_{\mathrm{cl}}^{n+p}$$
$$Z_c^*\left(S\times\mathbb{R}\times\Delta^\bullet;\mathcal{V}\right)^{n+p+1}\to Z^*\left(S\times\Delta^\bullet;\mathcal{V}\right)^{n+p}$$

given by "integration over $\mathbb{R}$" and "slant product with the fundamental class of $\bar{\mathbb{R}}$." By Corollary D.18 these maps are simplicial homotopy equivalences. Passing to homotopy pullbacks then gives a homotopy equivalence

$$\operatorname{filt}_{n+s+1}(E_{n+1,c};\iota_{n+1})^{S\times\mathbb{R}}\xrightarrow{\sim}\operatorname{filt}_{s+n}(E_n;\iota_n)^S. \tag{4.42}$$

For the weak equivalence

$$\operatorname{filt}_{n+s+1}(E_{n+1,c};\iota_{n+1})^{S\times\mathbb{R}}\to\Omega^{\mathrm{simp}}\operatorname{filt}_{n+s+1}(E_{n+1};\iota_{n+1})^S \tag{4.43}$$

first apply the "simplicial loops" to the diagram defining $\operatorname{filt}_{n+s+1}(E_{n+1};\iota_{n+1})^S$ to get a homotopy Cartesian square

$$\begin{array}{ccc} \Omega^{\mathrm{simp}}\operatorname{filt}_{n+s+1}(E_{n+1};\iota_{n+1})^S & \longrightarrow & \Omega^{\mathrm{simp}}\operatorname{filt}_{n+s+1}\Omega^*\left(S\times\Delta^\bullet;\mathcal{V}\right)_{\mathrm{cl}}^{n+p+1} \\ \downarrow & & \downarrow \\ \Omega^{\mathrm{simp}}\operatorname{sing}E_{n+1}^S & \longrightarrow & \Omega^{\mathrm{simp}}Z^*\left(S\times\Delta^\bullet;\mathcal{V}\right)^{n+p+1}. \end{array}$$

The adjunction between sing and "geometric realization" gives an isomorphism

$$\operatorname{sing} E^{S\times\mathbb{R}}_{n+1,c} \to \Omega^{\mathrm{simp}} \operatorname{sing} E^{S}_{n+1}. \tag{4.44}$$

A k-simplex of $\Omega^{\mathrm{simp}} Z^*\left(S\times\Delta^\bullet;\mathcal{V}\right)^{n+p+1}$ consists of a sequence of cocycles

$$c_0,\dots c_{k+1} \in Z^*\left(S\times\Delta^k;\mathcal{V}\right)^{n+p+1}$$

satisfying the relations corresponding to (4.38). On the other hand, a k-simplex of $Z^*_c\left(S\times\mathbb{R}\times\Delta^\bullet;\mathcal{V}\right)^{n+p+1}$ can be identified with a cocycle

$$c \in Z^*\left(S\times\Delta^1\times\Delta^k;\mathcal{V}\right)^{n+p+1}$$

which vanishes on $S\times\partial\Delta^1\times\Delta^k$. Restricting c to the standard triangulation of $\Delta^1\times\Delta^k$ leads to a map of simplicial sets

$$Z^*_c\left(S\times\mathbb{R}\times\Delta^\bullet;\mathcal{V}\right)^{n+p+1} \to \Omega^{\mathrm{simp}} Z^*\left(S\times\Delta^\bullet;\mathcal{V}\right)^{n+p+1}, \tag{4.45}$$

which by explicit computation is easily checked to be a weak equivalence. Similarly, a k-simplex of

$$\Omega^{\mathrm{simp}} \operatorname{filt}_{n+s+1} \Omega^*\left(S\times\Delta^\bullet;\mathcal{V}\right)^{n+p+1}_{\mathrm{cl}}$$

consists of a sequences of forms

$$\omega_0,\dots\omega_{k+1} \in \operatorname{filt}_{n+s+1} \Omega^*\left(S\times\Delta^k;\mathcal{V}\right)^{n+p+1}_{\mathrm{cl}} \tag{4.46}$$

satisfying the analogue of (4.38). A k-simplex of

$$\operatorname{filt}_{n+s+1} \Omega^*_c\left(S\times\mathbb{R}\times\Delta^\bullet;\mathcal{V}\right)^{n+p+1}_{\mathrm{cl}}$$

can be identified with a form ω on $S\times\Delta^1\times\Delta^k$ whose restriction to $S\times\partial\Delta^1\times\Delta^k$ vanishes, and whose Kunneth components on $\Delta^1\times\Delta^k$of degrees $< n+s+1$ vanish. Restricting ω to the simplices in the standard triangulation of $\Delta^1\times\Delta^k$ leads to a sequence (4.46), and hence to a map of simplicial abelian groups

$$\operatorname{filt}_{n+s+1} \Omega^*_c\left(S\times\mathbb{R}\times\Delta^\bullet;\mathcal{V}\right)^{n+p+1}_{\mathrm{cl}} \to \Omega^{\mathrm{simp}} \operatorname{filt}_{n+s+1} \Omega^*\left(S\times\Delta^\bullet;\mathcal{V}\right), \tag{4.47}$$

which, also by explicit computation, is easily checked to be a weak equivalence.

The equivalences (4.44), (4.45), and (4.47) are compatible with pullback of cochains, and the inclusion of forms into cochains, and so patch together, via homotopy pullback, to give the desired weak equivalence (4.43).

4.7. Naturality and Homotopy for Spectra. We now turn to some further constructions on spectra, and the analogues of the results of §4.2.

Definition 4.48. A *map* between spectra

$$E = \{E_n, t_n^E\} \quad \text{and} \quad F = \{F_n, t_n^F\}$$

consists of a collection of maps $f_n : E_n \to F_n$, which are compatible with the structure maps (4.27), in the sense that the following diagram commutes

$$\begin{array}{ccc} E_n & \xrightarrow{f_n} & F_n \\ {\scriptstyle t_n^E}\downarrow & & \downarrow{\scriptstyle t_n^F} \\ \Omega E_{n+1} & \xrightarrow[\Omega f_{n+1}]{} & \Omega F_{n+1}. \end{array}$$

Remark 4.49. The set of maps is topologized as a subspace of

$$\prod F_n^{E_n}.$$

Given $f : E \to F$ and a cocycle

$$\iota \in Z^p(F; \mathbb{R}),$$

composition with f gives a map

$$\check{f} : \mathrm{filt}_s(E; f^*\iota)^S \to \mathrm{filt}_s(F; \iota)^S.$$

Proposition 4.50. *Suppose that $f : E \to F$ is a (weak) homotopy equivalence, and $\iota \in Z^p(F; \mathbb{R})$ is a cocycle. Then for each manifold S, the map*

$$\check{f} : \mathrm{filt}_s(E; f^*\iota)^S \to \mathrm{filt}_s(F; \iota)^S$$

is a (weak) homotopy equivalence.

Proof. This is a consequence of Proposition 4.13. q.e.d.

As in §4.2, Proposition 4.50 implies that a homotopy between maps of spectra $f, g : E \to F$ leads to a filtration preserving homotopy between maps of differential function spectra.

Given two cocycles $\iota^1, \iota^2 \in Z^p(E; \mathbb{R})$, and a cochain $b \in C^{p-1}(E; \mathbb{R})$ with $\delta b = \iota^2 - \iota^1$, the isomorphisms (4.16)

$$\mathrm{filt}_{s+n}(E_n; \iota_n^1)^S \to \mathrm{filt}_{s+n}(E_n; \iota_n^2)^S$$

fit together to give an isomorphism

$$\mathrm{filt}_s(E; \iota^1)^S \to \mathrm{filt}_s(E; \iota^2)^S.$$

It follows that given a map $f : E \to F$, and cocycles $\iota_E \in Z^p(E; \mathbb{R})$, $\iota_F \in Z^p(F; \mathbb{R})$, and a cochain $b \in C^{p-1}(E; \mathbb{R})$ with $\delta b = f^*\iota_F - \iota_E$ we get a map of differential function spectra

$$\mathrm{filt}_s(E; \iota_E)^S \to \mathrm{filt}_s(F; \iota_F)^S$$

which is a weak equivalence if f is. It also follows that the (weak) homotopy type of

$$\mathrm{filt}_s(E,\iota)^S$$

depends only on the cohomology class of ι and and the homotopy type of E in the sense described in Remark 4.17.

Remark 4.51. Associated to a space X is its suspension spectrum, $\Sigma^\infty X$, with

$$(\Sigma^\infty X)_n = \varinjlim \Omega^k \Sigma^{n+x} X.$$

One can easily check that the space of maps from $\Sigma^\infty X$ to a spectrum E is simply the space E_0^X, with components, the cohomology group $E^0(X)$.

Remark 4.52. If $E = \{E_n\}$ is a spectrum, one can construct a new spectrum by simply shifting the indices. These are known as the *shift suspensions* of E, and we will indicate them with the notation $\Sigma^k E$. To be specific

$$\left(\Sigma^k E\right)_n = E_{n+k}.$$

Note that k may be any integer, and that $\Sigma^k E$ is the spectrum representing the cohomology theory

$$X \mapsto E^{*+k}(X).$$

Remark 4.53. Sometimes the symbol $\Sigma^k E$ is used to denote the spectrum with

$$\left(\Sigma^k E\right)_n = \varinjlim \Omega^n \Sigma^k E_n. \tag{4.54}$$

The spectrum described by (4.54) is canonically homotopy equivalent, but not equal to $\Sigma^k E$.

4.8. The fundamental cocycle. Recall that for any compact S, and any cohomology theory E there is a canonical isomorphism

$$E^*(S) \otimes \mathbb{R} = H^*(S; \pi_* E \otimes \mathbb{R}). \tag{4.55}$$

When E is K-theory, this isomorphism is given by the Chern character. The isomorphism (4.55) arises from a universal cohomology class

$$i_E \in H^0(E; \pi_* E \otimes \mathbb{R}) = \varprojlim H^n(E_n; \pi_{*+n} E \otimes \mathbb{R}),$$

and associates to a map $f : S \to E_n$ representing an element of $E^n(S)$ the cohomology class $f^* i_n$, where i_n is the projection of i_E to $H^n(E_n; \pi_{*+n} E)$. We will call the class i_E a *fundamental cohomology class*–the term used in the case of the Eilenberg-MacLane spectrum, with $E_n = K(\mathbb{Z}, n)$. A cocycle representative of i_E will be called a *fundamental cocycle*. More precisely,

Definition 4.56. Let E be a spectrum. A *fundamental cocycle* on E is a cocycle

$$\iota^E \in Z^0(E; \pi_* E \otimes \mathbb{R})$$

representing the cohomology class corresponding to the map

$$\begin{aligned} \pi_* E &\to \pi_* E \otimes \mathbb{R} \\ a &\mapsto a \otimes 1 \end{aligned}$$

under the Hurewicz isomorphism

$$H^0(E; \mathcal{V}) \xrightarrow{\approx} \hom(\pi_* E, \mathcal{V}).$$

Any two choices of fundamental cocycle are cohomologous, and so lead to isomorphic differential cohomology theories. Unless otherwise specified, we will use $\check{E}(n)$ to denote the differential cohomology theory associated to a choice of fundamental cocycle.

When ι is a choice of fundamental cocycle, the square (4.12) leads to short exact sequences

(4.57)

$$\begin{gathered} 0 \to E^{q-1}(S; \mathbb{R}/\mathbb{Z}) \to \check{E}(q)^q(S) \to \Omega^*(S; \pi_* E)^q_{\mathrm{cl}} \to 0 \\ 0 \to E^{q-1}(S) \otimes \mathbb{R}/\mathbb{Z} \to \check{E}(q)^q(S) \to A^q_E(S) \to 0 \\ 0 \to \Omega^*(S; \pi_* E)^{q-1}/\Omega^*(S; \pi_* E)^{q-1}_0 \to \check{E}(q)^q(S) \to E^q(M) \to 0. \end{gathered}$$

The group $A^q_E(S)$ is defined by the pullback square

$$\begin{CD} A^q_E(S) @>>> \Omega(S; \pi_* E \otimes \mathbb{R})^q_{\mathrm{cl}} \\ @VVV @VVV \\ E^q(S) @>>> H^q(S; \pi_* E \otimes \mathbb{R}). \end{CD}$$

Remark 4.58. Any cocycle in E is a specialization of the fundamental cocycle in the sense that given any cocycle $\iota_1 \in Z^*(E; \mathcal{V})^0$, there is a map (the Hurewicz homomorphism)

$$t : \pi_* E \otimes \mathbb{R} \to V$$

and a cocycle $b \in C^{-1}(E; \mathcal{V})$ with

$$\delta b = \iota_1 - t \circ \iota.$$

4.9. Differential bordism.

4.9.1. Thom spectra. Let $\mathbb{R}^\infty$ be the vector space with basis $\{e_1, e_2, \dots\}$, and

$$MO\,(k) = \mathrm{Thom}\,(Gr_k\,(\mathbb{R}^\infty)\,, V_k)$$

the Thom complex of the natural k-plane bundle (V_k) over the Grassmannian $Gr_k\,(\mathbb{R}^\infty)$. The "translation map"

$$\begin{aligned} T : \mathbb{R}^\infty &\to \mathbb{R}^\infty \\ T(e_i) &= e_{i+1} \end{aligned}$$

induces a map

$$\begin{aligned} Gr_k(T) : Gr_k(\mathbb{R}^\infty) &\to Gr_{k+1}(\mathbb{R}^\infty) \\ V &\mapsto \mathbb{R}e_1 \oplus V. \end{aligned}$$

The space $\varinjlim_k Gr_k(\mathbb{R}^\infty)$ is a classifying space for the stable orthogonal group, and will be denoted BO.

The pullback of V_{k+1} along $Gr_k(T)$ is $\mathbb{R}e_1 \oplus V_k$, giving closed inclusions

$$\begin{aligned} S_k &: \Sigma\, MO(k) \to MO(k+1) \\ T_k &: MO(k) \to \Omega\, MO(k+1). \end{aligned}$$

The spaces

$$MO_k = \varinjlim_n \Omega^n\, MO(k+n)$$

form a spectrum–the *unoriented bordism* spectrum, MO.

More generally, given a sequence of closed inclusions

$$t_k : B(k) \to B(k+1),$$

fitting into a diagram

$$\begin{tikzcd} \cdots \arrow[r] & B(k) \arrow[r] \arrow[d, "\xi_k"] & B(k+1) \arrow[r] \arrow[d, "\xi_{k+1}"] & \cdots \\ \cdots \arrow[r] & Gr_k(\mathbb{R}^\infty) \arrow[r, "Gr_k(T)"'] & Gr_{k+1}(\mathbb{R}^\infty) \arrow[r] & \cdots \end{tikzcd} \tag{4.59}$$

the Thom complexes

$$B(k)^{\xi_k^* V_k}$$

come equipped with closed inclusions

$$\Sigma B(k)^{\xi_k} \to B(k+1)^{\xi_{k+1}}$$

leading to a spectrum $X = \mathrm{Thom}(B;\xi)$ with

$$X_k = \varinjlim \Omega^n\, \mathrm{Thom}(B(n+k);\xi_{n+k}). \tag{4.60}$$

For instance, $B(k)$ might be the space $\widetilde{Gr}_k(\mathbb{R}^\infty)$ of oriented k-planes in $\mathbb{R}^\infty$, in which case the resulting spectrum is the oriented cobordism spectrum MSO.

Spaces $B(k)$ can be constructed from a single map $\xi : B \to BO$, by forming the homotopy pullback square

$$\begin{tikzcd} B(k) \arrow[r] \arrow[d, "\xi_k"'] & B \arrow[d, "\xi"] \\ Gr_k(\mathbb{R}^\infty) \arrow[r] & BO. \end{tikzcd}$$

The resulting spectrum $\mathrm{Thom}(B;\xi)$ is the *Thom spectrum* of ξ.

4.9.2. Differential B-oriented maps. Let $X = \mathrm{Thom}(B;\xi)$ be the Thom spectrum of a map $\xi : B \to BO$, and S a compact manifold. Suppose that we are given a cocycle $\iota \in Z^n(X;\mathcal{V})$ for some real vector space $\mathcal{V}$.

Definition 4.61. Let $p : E \to S$ be a (neat) map of manifolds of relative dimension n. A *B-orientation* of p consists of a neat embedding $p_N = (p, p'_N) : E \hookrightarrow S \times \mathbb{R}^N$, for some N, a tubular neighborhood $W_N \hookrightarrow S \times \mathbb{R}^N$, and a vector bundle map

$$t_N : W_N \to \xi_{N-n}$$

classifying W_N.

Remark 4.62. Two B-orientations are equivalent if they are in the equivalence relation generated by identifying

$$\xi_{N-n} \leftarrow W_N \subset S \times \mathbb{R}^N$$

with

$$\xi_{N+1-n} \approx \xi_{N+1} \times \mathbb{R}^1 \leftarrow (W_N \times \mathbb{R}^1) \approx W_{N+1} \subset S \times \mathbb{R}^{N+1}.$$

Definition 4.63. A *B-oriented map* is a neat map $p : E \to S$ equipped with an equivalence class of B-orientations.

Definition 4.64. Let $p : E \to S$ be a (neat) map of manifolds of relative dimension n. A *differential B-orientation* of p consists of a neat embedding $p_N = (p, p'_N) : E \hookrightarrow S \times \mathbb{R}^N$, for some N, a tubular neighborhood $W_N \hookrightarrow S \times \mathbb{R}^N$ and a differential vector bundle map

$$t_N : W_N \to (\xi_{N-n}; \iota_{N-n})$$

classifying W_N.

Remark 4.65. In short, a differential B-orientation of $p : E \to S$ is a lift of p to a differential $B(N-n)$-oriented embedding $E \subset \mathbb{R}^N \times S$.

Remark 4.66. Two differential B-orientations are equivalent if they are in the equivalence relation generated by identifying

$$(\xi_{N-n}; \iota_{N-n}) \xleftarrow{t_N} W_N \subset S \times \mathbb{R}^N$$

with

$$(\xi_{N+1-n}; \iota_{N+1-n}) \xleftarrow{t_{N+1}} (W_N \times \mathbb{R}^1) \approx W_{N+1} \subset S \times \mathbb{R}^{N+1}$$

where $t_N = (c_N, h_N, \omega_N)$, $t_{N+1} = (c_N \times \mathrm{Id}, h_{N+1}, \omega_{N+1})$, and

$$h_N = h_{N+1}/Z_{\mathbb{R}^1}$$
$$\omega_N = \int_{\mathbb{R}^1} \omega_N.$$

Definition 4.67. A *differential B-oriented map* is a neat map

$$E \to S$$

together with an equivalence class of differential B-orientations.

Remark 4.68. Let $p : E \to S$ be a differential B-oriented map of relative dimension n. Using §4.3, the construction of Pontryagin-Thom gives a differential function

$$S \to (\mathrm{Thom}(B, \xi)_n, \iota_{-n}) . \tag{4.69}$$

In fact the homotopy type of

$$\mathrm{filt}_s \left(\mathrm{Thom}(B, \xi)_n, \iota_{-n}\right)^S$$

can be described entirely in terms of differential B-oriented maps

$$E \to S \times \Delta^k.$$

The proof is basically an elaboration of the ideas of Thom [**63**], and we omit the details. We will refer to this correspondence by saying that the differential B-oriented map $p : E \to S$ is *classified* by the differential function (4.69).

Remark 4.70. Our definition of differential B-orientation depends on many choices (a tubular neighborhood, a differential function, etc.) which ultimately affect the push-forward or integration maps to be defined in §4.10. These choices are all homotopic, and so aren't made explicit in the purely topological approaches. A homotopy between two differential B-orientations can be thought of as a differential B-orientation of $E \times \Delta^1 \to S \times \Delta^1$, and the effect of a homotopy between choices can be described in terms of integration along this map.

4.9.3. BSO-orientations and $\check{H}$-orientations.

Let $BSO = \varinjlim \widetilde{Gr}_k(\mathbb{R}^\infty)$ be the stable oriented Grassmannian, as described in §4.9.1, and choose a Thom cocycle $U \in Z^0(MSO)$. The resulting notion of a BSO-oriented map is a refinement to differential algebraic topology of the topological notion of an oriented map. We there have formulated two slightly different notions of a differential orientation for a map $E \to S$: that of a differential BSO-orientation and that of an $\check{H}$-orientation. For practical purposes these two notions are equivalent, and we now turn to making precise the relationship between them.

Fix a manifold S.

Definition 4.71. The space of *$\check{H}$-oriented-maps* of relative dimension n is the simplicial set whose k-simplices are $\check{H}$-oriented maps $E \to S \times \Delta^k$ of relative dimension n. We denote this simplicial set $A(S)$, and define $\mathrm{filt}_t\, A(S)$ by restricting the Kunneth component of the Thom form along the simplices.

Definition 4.72. The space of *differential BSO-oriented maps* of relative dimension n, is the simplicial set with k-simplices the differential BSO-oriented maps $E \to S \times \Delta^k$ of relative dimension n. We denote this space $B(S)$ and define $\operatorname{filt}_s B(S)$ similarly.

We also let $\tilde{B}(S)$ be the space whose k-simplices are the k-simplices of $B(S)$, together with a differential cochain

$$(b, k, 0) \in C(N-n)^{N-n-1}(E),$$

and define $\operatorname{filt}_t \tilde{B}(S)$ by restricting the Kunneth components as above. Forgetting about $(b, k, 0)$ defines a filtration preserving function

$$\tilde{B}(S) \to B(S)$$

and taking $(b, k, 0) = (0, 0, 0)$ defines a filtration preserving function

$$B(S) \to \tilde{B}(S).$$

Suppose that $E/S \times \Delta^k$ is a k-simplex of $B(S)$. That is, we are given

$$(\xi_{N-n}; U_{N-n}) \xleftarrow{(c,h,\omega)} W \hookrightarrow S \times \Delta^k \times \mathbb{R}^N.$$

Then $W \hookrightarrow S \times \Delta^k \times \mathbb{R}^N$ together with $(c^* U_{N-n}, h, \omega)$ is a k-simplex of $A(S)$. This defines a forgetful map

$$\operatorname{filt}_s B(S) \to \operatorname{filt}_s A(S).$$

We define a map

$$\operatorname{filt}_s \tilde{B}(S) \to \operatorname{filt}_s A(S)$$

in a similar way, but with the Thom cocycle

$$(c^* U_{N-n}, h, \omega) + d(b, k, 0).$$

Note that there is a factorization

$$\operatorname{filt}_s B(S) \to \operatorname{filt}_s \tilde{B}(S) \to \operatorname{filt}_s A(S),$$

and that

$$\operatorname{filt}_s B(S) \to \operatorname{filt}_s \tilde{B}(S) \to \operatorname{filt}_s B(S)$$

is the identity map.

Lemma 4.73. *For each t, the maps*

$$\operatorname{filt}_s \tilde{B}(S) \to \operatorname{filt}_s A(S)$$
$$\operatorname{filt}_s \tilde{B}(S) \to \operatorname{filt}_s B(S)$$

are acyclic fibrations of simplicial sets. In particular they are homotopy equivalences.

Corollary 4.74. *For each s, the map $\operatorname{filt}_s B(S) \to \operatorname{filt}_s A(S)$ is a simplicial homotopy equivalence.*

Proof of Lemma 4.73. We'll give the proof for

$$\operatorname{filt}_s \tilde{B}(S) \to \operatorname{filt}_s A(S).$$

The situation with the other map is similar. By definition, we must show that a lift exists in every diagram of the form

$$\begin{tikzcd} \partial\Delta^k_\bullet \arrow[r] \arrow[d] & \operatorname{filt}_s \tilde{B}(S) \arrow[d] \\ \Delta^k_\bullet \arrow[r] \arrow[ur, dashed, "\exists"] & \operatorname{filt}_s A(S). \end{tikzcd} \tag{4.75}$$

The bottom k-simplex classifies an $\check{H}$-oriented map

$$p : E \to S \times \Delta^k$$
$$W \hookrightarrow S \times \Delta^k \times \mathbb{R}^N$$
$$(c, h, \omega) \in Z(N-n)^{N-n}(W),$$

and the top map gives compatible BSO-orientations to the boundary faces

$$\partial_i E \to S \times \partial_i \Delta^n$$
$$(\xi_{N-n}; U_{N-n}) \xleftarrow{(f_i, h_i, \omega_i)} \partial_i W \hookrightarrow S \times \partial_i \Delta^k \times \mathbb{R}^{N-n}$$

and compatible differential cocycles

$$(b_i, k_i, 0) \in C(N-n)^{N-n}(f_i^* \xi),$$

satisfying

$$(c, h, \omega)|_{\partial_i W} = (f_i^* U_{N-n}, h_i, \omega_i) + d(b_i, k_i, 0).$$

Write

$$\partial E = \bigcup_i \partial_i E \qquad \partial W = \bigcup_i \partial_i W = \bigcup_i W|_{\partial_i E}.$$

The compatibility conditions imply that the functions f_i together form a vector bundle map

$$\partial f : \partial W \to \xi_{N-n}.$$

By the universal property of $BSO(N-n)$, ∂f extends to a vector bundle map

$$f : W \to \xi_{N-n},$$

inducing the same (topological) orientation on W as the one given by the Thom cocycle (c, h, ω). In particular, this implies that $f^* U(N-n)$ and c represent the same cohomology class.

To construct a lift in (4.75), it suffices to find a cocycle $h' \in C^{N-n-1}(\bar{W};\mathbb{R})$ with

$$\begin{gathered}\delta h' = \omega - f^* U_{N-n} \\ h'|_{\partial_i W} = h_i,\end{gathered} \tag{4.76}$$

and a differential cochain $(b,k,0) \in C(N-n)^{N-n-1}(\bar{W})$ satisfying

$$\begin{gathered}(b,k,0)|_{\partial_i W} = (b_i,k_i,0) \\ (c,h,\omega) = (f^* U_{N-n}, h', \omega) + d(b,k,0).\end{gathered} \tag{4.77}$$

To construct h', first choose any $h'' \in C^{N-n-1}(\bar{W};\mathbb{R})$ with $h''|_{\partial_i W} = h_i$ for all i. Then the cocycle

$$\delta h'' - \omega + f^* U_{N-n} \tag{4.78}$$

represents a relative cohomology class in

$$H^{N-n}(\bar{W},\partial\bar{W};\mathbb{R}) \xrightarrow{\approx} H^0(E,\partial E;\mathbb{R}).$$

Now this latter group is simply the ring of real-valued functions on the set of path components of E which do not meet ∂E. But on those components, the expression (4.78) represents 0, since the image of $f^* U_{N-n}$ in $Z^{N-n}(\bar{W};\mathbb{R})$ and ω represent the same cohomology class. It follows that (4.78) is the coboundary of a relative cochain

$$h''' \in C^{N-n-1}(\bar{W},\partial\bar{W};\mathbb{R}).$$

We can then take

$$h' = h'' - h'''.$$

A similar argument, using the fact that $H^{N-n-1}(\bar{W},\partial\bar{W};\mathbb{R}/\mathbb{Z}) = 0$ leads to the existence of $(b,k,0)$ satisfying (4.77). q.e.d.

The fact that acyclic fibrations are preserved under change of base makes the result of Proposition 4.73 particularly convenient. For instance, a single $\check{H}$-oriented map E/S defines a 0-simplex in $A(S)$, and Proposition 4.73 asserts that the inverse image of this 0-simplex in $\operatorname{filt}_s \tilde{B}(S)$ is contractible. This means that for all practical purposes the notions of an $\check{H}$-oriented map and a BSO-oriented are equivalent. This same discussion applies to many combinations of geometric data.

4.10. Integration. In this section we define *integration* or *push-forward* in differential cohomology theories. We will see that because of the results of §4.9.3 and Appendix D this recovers our theory of integration of differential cocycles (§3.4) in the case of ordinary differential cohomology. At the end of this section we discuss the integration in differential K-theory.

The topological theory of integration is simply the interpretation in terms of manifolds of a map from a Thom spectrum to another spectrum. Thus let $X = \mathrm{Thom}(B; \xi)$ be a Thom spectrum, and R a spectrum. Consider

$$\mathrm{Thom}(B \times R_m; \xi \oplus 0) = X \wedge (R_m)_+$$

and suppose there is given a map

$$\mu : X \wedge (R_m)_+ \to \Sigma^m R.$$

In geometric terms, a map

$$S \to (X \wedge (R_m)_+)_{-n}$$

arises from a $B \times R_m$-oriented map $E \to S$ of relative dimension n, or what amounts to the same thing, a B-oriented map, together with a map $x : E \to R_m$, thought of as a "cocycle" representing an element of $R^m(E)$. The composition

$$S \to (X \wedge (R_m)_+)_{-n} \xrightarrow{\mu} (\Sigma^m R)_{-n} \approx (R)_{m-n}$$

is a map y representing an element of $R^{m-n}(S)$. Thus the geometric interpretation of the map μ is an operation which associates to every B-oriented map E/S of relative dimension n and every $x : E \to R_m$, a map $y : S \to R_{m-n}$. We think of y as the *integral* of x.

Remark 4.79. The notation for the spectra involved in discussing integration tends to become compounded. Because of this, given a spectrum R and a cocycle $\iota \in C^*(R; \mathcal{V})$ we will denote pair $(R_n; \iota_n)$ as

$$(R_n; \iota_n) = (R; \iota)_n.$$

For the *differential* theory of integration, suppose that cocycles ι_1, ι_2 have been chosen so as to refine μ to a map of differential cohomology theories

$$\check{\mu} : (X \wedge (R_m)_+; \iota_1) \to (R; \iota_2)\,.$$

The map $\check{\mu}$ associates to every differential B-oriented map $E \to S$ of relative dimension n together with a differential function $x : E \to R_m$, a differential function $y : S \to R_{m-n}$. We will refer to y as the *push-forward* of x and write $y = p_!(x)$, or

$$y = \int_{E/S}^{\mu} x$$

to emphasize the analogy with integration of differential cocycles. When the map μ is understood, we will simply write

$$y = \int_{E/S} x.$$

Remark 4.80. The construction $p_!$ is a map of differential function spaces

$$\int^{\mu} : (X \wedge (R_m)_+; \iota_1)^S_{-n} \to (R; \iota_2)^S_{m-n} .$$

The weight filtration of $\int^{\mu}$ can be controlled geometrically, but in the cases that come up in this paper it is something that can be computed after the fact.

The push-forward construction is compatible with changes in R. Suppose $f : P \to R$ is a map of spectra, and that there are maps

$$\mu_P : X \wedge (P_m)_+ \to \Sigma^m P$$
$$\mu_R : X \wedge (R_m)_+ \to \Sigma^m R$$

and a homotopy

$$H : X \wedge (P_m)_+ \wedge \Delta^1_+ \to \Sigma^m R$$

between the two ways of going around

$$\begin{array}{ccc} X \wedge (P_m)_+ & \xrightarrow{\mu_P} & \Sigma^m P \\ \downarrow & & \downarrow \\ X \wedge (R_m)_+ & \xrightarrow{\mu_R} & \to \Sigma^m R. \end{array} \tag{4.81}$$

Suppose also that cocycles

$$\iota_P \in Z^k(P; \mathcal{V}) \qquad \iota_R \in Z^k(R; \mathcal{W})$$

have been chosen so that f refines to a map of differential cohomology theories

$$\check{f} : (P; \iota_P) \to (R; \iota_R) .$$

Then differential pushforward maps

$$\int^{\mu_P} (\,-\,) : (X \wedge (P_m)_+; \mu_P^* \iota_P)^S_{-n} \to (P; \iota_P)^S_{m-n}$$
$$\int^{\mu_R} (\,-\,) : (X \wedge (R_m)_+; \mu_R^* \iota_R)^S_{-n} \to (R; \iota_R)^S_{m-n}$$

are defined, and the results of §4.7 give a canonical (weight filtration preserving) homotopy between

$$\check{f} \circ \int^{\mu_P} (\,-\,) \quad \text{and} \quad \int^{\mu_P} \check{f}_*(\,-\,).$$

Remark 4.82. In case $B = \mathrm{pt}$, the Thom spectrum $X = S^0$ represents framed cobordism, and every spectrum R comes equipped with a natural map

$$X \wedge (R_m)_+ \to \Sigma^m R,$$

namely the structure map of the spectrum. This map is compatible with every map of spectra $P \to R$ in the sense described above. *Thus* *pushforward maps exist in every cohomology theory for a* differential framed *map $E \to S$, and these are compatible with all maps between the cohomology theories.*

Example 4.83 (Ordinary differential cohomology)**.** Let X be the oriented bordism spectrum MSO and R the Eilenberg-MacLane spectrum $H\mathbb{Z}$, with $R_m = K(\mathbb{Z}, m)$. Choose a Thom cocycle $U \in Z^0(MSO;\mathbb{Z}) \subset Z^0(MSO;\mathbb{R})$ and a fundamental cocycle $x \in Z^0(H\mathbb{Z};\mathbb{Z}) \subset Z^0(H\mathbb{Z};\mathbb{R})$. We take

$$\mu : MSO \wedge K(\mathbb{Z}, m)_+ \to \Sigma^m H\mathbb{Z}$$

to be a map representing

$$\iota = u \cup x_m \in Z^m(MSO \wedge K(\mathbb{Z}, m)_+;\mathbb{Z}) \subset Z^m(MSO \wedge K(\mathbb{Z}, m)_+;\mathbb{R}).$$

Let

$$\begin{gathered} E \to S \\ (\xi_{N-n}; U_{N-n}) \xleftarrow{\check{U}} W \subset S \times \mathbb{R}^N \end{gathered}$$

be a differential BSO-oriented map. Given a differential function

$$\check{x} : E \to (K(\mathbb{Z}, m); \iota_m)$$

form the differential $BSO \times K(\mathbb{Z}, m)$-oriented map

$$(\xi_{N-n} \oplus 0; U_{N-n} \cup \iota_m) \xleftarrow{\check{U} \cup \check{x}} W \subset S \times \mathbb{R}^N. \tag{4.84}$$

The map (4.84) is classified by a differential function

$$S \to (MSO \wedge K(\mathbb{Z}, m)_+; U \cup \iota_m)_{-n}\,,$$

and composing this with $\int^\mu$ gives a differential function

$$\int_{E/S}^{\mu} \check{x} : S \to (K(\mathbb{Z}, m-n); \iota_{m-n})\,.$$

Replacing $E \to S$ with $E \times \Delta^k \to S \times \Delta^k$ leads to a map of differential function complexes

$$\int^{\mu} : \operatorname{filt}_s (K(\mathbb{Z}, m); \iota_m)^E \to \operatorname{filt}_s (K(\mathbb{Z}, m-n); \iota_{m-n})^S\,.$$

On the other hand, associated to the differential BSO-orientation of E/S is an $\check{H}$-orientation of E/S, and, as described in Appendix D the differential function complexes

$$\operatorname{filt}_s (K(\mathbb{Z}, m); \iota_m)^E \quad \text{and} \quad \operatorname{filt}_s (K(\mathbb{Z}, m-n); \iota_{m-n})^S$$

are homotopy equivalent to the simplicial abelian groups underlying the chain complexes

$$\check{Z}(m-s)^m(E) \leftarrow \check{C}(n)^{m-1}(E) \ldots \leftarrow \check{C}(m-s)^0(E) \quad \text{and}$$
$$\check{Z}(m-n-s)^{m-n}(S) \leftarrow \check{C}(m-n-s)^{m-n-1}(S) \cdots \leftarrow \check{C}(m-n-s)^0(S),$$

respectively. As the reader can check, our conventions have been chosen so that integration of differential cocycles and integration of differential functions to an Eilenberg-MacLane space agree under this correspondence.

Example 4.85 (The case of differential K-theory)**.** We now turn to the pushforward map in differential K-theory. Let $p : E \to S$ be a map of relative dimension $2n$ with a "differential Spin^c-structure" on the relative normal bundle. The theory described above gives a map

$$p_! : \check{K}^0(E) \to \check{K}^{-2n}(S) \approx \check{K}^0(S).$$

Actually $p_!$ sends a *differential vector bundle* $\bar{V}$ on E, i.e., a differential function

$$\bar{V} = (c, h, \omega) : E \to (\mathcal{F}; \iota),$$

to a differential vector bundle on S. In this section we will describe this map in some detail.

If we drop the "differential" apparatus, we get the topological pushforward map

$$p_!^{\text{top}} : K^0(E) \to K^0(S).$$

We remind the reader how that goes.

To define the topological pushforward, embed E in $\mathbb{R}^{2N} \times S$, and let

$$\pi : \nu \to E$$

be the normal bundle. Clifford multiplication

$$\nu \times S^+(\nu) \to \nu \times S^-(\nu)$$

defines a K-theory class

$$\Delta \in K^0(\nu, \nu - E).$$

Let

$$\nu \approx D \subset \mathbb{R}^{2N} \times S$$

be a tubular neighborhood of E. We have canonical isomorphisms

$$K^0(\nu, \nu - E) \approx K^0(D, D - E) \xleftarrow{\approx} K^0(\mathbb{R}^{2N} \times S, R^{2N} \times S - E).$$

If V is a vector bundle representing an element of $K^0(E)$, then

$$\Delta \cdot \pi^* V$$

is an element of $K^0(\nu,\nu-E)$, and the topological pushforward of V is the image of $\Delta\cdot\pi^*V$ under

$$K^0(\nu,\nu-E)\approx K^0(\mathbb{R}^{2N}\times S, R^{2N}\times S-E)\to K^0(\bar{\mathbb{R}}^{2N}\times S)\to K^0(S).$$

We can now turn to differential K-theory. To get a "differential $p_!$" we need a differential Spin^c-structure–that is, a refinement of Δ to

$$\hat{\Delta}=(c_\Delta,h_\Delta,\omega_\Delta)$$

where

$$c_\Delta:\bar{\nu}\to\mathcal{F}$$

is a map classifying Δ, and

$$h_\Delta\in C^{\mathrm{odd}}(\nu;\mathbb{R}),\quad \omega_\Delta\in\Omega^{\mathrm{ev}}_{\mathrm{cl}}(\bar{\nu})$$

satisfy

$$\delta h_\Delta=\omega_\Delta-c_\Delta^*\iota,$$

where ι is a choice of Chern character cocycles as in §4.4. We now imitate the topological construction. Let $\check{V}=(c,h,\omega)$ represent an element of $\check{K}^0(E)$. Then $\pi^*\check{V}\in\check{K}^0(\nu)$, and we define $p_!\check{V}$ to be the image of $\check{\Delta}\cdot\pi^*\check{V}$ under

$$\check{K}^0(\bar{\nu})\approx\check{K}^0(\mathbb{R}^{2N}\times S, R^{2N}\times S-D)\to\check{K}^0(\bar{\mathbb{R}}^{2N}\times S)\to\check{K}^0(S).$$

5. The topological theory

5.1. Proof of Theorem 2.17. We now turn to the proof of Theorem 2.17. Our approach will be to reformulate the result in terms of a transformation of differential cohomology theories. In fact we need only construct a transformation of *cohomology theories*–the refinement to the differential versions is more or less automatic. It is rather easy to show that many transformations exist which will do the job (see Proposition 5.8). But it turns out that results of Milgram [**47**], and Morgan–Sullivan [**51**] make it possible to single one out. In this first section we prove Theorem 2.17, but do not single out a particular κ. The remaining sections are devoted to making a particular choice.

The input of the functor κ is an $\check{H}$-oriented map $E\to S$ equipped with a "twisted" differential cocycle giving an integral Wu-structure, and a differential L-cocycle. We first turn to interpreting this data in terms of cobordism.

Choose a map

$$BSO\to K(\mathbb{Z}/2,2k)$$

representing the Wu class, ν_{2k}, of the universal bundle, and $BSO\langle\beta\nu_{2k}\rangle$ by the homotopy pullback square

$$\begin{array}{ccc} BSO\langle\beta\nu_{2k}\rangle & \xrightarrow{\lambda} & K(\mathbb{Z},2k) \\ \big\downarrow & & \big\downarrow \\ BSO & \longrightarrow & K(\mathbb{Z}/2,2k). \end{array}$$

Let $MSO\langle\beta\nu_{2k}\rangle$ be the associated Thom spectrum. The group

$$MSO\langle\beta\nu_{2k}\rangle^{i-4k}(S)$$

is the cobordism group of maps $E\to S$ of relative dimension $(4k-i)$ equipped with an orientation of the relative stable normal bundle, and an integral Wu-structure.

The Thom spectrum of the vector bundle classified by the projection map

$$BSO\langle\beta\nu_{2k}\rangle\times K(\mathbb{Z},2k)\to BSO$$

is the smash product

$$MSO\langle\beta\nu_{2k}\rangle\wedge K(\mathbb{Z},2k)_+.$$

The group

$$\left(MSO\langle\beta\nu_{2k}\rangle\wedge K(\mathbb{Z},2k)_+\right)^{i-4k}(S)$$

is the cobordism group of maps $E\to S$ of relative dimension $(4k-i)$ equipped with an orientation of the relative stable normal bundle, an integral Wu-structure, and a cocycle $x\in Z^{2k}(E)$.

The functors κ and q^λ in Theorem 2.17 are constructed out of maps from $MSO\langle\beta\nu_{2k}\rangle$ and $MSO\langle\beta\nu_{2k}\rangle\wedge K(\mathbb{Z},2k)_+$ to some other spectrum. There are many possible choices of this other spectrum, and we will work with one which is "universal" in the sense that it *receives* a map from any other choice. This universal spectrum theory is the *Anderson dual* of the sphere [**4**, **68**], which we denote $\tilde{I}$. Roughly speaking, Anderson duality is like Pontryagin duality, and for any spectrum X, any homomorphism $\pi_nX\to\mathbb{Z}$ is represented by a map $X\to\Sigma^n\tilde{I}$. More precisely, there is a (splittable) short exact sequence

$$\operatorname{Ext}\left(\pi_{n-1}X,\mathbb{Z}\right)\rightarrowtail[X,\Sigma^n\tilde{I}]=\tilde{I}^n(X)\twoheadrightarrow\hom\left(\pi_nX,\mathbb{Z}\right). \tag{5.1}$$

The spectrum $\tilde{I}$ is defined in Appendix B. By (5.1), one has

$$\tilde{I}^k(\mathrm{pt})=\begin{cases}0 & \text{if } k<0\\ \mathbb{Z} & \text{if } k=0\\ 0 & \text{if } k=1\\ \hom(\pi_{k-1}^{\mathrm{st}}S^0,\mathbb{Q}/\mathbb{Z}) & k>1\end{cases}$$

where

$$\pi_{k-1}^{\mathrm{st}}S^0=\varinjlim\pi_{k-1+N}S^N$$

denotes the $(k-1)^{\mathrm{st}}$ stable homotopy group of the sphere. It follows from the Atiyah-Hirzebruch spectral sequence that for any space M, one has

$$\begin{aligned}\tilde{I}^0(M) &= H^0(M;\mathbb{Z})\\ \tilde{I}^1(M) &= H^1(M;\mathbb{Z})\\ \tilde{I}^2(M) &= H^2(M;\mathbb{Z}) \times H^0(M;\mathbb{Z}/2).\end{aligned}$$

In particular, the group $\tilde{I}^2(M)$ contains the group of complex line bundles. In fact, thinking of $\mathbb{Z}/2$ as the group ± 1, the whole group $\tilde{I}^2(M)$ can be identified with the group of *graded* line bundles, the element of $H^0(M;\mathbb{Z}/2)$ corresponding to the degree. Since the map $H\mathbb{Z} \to \tilde{I}$ is a rational equivalence, we can choose a cocycle

$$\tilde{\iota} \in Z^0\left(\tilde{I};\mathbb{R}\right)$$

restricting to a fundamental cocycle in $Z^0\left(H\mathbb{Z};\mathbb{R}\right)$. This makes $\tilde{I}$ into a differential cohomology theory. The differential cohomology group $\tilde{I}^2(2)(M)$ can be interpreted as the group of horizontal isomorphism classes of graded $U(1)$-bundles with connection.

To a first approximation, the functor κ is derived from a map of spectra

$$MSO\langle\beta\nu_{2k}\rangle \to \Sigma^{4k}\tilde{I}. \tag{5.2}$$

But there is one more complication. In order to establish the symmetry

$$\kappa(-\lambda) \approx \kappa(\lambda) \tag{5.3}$$

we need to put a $\mathbb{Z}/2$-action on $MSO\langle\beta\nu_{2k}\rangle$ in such a way that it corresponds to the symmetry

$$\lambda \mapsto -\lambda.$$

In §5.4 we will describe a $\mathbb{Z}/2$-equivariant Eilenberg-MacLane space, $K(\mathbb{Z}(1),2k)$, with the following properties:

(1) The involution

$$\tau : K\left(\mathbb{Z}(1),2k\right) \to K\left(\mathbb{Z}(1),2k\right)$$

has degree -1;

(2) there is an equivariant map

$$K\left(\mathbb{Z}(1),2k\right) \to K(\mathbb{Z}/2,2k),$$

with $\mathbb{Z}/2$ acting trivially on $K(\mathbb{Z}/2,2k)$, corresponding to reduction modulo 2.

Using this, we define a $\mathbb{Z}/2$-equivariant $BSO\langle\beta\nu_{2k}\rangle$ by the homotopy pullback square

$$\begin{array}{ccc} BSO\langle\beta\nu_{2k}\rangle & \xrightarrow{\lambda} & K(\mathbb{Z}(1),2k) \\ \downarrow & & \downarrow \\ BSO & \xrightarrow[\nu_{2k}]{} & K(\mathbb{Z}/2,2k)\,. \end{array}$$

The associated Thom spectrum $MSO\langle\beta\nu_{2k}\rangle$ then acquires a $\mathbb{Z}/2$-action, and the existence of the symmetry isomorphism (5.3) can be guaranteed by factoring (5.2) through a map

$$MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2} \to \Sigma^{4k}\tilde{I}. \tag{5.4}$$

We have used the notation

$$X_{h\mathbb{Z}/2} = X \wedge_{\mathbb{Z}/2} E\mathbb{Z}/2_+$$

to denote the *homotopy orbit spectrum of* of a spectrum X with a $\mathbb{Z}/2$-action.

The spectrum $MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2}$ is also a Thom spectrum

$$MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2} = \mathrm{Thom}\left(BSO\langle\beta\nu_{2k}\rangle \times_{\mathbb{Z}/2} E\mathbb{Z}/2; V\right),$$

with V the stable vector bundle classified by

$$BSO\langle\beta\nu_{2k}\rangle \times_{\mathbb{Z}/2} E\mathbb{Z}/2 \to BSO \times B\mathbb{Z}/2 \to BSO\,. \tag{5.5}$$

The group

$$\pi_{4k}\, MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2}$$

is the cobordism group of $4k$-dimensional oriented manifolds M, equipped with a map $t : \pi_1 M \to \mathbb{Z}/2$ classifying a local system $\mathbb{Z}(1)$, and a cocycle $\lambda \in Z^{2k}(M;\mathbb{Z}(1))$ whose mod 2-reduction represents the Wu-class ν_{2k}.

Fix a cocycle $L_{4k} \in Z^{4k}(BSO;\mathbb{R})$ representing the component of the Hirzebruch L-polynomial of degree $4k$. By abuse of notation we will write

$$U \in \mathbb{Z}^0(MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2};\mathbb{R}) \qquad L_{4k} \in \mathbb{Z}^0(BSO\langle\beta\nu_{2k}\rangle \times_{\mathbb{Z}/2} E\mathbb{Z}/2;\mathbb{R})$$

for the pullback along the maps derived from (5.5) of the Thom cocycle and L_{4k}, respectively.

The group of maps (5.4) sits in a (splittable) short exact sequence

$$\begin{aligned} \mathrm{Ext}\left(\pi_{4k-1}\, MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2},\mathbb{Z}\right) &\rightarrowtail \tilde{I}^{4k}\left(MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2}\right) \\ &\twoheadrightarrow \hom\left(\pi_{4k}\, MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2},\mathbb{Z}\right), \end{aligned} \tag{5.6}$$

whose rightmost terms are the group of integer-valued cobordism invariants of manifolds M of the type just described. One example is the signature $\sigma = \sigma(M)$ of the non-degenerate bilinear form

$$B(x,y) = \int_M x \cup y : H^{2k}\left(M; \mathbb{Q}(1)\right) \times H^{2k}\left(M; \mathbb{Q}(1)\right) \to \mathbb{Q}. \tag{5.7}$$

Another is

$$\int_M \lambda^2.$$

By definition, λ is a characteristic element for the bilinear form (5.7), and so

$$\int_M \lambda^2 \equiv \sigma(M) \mod 8.$$

Proposition 5.8. *Any map of spectra*

$$\kappa : MSO\langle \beta\nu_{2k}\rangle_{h\mathbb{Z}/2} \to \Sigma^{4k}\tilde{I}$$

whose underlying homomorphism

$$\pi_{4k}\, MSO\langle \beta\nu_{2k}\rangle_{h\mathbb{Z}/2} \to \mathbb{Z}$$

associates to M^{4k}, $\lambda \in Z^{2k}(M; \mathbb{Z}(1))$, the integer

$$\frac{1}{8}\left(\int_M \lambda^2 - \sigma\right), \tag{5.9}$$

gives a family of functors

$$\kappa_{E/S} : \check{\mathcal{H}}^{2k}_{\nu}(E) \to \check{\mathcal{H}}^{i}(S)$$

having the properties listed in Theorem 2.17 *and Corollary* 2.18.

Because of the sequence (5.6) the set of maps κ satisfying the condition of Proposition 5.10 is a non-empty principal homogeneous space for

$$\operatorname{Ext}\left(\pi_{4k-1}\, MSO\langle \beta\nu_{2k}\rangle_{h\mathbb{Z}/2}, \mathbb{Z}.\right)$$

Before going through the proof we need one more topological result. The change of an integral Wu-structure by a $2k$-cocycle is represented by a map

$$(BSO\langle \beta\nu_{2k}\rangle \times K(\mathbb{Z}, 2k)) \times_{\mathbb{Z}/2} E\mathbb{Z}/2 \to BSO\langle \beta\nu_{2k}\rangle \times_{\mathbb{Z}/2} E\mathbb{Z}/2.$$

We'll write the induced map of Thom spectra as

$$(\lambda - (2)x) : (MSO\langle \beta\nu_{2k}\rangle \wedge K(\mathbb{Z}(1), 2k)_+)_{h\mathbb{Z}/2} \to MSO\langle \beta\nu_{2k}\rangle_{h\mathbb{Z}/2}.$$

Of course, the map $(\lambda - (2)x) - (\lambda)$ factors through

$$MSO\langle \beta\nu_{2k}\rangle \wedge K(\mathbb{Z}(1), 2k)_{h\mathbb{Z}/2} \to MSO\langle \beta\nu_{2k}\rangle_{h\mathbb{Z}/2}.$$

We will be interested in

$$b(x,y) = (\lambda - (2)(x+y)) - (\lambda - (2)x) - (\lambda - (2)y) + (\lambda)$$

which is a map

$$MSO\langle \beta\nu_{2k}\rangle \wedge K(\mathbb{Z}(1), 2k) \wedge K(\mathbb{Z}(1), 2k)_{h\mathbb{Z}/2} \to MSO\langle \beta\nu_{2k}\rangle_{h\mathbb{Z}/2}.$$

Lemma 5.10. *If* $\kappa : MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2} \to \Sigma^{4k}\tilde{I}$ *is any map satisfying the condition of Proposition* 5.8*, then the following diagram commutes up to homotopy*

$$\begin{array}{ccc} MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z}(1),2k) \wedge K(\mathbb{Z}(1),2k)_{h\mathbb{Z}/2} & \xrightarrow{U\cup x\cup y} & \Sigma^{4k}\, H\mathbb{Z} \\ {\scriptstyle b(x,y)}\big\downarrow & & \big\downarrow \\ MSO\langle\beta\nu_{2k}\rangle & \xrightarrow[\kappa]{} & \Sigma^{4k}\tilde{I}. \end{array}$$

Proof. Since

$$\pi_{4k-1}\, MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z}(1),2k) \wedge K(\mathbb{Z}(1),2k)_{h\mathbb{Z}/2} = 0,$$

it suffices to check that the two ways of going around the diagram agree after passing to π_{4k}. The clockwise direction associates to (M,λ,x,y) the integer

$$q^{\lambda}(x+y) - q^{\lambda}(x) - q^{\lambda}(y) + q^{\lambda}(0),$$

where we have written

$$q^{\lambda}(x) = \kappa(\lambda - (2)x) = \tfrac{1}{2}\int_M (x^2 - x\lambda).$$

The counter-clockwise direction is

$$\int_M x \cup y.$$

The result follows easily. q.e.d.

Proof of Proposition 5.8. By construction, $\tilde{I}$ comes equipped with a fundamental cocycle $\tilde{\iota} \in Z^0(\tilde{I};\mathbb{R})$ which necessarily restricts to a fundamental cocycle in $H\mathbb{Z}$. Now the signature of the bilinear form (5.7) coincides with the signature of M. Indeed, let $\tilde{M} \to M$ be the double cover classified by the homomorphism $t : \pi_1 M \to \mathbb{Z}/2$. Then by the signature theorem, $\sigma(\tilde{M}) = 2\sigma(M)$, and the claim then follows by decomposing $H^{2k}(\tilde{M};\mathbb{Q})$ into eigenspaces under the action of $\mathbb{Z}/2$. It follows that the cohomology class of $\kappa^*\tilde{\iota}$ coincides with that of

$$\alpha = U \cdot \frac{(\lambda^2 - L_{4k})}{8}, \tag{5.11}$$

so after choosing a cochain whose coboundary is the difference between $\kappa^*\tilde{\iota}$ and α, we have a map of differential cohomology theories

$$\left(MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2};\alpha\right) \to \Sigma^{4k}\left(\tilde{I};\tilde{\iota}\right).$$

Since $H^{4k-1}\left(MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2};\mathbb{R}\right) = 0$, this choice of cochain has no effect on the maps of fundamental groupoids we derive from it.

Choose a point $b \in E\mathbb{Z}/2$. We use b to define a map

$$i_b : MSO\langle\beta\nu_{2k}\rangle = MSO\langle\beta\nu_{2k}\rangle \wedge \{b\}_+ \hookrightarrow MSO\langle\beta\nu_{2k}\rangle \wedge E\mathbb{Z}/2_+.$$

And to keep the notation simple, we will not distinguish in notation between α and $\iota_b^*\alpha$. Suppose that E/S is an $\check{H}$-oriented map of manifolds of relative dimension $(4k-i)$. The functor $\kappa_{E/S}$ is constructed from the map of fundamental groupoids of differential function complexes

$$\pi_{\leq 1}\,\mathrm{filt}_0\left(MSO\langle\beta\nu_{2k}\rangle_{i-4k};\alpha\right)^S \to \pi_{\leq 1}\,\mathrm{filt}_0\left(\tilde{I}_i;\tilde{\iota}\right)^S. \tag{5.12}$$

In principle, a differential integral Wu-structure on E/S defines a 0-simplex of the differential function space

$$\mathrm{filt}_0\left(MSO\langle\beta\nu_{2k}\rangle_{i-4k};\alpha\right)^S,$$

in such a way as to give a functor

$$\check{\mathcal{H}}^{2k}_{\nu}(E) \to \pi_{\leq 1}\,\mathrm{filt}_0\left(MSO\langle\beta\nu_{2k}\rangle_{i-4k};\alpha\right)^S. \tag{5.13}$$

The functor $\kappa_{E/S}$ is just the composition of (5.13) with (5.12). We have said "in principle" because the geometric data we specified in the statement of Theorem 2.17 is not quite the data which is classified by a differential function from S to $(MSO\langle\beta\nu_{2k}\rangle;\alpha)_{i-4k}$. To remedy this, we use the technique described in §4.9.3 to produce a diagram

$$\check{\mathcal{H}}^{2k}_{\nu}(E) \xleftarrow{\sim} \check{\mathcal{H}}^{2k}_{\nu}(E)_{\mathrm{geom}} \to \pi_{\leq 1}\,\mathrm{filt}_0\left(MSO\langle\beta\nu_{2k}\rangle_{i-4k};\alpha\right)^S,$$

in which the left map is an equivalence of groupoids. The functor $\kappa_{E/S}$ is then constructed as described, after choosing an inverse to this equivalence.

The category $\check{\mathcal{H}}^{2k}_{\nu}(E)_{\mathrm{geom}}$ is the fundamental groupoid of a certain simplicial set $\mathcal{S}^{2k}_{\nu}(E)$. To describe it, first note that by the results of §4.9.3 we may assume that the $\check{H}$-orientation of E/S comes from a differential BSO-orientation

$$(\xi_{N-i+4k}, U_{N-i+4k}) \xleftarrow{(c,h,\omega)} W \subset S\times\mathbb{R}^N,$$

and that the differential cocycle[11] L_{4k} refines the map classifying W to a differential function

$$E \xrightarrow{(c_0,h_0,\omega_0)} (BSO;L_{4k}).$$

A k-simplex of $\mathcal{S}^{2k}_{\nu}(E)$ consists of a differential function

$$E\times\Delta^k \xrightarrow{(c_1,h_1,\omega_1)} (BSO\langle\beta\nu_{2k}\rangle;\lambda) \tag{5.14}$$

[11] We are using the symbol L_{4k} to denote both the universal signature cocycle and the chosen refinement to a differential cocycle on E. We hope this causes no confusion.

of weight filtration 0, for which the map c_1 fits into the commutative diagram

$$\begin{tikzcd}
E \times \Delta^k \arrow[rr, "c_1"] \arrow[d] & & BSO\langle \beta\nu_{2k}\rangle \arrow[d] \\
E \arrow[r] & BSO(N-i+4k) \arrow[r] & BSO\,.
\end{tikzcd}$$

The functor

$$\check{\mathcal{H}}^{2k}_{\nu}(E)_{\mathrm{geom}} = \pi_{\le 1}\mathcal{S}^{2k}_{\nu}(E) \to \check{\mathcal{H}}^{2k}_{\nu}(E) \tag{5.15}$$

sends (5.14) to the slant product of $(c_1^*\lambda, h_1, \omega_1)$ with the fundamental class of Δ^k. Using the results of Appendix D one checks that (5.15) is an equivalence of groupoids.

Write

$$\begin{aligned}
\check{U} &= (c^* U_{N-i+4k}, h, \omega) \\
\check{\lambda} &= (c_1^*\lambda, h_1, \omega_1)\,.
\end{aligned}$$

The lift c_1 of the map classifying W, and the differential cocycle

$$\check{U} \cdot \frac{\left(\check{\lambda}^2 - L_{4k}\right)}{8}$$

combine to make

$$E \times \Delta^k \to S \times \Delta^k$$

into a differential $BSO\langle \beta\nu_{2k}\rangle$-oriented map, with respect to the cocycle α. This defines

$$\mathcal{S}^{2k}_{\nu}(E) \to \mathrm{filt}_0\left(MSO\langle \beta\nu_{2k}\rangle_{i-4k}; \alpha\right)^S,$$

and hence $\kappa_{E/S}$ as described above.

It is now a fairly routine exercise to verify that this functor $\kappa_{E/S}$ and the resulting q have the properties claimed by Theorem 2.17. Property i) is immediate by assumption. For property ii), write the action of $\mathbb{Z}/2$ on $E\mathbb{Z}/2_+$ as $b \mapsto -b$. Consider the diagram of differential function complexes, in which the vertical maps are obtained by smashing the identity map with the inclusions $\{b\} \to E\mathbb{Z}/2$ and $\{-b\} \to E\mathbb{Z}/2$

$$\begin{tikzcd}
& \mathrm{filt}_0\left(MSO\langle \beta\nu_{2k}\rangle_{i-4k}; \alpha\right)^S \arrow[d, "b"] \arrow[dr] & \\
\mathcal{S}^{2k}_{\nu}(E) \arrow[ur] \arrow[r] \arrow[dr] & \mathrm{filt}_0\left(\left(MSO\langle \beta\nu_{2k}\rangle \wedge E\mathbb{Z}/2_+; \alpha\right)_{i-4k}\right)^S \arrow[r, "\kappa"] & \mathrm{filt}_0\left(\tilde{I}_i; \tilde{\iota}\right)^S \\
& \mathrm{filt}_0\left(MSO\langle \beta\nu_{2k}\rangle_{i-4k}\right)^S \arrow[u, "-b"'] \arrow[ur] &
\end{tikzcd}.$$

The map of fundamental groupoids induced by the upper composition is $\kappa_{E/S}(\lambda)$, by definition. Since κ factors through the quotient by the diagonal $\mathbb{Z}/2$-action, the bottom composition is $\kappa_{E/S}(-\lambda)$. The fact that the vertical arrows are homotopy equivalences means that there is a homotopy between the two ways of going around. Choose one. By definition this homotopy is an isomorphism

$$\tau(\lambda) : \kappa(-\lambda) \xrightarrow{\approx} \kappa(\lambda)$$

in the fundamental groupoid

$$\pi_{\leq 1}\left(\tilde{I}_i, \tilde{\iota}\right)^S.$$

Any two homotopies extend over a disk, and so define the same isomorphism in the fundamental groupoid. The compositions of the homotopies $\tau(\lambda)$ and $\tau(-\lambda)$ also extend over the disk, and so is the identity map

$$\tau(\lambda) \circ \tau(-\lambda) = \text{identity map of } \kappa(\lambda).$$

For the base change property iii) note that if E/S is classified by a differential function

$$S \to (MSO\langle\beta\nu_{2k}\rangle_{i-4k}; \alpha),$$

then the map

$$S' \to S \to (MSO\langle\beta\nu_{2k}\rangle_{i-4k}; \alpha)$$

is the map classifying a differential $BSO\langle\beta\nu_{2k}\rangle$-orientation of E'/S'. The result then follows easily.

Now for the transitivity property iv). Suppose that E/B has relative dimension m and is classified by a differential function

$$B \to (MSO\langle\beta\nu_{2k}\rangle, \alpha)_{-m}, \tag{5.16}$$

and that B/S has relative dimension ℓ. Let

$$S^0 \wedge (MSO\langle\beta\nu_{2k}\rangle_{-m})_+ \tag{5.17}$$

denote the unreduced suspension spectrum of $MSO\langle\beta\nu_{2k}\rangle_{-m}$. Interpreting (5.17) as the Thom spectrum $\mathrm{Thom}(MSO\langle\beta\nu_{2k}\rangle_{-m}; 0)$, we can regard the differential framing on B/S together with the map (5.16) as classified by a differential function

$$S \to \left(S^0 \wedge (MSO\langle\beta\nu_{2k}\rangle_{-m})_+, \alpha_{-m}\right)_{-\ell}. \tag{5.18}$$

We will use the structure map

$$\left(S^0 \wedge (MSO\langle\beta\nu_{2k}\rangle_{-m})_+; \alpha_{-m}\right) \to \Sigma^{-m}(MSO\langle\beta\nu_{2k}\rangle; \alpha) \tag{5.19}$$

of the differential spectrum $(MSO\langle\beta\nu_{2k}\rangle; \alpha)$. Composing (5.18) with (5.19) gives a differential function

$$S \to (MSO\langle\beta\nu_{2k}\rangle; \alpha)_{-m-\ell}$$

classifying a differential $BSO\langle\beta\nu_{2k}\rangle$-orientation on E/S. In this way E/S acquires an $\check{H}$-orientation. One easily checks that the differential cocycle refining the pullback of α is

$$\frac{1}{8}\check{U}\cup(\check{\lambda}^2-\check{L}_{4k})$$

The transitivity isomorphism is then derived from a choice of homotopy between the two ways of going around the diagram of differential function spectra

$$\begin{array}{ccc}
\Sigma^{-\ell}\left(S^0\wedge(MSO\langle\beta\nu_{2k}\rangle_{-m})_+,\alpha_{-m}\right) & \longrightarrow & \Sigma^{-m-\ell}\left(MSO\langle\beta\nu_{2k}\rangle;\alpha\right) \\
\Big\downarrow & & \Big\downarrow \\
\Sigma^{-\ell}\left(S^0\wedge(\tilde{I}_{4k-m})_+,(\tilde{\iota})_{4k-m}\right) & \longrightarrow & \Sigma^{4k-m-\ell}\left(I,\tilde{\iota}\right).
\end{array}$$

The clockwise composition associates to the differential function classifying E/B the value

$$\kappa_{E/S}(\lambda),$$

and the counter-clockwise composition gives the value

$$\int_{B/S}\kappa_{E/B}(\lambda).$$

It remains to establish the properties of q stated in Corollary 2.18. Properties ii)– iv) are formal consequences of the corresponding properties of κ. Property i) follows from Lemma 5.10 using methods similar to those we've been using for the properties of κ. q.e.d.

Remark 5.20. The proofs of the symmetry, base change, and transitivity properties didn't make use of the condition on κ stated in Proposition 5.8. These properties are built into the formalism of differential bordism theories, and would have held for any κ.

5.2. The topological theory of quadratic functions. We now turn to the construction of a particular topological κ

$$\kappa: MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2}\to\Sigma^{4k}\tilde{I}$$

satisfying (5.9). To describe a map to $\tilde{I}$ requires a slightly more elaborate algebraic object than an abelian group. There are several approaches, but for our purposes, the most useful involves the language of *Picard categories* (see Appendix B.) Here we state the main result without this language.

Let $\bar{\nu}_{2k}$ be the composite

$$BSO\xrightarrow{\nu_{2k}}K(\mathbb{Z}/2,2k)\to K(\mathbb{Q}/\mathbb{Z}(1),2k),$$

and $BSO\langle\bar\nu_{2k}\rangle$ its (equivariant) homotopy fiber. The space $BSO\langle\bar\nu_{2k}\rangle$ fits into a homotopy Cartesian square

$$\begin{tikzcd} BSO\langle\bar\nu_{2k}\rangle \arrow[r] \arrow[d] & K(\mathbb{Q}/\mathbb{Z}(1), 2k-1) \arrow[d, "\beta"] \\ BSO \arrow[r, "\nu_{2k}"'] & K(\mathbb{Z}/2, 2k). \end{tikzcd}$$

The $\mathbb{Z}/2$-action corresponds to sending η to $-\eta$, and the associated Thom spectrum $MSO\langle\bar\nu_{2k}\rangle_{h\mathbb{Z}/2}$ is the bordism theory of manifolds N equipped with a homomorphism $t : \pi_1 N \to \mathbb{Z}/2$ classifying a local system $\mathbb{Z}(1)$ and a cocycle $\eta \in Z^{2k-1}(N; \mathbb{Q}/\mathbb{Z}(1))$ for which $\beta\eta \in Z^{2k}(N; \mathbb{Z}(1))$ is an integral Wu-structure.

To ease the notation, set

$$\begin{aligned} B &= BSO\langle\beta\nu_{2k}\rangle \times_{\mathbb{Z}/2} E\mathbb{Z}/2 \\ \bar B &= BSO\langle\bar\nu_{2k}\rangle \times_{\mathbb{Z}/2} E\mathbb{Z}/2. \end{aligned}$$

Let $\mathcal{C}$ be the groupoid whose objects are closed $\bar B$-oriented manifolds (M, η) of dimension $(4k-1)$, and whose morphisms are equivalence classes of B-oriented maps

$$p : M \to \Delta^1 \tag{5.21}$$

equipped with $\bar B$-orientations η_0 and η_1 of $\partial_0 M = p^{-1}(0)$ and $\partial_1 M = p^{-1}(1)$ which are compatible with the B-orientation in the sense that

$$\beta\eta_i = \lambda|_{\partial_i M}, \quad i = 0, 1.$$

The equivalence relation and the composition law are described in terms of B-oriented maps to $f : E \to \Delta^2$ equipped with compatible $\bar B$ orientations of $f^{-1}(e_i)$, $i = 0, 1, 2$. Write

$$\partial_i E = f|_{p^{-1}\partial_i \Delta^2}.$$

Then for each such E/Δ^2 we set

$$\partial_0 E \circ \partial_2 E \sim \partial_1 E. \tag{5.22}$$

Remark 5.23. Strictly speaking we define $\mathcal{C}$ to be the quotient of the category freely generated by the maps (5.21), by the relations (5.22). This category happens to be a groupoid; every morphism is represented by some map (5.21), and the composition law can be thought of as derived from the operation of gluing together manifolds along common boundary components. It works out that the identity morphism Id_N is represented by the projection

$$N \times \Delta^1 \to \Delta^1.$$

Proposition 5.24. *The set of homotopy classes of maps*

$$MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2} \to \Sigma^{4k}\tilde I \tag{5.25}$$

can be identified with the set of equivalence classes of pairs of invariants

$$\kappa_{4k} : \{\textit{morphisms of } \mathcal{C}\} \to \mathbb{Q}$$
$$\kappa_{4k-1} : \{\textit{objects of } \mathcal{C}\} \to \mathbb{Q}/\mathbb{Z}$$

satisfying

$$\begin{aligned} \kappa_{4k}(M_1/\Delta^1 \amalg M_2/\Delta^1) &= \kappa_{4k}(M_1/\Delta^1) + \kappa_{4k}(M_2/\Delta^1) \\ \kappa_{4k-1}(N_1 \amalg N_2) &= \kappa_{4k-1}(N_1) + \kappa_{4k-1}(N_2), \end{aligned} \tag{5.26}$$

$$\kappa_{4k}(M/\Delta^1) \equiv \kappa_{4k-1}(\partial_0 M) - \kappa_{4k-1}(\partial_1 M) \mod \mathbb{Z}, \tag{5.27}$$

and for each B-oriented $f : E \to \Delta^2$ equipped with compatible $\bar{B}$ orientations of $f^{-1}(e_i)$, $i = 0, 1, 2$,

$$\kappa_{4k}(\partial_0 E) - \kappa_{4k}(\partial_1 E) + \kappa_{4k}(\partial_2 E) = 0. \tag{5.28}$$

Two pairs $(\kappa_{4k}, \kappa_{4k-1})$ and $(\kappa'_{4k}, \kappa'_{4k-1})$ are equivalent if there is a map

$$h : \{\textit{objects of } \mathcal{C}\} \to \mathbb{Q}$$

with

$$\begin{aligned} h(M_1 \amalg M_2) &= h(M_1) + h(M_2), \\ h(N) &\equiv \kappa'_{4k-1}(N) - \kappa_{4k-1}(N) \mod \mathbb{Z} \end{aligned}$$

and for each M/Δ^1,

$$h(\partial_0 M) - h(\partial_1 M) = \kappa_{4k}(M/\Delta^1) - \kappa_{4k}(M/\Delta^1).$$

Remark 5.29. In order to keep the statement simple, we have been deliberately imprecise on one point in our statement of Proposition 5.24. With our definition, the disjoint union of two B-oriented manifolds doesn't have a canonical B-orientation. In order to construct one, we need to choose a pair of disjoint cubes embedded in $\mathbb{R}^\infty$. The more precise statement is that for *any B-orientation arising from any such choice of pair of cubes*, one has

$$\begin{aligned} \kappa_{4k}(M_1 \amalg M_2) &= \kappa_{4k}(M_1) + \kappa_{4k}(M_2) \\ \kappa_{4k-1}(M_1 \amalg M_2) &= \kappa_{4k-1}(M_1) + \kappa_{4k}(M_2). \end{aligned}$$

In particular, the value of κ_i on a disjoint union is assumed to be independent of this choice. For more on this, see Example B.11, and the remark preceding it.

Remark 5.30. Proposition 5.24 can be made much more succinct. Let $(\mathbb{Q} \to \mathbb{Q}/\mathbb{Z})$ denote the category with objects $\mathbb{Q}/\mathbb{Z}$ and in which a map from a to b is a rational number r satisfying

$$r \equiv b - a \mod \mathbb{Z}.$$

Then the assertion of Proposition 5.24 is that the set of homotopy classes (5.25) can be identified with the set of "additive" functors

$$\kappa : \mathcal{C} \to (\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$$

modulo the relation of "additive" natural equivalence. The language of Picard categories is needed in order to make precise this notion of "additivity." We have chosen to spell out the statement of Proposition 5.24 in the way we have in order to make clear the exact combination of invariants needed to construct the map (5.25).

Aside from the fact that the integral Wu-structure on the boundary has a special form, Proposition 5.24 is a fairly straightforward consequence of Corollary B.17. We now turn to showing that this boundary condition has no real effect.

Lemma 5.31. *The square*

$$\begin{array}{ccc} BSO\langle\bar{\nu}_{2k}\rangle & \longrightarrow & BSO \\ \big\downarrow & & \big\downarrow \\ BSO\langle\beta\nu_{2k}\rangle & \longrightarrow & BSO\times K(\mathbb{Q}(1),2k), \end{array} \tag{5.32}$$

is homotopy co-Cartesian. The components of the bottom map are the defining projection to BSO and the image of λ in

$$Z^{2k}(BSO\langle\beta\nu_{2k}\rangle;\mathbb{Q}(1)).$$

Proof. It suffices to prove the result after profinite completion and after localization at $\mathbb{Q}$. Since

$$\begin{aligned} K(\mathbb{Q}/\mathbb{Z},2k-1)\to K(\mathbb{Z},2k) \quad &\text{and} \\ \mathrm{pt}\to K(\mathbb{Q},2k) \quad & \end{aligned}$$

are equivalences after profinite completion, so are the vertical maps in (5.32) and so the square is homotopy co-Cartesian after profinite completion. The horizontal maps become equivalences after $\mathbb{Q}$-localization, and so the square is also homotopy co-Cartesian after $\mathbb{Q}$-localization. This completes the proof. q.e.d.

Passing to Thom spectra gives

Corollary 5.33. *The square*

$$\begin{array}{ccc} MSO\langle\bar{\nu}_{2k}\rangle & \longrightarrow & MSO \\ \big\downarrow & & \big\downarrow \\ MSO\langle\beta\nu_{2k}\rangle & \longrightarrow & MSO\wedge(K(\mathbb{Q}(1),2k)_+) \end{array}$$

is homotopy co-Cartesian.

Finally, passing to homotopy orbit spectra, and using the fact that

$$\iota^2\times 1: K(\mathbb{Q}(1),2k)\times_{\mathbb{Z}/2} E\mathbb{Z}/2\to K(\mathbb{Q},4k)\times B\mathbb{Z}/2$$

is a stable weak equivalence (its cofiber has no homology), gives

Corollary 5.34. *The square*

$$
\begin{array}{ccc}
MSO\langle\bar{\nu}_{2k}\rangle_{h\mathbb{Z}/2} & \longrightarrow & MSO\wedge B\mathbb{Z}/2_+ \\
\big\downarrow & & \big\downarrow \\
MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2} & \longrightarrow & MSO\wedge(K(\mathbb{Q},4k)\times B\mathbb{Z}/2)_+
\end{array}
$$

is homotopy co-Cartesian.

Corollary 5.35. *The map*

$$MSO\langle\bar{\nu}_{2k}\rangle_{h\mathbb{Z}/2}\to MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2}$$

is $(4k-1)$*-connected. In particular, there are cell decompositions of both spectra, for which the map of* $(4k-1)$*-skeleta*

$$\big(MSO\langle\bar{\nu}_{2k}\rangle_{h\mathbb{Z}/2}\big)^{(4k-1)}\to\big(MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2}\big)^{(4k-1)} \tag{5.36}$$

is a weak equivalence.

Proof of Proposition 5.24. We will freely use the language of Picard categories, and the results of Appendix B.

Choose a cell decomposition of $\bar{B}$, and take

$$MSO\langle\bar{\nu}_{2k}\rangle^{(t)}_{h\mathbb{Z}/2}=\mathrm{Thom}\left(\bar{B}^{(t)};V\right).$$

Let $MSO\langle\beta\nu_{2k}\rangle^{(t)}_{h\mathbb{Z}/2}$ be any cell decomposition satisfying (5.36). Finally, let

$$\pi'_{\le 1}(MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2})_{(1-4k)}$$

be the Picard category whose objects are transverse maps

$$S^{4k-1}\to(MSO\langle\beta\nu_{2k}\rangle^{(4k-1)}_{h\mathbb{Z}/2}=MSO\langle\bar{\nu}_{2k}\rangle^{(4k-1)}_{h\mathbb{Z}/2},$$

and whose morphisms are homotopy classes, relative to

$$S^{4k-1}\wedge\partial\Delta^1_+$$

of transverse maps of pairs

$$\left(S^{4k-1}\wedge\Delta^1_+,S^{4k-1}\wedge\partial\Delta^1_+\right)\to\left(MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2},MSO\langle\beta\nu_{2k}\rangle^{(4k-1)}_{h\mathbb{Z}/2}\right).$$

By the cellular approximation theorem, and the geometric interpretation of the homotopy groups of Thom spectra, the Pontryagin-Thom construction gives an equivalence of Picard categories.

$$\pi'_{\le 1}(MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2})_{(1-4k)}\to\pi_{\le 1}\,\mathrm{sing}\left(\big(MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2}\big)_{(1-4k)}\right).$$

Now, by definition, a pair of invariants $(\kappa_{4k},\kappa_{4k-1})$ is a functor (of Picard categories)

$$\pi'_{\le 1}(MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2})_{(1-4k)}\to(\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}). \tag{5.37}$$

It is easy to check, using the exact sequences

$$\mathrm{Ext}\left(\pi_{4k-1}\,MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2},\mathbb{Z}\right)\rightarrowtail A\twoheadrightarrow\hom\left(\pi_{4k}\,MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2},\mathbb{Z}\right),$$

(where A either the group of homotopy classes of maps (5.25), or the natural equivalences classes of functors (5.37)) that this gives an isomorphism of the group of equivalence classes of pairs $(\kappa_{4k}, \kappa_{4k-1})$ and the group of natural equivalence classes of functors of Picard categories (5.37). The result now follows from Corollary B.17. q.e.d.

5.3. The topological κ. By Proposition 5.24, in order to define the map

$$\kappa : MSO\langle\beta\nu_{2k}\rangle_{h\mathbb{Z}/2} \to \Sigma^{4k}\tilde{I},$$

we need to construct invariants $(\kappa_{4k}, \kappa_{4k-1})$, satisfying (5.26)–(5.28). In the case $\lambda = 0$, these invariants appear in surgery theory in the work of Milgram [**47**], and Morgan–Sullivan [**51**]. The methods described in [**47**, **51**] can easily be adapted to deal with general λ. We will adopt a more homotopy theoretic formulation, which is more convenient for our purposes.

Suppose that M/Δ^1 is a morphism of $\mathcal{C}$, and let σ be the signature of the non-degenerate bilinear form $\int_M x \cup y$ on the subgroup of $H^{2k}(M; \mathbb{Q}(1))$ consisting of elements which vanish on the boundary. Because of the boundary condition the image of λ in $H^{2k}(M; \mathbb{Q}(1))$ is in this subgroup, and so

$$\int_M \lambda^2$$

is also defined. We set

$$\kappa_{4k}(M) = \frac{1}{8}\left(\int_M \lambda^2 - \sigma\right). \tag{5.38}$$

We will show in Proposition 5.66 of §5.4 that the "Wu-structure"

$$\eta \in Z^{2k-1}(N; \mathbb{Q}/\mathbb{Z}(1))$$

on an object $N \in \mathcal{C}$ gives rise to a quadratic refinement

$$\phi = \phi_N = \phi_{N,\eta} : H^{2k}(N; \mathbb{Z}(1))_{\text{tor}} \to \mathbb{Q}/\mathbb{Z}$$

of the link pairing

$$H^{2k}(N; \mathbb{Z}(1))_{\text{tor}} \times H^{2k}(N; \mathbb{Z}(1))_{\text{tor}} \to \mathbb{Q}/\mathbb{Z}.$$

The function ϕ is a cobordism invariant in the sense that if M/Δ^1 is a morphism in $\mathcal{C}$, and x is an element of $H^{2k}(M; \mathbb{Z}(1))$ whose restriction to ∂M is torsion, then

$$\begin{gathered} \phi_{\partial_0 M}(\partial_0 x) - \phi_{\partial_1 M}(\partial_1 x) \equiv \frac{1}{2}\int_M (x^2 - x\lambda) \mod \mathbb{Z} \\ \partial_i x = x|_{\partial_i M}. \end{gathered} \tag{5.39}$$

Note that the integral is well-defined since it depends only on the image of x in $H^{2k}(M; \mathbb{Q}(1))$, which vanishes on ∂M. Note also that the integral vanishes when x is itself torsion.

We define $\kappa_{4k-1}(N)$ by

$$
\begin{aligned}
e^{2\pi i\,\kappa_{4k-1}(N)} &= \frac{1}{\sqrt{d}} \sum_{x\in H^{2k}(N;\mathbb{Z}(1))_{\mathrm{tor}}} e^{-2\pi i \phi(x)} \\
d &= \# H^{2k}(N;\mathbb{Z}(1))_{\mathrm{tor}}.
\end{aligned}
\tag{5.40}
$$

With these definitions property (5.26) is immediate, and property (5.28) is nearly so. In the situation of (5.28), note that ∂E is homeomorphic to a smooth manifold whose signature is zero since it bounds an oriented manifold. Novikov's additivity formula for the signature then gives

$$
\sigma\left(\partial_0 E\right) - \sigma\left(\partial_1 E\right) + \sigma\left(\partial_2 E\right) = \sigma\left(\partial E\right) = 0
$$

which is property (5.28).

For (5.27) we need an algebraic result of Milgram [**47**, **51**, **50**]. Suppose that V is a vector space over $\mathbb{Q}$ of finite dimension, and $q_V : V \to \mathbb{Q}$ is a quadratic function (not necessarily even) whose underlying bilinear form

$$
B(x,y) = q_V(x+y) - q_V(x) - q_V(y)
$$

is non-degenerate. One easily checks that $q_V(x) - q_V(-x)$ is linear, and so there exists a unique $\lambda \in V$ with

$$
q_V(x) - q_V(-x) = -B(\lambda, x).
$$

By definition

$$
q_V(x) + q_V(-x) = B(x,x).
$$

Adding these we have

$$
q_V(x) = \frac{B(x,x) - B(x,\lambda)}{2}. \tag{5.41}
$$

The class λ is a *characteristic element* of $B(x,y)$.

Suppose that L is a lattice in V on which q_V takes integer values, and let

$$
L^* = \{x \in V \mid B(x,y) \in \mathbb{Z} \quad \forall y \in L\}
$$

be the dual lattice. Then $L \subset L^*$, and q_V descends to a non-degenerate quadratic function

$$
\begin{aligned}
q : L^*/L &\to \mathbb{Q}/\mathbb{Z} \\
x &\mapsto q_V(x) \mod \mathbb{Z}.
\end{aligned}
$$

We associate to q the Gauss sum

$$
\begin{aligned}
g(q) &= \frac{1}{\sqrt{d}} \sum_{x \in L^*/L} e^{-2\pi i\, q(x)} \\
d &= \left| L^*/L \right|.
\end{aligned}
$$

Finally, let $\sigma = \sigma(q_V)$ denote the signature of B.

Proposition 5.42. *With the above notation,*

$$g(q) = e^{2\pi i(B(\lambda,\lambda)-\sigma)/8}. \tag{5.43}$$

Proof. The result is proved in Appendix 4 of [**50**] in case $\lambda = 0$. To reduce to this case, first note that both the left and right hand sides of (5.43) are independent of the choice of L. Indeed, the right hand side doesn't involve the choice of L, and the fact that the left hand side is independent of L is Lemma 1, Appendix 4 of [**50**] (the proof doesn't make use of the assumption that q_V is even). Replacing L by $4L$, if necessary, we may assume that

$$B(x,x) \equiv 0 \mod 2 \qquad x \in L,$$

or, equivalently that

$$\eta = \frac{1}{2}\lambda$$

is in L^*. Set

$$q'_V(x) = \frac{1}{2}B(x,x).$$

The function q_V and q'_V are quadratic refinements of the same bilinear form, q'_V takes integer values on L, and

$$q_V(x) = q'_V(x-\eta) - q'_V(\eta).$$

The case $\lambda = 0$ of (5.43) applies to q'_V giving

$$g(q') = e^{2\pi i\sigma/8}.$$

But

$$\begin{aligned} g(q') &= \frac{1}{\sqrt{d}} \sum_{x\in L^*/L} e^{-2\pi i\, q'(x)} \\ &= \frac{1}{\sqrt{d}} \sum_{x\in L^*/L} e^{-2\pi i\, q'(x-\eta)} \\ &= \frac{1}{\sqrt{d}} \sum_{x\in L^*/L} e^{-2\pi i\, q(x)-q'(\eta)}, \end{aligned}$$

and the result follows since

$$q'(\eta) = \frac{B(\lambda,\lambda)}{8}.$$

q.e.d.

Proposition 5.44. *The invariants* $(\kappa_{4k}, \kappa_{4k-1})$ *defined above satisfy the conditions of Proposition* 5.24*:*

$$\begin{gathered} \kappa_{4k}(M_1/\Delta^1 \amalg M_2/\Delta^1) = \kappa_{4k}(M_1/\Delta^1) + \kappa_{4k}(M_2/\Delta^1) \\ \kappa_{4k-1}(N_1 \amalg N_2) = \kappa_{4k-1}(N_1) + \kappa_{4k-1}(N_2), \end{gathered} \tag{5.45}$$

$$\kappa_{4k}(M/\Delta^1) \equiv \kappa_{4k-1}(\partial_0 M) - \kappa_{4k-1}(\partial_1 M) \mod \mathbb{Z}, \tag{5.46}$$

and for each B-oriented $f : E \to \Delta^2$ equipped with compatible $\bar{B}$ orientations of $f^{-1}(e_i)$, $i = 0, 1, 2$,

$$\kappa_{4k}\left(\partial_0 E\right) - \kappa_{4k}\left(\partial_1 E\right) + \kappa_{4k}\left(\partial_2 E\right) = 0. \tag{5.47}$$

Proof. As we remarked just after defining κ_{4k-1} (equation (5.40)), property (5.45) is immediate, and (5.47) follows from Novikov's additivity formula for the signature. For property (5.46), let M'/Δ^1 be a morphism in $\mathcal{C}$, and M/Δ^1 any morphism for which

$$M/\Delta^1 + \mathrm{Id}_{\partial_1 M'} \sim M'/\Delta^1.$$

Note that this identity forces $\partial_1 M$ to be empty. Applying (5.47) to $N \times \Delta^2 \to \Delta^2$ one easily checks that both the left and right sides of (5.46) vanish for $\mathrm{Id}_{\partial_1 M'}$. It then follows from (5.45) that to check (5.46) for general M'/Δ^1, it suffices to check (5.46) for morphisms M/Δ^1 with the property that

$$\partial_1 M = \phi.$$

We apply Milgram's result to the situation

$$\begin{aligned} V &= \text{image}\left(H^{2k}(M, \partial M; \mathbb{Q}(1)) \to H^{2k}(M; \mathbb{Q}(1))\right) \\ &= \ker\left(H^{2k}(M; \mathbb{Q}(1)) \to H^{2k}(\partial M; \mathbb{Q}(1))\right), \\ L &= \text{image}\left(H^{2k}(M, \partial M; \mathbb{Z}(1)) \to V\right), \end{aligned}$$

with q_V the quadratic function

$$\phi(x) = \phi_M(x) \int_M \frac{x^2 - x\lambda}{2}.$$

Poincaré duality gives

$$L^* = V \cap \text{image}\left(H^{2k}(M; \mathbb{Z}(1)) \to H^{2k}(M; \mathbb{Q}(1))\right).$$

The right hand side of (5.43) is by definition

$$e^{2\pi i\, \kappa_{4k}(M)}.$$

We need to identify the left hand side with

$$\kappa_{4k-1}\left(\partial_0 M\right) - \kappa_{4k-1}\left(\partial_1 M\right) = \kappa_{4k-1}\left(\partial_0 M\right).$$

Write

$$\begin{aligned} A &= H^{2k}(\partial M; \mathbb{Z}(1))_{\mathrm{tor}}, \\ A_0 &= \text{image}\left(H^{2k}(M)_{\mathrm{tor}} \to H^{2k}(\partial M)_{\mathrm{tor}}\right). \end{aligned}$$

and let B be the torsion subgroup of the image of

$$H^{2k}(M) \to H^{2k}(\partial M).$$

Then

$$L^*/L = B/A_0.$$

By cobordism invariance (5.39) the restriction of $\phi_{\partial M}$ to B is compatible with this isomorphism, and so

$$\frac{1}{\sqrt{|L^*/L|}}\sum_{x\in L^*/L} e^{-2\pi i\, q(x)} = \frac{1}{\sqrt{|B/A_0|}}\sum_{x\in B/A_0} e^{-2\pi i\, \phi_{\partial M}(x)}.$$

By Lemma 5.49 below, the subgroup B coincides with the annihilator A_0^* of A_0, and so we can further re-write this expression as

$$\frac{1}{\sqrt{|A_0^*/A_0|}}\sum_{x\in A_0^*/A_0} e^{-2\pi i\, \phi_{\partial M}(x)}.$$

The identification of this with

$$\kappa_{4k-1}(\partial M) = \kappa_{4k-1}(\partial_0 M)$$

is then given by Lemma 5.48 below. q.e.d.

Lemma 5.48. *Suppose that A is a finite abelian group, and*

$$q : A \to \mathbb{Q}/\mathbb{Z}$$

a (non-degenerate) quadratic function with underlying bilinear form B. Given a subgroup $A_0 \subset A$ on which q vanishes, let $A_0^* \subset A$ be the dual of A_0:

$$A_0^* = \{x \in A \mid B(x,a) = 0, a \in A_0\}\,.$$

Then q descends to a quadratic function on A_0^*/A_0, and

$$\frac{1}{\sqrt{|A|}}\sum_{x\in A} e^{-2\pi i\, q(a)} = \frac{1}{\sqrt{|A'/A_0|}}\sum_{x\in A_0^*/A_0} e^{-2\pi i\, q(a)}.$$

Proof. Since $q(a) = 0$ for $a \in A_0$, we have

$$q(a+x) = q(a) + B(x,a).$$

Choose coset representatives $S \subset A$ for A/A_0, and write $S_0 = S \cap A_0^{ast}$. The set S_0 is a set of coset representatives for A_0^*/A_0. Now write

$$\begin{aligned}\frac{1}{\sqrt{|A|}}\sum_{x\in A} e^{-2\pi i\, q(a)} &= \frac{1}{\sqrt{|A|}}\sum_{x\in S}\sum_{a\in A_0} e^{-2\pi i\, q(x+a)} \\ &= \frac{1}{\sqrt{|A|}}\sum_{x\in S}\left(e^{-2\pi i\, q(x)}\sum_{a\in A_0} e^{-2\pi i\, B(x,a)}\right).\end{aligned}$$

By the linear independence of characters

$$\sum_{a\in A_0} e^{-2\pi i\, B(x,a)} = \begin{cases} 0 & B(x,a) \neq 0 \\ |A_0| & B(x,a) = 0.\end{cases}$$

It follows that

$$\frac{1}{\sqrt{|A|}}\sum_{x\in S}\left(e^{-2\pi i\, q(x)}\sum_{a\in A_0} e^{-2\pi i\, B(x,a)}\right) = \frac{|A_0|}{\sqrt{|A|}}\sum_{x\in S_0}\left(e^{-2\pi i\, q(x)}\right).$$

This proves the result, since the bilinear form B identifies A/A_0^* with the character group of A_0, giving

$$|A/A_0^*| = |A_0|$$

and

$$|A| = |A_0|\cdot|A_0^*/A_0|\cdot|A/A_0^*| = |A_0|^2\,|A_0^*/A_0|\,.$$

q.e.d.

We have also used

Lemma 5.49. *With the notation Lemma* 5.48*, the subgroup A_0^* coincides with the torsion subgroup of the image of*

$$H^{2k}(M)\to H^{2k}(\partial M).$$

Proof. Poincaré duality identifies the $\mathbb{Q}/\mathbb{Z}$ dual of

$$H^{2k}(\partial M)\leftarrow H^{2k}(M) \tag{5.50}$$

with

$$H^{2k-1}(\partial M;\mathbb{Q}/\mathbb{Z})\to H^{2k}(M,\partial M;\mathbb{Q}/\mathbb{Z}). \tag{5.51}$$

Thus the orthogonal complement of the image of (5.50) is the image of (5.51). The claim follows easily. q.e.d.

5.4. The quadratic functions. We now turn to the relationship between integral Wu-structures and quadratic functions. That there is a relationship at all has a simple algebraic explanation. Suppose that L is a finitely generated free abelian group equipped with a non-degenerate symmetric bilinear form

$$B: L\times L\to\mathbb{Z}.$$

A *characteristic element* of B is an element $\lambda\in L$ with the property

$$B(x,x)\equiv B(x,\lambda)\mod 2.$$

If λ is a characteristic element, then

$$q(x)=\frac{B(x,x)-B(x,\lambda)}{2} \tag{5.52}$$

is a quadratic refinement of B. Conversely, if q is a quadratic refinement of B,

$$q(x+y)-q(x)-q(y)+q(0)=B(x,y),$$

then

$$q(x)-q(-x): L\to\mathbb{Z}$$

is linear, and so there exists $\lambda \in L$ with

$$q(x) - q(-x) = -B(x, \lambda).$$

We also have

$$q(x) + q(-x) = 2\,q(0) + B(x, x).$$

If we assume in addition that $q(0) = 0$ then it follows that $q(x)$ is given by (5.52). Thus the set of quadratic refinements q of B, with $q(0) = 0$ are in one to one correspondence with the characteristic elements λ of B.

We apply the above discussion to the situation in which M is an oriented manifold of dimension $4k$,

$$L = H^{2k}(M; \mathbb{Z})/\text{torsion},$$

and

$$B(x, y) = \int_M x \cup y.$$

By definition, the Wu-class

$$\nu_{2k} \in H^{2k}(M; \mathbb{Z}/2)$$

satisfies

$$\int_M x^2 = \int_M x\,\nu_{2k} \in \mathbb{Z}/2, \quad x \in H^{2k}(M; \mathbb{Z}/2).$$

Thus the integer lifts λ are exactly the characteristic elements of B, and correspond to quadratic refinements q^λ of the intersection pairing.

Lemma 5.53. *The function*

$$q^\lambda(x) = \tfrac{1}{2} \int_M (x^2 - x\lambda)$$

defines a homomorphism

$$\pi_{4k}\, MSO\langle \beta\nu_{2k}\rangle \wedge K(\mathbb{Z}, 2k) \to \mathbb{Z}.$$

Proof. The group $\pi_{4k}\, MSO\langle \beta\nu_{2k}\rangle \wedge K(\mathbb{Z}, 2k)_+$ is the cobordism group of triples (M, x, λ) with M an oriented manifold of dimension $4k$, $\lambda \in Z^{2k}(M; \mathbb{Z})$ a lift of ν_{2k} and $x \in Z^{2k}(M; \mathbb{Z})$. The group $\pi_{4k}\, MSO\langle \beta\nu_{2k}\rangle \wedge K(\mathbb{Z}, 2k)$ is the quotient by the subgroup in which $x = 0$. Let (M_1, λ_1, x_1) and (M_2, λ_2, x_2) be two such manifolds. Since

$$x_1 \cup x_2 = 0 \in H^{4k}(M_1 \amalg M_2)$$

we have

$$q^{\lambda_1 + \lambda_2}(x_1 + x_2) = q^{\lambda_1}(x_1) + q^{\lambda_2}(x_2),$$

and so q^λ is additive. If $M = \partial N$ and both λ and x extend to N, then

$$q^\lambda(x) = \frac{1}{2} \int_M x^2 - x\lambda = \frac{1}{2} \int_N d(x^2 - x\lambda) = 0,$$

and q is a cobordism invariant. The result now follows since $q(0) = 0$. q.e.d.

We will now show that the homomorphism

$$\pi_{4k}\, MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z},2k) \to \mathbb{Z}$$
$$(M,\lambda,x) \mapsto \tfrac{1}{2}\int_M x^2 - x\lambda$$

has a canonical lift to a map

$$MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z},2k) \to \Sigma^{4k}\tilde{I}. \tag{5.54}$$

It suffices to produce the adjoint to (5.54) which is a map

$$MSO\langle\beta\nu_{2k}\rangle \to \Sigma^{4k}\tilde{I}\,(K(\mathbb{Z},2k))\,.$$

Since $MSO\langle\beta\nu_{2k}\rangle$ is (-1)-connected this will have to factor through the (-1)-connected cover[12]

$$MSO\langle\beta\nu_{2k}\rangle \to \Sigma^{4k}\tilde{I}\,(K(\mathbb{Z},2k))\,\langle 0,\dots,\infty\rangle.$$

To work out the homotopy type of $\Sigma^{4k}\tilde{I}\,(K(\mathbb{Z},2k))\,\langle 0,\dots,\infty\rangle$ we will follow the approach of Browder and Brown [**10**, **12**, **14**, **13**].

For a space X and a cocycle $x \in Z^{2k}(X;\mathbb{Z}/2)$ the theory of Steenrod operations provides a universal $(2k-1)$-cochain h with the property

$$\delta h = x \cup x + \mathrm{Sq}^{2k}(x).$$

Taking the universal case $X = K(\mathbb{Z},2k)$ $x = \iota$, this can be interpreted as as an explicit homotopy making the diagram

$$\begin{array}{ccc} \Sigma^\infty K(\mathbb{Z},2k) & \xrightarrow{\iota\cup\iota} & \Sigma^{4k}\,H\mathbb{Z} \\ {\scriptstyle\iota}\downarrow & & \downarrow \\ \Sigma^{2k}\,H\mathbb{Z} & \xrightarrow[\mathrm{Sq}^{2k}]{} & \Sigma^{4k}\,H\mathbb{Z}\,/2. \end{array}$$

Passing to Postnikov sections gives an explicit homotopy making

$$\begin{array}{ccc} \Sigma^\infty K(\mathbb{Z},2k)\langle-\infty,4k\rangle & \xrightarrow{\iota\cup\iota} & \Sigma^{4k}\,H\mathbb{Z} \\ {\scriptstyle\iota}\downarrow & & \downarrow \\ \Sigma^{2k}\,H\mathbb{Z} & \xrightarrow[\mathrm{Sq}^{2k}]{} & \Sigma^{4k}\,H\mathbb{Z}\,/2 \end{array} \tag{5.55}$$

commute.

Proposition 5.56. *The square* (5.55) *is homotopy Cartesian.*

Proof. This is an easy consequence of the Cartan–Serre computation of the cohomology of $K(\mathbb{Z},2k)$. q.e.d.

[12] The symbol $X\langle n,\dots,m\rangle$ indicates the Postnikov section of X having homotopy groups only in dimension $n \le i \le m$. The angled brackets are given lower precedence than suspensions, so that the notation $\Sigma X\langle n,\dots,m\rangle$ means $(\Sigma X)\,\langle n,\dots,m\rangle$ and coincides with $\Sigma\,(X\langle n+1,\dots,m+1\rangle)$.

It will be useful to re-write (5.55). Factor the bottom map as

$$\Sigma^{2k} H\mathbb{Z} \to \Sigma^{2k} H\mathbb{Z}/2 \to \Sigma^{4k} H\mathbb{Z}/2,$$

and define a spectrum X by requiring that the squares in

$$\begin{tikzcd}
\Sigma^\infty K(\mathbb{Z},2k)\langle -\infty, 4k\rangle \arrow[r] \arrow[d, "\iota"'] & X \arrow[r] \arrow[d] & \Sigma^{4k} H\mathbb{Z} \arrow[d] \\
\Sigma^{2k} H\mathbb{Z} \arrow[r] & \Sigma^{2k} H\mathbb{Z}/2 \arrow[r, "\mathrm{Sq}^{2k}"'] & \Sigma^{4k} H\mathbb{Z}/2
\end{tikzcd}$$

be homotopy Cartesian. Then (5.55) determines a homotopy Cartesian square

$$\begin{tikzcd}
\Sigma^{2k-1} H\mathbb{Z}/2 \arrow[r, "\beta"] \arrow[d, "\beta\,\mathrm{Sq}^{2k}"'] & \Sigma^{2k} H\mathbb{Z} \arrow[d] \\
\Sigma^{4k} H\mathbb{Z} \arrow[r] & \Sigma^\infty K(\mathbb{Z},2k)\langle -\infty, \dots, 0\rangle,
\end{tikzcd}$$

in which the horizontal map is the inclusion of the fiber of ι, and the vertical map is the inclusion of the fiber of the map to X. Taking Anderson duals then gives

Proposition 5.57. *The diagram* (5.55) *determines a homotopy Cartesian square*

$$\begin{tikzcd}
\Sigma^{4k}\tilde{I}K(\mathbb{Z},2k)\langle 0,\dots,\infty\rangle \arrow[r, "b"] \arrow[d, "a"'] & \Sigma^{2k}H\mathbb{Z} \arrow[d] \\
H\mathbb{Z} \arrow[r, "\chi\,\mathrm{Sq}^{2k}"] & \Sigma^{2k}H\mathbb{Z}/2.
\end{tikzcd} \tag{5.58}$$

By Proposition 5.57, to give a map

$$X \to \Sigma^{4k}\tilde{I}K(\mathbb{Z},2k)\langle 0,\dots,\infty\rangle$$

is to give cocycles $a \in Z^0(X;\mathbb{Z})$, $b \in Z^{2k}(X;\mathbb{Z})$, and a cochain $c \in C^{2k-1}(X;\mathbb{Z}/2)$ satisfying

$$\chi Sq^{2k}(a) - b \equiv \delta c \mod 2.$$

In case $X = MSO\langle\beta\nu_{2k}\rangle$ and $a = U$ is the Thom cocycle, the theory of Wu classes gives a universal c for which

$$\delta c = \chi\,\mathrm{Sq}^{2k}\, a + \nu_{2k}\cdot U.$$

Taking $b = -\lambda \cdot U$ leads to a canonical map

$$MSO\langle\beta\nu_{2k}\rangle \to X \to \Sigma^{4k}\tilde{I}K(\mathbb{Z},2k). \tag{5.59}$$

Proposition 5.60. *The adjoint to* (5.59)

$$MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z},2k) \to \Sigma^{4k}\tilde{I} \tag{5.61}$$

is a lift of the homomorphism

$$\tfrac{1}{2}\int_M \left(x^2 - x\lambda\right).$$

Proof. We'll use the notation of (5.55) and (5.58). The maps

$$\Sigma^\infty K(\mathbb{Z},2k)\langle -\infty,\dots,0\rangle \xrightarrow{(\iota,\iota^2)} \Sigma^{2k} H\mathbb{Z} \vee \Sigma^{4k} H\mathbb{Z}$$

$$\Sigma^{4k}\tilde{I}\left(K(\mathbb{Z},2k)\right)\langle 0,\dots,\infty\rangle \xrightarrow{(a,b)} H\mathbb{Z}\vee\Sigma^{2k} H\mathbb{Z},$$

are rational equivalences, and by construction

$$\iota\circ\Sigma^{4k}\tilde{I}(b) = 2 \qquad \iota\circ\Sigma^{4k}\tilde{I}(a) = 0$$

$$\iota\cup\iota\circ\Sigma^{4k}\tilde{I}(b) = 0 \qquad \iota\cup\iota\circ\Sigma^{4k}\tilde{I}(a) = 2.$$

It follows that the rational evaluation map

$$\Sigma^{4k}\tilde{I}\left(K(\mathbb{Z},2k)\right)\langle 0,\dots,\infty\rangle \wedge \Sigma^\infty K(\mathbb{Z},2k)\langle -\infty,\dots,0\rangle \to \Sigma^{4k}\tilde{I} \to H\mathbb{Q}$$

is $(a\iota^2 + b\iota)/2$. This means that

$$MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z},2k) \to \Sigma^{4k}\tilde{I}K(\mathbb{Z},2k)\wedge K(\mathbb{Z},2k) \to \Sigma^{4k}\tilde{I} \to \Sigma^{4k} H\mathbb{Q}$$

is given by

$$(U\cdot x^2 - U\cdot\lambda\cdot x)/2,$$

and the claim follows. q.e.d.

Remark 5.62. The topological analogue of a characteristic element is an integral Wu-structure, and the object in topology corresponding to an integer invariant is a map to $\tilde{I}$. We've shown that an integral Wu-structure gives a function to $\tilde{I}$. That this function is quadratic is expressed in topology by the following diagram which is easily checked to be homotopy commutative (and in fact to come with an explicit homotopy):

$$\begin{tikzcd} MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z},2k)\wedge K(\mathbb{Z},2k) \arrow[r, "\int x\cup y"] \arrow[d, "(x+y)-(x)-(y)"'] & \Sigma^{4k} H\mathbb{Z} \arrow[d] \\ MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z},2k) \arrow[r] & \Sigma^{4k}\tilde{I}. \end{tikzcd} \tag{5.63}$$

There is more information in the map (5.61). Consider the following diagram in which the bottom vertical arrows are localization at $H\mathbb{Q}$ and the vertical sequences are fibrations

$$\begin{tikzcd} MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Q}/\mathbb{Z},2k-1) \arrow[r] \arrow[d] & \Sigma^{4k-1} I \arrow[d] \\ MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z},2k) \arrow[r] \arrow[d] & \Sigma^{4k}\tilde{I} \arrow[d] \\ MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Q},2k) \arrow[r] & \Sigma^{4k} H\mathbb{Q}. \end{tikzcd} \tag{5.64}$$

The top map is classified by a homomorphism

$$\pi_{4k-1}\, MSO\langle \beta\nu_{2k}\rangle \wedge K(\mathbb{Q}/\mathbb{Z}, 2k-1) \to \mathbb{Q}/\mathbb{Z}.$$

In geometric terms, this associates to each integral Wu-structure on an oriented manifold of N of dimension $(4k-1)$ a function

$$\phi_{4k-1} = \phi_{4k-1}(N, -\,) : H^{2k-1}(N; \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}.$$

One easily checks (using (5.63)) that this map is a quadratic refinement of the link pairing

$$\phi_{4k-1}(x+y) - \phi_{4k-1}(x) - \phi_{4k-1}(y) = \int_N x \cup \beta y.$$

The bottom map is classified by a homomorphism

$$\phi_{4k} : \pi_{4k}\, MSO\langle \beta\nu_{2k}\rangle \wedge K(\mathbb{Q}, 2k) \to \mathbb{Q}.$$

Thinking of $\pi_{4k}\, MSO\langle \beta\nu_{2k}\rangle \wedge K(\mathbb{Q}, 2k)$ as the relative homotopy group

$$\pi_{4k}\left(MSO\langle \beta\nu_{2k}\rangle \wedge K(\mathbb{Z}, 2k), MSO\langle \beta\nu_{2k}\rangle \wedge K(\mathbb{Q}/\mathbb{Z}, 2k-1)\right)$$

we interpret it in geometric terms as the bordism group of manifolds with boundary M, together with cocycles

$$\begin{aligned} x &\in Z^{2k}(M; \mathbb{Z}) \\ y &\in Z^{2k-1}(M; \mathbb{Q}/\mathbb{Z}) \end{aligned}$$

satisfying:

$$\beta(y) = x|_{\partial M}.$$

Because of this identity, the cocycle x defines a class $x \in H^{2k}(M, \partial M; \mathbb{Q})$ and in fact

$$\phi_{4k}(M, x, y) = \phi_{4k}(M, x) = \int_M (x^2 - x\lambda)/2.$$

The compatibility of the maps ϕ with the connecting homomorphism gives

$$\phi_{4k-1}(\partial M, y) \equiv \phi_{4k}(M, x) \mod \mathbb{Z}.$$

The functions ϕ_{4k} and ϕ_{4k-1} share the defect that they are not quite defined on the correct groups. For instance ϕ_{4k} is defined on the middle group in the exact sequence

$$\cdots \to H^{2k-1}(\partial M; \mathbb{Q}) \xrightarrow{\delta} H^{2k}(M, \partial M; \mathbb{Q}) \to H^{2k}(M; \mathbb{Q}) \to \cdots$$

but does not necessarily factor through the image of this group in

$$H^{2k}(M; \mathbb{Q}).$$

The obstruction is

$$\phi_{4k}(M, \delta x) = \frac{1}{2}\int_{\partial M} x \cdot \lambda.$$

Similarly, ϕ_{4k-1} is defined on $H^{2k-1}(N;\mathbb{Q}/\mathbb{Z})$, but does not necessarily factor through $H^{2k}(N;\mathbb{Z})_{\text{torsion}}$, and the obstruction is

$$\phi_{4k-1}(N,x) = \frac{1}{2}\int_N x\cdot\lambda, \quad x\in H^{2k-1}(N;\mathbb{Q}).$$

Both of these obstructions vanish if it happens that the integral Wu-structures in dimension $(4k-1)$ are torsion. This is the situation considered in §5.2.

We still haven't quite constructed functions $\phi_{N,\eta}$ needed in §5.3. For one thing, we've only described the compatibility relation in the case of a manifold with boundary. To get a formula like (5.39) one simply considers maps of pairs

$$\begin{aligned}&\left(S^{4k-1}\wedge\Delta^1_+, S^{4k-1}\wedge\partial\Delta^1_+\right)\\ &\qquad\to (MSO\langle\beta\nu_{2k}\rangle\wedge K(\mathbb{Z},2k), MSO\langle\beta\nu_{2k}\rangle\wedge K(\mathbb{Q}/\mathbb{Z},2k-1))\end{aligned}$$

instead of relative homotopy groups. We also haven't incorporated the symmetry. We now indicate the necessary modifications.

We need to work with spaces and spectra[13] equipped with an action of the group $\mathbb{Z}/2$, with the convention that an equivariant map is regarded as a a weak equivalence if the underlying map of spaces or spectra is a weak equivalence.[14] With this convention the map

$$X\times E\mathbb{Z}/2\to E\mathbb{Z}/2$$

is a weak equivalence, and a spectrum E with the trivial $\mathbb{Z}/2$-action represents the equivariant cohomology theory

$$X\mapsto E^*_{\mathbb{Z}/2}(X) = E^*(E\mathbb{Z}/2\times_{\mathbb{Z}/2}X).$$

Let $\mathbb{Z}(1)$ denote the local system which on each $\mathbb{Z}/2$-space $E\mathbb{Z}/2\times X$ is locally $\mathbb{Z}$, and has monodromy given by the homomorphism

$$\pi_1 E\mathbb{Z}/2\times X\to\pi_1 B\mathbb{Z}/2=\mathbb{Z}/2.$$

We'll write $H\mathbb{Z}(1)$ for the equivariant spectrum representing the cohomology theory

$$X\mapsto H^*_{\mathbb{Z}/2}(X;\mathbb{Z}(1)) = H^*(E\mathbb{Z}/2\times_{\mathbb{Z}/2}X;\mathbb{Z}(1)),$$

and more generally $HA(1)$ for the equivariant spectrum representing

$$H^*(E\mathbb{Z}/2\times_{\mathbb{Z}/2}X;A\otimes\mathbb{Z}(1)).$$

The associated Eilenberg-MacLane spaces are denoted $K(\mathbb{Z}(1),n)$, etc. Since

$$\mathbb{Z}/2(1)=\mathbb{Z}/2,$$

[13] In the terminology of [**40**] these are *naive* $\mathbb{Z}/2$-spectrum.

[14] This is sometimes known as the "coarse" model category structure on equivariant spaces or spectra. The alternative is to demand that the map of fixed points be a weak equivalence as well.

the spectrum $H\mathbb{Z}/2(1)$ is just $H\mathbb{Z}/2$, and reduction modulo 2 is represented by a map

$$H\mathbb{Z}(1) \to H\mathbb{Z}/2(1).$$

Note that if M is a space and $t : \pi_1 M \to \mathbb{Z}/2$ classifies a double cover $\tilde{M} \to M$, and local system $\mathbb{Z}(1)$ on M, then

$$E\mathbb{Z}/2 \times_{\mathbb{Z}/2} \tilde{M} \to M$$

is a homotopy equivalence (it is a fibration with contractible fibers) and we have an isomorphism

$$H^*(M; \mathbb{Z}(1)) = H^*_{\mathbb{Z}/2}(\tilde{M}; \mathbb{Z}(1)).$$

As described in §5.1 we define an equivariant $BSO\langle\beta\nu_{2k}\rangle$ by the homotopy pullback square

$$\begin{array}{ccc} BSO\langle\beta\nu_{2k}\rangle & \xrightarrow{\lambda} & K(\mathbb{Z}(1), 2k) \\ \big\downarrow & & \big\downarrow \\ BSO & \longrightarrow & K(\mathbb{Z}/2, 2k), \end{array}$$

and in this way give the associated Thom spectra $MSO\langle\beta\nu_{2k}\rangle$ and

$$\begin{aligned} &MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z}(1), 2k)_+ \\ &\qquad = \mathrm{Thom}\,(BSO\langle\beta\nu_{2k}\rangle \times K(\mathbb{Z}(1), 2k), \xi \oplus 0) \end{aligned}$$

$\mathbb{Z}/2$-actions. The group

$$\begin{aligned} &\pi_{4k}\,(MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z}(1), 2k)_+)_{h\mathbb{Z}/2} \\ &\qquad = \pi_{4k}\,\mathrm{Thom}\,\big((BSO\langle\beta\nu_{2k}\rangle \times K(\mathbb{Z}(1), 2k)) \times_{Z/2} E\mathbb{Z}/2, \xi \oplus 0\big) \end{aligned}$$

is the cobordism group of $4k$-dimensional oriented manifolds M, equipped with a map $t : \pi_1 M \to \mathbb{Z}/2$ classifying a local system $\mathbb{Z}(1)$, a cocycle $\lambda \in Z^{2k}(M; \mathbb{Z}(1))$ whose mod 2-reduction represents the Wu-class ν_{2k}, and a cocycle $x \in Z^{2k}(M; \mathbb{Z}(1))$. As in the non-equivariant case, the integral Wu-structure $\lambda \in H^{2k}(M; \mathbb{Z}(1))$ defines the quadratic function

$$\phi_{4k} = \phi_{M,\lambda}(x) = \tfrac{1}{2} \int_M (x^2 - x\lambda) \qquad x \in H^{2k}(M; \mathbb{Z}(1)),$$

and a homomorphism

$$\pi_{4k}\, MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z}(1), 2k)_{h\mathbb{Z}/2} \to \mathbb{Z}.$$

An analysis of the Anderson dual of $\Sigma^\infty K(\mathbb{Z}(1), 2k)$ leads to a canonical refinement of this to a map

$$MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z}(1), 2k)_{h\mathbb{Z}/2} \to \tilde{I},$$

and checking the rationalization sequence leads to the desired quadratic refinement of the link pairing in dimension $(4k-1)$.

The the analysis of $\tilde{I}(K(\mathbb{Z}(1),2k))$ works more or less the same as in the non-equivariant case. Here are the main main points:

(1) With our conventions, a homotopy class of maps

$$MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z}(1),2k)_{h\mathbb{Z}/2} \to \Sigma^{4k}\tilde{I}$$

is the same as an *equivariant* homotopy class of maps

$$MSO\langle\beta\nu_{2k}\rangle \wedge K(\mathbb{Z},2k) \to \Sigma^{4k}\tilde{I}.$$

We are therefore looking for an equivariant map

$$MSO\langle\beta\nu_{2k}\rangle \to \Sigma^{4k}\tilde{I}\left(K(\mathbb{Z}(1),2k)\right). \tag{5.65}$$

(2) By attaching cells of the form $D^n \times \mathbb{Z}/2$, the Postnikov section $X\langle n,\dots,m\rangle$ of an equivariant spectrum or space can be formed. Its non-equivariant homotopy groups vanish outside of dimensions n through m, and coincide with those of X in that range. If E is a space or spectrum with a free $\mathbb{Z}/2$-action, and having only cells in dimensions less than n, then

$$[E, X\langle n,\dots,m\rangle] = 0.$$

The same is true if E is an equivariant space or spectrum with $\pi_k E = 0$ for $k > m$. Since the equivariant spectrum $MSO\langle\beta\nu_{2k}\rangle$ is (-1)-connected, this means, as before, that we can replace (5.65) with

$$\begin{aligned} MSO\langle\beta\nu_{2k}\rangle &\to \Sigma^{4k}\tilde{I}\left(K(\mathbb{Z}(1),2k)\right)\langle 0,\dots,\infty\rangle \\ &= \Sigma^{4k}\tilde{I}\left(\Sigma^{\infty}K(\mathbb{Z}(1),2k)\langle-\infty,\dots,4k\rangle\right). \end{aligned}$$

(3) The cup product goes from

$$H^*_{\mathbb{Z}/2}(X;\mathbb{Z}(1)) \times H^*_{\mathbb{Z}/2}(X;\mathbb{Z}(1)) \to H^*_{\mathbb{Z}/2}(X;\mathbb{Z})$$

and is represented by a map of spectra

$$H\mathbb{Z}(1) \wedge H\mathbb{Z}(1) \to H\mathbb{Z}.$$

(4) The adjoint of

$$H\mathbb{Z}(1) \wedge H\mathbb{Z}(1) \to H\mathbb{Z} \to \tilde{I}$$

is an equivariant weak equivalence

$$H\mathbb{Z}(1) \to \tilde{I}\, H\mathbb{Z}(1).$$

(5) The square

$$\begin{CD} \Sigma^{\infty}K(\mathbb{Z}(1),2k)\langle-\infty,4k\rangle @>{\iota\cup\iota}>> \Sigma^{4k}\, H\mathbb{Z} \\ @V{\iota}VV @VVV \\ \Sigma^{2k}\, H\mathbb{Z}(1) @>>{\mathrm{Sq}^{2k}}> \Sigma^{4k}\, H\mathbb{Z}/2 \end{CD}$$

is a homotopy pullback square.

(6) As a consequence, so is

$$\begin{tikzcd} \Sigma^{4k}\tilde{I}\left(K(\mathbb{Z}(1),2k)\right)\langle 0,\infty\rangle \arrow[r, "\iota\cup\iota"] \arrow[d, "\iota"'] & \Sigma^{2k} H\mathbb{Z}(1) \arrow[d] \\ H\mathbb{Z} \arrow[r, "\chi \operatorname{Sq}^{2k}"'] & \Sigma^{2k} H\mathbb{Z}/2. \end{tikzcd}$$

The following proposition summarizes the main result of this discussion. As in §5.2 we'll use the notation

$$\begin{aligned} B &= BSO\langle \beta\nu_{2k}\rangle \times_{\mathbb{Z}/2} E\mathbb{Z}/2 \\ \bar{B} &= BSO\langle \bar{\nu}_{2k}\rangle \times_{\mathbb{Z}/2} E\mathbb{Z}/2. \end{aligned}$$

Proposition 5.66. *Let N be a $\bar{B}$-oriented manifold of dimension $(4k-1)$. Associated to the "Wu-cocycle" $\eta \in Z^{2k-1}(N;\mathbb{Q}/\mathbb{Z})$ is a quadratic function*

$$\phi = \phi_N = \phi_{N,\eta} : H^{2k}(N;\mathbb{Z}(1))_{tor} \to \mathbb{Q}/\mathbb{Z}$$

whose associated bilinear form is the link pairing

$$H^{2k}(N;\mathbb{Z}(1))_{tor} \times H^{2k}(N;\mathbb{Z}(1))_{tor} \to \mathbb{Q}/\mathbb{Z}.$$

If M/Δ^1 is a B-oriented map of relative dimension $(4k-1)$, equipped with compatible $\bar{B}$ orientations on $\partial_0 M$ and $\partial_1 M$, and x is an element of $H^{2k}(M;\mathbb{Z}(1))$ whose restriction to ∂M is torsion, then

$$\begin{aligned} \phi_{\partial_0 M}(\partial_0 x) - \phi_{\partial_1 M}(\partial_1 x) &\equiv \frac{1}{2}\int_M (x^2 - x\lambda) \mod \mathbb{Z} \\ \partial_i x &= x|_{\partial_i M}. \end{aligned} \tag{5.67}$$

Appendix A. Simplicial methods

A.1. Simplicial set and simplicial objects. A *simplicial set* $X_\bullet$ consists of a sequence of sets X_n, $n \geq 0$ together with "face" and "degeneracy" maps

$$\begin{aligned} d_i &: X_n \to X_{n-1} & i &= 0,\dots,n \\ s_i &: X_{n-1} \to X_n & i &= 0,\dots,n-1 \end{aligned}$$

satisfying

$$\begin{aligned} d_j d_i &= d_i d_{j+1} & j &\geq i \\ s_j s_i &= s_i s_{j-1} & j &> i \end{aligned}$$

$$d_j s_i = \begin{cases} s_{i-1} d_j & j < i \\ \text{identity} & j = i, i+1 \\ s_i d_{j-1} & j > i+1. \end{cases}$$

The set X_n is called the set of n-simplices of $X_\bullet$.

Remark A.1. Let Δ be the category whose objects are the finite ordered sets

$$[n] = \{0 \leq 1 \cdots \leq n\}, \qquad n \geq 0$$

and whose morphisms are the order preserving maps. The data describing a simplicial set $X_\bullet$ is equivalent to the data describing a contravariant functor

$$X : \Delta \to \underline{\text{Sets}}$$

with $X[n]$ corresponding to the set X_n of n-simplices, and the face and degeneracy maps d_i and s_i corresponding to the values of X on the maps

$$\begin{aligned} [n-1] &\hookrightarrow [n] \\ [n] &\twoheadrightarrow [n-1] \end{aligned} \tag{A.2}$$

which, respectively, "skip i" and "repeat i."

Remark A.3. A simplicial object in a category $\mathcal{C}$ is a contravariant functor

$$X_\bullet : \Delta \to \mathcal{C}.$$

Thus one speaks of simplicial abelian groups, simplicial Lie algebras, etc.

One basic example of a simplicial set is the *singular complex*, $\operatorname{sing} S$ of a space S, defined by

$$(\operatorname{sing} S)_n = \text{set of maps from } \Delta^n \text{ to } S,$$

where

$$\Delta^n = \{(t_0, \dots, t_n) \mid 0 \leq t_i \leq 1, \sum t_i = 1\}$$

is the standard n-simplex with vertices

$$e_i = (0, \dots, \overset{i}{1}, \dots, 0) \qquad i = 0, \dots, n.$$

The face and degeneracy maps are derived from the linear extensions of (A.2).

Definition A.4. The *geometric realization* $|X_\bullet|$ of a simplicial set $X_\bullet$ is the space

$$\coprod X_n \times \Delta^n / \sim$$

where $\sim$ is the equivalence relation generated by

$$\begin{aligned} (d_i x, t) &= (x, d^i t) \\ (s_i x, t) &= (x, s^i t) \end{aligned}$$

and

$$\begin{aligned} d^i &: \Delta^{n-1} \to \Delta^n \\ s^i &: \Delta^n \to \Delta^{n-1} \end{aligned}$$

are the linear extensions of the maps (A.2).

The evaluation maps assemble to a natural map

$$|\operatorname{sing} S| \to S \tag{A.5}$$

which induces an isomorphism of both singular homology and homotopy groups.

The *standard (simplicial) n-simplex* is the simplicial set $\Delta^n_\bullet$, given as a contravariant functor by

$$\Delta^n([m]) = \Delta([m],[n]).$$

One easily checks that

$$|\Delta^n_\bullet| = \Delta^n.$$

A *simplicial homotopy* between maps $f : X_\bullet \to Y_\bullet$ is a simplicial map

$$h : X_\bullet \times \Delta^1_\bullet \to Y_\bullet$$

for which $\partial_1 h = f$, and $\partial_0 h = g$.

A.2. Simplicial homotopy groups.

Definition A.6. Let $X_\bullet$ be a pointed simplicial set. The set $\pi_n^{\mathrm{simp}} X_\bullet$ is the set of simplicial maps

$$(\Delta^n_\bullet, \partial\Delta^n_\bullet) \to (X_\bullet, *)$$

modulo the relation of simplicial homotopy.

There is a map

$$\pi_n^{\mathrm{simp}} X_\bullet \to \pi_n(|X_\bullet|). \tag{A.7}$$

The *k-horn* of $\Delta^n_\bullet$ is the simplicial set

$$V^{n,k}_\bullet = \bigcup_{i \neq k} \partial_i \Delta^n_\bullet \subset \partial\Delta^n_\bullet.$$

Definition A.8. A simplicial set $X_\bullet$ satisfies the *Kan extension condition* if for every n, k, every map $V^{n,k}_\bullet \to X_\bullet$ extends to a map $\Delta^n_\bullet \to X_\bullet$.

Proposition A.9. *If $X_\bullet$ satisfies the Kan extension condition then* (A.7) *is an isomorphism.*

A.3. Simplicial abelian groups. The category of simplicial abelian groups is equivalent to the category of chain complexes, making the linear theory of simplicial abelian groups particularly simple. The equivalence associates to a simplicial abelian group A the chain complex NA with

$$\begin{aligned} NA_n &= \{a \in A_n \mid d_i a = 0, \quad i = 1, \dots n\} \\ d &= d_0. \end{aligned}$$

The inverse correspondence associates to a chain complex C_*, the simplicial abelian group with n-simplices

$$\bigoplus_{\substack{f:[n]\to[k] \\ f \text{ surjective}}} C_k.$$

Remark A.10. The normalized complex NA of a simplicial abelian group A is a subcomplex of the simplicial abelian group A regarded as a chain complex with differential $\sum(-1)^i\partial_i$. In fact NA is a retract of this chain complex, and the complementary summand is contractible. It is customary not to distinguish in notation between a simplicial abelian group A and the chain complex just described.

It follows immediately from the definition that

$$\pi_n^{\mathrm{simp}} A = H_n(NA).$$

Moreover, any simplicial abelian group satisfies the Kan extension condition. Putting this together gives

Proposition A.11. *For a simplicial abelian group $A_\bullet$, the homotopy groups of $|A_\bullet|$ are given by*

$$\pi_n|A_\bullet| = H_n(NA).$$

Fix a fundamental cocycle $\iota \in Z^n\left(K(\mathbb{Z},n);\mathbb{Z}\right)$.

Proposition A.12. *The map*

$$\begin{aligned} \operatorname{sing} K(A,n)^X &\to Z^n(X\times\Delta^\bullet) \\ f &\mapsto f^*\iota \end{aligned}$$

is a simplicial homotopy equivalence.

Proof. On path components, the map induces the isomorphism

$$[X, K(\mathbb{Z},n)] \approx H^n(X;\mathbb{Z}).$$

One can deduce from this that the map is an isomorphism of higher homotopy groups by replacing X with $\Sigma^k X$, and showing that the map "integration along the suspension coordinates" gives a simplicial homotopy equivalence

$$Z^n(\Sigma^k X\times\Delta^\bullet) \to Z^{n-k}(X\times\Delta^\bullet).$$

This latter equivalence is a consequence of Corollary D.13. q.e.d.

Appendix B. Picard categories and Anderson duality

B.1. Anderson duality. In this appendix we will work entirely with spectra, and we will use the symbol $\pi_k X$ to denote the k^{th} homotopy group of the spectrum. In case X is the suspension spectrum of a space M this group is the k^{th} *stable* homotopy of M.

To define the Anderson dual of the sphere, first note that the functors

$$X \mapsto \hom(\pi_*^{\mathrm{st}} X, \mathbb{Q})$$
$$X \mapsto \hom(\pi_*^{\mathrm{st}} X, \mathbb{Q}/\mathbb{Z})$$

satisfy the Eilenberg-Steenrod axioms and so define cohomology theories which are represented by spectra. In the first case this cohomology theory is just ordinary cohomology with coefficients in the rational numbers, and the representing spectrum is the Eilenberg-MacLane spectrum $H\mathbb{Q}$. It the second case the spectrum is known as the *Brown-Comenetz dual of the sphere*, and denoted I [**11**]. There is a natural map

$$H\mathbb{Q} \to I \tag{B.1}$$

representing the transformation

$$\hom(\pi_*^{\mathrm{st}} M, \mathbb{Q}) \to \hom(\pi_*^{\mathrm{st}} M, \mathbb{Q}/\mathbb{Z}).$$

Definition B.2. The *Anderson dual of the sphere*, $\tilde{I}$, is the homotopy fiber of the map (B.1).

By definition there is a long exact sequence

$$\begin{aligned}\cdots \hom(\pi_{n-1} X, \mathbb{Q}/\mathbb{Z}) \to \tilde{I}^n(X) \\ \to \hom(\pi_n X, \mathbb{Q}) \to \hom(\pi_n X, \mathbb{Q}/\mathbb{Z}) \cdots ,\end{aligned}$$

from which one can extract a short exact sequence

$$\operatorname{Ext}(\pi_{n-1} X, \mathbb{Z}) \rightarrowtail \tilde{I}^n(X) \twoheadrightarrow \hom(\pi_n X, \mathbb{Z}). \tag{B.3}$$

The sequence (B.3) always splits, but not canonically.

By means of (B.3), the group $\hom(\pi_n X, \mathbb{Z})$ gives an algebraic approximation to $\tilde{I}^n X$. In the next two sections we will refine this to an algebraic description of all of $\tilde{I}^n(X)$ (Corollary B.17).

For a spectrum E, let $\tilde{I}(E)$ denote the function spectrum of maps from E to $\tilde{I}$. Since

$$\left[X, \tilde{I}(E)\right] = \left[X \wedge E, \tilde{I}\right],$$

there is a splittable short exact sequence

$$\operatorname{Ext}(\pi_{n-1} X \wedge E, \mathbb{Z}) \rightarrowtail \tilde{I}^n(X \wedge E) \twoheadrightarrow \hom(\pi_n X \wedge E, \mathbb{Z}).$$

If $E = H\mathbb{Z}$, then $\tilde{I}(E) = H\mathbb{Z}$, and the above sequence is simply the universal coefficient sequence. One can also check that $\tilde{I}(H\mathbb{Z}/2) = \Sigma^{-1} H\mathbb{Z}/2$, that the Anderson dual of mod 2 reduction

$$H\mathbb{Z} \to H\mathbb{Z}/2$$

is the Bockstein

$$\Sigma^{-1} H\mathbb{Z}/2 \to H\mathbb{Z},$$

and that the effect of Anderson duality on the Steenrod algebra

$$[H\mathbb{Z}/2, H\mathbb{Z}/2]_*$$

is given by the canonical anti-automorphism χ.

Remark B.4. If X is a pointed space, we will abbreviate $\tilde{I}\,(\Sigma^\infty X)$ to $\tilde{I}(X)$.

B.2. Picard categories.

Definition B.5. A *Picard category* is a groupoid equipped with the structure of a symmetric monoidal category in which each object is invertible.

A functor between Picard categories is a functor f together with a natural transformation

$$f(a\otimes b)\to f(a)\otimes f(b)$$

which is compatible with the commutativity and associativity laws. The collection of Picard categories forms a 2-category[15] . If $\mathcal{C}$ is a Picard category, then the set $\pi_0\mathcal{C}$ of isomorphism classes of objects in $\mathcal{C}$ is an abelian group. For every $x\in\mathcal{C}$, the map

$$\begin{aligned}\operatorname{Aut}(e)&\to \operatorname{Aut}(e\otimes x)\approx\operatorname{Aut}(x)\\ f&\mapsto f\otimes \operatorname{Id}\end{aligned}$$

is an isomorphism (e is the unit for $\otimes$). This group is also automatically abelian, and is denoted $\pi_1\mathcal{C}$.

For each $a\in\mathcal{C}$, the symmetry of $\otimes$ gives an isomorphism

$$(a\otimes a\to a\otimes a)\in\operatorname{Aut}(a\otimes a)\approx\operatorname{Aut}(e)=\pi_1\mathcal{C}$$

which squares to 1. Write $\epsilon(a)$ for this element. The invariant ϵ descends to a homomorphism

$$\pi_0\mathcal{C}\otimes\mathbb{Z}/2\to\pi_1\mathcal{C},$$

known as the *k-invariant* of $\mathcal{C}$. The Picard category $\mathcal{C}$ is determined by this invariant, up to equivalence of Picard categories (see Proposition B.12 below). A Picard category with $\epsilon=0$ is said to be a *strict* Picard category.

Example B.6. A map $\partial: A\to B$ of abelian groups determines a strict Picard category with B as the set of objects, and in which a morphism from b_0 to b_1 is a map for which $\partial a=b_1-b_0$. The operation $+$ comes from the group structure, and the natural transformations ψ and α are the identity maps. We'll denote this groupoid with the symbol

$$(A\to B)=\left(A\xrightarrow{\partial}B\right).$$

It is immediate from the definition that

$$\begin{aligned}\pi_0\left(A\xrightarrow{\partial}B\right)&=\operatorname{coker}\partial\\ \pi_1\left(A\xrightarrow{\partial}B\right)&=\ker\partial.\end{aligned}$$

[15]A 2-category is a category in which the morphisms form a category

One can easily check that the Picard category $(A \to B)$ is equivalent, as a Picard category, to $\left(\pi_1 \xrightarrow{0} \pi_0\right)$. Every strict Picard category is equivalent to one of these forms.

Example B.7. If E is a spectrum, then each of the fundamental groupoids

$$\pi_{\leq 1} E_n$$

is a Picard category, in which the $\otimes$ structure comes from the "loop multiplication" map

$$E_n \times E_n \to E_n.$$

To make this precise, first note that the space E_n is the space of maps of spectra

$$S^0 \to \Sigma^n E,$$

and a choice of "loop multiplication" amounts to a choice of deformation of the diagonal $S^0 \to S^0 \times S^0$ to a map $S^0 \to S^0 \vee S^0$.

Let $\mathcal{L}_k(\mathbb{R}^n)$ denote the space of k-tuples of linearly embedded n-cubes in I^n. The "Pontryagin-Thom collapse" gives a map from $\mathcal{L}_k(\mathbb{R}^n)$ to the space of deformations of the diagonal (here S^n denotes the *space* S^n)

$$S^n \to \prod^k S^n$$

to a map

$$S^n \to \bigvee^k S^n.$$

Set

$$\mathcal{L}_k = \varinjlim_{n \to \infty} L_k(\mathbb{R}^n).$$

The space L_k is contractible, and by passing to the limit from the above, parameterizes deformations of the iterated diagonal map of *spectra*

$$S^0 \to \prod^k S^0 \tag{B.8}$$

to

$$S^0 \to \bigvee^k S^{0.} \tag{B.9}$$

It is easy to see that a choice of point $x \in \mathcal{L}_2$ for the monoidal structure, a path $x \to \tau x$ (τ the transposition in Σ_2) for the symmetry, and a path in $\mathcal{L}_3$ for the associativity law give the fundamental groupoid

$$\pi_{\leq 1} E_n$$

the structure of a Picard category. This Picard category structure is independent, up to equivalence of Picard categories, of the choice of point x, and the paths.

Remark B.10. The space of all deformations of (B.8) to (B.9) is contractible, and can be used for constructing the desired Picard category structure. The "little cubes" spaces were introduced in [**44**, **9**], and have many technical advantages. As we explain in the next example, in the case of Thom spectra they give compatible Picard category structures to the the fundamental groupoids of both the transverse and geometric singular complexes.

Example B.11. Suppose $X = \operatorname{Thom}(B; V)$ is a Thom spectrum. Let $\operatorname{sing}^{\mathrm{geom}} X^S_{-n}$ be the simplicial set with k-simplices the set of B-oriented maps $E \to S \times \Delta^k$. The groupoid

$$\pi_{\leq 1} \operatorname{sing}^{\mathrm{geom}} X^S_{-n}$$

then has objects the n-dimensional B-oriented maps $E \to S$, and in morphisms from E_0 to E_1 equivalence classes of B-oriented maps

$$p : E \to S \times \Delta^1$$

with $p|_{\partial_i \Delta^1} = E_i/S$, $i = 0, 1$. In principle $\pi_{\leq 1} \operatorname{sing}^{\mathrm{geom}} X^S_{-n}$ is a Picard category with the the $\otimes$-structure coming from disjoint union of manifolds. But this requires a little care. Recall that a B-orientation of E/S consists of an embedding $E \subset S \times \mathbb{R}^N \subset S \times \mathbb{R}^\infty$, a tubular neighborhood $W \subset S \times \mathbb{R}^N$, and a map $E \to B$ classifying W. To give a B-orientation to the disjoint union of two B-oriented maps, $E_1 \subset S \times \mathbb{R}^N$ and $E_2 \subset S \times \mathbb{R}^M$ we need to construct an embedding

$$E_1 \amalg E_2 \subset W_1 \amalg W_2 \subset S \times \mathbb{R}^P$$

for some P. Regard $\mathbb{R}^k$ as the interior of I^k, and choose a point

$$I^P \amalg I^P \subset I^P \qquad P \gg 0$$

in $\mathcal{L}_2 = \varinjlim L_2(\mathbb{R}^P)$. Making sure $P > N, M$, we can then use

$$W_1 \amalg W_2 \subset I^P \amalg I^P \subset I^P.$$

The rest of the data needed to give $E_1 \amalg E_2$ a B-orientation is then easily constructed. The Pontryagin-Thom construction gives an equivalence of simplicial sets

$$\operatorname{sing}^{\mathrm{geom}} X^S_{-n} \to \operatorname{sing} X^S_{-n}$$

and an equivalence of Picard categories

$$\pi_{\leq 1} \operatorname{sing}^{\mathrm{geom}} X^S_{-n} \to \pi_{\leq 1} \operatorname{sing} X^S_{-n} = \pi_{\leq 1} X^S_n.$$

All Picard categories arise as the fundamental groupoid of a spectrum:

Proposition B.12. *The correspondence*

$$E \mapsto \pi_{\leq 1} E_0$$

is an equivalence between the 2-category of spectra E satisfying

$$\pi_i E = 0 \quad i \neq 0, 1$$

and the 2*-category of Picard categories.* (*The collection of such spectra* E *is made into a* 2*-category by taking as morphisms the fundamental groupoid of the space of maps.*)

Proof. We merely sketch the proof. The homotopy type of such a spectrum E is determined by its k-invariant

$$H\pi_0 E \to \Sigma^2 H\pi_1 E. \tag{B.13}$$

Now the set of homotopy classes of maps (B.13) is naturally isomorphic to

$$\hom\left(A \otimes \mathbb{Z}/2, B\right).$$

One can associate to a Picard category with k-invariant ϵ, the spectrum E with the same k-invariant. Using this it is not difficult to then check the result. q.e.d.

Remark B.14. Clearly, the functor

$$E \mapsto \pi_{\leq 1} E_{n-1}$$

gives an equivalence between the 2-category of Picard categories, and the 2-category of spectra E whose only non-zero homotopy groups are $\pi_{n-1}E$ and $\pi_n E$.

B.3. Anderson duality and functors of Picard categories. We now relate this discussion to Anderson duality. First a couple of easy observations.

Lemma B.15. *The maps*

$$[X, \Sigma^n \tilde{I}] \to \left[X\langle n-1, \infty\rangle, \Sigma^n \tilde{I}\right] \leftarrow \left[X\langle n-1, n\rangle, \Sigma^n \tilde{I}\right]$$
$$\left[X\langle n-1, n\rangle, \Sigma^n \tilde{I}\langle n-1, \infty\rangle\right] \to \left[X\langle n-1, 0\rangle, \Sigma^n \tilde{I}\right]$$

are isomorphisms.

Proof. This follows easily from the exact sequence

$$\operatorname{Ext}\left(\pi_{n-1}X, \mathbb{Z}\right) \rightarrowtail \tilde{I}^n(X) \twoheadrightarrow \hom\left(\pi_n X, \mathbb{Z}\right).$$

q.e.d.

Since

$$\Sigma^n \tilde{I}\langle n-1, \dots, \infty\rangle \approx \Sigma^n H\mathbb{Z},$$

we have

Corollary B.16. *There is a natural isomorphism*

$$\tilde{I}^n(X) \approx H^n\left(X\langle n-1, n\rangle; \mathbb{Z}\right).$$

Now the Picard category

$$\pi_{\le 1}\left(\Sigma^n \tilde{I}\right)_{n-1} \approx \pi_{\le 1}\left(\Sigma^n H\mathbb{Z}\right)_{n-1}) = \pi_{\le 1} K(\mathbb{Z},1)$$

is canonically equivalent to

$$(\mathbb{Z} \to 0) \approx (\mathbb{Q} \to \mathbb{Q}/\mathbb{Z})\,.$$

Combining this with Proposition B.12 then gives

Corollary B.17. *The group $\tilde{I}^n(X)$ is naturally isomorphic to the group of natural equivalences classes of functors of Picard categories*

$$\pi_{\le 1} X_{n-1} \to (\mathbb{Q} \to \mathbb{Q}/\mathbb{Z})\,.$$

Appendix C. Manifolds with corners

In this section we briefly review the basics of the theory of manifolds with corners. For more information see [**18**, **26**, **19**] and for a discussion in connection with cobordism see [**39**]. In our discussion we have followed Melrose [**46**].

C.1. t-manifolds. Let $\mathbb{R}^n_k \subset \mathbb{R}^n$ be the subspace

$$\mathbb{R}^n_k = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \ge 0, 1 \le i \le k\}\,.$$

Definition C.1. The set of points $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_k$ for which exactly j of $\{x_1, \ldots, x_k\}$ are 0 is the set of *j-corner points* of $\mathbb{R}^n_k$.

Definition C.2. A function f on an open subset $U \subset \mathbb{R}^n_k$ is *smooth* if and only if f extends to a C^∞ function on a neighborhood U' with

$$U \subset U' \subset \mathbb{R}^n.$$

A map g from an open subset $U \subset \mathbb{R}^n_k$ to an open subset $U' \subset \mathbb{R}^m_l$ is *smooth* if $f \circ g$ is smooth whenever f is smooth (this only need be checked when f is one of the coordinate functions).

Remark C.3. The definition of smoothness can be characterized without reference to U'. It is equivalent to requiring that f be C^∞ on $(0,\infty)^k \times \mathbb{R}^{n-k} \cap U$ with all derivatives bounded on all subsets of the form $K \cap U$ with $K \subset \mathbb{R}^n_k$ compact.

Definition C.4. A *chart with corners* on a space U is a homeomorphism $\phi : U \to \mathbb{R}^n_k$ of U with an open subset of $\mathbb{R}^n_k$. Two charts ϕ_1, ϕ_2 on U are said to be *compatible* if the function $\phi_1 \circ \phi_2^{-1}$ is smooth.

Lemma C.5. *Suppose that ϕ_1 and ϕ_2 are two compatible charts on a space U. If $x \in U$, and $\phi_1(x)$ is a j-corner point, then $\phi_2(x)$ is a j-corner point.*

Definition C.6. An *atlas* (*with corners*) on a space X is a collection of pairs (ϕ_a, U_a) with ϕ_a a chart on U_a, U_a a cover of X, and for which ϕ_a and $\phi_{a'}$ are compatible on $U_a \cap U_{a'}$. A *C^∞-structure with corners* on X is a maximal atlas.

Definition C.7. A *t-manifold* is a paracompact Hausdorff space X together with a C^∞-structure with corners.

Definition C.8. A *j-corner point* of a t-manifold X is a point x which in some (hence any) chart corresponds to a j-corner point of $\mathbb{R}^n_k$.

Let U be an open subset of $\mathbb{R}^n_k$. The *tangent* bundle to U is the restriction of the tangent bundle of $\mathbb{R}^n$ to U, and will be denoted TU. A smooth map $f : U_1 \to U_2$ has a derivative $df : TU_1 \to TU_2$. It follows that if X is a t-manifold, the tangent bundles to the charts in an atlas patch together to form a vector bundle over X, the tangent bundle to X.

If $x \in X$ is a j-corner point, then the tangent space to X at x contains j distinguished hyperplanes $H_1, \dots, H_j$ in general position, equipped with orientations of T_xX/H_i. The intersection of these hyperplanes is the *tangent space to the corner at x* and will be denoted bT_x. In a general t-manifold, these hyperplanes need not correspond to distinct components of the space of 1-corner points, as for example happens with the polar coordinate region

$$0 \leq \theta \leq \pi/2$$
$$0 \leq r \leq \sin(2\theta).$$

In a *manifold with corners* (Definition C.12) a global condition is imposed, which guarantees that the tangent hyperplanes do correspond to distinct components.

C.2. Neat maps and manifolds with corners.

Definition C.9. A smooth map of t-manifolds $f : X \to Y$ is *neat* if f maps j-corner points to j-corner points, and if for each j-corner point, the map

$$df : T_x/bT_x \to T_{f(x)}/bT_{f(x)}$$

is an isomorphism.

The term *neat* is due to Hirsch [**34**, p. 30], who considered the case of embeddings of manifolds with boundary.

Remark C.10. The condition on tangent spaces is simply the requirement that f be transverse to the corners.

Example C.11. The map

$$I^m \to I^{m+1}$$
$$(x) \mapsto (x, t)$$

is "neat" if and only if $t \neq 0, 1$.

Definition C.12. A t-manifold X is a *manifold with corners* if there exists a neat map $X \to \Delta^n$ for some n.

Remark C.13. The existence of a neat map $X \to \Delta^n$ for some n is equivalent to the existence of a neat map $X \to I^m$ for some m.

Remark C.14. In this paper we have restricted our discussion to manifolds with corners, for sake of simplicity, but in fact our results apply to neat maps of t-manifolds.

Proposition C.15. *The product of manifolds with corners is a manifold with corners. If $X \to M$ is a neat map, and M is a manifold with corners, then so is X.*

C.3. Normal bundles and tubular neighborhoods. Suppose that $f : X \to Y$ is a neat embedding of t-manifolds. The relative *normal bundle* of f is the vector bundle over X whose fiber at x is $T_{f(x)}Y/df\,(T_xX)$. An embedding of t-manifolds does not necessarily admit a tubular neighborhood, as the example

$$\begin{gathered}[0,1] \to [0,1] \times \mathbb{R} \\ x \mapsto (x, \sqrt{x})\end{gathered}$$

shows. However, when f is a neat embedding, the fiber of the relative normal bundle at x can be identified with $bT_{f(x)}Y/df(bT_xX)$, and the usual proof of the existence of a normal bundle applies.

Proposition C.16. *Suppose that X is a compact t-manifold, and $f : X \to Y$ is a neat embedding. There exists a neighborhood U of $f(X)$, a vector bundle W over X, and a diffeomorphism of W with U carrying the zero section of W to f.*

The neighborhood W is a tubular neighborhood of $f(X)$, and the derivative of the embedding identifies W with the relative normal bundle of f

$$W_x \approx T_{f(x)}Y/df\,(T_xX) \approx bT_{f(x)}Y/df(bT_xX).$$

Proposition C.17. *Let $f : X \to Y$ be a neat map of compact t-manifolds. There exists $N \gg 0$ and a factorization*

$$X \to Y \times \mathbb{R}^N \to Y$$

of f through a neat embedding $X \hookrightarrow Y \times \mathbb{R}^N$.

C.4. Transversality.

Definition C.18. A *vector space with j-corners* is a real vector space V together with j codimension 1, relatively oriented subspaces $\{H_1, \dots, H_j\}$ in general position:

i) each of the 1-dimensional vector spaces V/H_i is equipped with an orientation;

ii) for all $\alpha \subseteq \{1, \dots, j\}$, $\operatorname{codim} H_\alpha = s$, where $H_\alpha = \bigcap_{i\in\alpha} H_i$.

Example C.19. The tangent space to each point of a t-manifold is a vector space with corners.

Suppose that $(V; H_1, \dots, H_j)$ and $(V'; H'_1, \dots, H'_l)$ are two vector spaces with corners, and W is a vector space.

Definition C.20. Two linear maps

$$V \to W \leftarrow V'$$

are *transverse* if for each $\alpha \subset \{1, \dots, j\}$, $\beta \subset \{1, \dots, l\}$ the map

$$H_\alpha \oplus H'_\beta \to W$$

is surjective.

The space of transverse linear maps is a possibly empty open subset of the space of all maps.

Definition C.21. Suppose that Z and M are t-manifolds, and that S is a closed manifold. Two maps

$$Z \xrightarrow{f} S \quad \text{and} \quad M \xrightarrow{g} S$$

are *transverse* if for each $z \in Z$ and $m \in M$ with $f(z) = g(m)$, the maps

$$T_z Z \xrightarrow{Df} T_s S \xleftarrow{Dg} T_m M \qquad s = f(z) = g(m)$$

are transverse.

Lemma C.22. *The class of "neat" maps is stable under composition and transverse change of base.*

Appendix D. Comparison of $H\mathbb{Z}(n)^k(S)$ and $\check{H}(n)^k(S)$

The purpose of this appendix to prove that the differential cohomology groups defined in §4.1 using the Eilenberg-MacLane spectrum coincide with the differential cohomology groups defined in §3.2 using the cochain complex.

The proof is in two steps. The first is to replace the differential function space

$$\mathrm{filt}_{k-n}\left(K(\mathbb{Z}, k); \iota_k\right)^S$$

with a simplicial abelian group. To describe the group, recall from §3.2 the cochain complex $\check{C}^*(S)$ with

$$\check{C}^k(S) = C^k(S; \mathbb{Z}) \times C^{k-1}(S; \mathbb{R}) \times \Omega^k(S),$$

and

$$d(c, h, \omega) = (\delta c, \omega - c - \delta h, d\omega).$$

Let $\check{Z}^k(S)$ be the group of k-cocycles in $\check{C}^*(S)$. We define

$$\mathrm{filt}_s\, \check{C}^*(S \times \Delta^k) = C^k(S \times \Delta^k; \mathbb{Z}) \times C^{k-1}(S \times \Delta^k; \mathbb{R}) \times \mathrm{filt}_s\, \Omega^k(S \times \Delta^k)$$

(see the discussion preceding Definition 4.5). The coboundary map in $\check{C}^*(S\times\Delta^k)$ carries the filt_s to filt_{s+1}. Let

$$\mathrm{filt}_s\,\check{Z}^k(S\times\Delta^m)=\mathrm{filt}_s\,\check{C}^k(S\times\Delta^m)\cap\check{Z}^k(S\times\Delta^m).$$

Thus $\mathrm{filt}_s\,\check{Z}^k(S\times\Delta^m)$ consists of triples (c,h,ω) for which c and ω are closed,

$$\delta h=\omega-c,$$

and for which the weight filtration of ω is less than or equal to s.

Associating to a k-simplex of $\mathrm{filt}_s\,(K(\mathbb{Z},k);\iota_k)^S$ its underlying differential cocycle gives a map of simplicial sets

$$\mathrm{filt}_s\,(K(\mathbb{Z},k);\iota_k)^S\to\mathrm{filt}_s\,\check{Z}^k\left(S\times\Delta^\bullet\right). \tag{D.1}$$

Lemma D.2. *The map* (D.1) *is a weak equivalence.*

Proof. By definition, the square

$$\begin{array}{ccc}\mathrm{filt}_s\,(K(\mathbb{Z},k);\iota_k)^S & \longrightarrow & \mathrm{filt}_s\,\check{Z}^k\left(S\times\Delta^\bullet\right)\\ \Big\downarrow & & \Big\downarrow\\ \operatorname{sing}K(\mathbb{Z},k)^S & \longrightarrow & Z^k(S\times\Delta^\bullet)\end{array}$$

is a pullback square. The left vertical map is a surjective map of simplicial abelian groups, hence a Kan fibration, so the square is in fact homotopy Cartesian. The bottom map is a weak equivalence by Proposition A.12. It follows that the top map is a weak equivalence as well. q.e.d.

For the second step note that "slant product with the fundamental class of the variable simplex" gives a map from the chain complex associated to the simplicial abelian group

$$\mathrm{filt}_{k-n}\,\check{Z}^k\left(S\times\Delta^\bullet\right) \tag{D.3}$$

to

$$\check{Z}(n)^k(S)\leftarrow\check{C}(n)^{k-1}(S)\ldots\leftarrow\check{C}(n)^0(S). \tag{D.4}$$

Proposition D.5. *The map described above, from* (D.3) *to* (D.4) *is a chain homotopy equivalence.*

Together Propositions D.2 and D.5 then give the following result:

Proposition D.6. *The map "slant product with the fundamental class of the variable simplex" gives an isomorphism*

$$H\mathbb{Z}(n)^k(S)\xrightarrow{\approx}\check{H}(n)^k(S).$$

Proof of Proposition D.5. Let A_* be the total complex of the following bicomplex:

(D.7)

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \\
\operatorname{filt}_{k-n} \check{Z}^k(S \times \Delta^2) \arrow[d, "\sum(-1)^i\partial_i^*"'] & \operatorname{filt}_{k-n-1} \check{C}^{k-1}(S \times \Delta^2) \arrow[l, "d"'] \arrow[d] & \cdots \arrow[l] \\
\operatorname{filt}_{k-n} \check{Z}^k(S \times \Delta^1) \arrow[d] & \operatorname{filt}_{k-n-1} \check{C}^{k-1}(S \times \Delta^1) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\operatorname{filt}_{k-n} \check{Z}^k(S \times \Delta^0) & \operatorname{filt}_{k-n-1} \check{C}^{k-1}(X \times \Delta^0) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

By Lemma D.9 below, the inclusion of the leftmost column

$$\operatorname{filt}_s \check{Z}^k \left(S \times \Delta^\bullet\right) \to A_*$$

is a quasi-isomorphism. By Lemma D.10 below, the inclusion $B_*^{(0)} \subset A_*$ is also a quasi-isomorphism. One easily checks that map "slant-product with the fundamental class of $\Delta^\bullet$" gives a retraction of A_* to $B_*^{(0)}$, which is therefore also a quasi-isomorphism. This means that the composite map

$$\operatorname{filt}_s \check{Z}^k \left(S \times \Delta^\bullet\right) \to A_* \to B_*^{(0)} \tag{D.8}$$

is a quasi-isomorphism. But the chain complex $B_*^{(0)}$ is exactly the chain complex (D.4), and the map (D.8) is given by slant product with the fundamental class of the variable simplex. This completes the proof. q.e.d.

We have used

Lemma D.9. *For each m and s the simplicial abelian group*

$$[n] \mapsto \operatorname{filt}_s \check{C}^m(S \times \Delta^n)$$

is contractible.

Lemma D.10. *Let $B_*^{(i)}$ be the i^{th} row of* (D.7)*. The map*

$$\Delta^0 \to \Delta^n$$

given by inclusion of any vertex induces an isomorphism of homology groups

$$H_*(B_*^{(0)}) \xrightarrow{\approx} H_*(B_*^{(i)}).$$

The inclusion of B_*^0 into the total complex associated to (D.7) *is therefore a quasi-isomorphism.*

Proof of Lemma D.9. It suffices to separately show that

$$[n] \mapsto C^m(S \times \Delta^n), \qquad [n] \mapsto C^{m-1}(S \times \Delta^n; \mathbb{R})$$

and

$$[n] \mapsto \operatorname{filt}_s \Omega^m(S \times \Delta^n)$$

are acyclic. These are given by Lemmas D.11 and D.12 below. q.e.d.

Lemma D.11. *For any abelian group A, the simplicial abelian group*

$$[n] \mapsto C^m(S \times \Delta^n; A)$$

is contractible.

Proof. It suffices to construct a contracting homotopy of the associated chain complex. The construction makes use of "extension by zero." If $f : X \to Y$ is the inclusion of a subspace, the surjection map

$$C^*Y \to C^*X$$

has a canonical section $f_!$ given by *extension by zero*:

$$f_!(c)(z) = \begin{cases} c(z') & \text{if } z = f \circ z' \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f_!$ is *not* a map of cochain complexes. Using the formula

$$\partial_i^*(\partial_0)_! c = \begin{cases} c & i = 0 \\ (\partial_0)_! \partial_{i-1}^* c & i \neq 0 \end{cases}$$

one easily checks that

$$h = \partial_{0!} : C^n(M \times \Delta^k) \to C^n(M \times \Delta^{k+1})$$

is a contracting homotopy. q.e.d.

Lemma D.12. *For each $s \geq 0$ and t chain complex*

$$\operatorname{filt}_s \Omega^t(S \times \Delta^\bullet)$$

is contractible.

Proof. We will exhibit a contracting homotopy. We will use barycentric coordinates on

$$\Delta^n = \{(t_0, \dots, t_n) \mid 0 \leq t_i \leq 1, \sum t_i = 1\},$$

and let v_i be the vertex for which $t_i = 1$. Let

$$\begin{aligned} p_n &: \Delta^n \setminus v_0 \to \Delta^{n-1} \\ &(t_0, \dots, t_n) \mapsto (t_1/(1 - t_0), \dots, t_n/(1 - t_0))) \end{aligned}$$

be radial projection from v_0 to the 0^{th} face. Then

$$\begin{aligned} p^n \circ \partial^0 &= \mathrm{Id} \\ p^n \circ \partial^i &= \partial^{i-1} \circ p^{n-1} \qquad i > 0. \end{aligned}$$

Finally, let

$$g : [0,1] \to \mathbb{R}$$

be a C^∞ bump function, vanishing in a neighborhood of 0, taking the value 1 in a neighborhood of 1. The map

$$\begin{aligned} h : \Omega^t(S \times \Delta^{m-1}) &\to \Omega^t(S \times \Delta^m) \\ \omega &\mapsto g(1-t_0)\,(1 \times p^m)^*\omega \end{aligned}$$

is easily checked to preserve filt_s and define a contracting homotopy. q.e.d.

Proof of Lemma D.10. It suffices to establish the analogue of Lemma D.10 with the complex $\check{C}^*$ replaced by C^*, $C^* \otimes \mathbb{R}$, and $\mathrm{filt}_s\,\Omega^*$. The cases of C^* and $C^* \otimes \mathbb{R}$ are immediate from the homotopy invariance of singular cohomology. For $\mathrm{filt}_s\,\Omega^*$, the spectral sequence associated to the filtration by powers of $\Omega^{*>0}(S)$ shows that the i^{th}-homology group of

$$\mathrm{filt}_s\,\Omega^k(S \times \Delta^m)_{\mathrm{cl}} \leftarrow \mathrm{filt}_{s-1}\,\Omega^{k-1}(S \times \Delta^m) \leftarrow \dots$$

is

$$\begin{cases} H^{k-i}_{\mathrm{DR}}(S;\mathbb{R}) \otimes H^0_{\mathrm{DR}}(\Delta^m) & i < s \\ \Omega^n_{\mathrm{cl}}(S) \otimes H^0_{\mathrm{DR}}(\Delta^m) & i = s, \end{cases}$$

so the result follows from the homotopy invariance of deRham cohomology. q.e.d.

The results in §4.6 require one other variation on these ideas. For a compact manifold S and a graded real vector space $\mathcal{V}$, let $C^*_c(S \times \mathbb{R}; \mathcal{V})$ denote the compactly supported cochains, $\Omega^*_c(S \times \mathbb{R}; \mathcal{V})$ the compactly supported forms, and $\mathrm{filt}_s\,\Omega^*_c(S \times \Delta^k \times \mathbb{R}; \mathcal{V})$ the subspace consisting forms whose Kunneth component on $\Delta^k \times \mathbb{R}$ has degree $\le s$. The following are easily verified using the techniques and results of this appendix.

Corollary D.13. *The map "slant product with the fundamental class of $\Delta^\bullet$" is a chain homotopy equivalence from the chain complex underlying*

$$Z^k(S \times \Delta^\bullet; \mathcal{V})$$

to

$$Z^k(S; \mathcal{V}) \leftarrow C^{k-1}(S; \mathcal{V}) \leftarrow \cdots \leftarrow C^0(S; \mathcal{V}).$$

Corollary D.14. *"Integration over $\Delta^\bullet$" is a chain homotopy equivalence between the chain complex underlying the simplicial abelian group* $\mathrm{filt}_s\,\Omega^*(S \times \Delta^\bullet; \mathcal{V})^k_{\mathrm{cl}}$ *and*

$$\Omega^*(S; \mathcal{V})^k_{\mathrm{cl}} \leftarrow \Omega^*(S; \mathcal{V})^{k-1} \leftarrow \dots \leftarrow \Omega^*(S; \mathcal{V})^{k-s}.$$

Corollary D.15. *The homotopy groups of* $\operatorname{filt}_s \Omega^*(S \times \Delta^\bullet; \mathcal{V})^0_{\mathrm{cl}}$ *are given by*

$$\pi_m \operatorname{filt}_s \Omega^*(S \times \Delta^\bullet; \mathcal{V})^0_{\mathrm{cl}} = \begin{cases} H^{-m}_{\mathrm{DR}}(S; \mathcal{V}) & m < s \\ \Omega^*(S; \mathcal{V})^{-s}_{\mathrm{cl}} & m = s \\ 0 & m > s. \end{cases}$$

Corollary D.16. *Let* $[\Delta^m \times \mathbb{R}]$ *be the product of the fundamental cycle of* Δ^m *with the fundamental cycle of the* 1*-point compactification of* $\mathbb{R}$*. The map from*

$$Z^k_c(S \times \Delta^\bullet \times \mathbb{R}; \mathcal{V})$$

to

$$Z^k_c(S \times \mathbb{R}; \mathcal{V}) \leftarrow C^{k-1}_c(S \times \mathbb{R}; \mathcal{V}) \leftarrow \cdots \leftarrow C^0_c(S \times \mathbb{R}; \mathcal{V})$$

sending $f \in Z^k(S \times \Delta^m \times \mathbb{R}; \mathcal{V})$ *to* $f/[\Delta^m \times \mathbb{R}]$ *is a chain homotopy equivalence.*

Corollary D.17. *"Integration over* $\Delta^\bullet \times \mathbb{R}$*" is a chain homotopy equivalence between chain complex underlying the simplicial abelian group* $\operatorname{filt}_s \Omega^*_c(S \times \Delta^\bullet \times \mathbb{R}; \mathcal{V})^k_{\mathrm{cl}}$ *and*

$$\Omega^*(S; \mathcal{V})^{k-1}_{\mathrm{cl}} \leftarrow \Omega^*(S; \mathcal{V})^{k-2} \leftarrow \ldots \leftarrow \Omega^*(S; \mathcal{V})^{k-s-1}.$$

Corollary D.18. *The maps "integration over* $\mathbb{R}$*" and "slant product with the fundamental class of* $\bar{\mathbb{R}}$*" give simplicial homotopy equivalences*

$$\operatorname{filt}_s \Omega^*_c \left(S \times \Delta^\bullet \times \mathbb{R}; \mathcal{V}\right)^k_{\mathrm{cl}} \to \operatorname{filt}_{s-1} \Omega^* \left(S \times \Delta^\bullet; \mathcal{V}\right)^{k-1}_{\mathrm{cl}}$$
$$Z^*_c \left(S \times \Delta^\bullet \times \mathbb{R}; \mathcal{V}\right)^k \to Z^* \left(S \times \Delta^\bullet; \mathcal{V}\right)^{k-1}.$$

Proof. We will do the case of forms. The result for singular cocycles is similar. It suffices to show that the map of underlying chain complexes is a chain homotopy equivalence. Now in the sequence

$$\operatorname{filt}_s \Omega^*_c(S \times \Delta^\bullet \times \mathbb{R}; \mathcal{V})^k_{\mathrm{cl}} \xrightarrow{\int_{\mathbb{R}}} \operatorname{filt}_{s-1} \Omega^*(S \times \Delta^\bullet; \mathcal{V})^{k-1}_{\mathrm{cl}}$$
$$\xrightarrow{\int_{\Delta^\bullet}} \left\{ \Omega^*(S; \mathcal{V})^{k-1}_{\mathrm{cl}} \leftarrow \Omega^*(S; \mathcal{V})^{k-2} \leftarrow \ldots \leftarrow \Omega^*(S; \mathcal{V})^{k-s-1} \right\},$$

the second map and the composition are chain homotopy equivalences by Corollaries D.14 and D.17 respectively. It follows that the first map is also a chain homotopy equivalence. q.e.d.

Appendix E. Integer Wu-classes for Spin-bundles

Let S be a space, and V and oriented vector bundle over S. Our aim in this appendix is to associate to a Spin-structure on V an integer lift of the total Wu-class of V, and to describe the dependence of this integer lift on the choice of Spin-structure. We begin by describing a family of integer cohomology characteristic classes $\nu^{\mathrm{Spin}}_k(V)$ for Spin-bundles V, whose mod 2-reduction are the Wu-classes of the underlying vector bundle. The classes $\nu_k^{\mathrm{Spin}}(V)$ are zero if $k \not\equiv 0 \bmod 4$. The ν_{4k}^{Spin} satisfy the Cartan formula

$$\nu_{4n}^{\mathrm{Spin}}(V \oplus W) = \sum_{i+j=n} \nu_{4k}^{\mathrm{Spin}}(V)\,\nu_{4j}^{\mathrm{Spin}}(W),$$

making the total Spin Wu-class

$$\nu_t^* = 1 + \nu_4^{\mathrm{Spin}} + \nu_8^{\mathrm{Spin}} + \dots$$

into a stable exponential characteristic class.

The characteristic classes ν_k^{Spin}, while very natural, don't quite constitute integral lifts of the Wu-classes. They are integer *cohomology classes* lifting the Wu-classes, defined by cocycles up to arbitrary coboundary, while an integral lift is represented by a cocycle chosen up to the coboundary of *twice* a cochain. We don't know of a natural way of making this choice in general, so instead we work with an arbitrary one. Fortunately, the formula for the effect of a change of Spin-structure does not depend on this choice of lift. We have decided to describe the characteristic classes ν_k^{Spin} partly because they seem to be interesting in their own right, and partly as a first step toward finding a natural choice of integer lift of the total Wu-class of Spin-bundles.

E.1. Spin Wu-classes.

E.1.1. Wu-classes. Let V be a real n-dimensional vector vector bundle over a space X, and write $U \in H^n(V, V - \{0\}; \mathbb{Z}/2)$ for the Thom class. The Wu-classes $\nu_i = \nu_i(V) \in H^i(X; \mathbb{Z}/2)$ are defined by the identity

$$\nu_t(V)\,U = \chi(\mathrm{Sq}_t)(U)$$

in which

$$\mathrm{Sq}_t = 1 + \mathrm{Sq}_1 + \mathrm{Sq}_2 + \dots$$

is the total mod 2 Steenrod operation, χ the canonical anti-automorphism (antipode) of the Steenrod algebra,

$$\nu_t = 1 + \nu_1 + \nu_2 + \dots$$

the total Wu-class of V. One checks from the definition that the Wu-classes satisfy

(1) (Cartan formula)

$$\nu_t(V \oplus W) = \nu_t(V)\,\nu_t(W);$$

(2) If V is a real line bundle with $w_1(V) = \alpha$ then

$$\nu_t(V) = \sum \alpha^{2^n - 1}.$$

Indeed, the first part follows from the fact the χ is compatible with the coproduct, and the second part follows from the formula

$$\chi(\mathrm{Sq}_t)(x) = \sum x^{2^n}.$$

These two properties characterize the Wu-classes as the mod 2 cohomology stable exponential characteristic class with characteristic series

$$1 + x + x^3 + \cdots + x^{2^n-1} + \cdots \in \mathbb{Z}/2[[x]] = H^*(RP^\infty; \mathbb{Z}/2).$$

E.1.2. Complex Wu classes. Now consider complex analogue of the above. We define

$$\nu_t^{\mathbb{C}} = 1 + \nu_1^{\mathbb{C}} + \nu_2^{\mathbb{C}} \cdots \in H^*(BU; \mathbb{Z})$$

to be the stable exponential characteristic class with characteristic series

$$1 + x + \cdots + x^{2^n-1} + \cdots \in \mathbb{Z}[[x]] = H^*(CP^\infty; \mathbb{Z}).$$

So for a complex line bundle L, with first Chern class x,

$$\nu_t^{\mathbb{C}}(L) = \sum_{n \geq 0} x^{2^n-1}.$$

One easily checks that

$$\nu^{\mathbb{C}}(L) \equiv \nu(L) \mod 2.$$

so the classes $\nu_t^{\mathbb{C}}$ is a stable exponential integer characteristic class lifting the total Wu class in case the vector bundle V has a complex structure.

E.1.3. Wu-classes for Spin bundles. Because of the presence of torsion in $H^*(\mathrm{BSpin}; \mathbb{Z})$, specifying a stable exponential characteristic class

$$\nu_t^{\mathrm{Spin}} = 1 + \nu_4^{\mathrm{Spin}} + \nu_8^{\mathrm{Spin}} + \cdots \in H^*(\mathrm{BSpin}; \mathbb{Z})$$

bundles is a little subtle. Things are simplified somewhat by the fact that the torsion in $H^*(\mathrm{BSpin})$ is all of order 2. It implies that

$$\begin{tikzcd} H^*(\mathrm{BSpin}; \mathbb{Z}) \arrow[r] \arrow[d] & H^*(\mathrm{BSpin}; \mathbb{Z})/\mathrm{torsion} \arrow[d] \\ H^*(\mathrm{BSpin}; \mathbb{Z}) \otimes \mathbb{Z}/2 \arrow[r] & (H^*(\mathrm{BSpin}; \mathbb{Z})/\mathrm{torsion}) \otimes \mathbb{Z}/2 \end{tikzcd}$$

is a pullback square and that $H^*(\mathrm{BSpin}; \mathbb{Z}) \otimes \mathbb{Z}/2$ is just the kernel of

$$\mathrm{Sq}^1 : H^*(\mathrm{BSpin}; \mathbb{Z}/2) \to H^{*+1}(\mathrm{BSpin}; \mathbb{Z}/2).$$

One also knows that $H^*(\mathrm{BSpin}; \mathbb{Z})/\mathrm{torsion}$ is a summand of $H^*(BSU; \mathbb{Z})$. So to specify an element of $H^*(\mathrm{BSpin}; \mathbb{Z})$ one needs to give $a \in H^*(\mathrm{BSpin}; \mathbb{Z}/2)$ and $b \in H^*(BSU; \mathbb{Z})$ with the properties

(1) $\mathrm{Sq}^1(a) = 0$;
(2) The image of b in $H^*(BSU; \mathbb{Q})$ is in the image of $H^*(\mathrm{BSpin}; \mathbb{Q})$;
(3) $b \equiv a \mod 2 \in H^*(BSU; \mathbb{Z}/2) = H^*(BSU; \mathbb{Z}) \otimes \mathbb{Z}/2$.

We take a to be (the restriction of) the total Wu-class $\nu_t \in H^*(\mathrm{BSpin}; \mathbb{Z}/2)$. Property (1) is then given by the following (well-known) result.

Lemma E.1. *If V is a* Spin*-bundle then $\nu_k(V) = 0$ if $n \not\equiv 0 \mod 4$, and $\mathrm{Sq}^1\, \nu_{4k} = 0$.*

Proof. Since V is a spin bundle $\mathrm{Sq}^k U = w_k U = 0$ for $k \leq 3$. Using the Adem relations

$$\begin{aligned}\mathrm{Sq}^1 \mathrm{Sq}^{2k} &= \mathrm{Sq}^{2k+1}\\ \mathrm{Sq}^2 \mathrm{Sq}^{4k} &= \mathrm{Sq}^{4k+2} + \mathrm{Sq}^{4k+1} \mathrm{Sq}^1\\ &= \mathrm{Sq}^{4k+2} + \mathrm{Sq}^1 \mathrm{Sq}^{4k} \mathrm{Sq}^1\end{aligned}$$

we calculate

$$\begin{aligned}\nu_{2k+1} U = \chi(\mathrm{Sq}^{2k+1}) U &= \chi(\mathrm{Sq}^1 \mathrm{Sq}^{2k})\\ &= \chi(Sq^{2k})\chi(Sq^1) U\\ &= \chi(Sq^{2k}) Sq^1 U = 0\end{aligned}$$

and

$$\begin{aligned}\nu_{4k+2} U = \chi(\mathrm{Sq}^{4k+2}) U &= \chi(\mathrm{Sq}^2 \mathrm{Sq}^{4k+1}) U + \chi(\mathrm{Sq}^1 \mathrm{Sq}^{4k} \mathrm{Sq}^1)\\ &= \chi(Sq^{4k})\chi(Sq^2) U + \chi(\mathrm{Sq}^1)\chi(\mathrm{Sq}^{4k})\chi(\mathrm{Sq}^1) U\\ &= \chi(Sq^{4k})\, Sq^2\, U + \mathrm{Sq}^1\, \chi(\mathrm{Sq}^{4k})\, \mathrm{Sq}^1\, U\\ &= 0.\end{aligned}$$

This gives the first assertion. For the second, use the Adem relation

$$\mathrm{Sq}^2 \mathrm{Sq}^{4k-1} = \mathrm{Sq}^{4k} \mathrm{Sq}^1$$

and calculate

$$\begin{aligned}\mathrm{Sq}^1 \nu_{4k} U = \mathrm{Sq}^1 \chi(\mathrm{Sq}^{4k}) U &= \chi(\mathrm{Sq}^{4k} \mathrm{Sq}^1) U\\ &= \chi(\mathrm{Sq}^2 \mathrm{Sq}^{4k-1}) U\\ &= \chi(\mathrm{Sq}^{4k-1})\, \mathrm{Sq}^2\, U = 0.\end{aligned}$$

q.e.d.

We would like to take for $b \in H^*(BSU;\mathbb{Z})$ the restriction of $\nu_t^{\mathbb{C}}$. Unfortunately that class is not in the image of $H^*(\mathrm{BSpin})$. Indeed, $\nu_t^{\mathbb{C}}$ is the stable exponential characteristic class with characteristic series

$$f(x) = 1 + x + x^3 + \cdots + x^{2^n-1} + \ldots.$$

which is not symmetric. But if we symmetrize f by setting

$$g(x) = \sqrt{f(x)f(-x)} = 1 - \frac{x^2}{2} - \frac{9\,x^4}{8} - \frac{17\,x^6}{16} + \cdots \in \mathbb{Z}[\tfrac{1}{2}][\![x]\!]$$

then $g(x)$, being even, will be the characteristic series of a stable exponential characteristic class

$$\chi \in H^*(BSO;\mathbb{Z}[\tfrac{1}{2}]).$$

Lemma E.2.
The restriction of χ *to* $H^*(BSU;\mathbb{Z}[\frac{1}{2}])$ *lies in* $H^*(BSU;\mathbb{Z})$.

Proof. We will use the result of [**36**, **6**] that a stable exponential characteristic class for SU-bundles is determined by its value on

$$(1-L_1)(1-L_2)$$

which can be any power series

$$h(x,y)\in H^*(CP^\infty\times CP^\infty)$$

satisfying

(1) $h(0,0)=1$;
(2) $h(x,y)=h(y,x)$;
(3) $h(y,z)h(x,y+z)=h(x+y,z)h(x,y)$.

We will call h the *SU-characteristic series*. The SU-characteristic series of χ is

$$\delta g(x,y):=\frac{g(x+y)}{g(x)g(y)}.$$

The lemma will follow once we show that $\delta g(x,y)$ has integer coefficients.

Now

$$g(x,y)^2=\delta f(x,y)\,\delta f(-x,-y).$$

Since the power series $\sqrt{1+4x}$ has integer coefficients, it suffices to show that

$$\delta f(x,y)\,\delta f(-x,-y)$$

is a square in $\mathbb{Z}/4[\![x]\!]$. For this, start with

$$\begin{aligned} f(x)-f(-x) &= 2xf(x^2)\\ &\equiv 2x\,f(x)^2 && \mod 4\,\mathbb{Z}[\![x]\!]\\ &\equiv 2x\,f(x)\,f(-x) && \mod 4\,\mathbb{Z}[\![x]\!] \end{aligned}$$

to conclude that

$$f(x)\equiv f(-x)\,(1+2xf(x))\mod 4\,\mathbb{Z}[\![x]\!].$$

Write

$$e(x)=1+2xf(x)=1+2\sum_{n\geq 0}x^{2^n}.$$

Then

$$e(x+y)\equiv e(x)e(y)\mod 4\,\mathbb{Z}[\![x]\!]$$

(in fact $e(x)\equiv e^{2x}\mod 4\,\mathbb{Z}[\![x]\!]$). This implies

$$\delta f(x,y)\equiv\delta f(-x,-y)\mod 4\,\mathbb{Z}[\![x]\!],$$

and so

$$\delta f(x,y)\,\delta f(-x,-y)=\delta f(x,y)^2\mod 4\,\mathbb{Z}[\![x]\!]. \tag{E.3}$$

q.e.d.

Continuing with the construction of ν_t^{Spin}, we now take the class b to be the stable exponential characteristic class with SU-characteristic series $g(x)$. It remains to check that $b \equiv a \in H^*(BSU;\mathbb{Z}/2)$, or, in terms of SU-characteristic series, that

$$\delta g(x,y) \equiv \delta f(x,y) \mod 2\,\mathbb{Z}[[x]].$$

But this identity obviously holds after squaring both sides, which suffices, since $\mathbb{Z}/2[[x]]$ is an integral domain over $\mathbb{Z}/2$.

To summarize, we have constructed a stable exponential characteristic class ν_t^{Spin} for Spin bundles, with values in integer cohomology, whose mod 2-reduction is the total Wu-class. Rationally, in terms of Pontryagin classes, the first few are

$$\begin{aligned}
\nu_4^{\mathrm{Spin}} &= -\frac{p_1}{2} \\
\nu_8^{\mathrm{Spin}} &= \frac{20\,p_2 - 9\,p_1^2}{8} \\
\nu_{12}^{\mathrm{Spin}} &= \frac{-80\,p_3 + 60\,p_1p_2 - 17\,p_1^3}{16}.
\end{aligned}$$

It frequently comes up in geometric applications that one wishes to express the value of a stable exponential characteristic class of the normal bundle in terms characteristic classes of the tangent bundle. We therefore also record

$$\begin{aligned}
\nu_4^{\mathrm{Spin}}(-T) &= \frac{p_1}{2} \\
\nu_8^{\mathrm{Spin}}(-T) &= \frac{-20\,p_2 + 11\,p_1^2}{8} \\
\nu_{12}^{\mathrm{Spin}}(-T) &= \frac{80\,p_3 - 100\,p_2p_1 + 37\,p_1^3}{16},
\end{aligned}$$

where in these expressions the Pontryagin classes are those of T (and not $-T$).

E.2. Integral Wu-structures and change of Spin**-structure.** Suppose S is a space, and $\nu \in Z^k(S;\mathbb{Z}/2)$ a cocycle. Recall from §2.5 that the category $\mathcal{H}_\nu^k(S)$ of *integer lifts* of ν is the category whose objects are cocycles $x \in Z^k(S;\mathbb{Z})$ whose reduction modulo 2 is ν. A morphism from x to y in $\mathcal{H}_\nu^k(S)$ is a $(k-1)$-cochain $c \in C^{k-1}(S;\mathbb{Z})$ with the property that

$$2\,\delta c = y - x.$$

We identify two morphisms c and c' if they differ by a $(k-1)$-cocycle. The set $H_\nu^k(S)$ of isomorphism classes of objects in $\mathcal{H}_\nu^k(S)$ is a torsor for $H^k(S;\mathbb{Z})$. We write the action of $b \in H^k(S)$ on $x \in H_\nu^k(S)$ as

$$x \mapsto x + (2)\,b.$$

There is a differential analogue of this notion when S is a manifold. We define a *differential integral lift* of ν to be a differential cocycle

$$x = (c, h, \omega) \in \check{Z}(k)^k(S)$$

with the property that $c \equiv \nu$ modulo 2. The set of differential integral lifts of ν forms a category $\check{\mathcal{H}}^k_\nu(S)$ in which a morphism from x to y is an element $c \in \check{C}(k)^{k-1}/\check{Z}(k)^{k-1}$ with the property that

$$2\,\delta c = y - x.$$

The set of isomorphism classes in $\check{\mathcal{H}}^k_\nu(S)$ is denoted $\check{H}(k)^k_\nu(S)$. It is a torsor for the differential cohomology group $\check{H}(k)^k(S)$. As above, we write the action of $b \in \check{H}(k)^k(S)$ on $x \in \check{H}(k)^k_\nu(S)$ as

$$x \mapsto x + (2)\,b.$$

Suppose now that V is an oriented vector bundle over S, and ν a cocycle representing the total Wu-class of V. By Proposition E.1, if $w_2(V) = 0$ the category $H^*_\nu(S)$ is non-empty, and an integer lift of the total Wu class can be associated to a Spin-structure. We wish to describe the effect of a change of Spin-structure on these integral lifts. In case S is a manifold, and V is equipped with a connection, we can associate a differential integral lift of the total Wu-class to the Spin-structure. We are also interested in the effect of a change of Spin-structure on these differential integral lifts.

Choose cocycles $z_{2k} \in Z^{2k}(BSO; \mathbb{Z}/2)$ representing the universal Wu-classes. To associate an integer lift of the total Wu-class to a Spin-structure we must choose cocycles $\bar{z}_{2k} \in Z^{2k}(\mathrm{BSpin}; \mathbb{Z})$ which reduce modulo 2 to the restriction of the z_{2k} cocycles to BSpin. The classes $\bar{z}$ could be taken to represent the classes ν^{Spin} constructed in the previous section, though this is not necessary. We represent the cocycles z_{2k} and $\bar{z}_{2k}$ by maps to Eilenberg-MacLane spaces, resulting in a diagram

$$\begin{array}{ccc}
\mathrm{BSpin} & \xrightarrow{\bar{z}=\prod \bar{z}_{2k}} & \prod_{k\geq 2} K(\mathbb{Z}, 2k) \\
\downarrow & & \downarrow \\
BSO & \xrightarrow[z=\prod z_{2k}]{} & \prod_{k\geq 1} K(\mathbb{Z}/2, 2k).
\end{array}$$

Choose a map $S \to BSO$ classifying V, $s_0 : S \to \mathrm{BSpin}$ a lift corresponding to a Spin-structure on V, and $\alpha : S \to K(\mathbb{Z}/2, 1)$ a 1-cocycle on S. Let s_1 be the Spin-structure on V gotten by changing s_0 by α. We want a formula relating $\bar{z}(s_0)$ and $\bar{z}(s_1)$, or, more precisely, a formula for the cohomology class represented by

$$\frac{\bar{z}(s_1) - \bar{z}(s_1)}{2}.$$

We first translate this problem into one involving the long exact sequence of homotopy groups of a fibration. Write $P_0 \to B_0$ for the fibration

$$\mathrm{BSpin}^S \to BSO^S \tag{E.4}$$

and $P_1 \to B_1$ for

$$\left(\prod_{k\geq 1} K(\mathbb{Z}, 2k)\right)^S \to \left(\prod_{k\geq 1} K(\mathbb{Z}/2, 2k)\right)^S. \tag{E.5}$$

Then (E.4) and (E.5) are principal bundles with structure groups

$$G_0 = K(\mathbb{Z}/2, 1)^S \quad \text{and} \quad \left(\prod_{k\geq 1} K(\mathbb{Z}, 2k)\right)^S$$

respectively. We give P_0 the basepoint corresponding to s_0, B_0 the basepoint corresponding to V, and P_1 and B_1 the basepoints corresponding to $\bar{z}(s_0)$ and $z(V)$. These choices identify the fibers over s_0 and $\bar{z}(s_0)$ with G_0 and G_1, and lead to a map of pointed fiber sequences

$$\begin{array}{ccccc} G_0 & \longrightarrow & P_0 & \longrightarrow & B_0 \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ G_1 & \longrightarrow & P_1 & \longrightarrow & B_1 \end{array}$$

derived from $(\bar{z}, z)$. Our problem is to describe the map

$$\pi_0 G_0 = H^1(S; \mathbb{Z}/2) \to \pi_0 G_1 = \prod_{k\geq 1} H^{2k}(S; \mathbb{Z}).$$

In general this can be difficult; but for elements in the image of $\pi_1 B_0 \to \pi_0 G_0$ the answer is given by the composite $\pi_1 B_0 \to \pi_1 B_1 \to \pi_0 G_0$. In [**48**] (stated in the language of Spin-structures) Milnor shows that $\pi_1 B_0 \to \pi_0 G_0$ is surjective. So in our case, every element of $\pi_0 G_0$ is in the image of $\pi_1 B_0$. This leads to our desired formula.

We now translate this discussion back into the language of Spin-structures and characteristic classes. The element of $\pi_0 G_0 = H^1(S; \mathbb{Z}/2)$ is the cohomology class represented by α. To lift this to an element of $\pi_1 B_0$ is equivalent to finding an oriented stable vector bundle W over

$$S \times S^1$$

satisfying

$$W|_{S\times\{1\}} = V$$
$$w_2(W) = \alpha \cdot U$$

where $U \in H^1(S^1; \mathbb{Z}/2)$ is the generator. This is easily done. We take

$$W = V \oplus (1 - L_\alpha) \otimes (1 - H)$$

in which L_α is the real line bundle whose first Stiefel-Whitney class is represented by α, and H is the non-trivial real line bundle over S^1.

(This proves Milnor's result on the surjectivity of $\pi_1 B_0 \to \pi_0 G_0$.) The image of this class in $\pi_1 B_1$ is the total Wu-class of W, which is

$$\nu_t(V)\,\frac{\nu(L_\alpha\otimes H)}{\nu(L_\alpha)\nu(H)}. \tag{E.6}$$

Writing

$$\begin{gathered} \nu_t(V) = 1+\nu_1+\nu_2+\dots \\ w_1(L_\alpha)=\alpha \\ w_1(H)=\epsilon \end{gathered}$$

and remembering that $\epsilon^2=0$, one evaluates (E.6) to be

$$\begin{aligned} \nu_t(V)\,\frac{\sum_{n\ge 0}(\alpha+\epsilon)^{2^n-1}}{(1+\epsilon)\sum_{n\ge 0}\alpha^{2^n-1}} &= \nu_t(V)\frac{\sum\alpha^{2^n-1}+\epsilon\left(\sum\alpha^{2^n-1}\right)^2}{(1+\epsilon)\sum\alpha^{2^n-1}} \\ &= \nu_t(V)\,\frac{1+\epsilon\sum\alpha^{2^n-1}}{(1+\epsilon)} \\ &= \nu_t(V)\,\left(1+\epsilon\sum_{n\ge 1}\alpha^{2^n-1}\right). \end{aligned}$$

The image of this class under $\pi_1 B_1 \to \pi_0 G_1$ is computed by taking the slant product with the fundamental class of S^1 and then applying the Bockstein homomorphism. This leads to

$$\beta\left(\nu_t(V)\sum_{n\ge 1}\alpha^{2^n-1}\right) = \bar\nu_t(V)\,\beta\left(\sum_{n\ge 1}\alpha^{2^n-1}\right)\in\prod H^{2n}(S;\mathbb{Z}). \tag{E.7}$$

This proves

Proposition E.8. *Suppose that V is a vector bundle over a space S, s a* Spin*-structure on V, $\alpha\in H^1(S;\mathbb{Z}/2)$. Let $\bar\nu_t$ be any integer lift of the restriction of the total Wu class to* BSpin. *Then*

$$\begin{aligned} \bar\nu_t(s+\alpha) &= \bar\nu(s)+(2)\,\beta\left(\nu_t(V)\sum_{n\ge 1}\alpha^{2^n-1}\right) \\ &= \bar\nu(s)+(2)\,\nu_t(V)\sum_{n\ge 1}\beta(\alpha^{2^n-1}) \end{aligned}$$

In this expression, the factor "(2)" is formal. It is written to indicate the action of $\prod H^{2n}(S;\mathbb{Z})$ on the set of integral Wu-structures. The number 2 serves as a reminder that the action of x on the cohomology class w underlying an integral Wu-structure is $w\mapsto w+2x$.

An analogous discussion, using differential cohomology, leads to the following

Proposition E.9. *Let S be a manifold and $s : S \to \mathrm{BSpin}$ represent a* Spin*-structure on a stable oriented vector bundle V with connection ∇. Write $\bar{\nu}(s,\nabla) \in \prod_{k\geq 0} \check{\mathcal{H}}^{2k}_{\nu}(S)$ for the twisted differential cocycle associated to s, ∇ and the cocycle $\bar{\nu}$ (see §3.3). If $\alpha \in Z^1(S;\mathbb{Z}/2)$, then*

$$\bar{\nu}(s+\alpha,\nabla) = \bar{\nu}(s,\nabla) + (2)\,\beta\left(\nu(V)\sum_{k\geq 1}\alpha^{2^k-1}\right)$$

where, again, the factor "(2)" is formal, indicating the action of $\prod \check{H}(2k)^{2k}(S)$ on the set of isomorphism classes of ν-twisted differential cocycles, and β denotes the map

$$\prod H^{2k-1}(S;\mathbb{Z}/2) \to \prod H^{2k-1}(S;\mathbb{R}/\mathbb{Z}) \subset \prod \check{H}(2k)^{2k}(S).$$

References

[1] J.F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, Chicago, 1974, MR 0402720, Zbl 0309.55016.

[2] O. Alvarez, *Cohomology and field theory*, Symposium on anomalies, geometry, topology (Chicago, Ill., 1985), World Sci. Publishing, Singapore, 1985, 3–21, MR 0850842.

[3] O. Alvarez & I.M. Singer, *Beyond the Elliptic Genus*, Nuclear Phys. B **633**(**3**) (2002) 309–344, MR 1910266, Zbl 0995.58016.

[4] D.W. Anderson, *Universal coefficient theorems for k-theory*, mimeographed notes, Univ. California, Berkeley, Calif., 1969.

[5] D.W. Anderson, E.H. Brown, Jr., & F.P. Peterson, su-*cobordism,* ko-*characteristic numbers, and the Kervaire invariant*, Ann. of Math. (2) **83** (1966) 54–67, MR 0189043, Zbl 0137.42802.

[6] M. Ando, M.J. Hopkins, & N.P. Strickland, *Elliptic spectra, the Witten genus and the theorem of the cube*, Invent. Math. **146**(**3**) (2001) 595–687, MR 1869850, Zbl 1031.55005.

[7] M.F. Atiyah, V.K. Patodi, & I.M. Singer, *Spectral asymmetry and Riemannian geometry*, III, Math. Proc. Cambridge Philos. Soc. **79**(**1**) (1976) 71–99, MR 0397799, Zbl 0325.58015.

[8] M.F. Atiyah, *Riemann surfaces and spin structures*, Ann. Sci. École Norm. Sup. (4) **4** (1971) 47–62, MR 0286136, Zbl 0212.56402.

[9] J.M. Boardman & R.M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Springer-Verlag, Berlin, 1973, Lecture Notes in Mathematics, **347**, MR 0420609, Zbl 0285.55012.

[10] W. Browder, *The Kervaire invariant of framed manifolds and its generalization*, Annals of Mathematics **90** (1969) 157–186, MR 0251736, Zbl 0198.28501.

[11] E.H. Brown & M. Comenetz, *Pontrjagin duality for generalized homology and cohomology theories*, American Journal of Mathematics **98** (1976) 1–27, MR 0405403, Zbl 0325.55008.

[12] E.H. Brown, Jr., *Note on an invariant of Kervaire*, Michigan Math. J. **12** (1965) 23–24, MR 0190941, Zbl 0129.39002.

[13] ———, *The Kervaire invariant of a manifold*, Algebraic topology (Proc. Sympos. Pure Math., Vol. XXII, Univ. Wisconsin, Madison, Wis., 1970), Amer. Math. Soc., Providence, RI, 1971, 65–71, MR 0322888, Zbl 0242.57017.

[14] ———, *Generalizations of the Kervaire invariant*, Ann. of Math. (2) **95** (1972) 368–383, MR 0293642, Zbl 0241.57014.

[15] E.H. Brown, Jr. & F.P. Peterson, *The Kervaire invariant of* $(8k+2)$*-manifolds*, Bull. Amer. Math. Soc. **71** (1965) 190–193, MR 0170346, Zbl 0129.39003.

[16] ———, *The Kervaire invariant of* $(8k+2)$*-manifolds*, Amer. J. Math. **88** (1966) 815–826, MR 0206970, Zbl 0148.17401.

[17] J.-L. Brylinski, *Loop spaces, characteristic classes and geometric quantization*, Birkhäuser Boston Inc., Boston, MA, 1993, MR 1197353, Zbl 0823.55002.

[18] J. Cerf, *Topologie de certains espaces de plongements*, Bull. Soc. Math. France **89** (1961) 227–380, MR 0140120, Zbl 0101.16001.

[19] ———, *La nullité de* $\pi_0\,(\mathrm{diff}\, S^3)$*. Théorèmes de fibration des espaces de plongements. Applications*, Séminaire Henri Cartan, 1962/63, Exp. 8, Secrétariat mathématique, Paris, 1962/1963, MR 0198483, Zbl 0121.39903.

[20] J. Cheeger & J. Simons, *Differential characters and geometric invariants*, Geometry and topology (College Park, Md., 1983/84), Lecture Notes in Math., **1167**, Springer, Berlin, 1985, 50–80, MR 0827262, Zbl 0621.57010.

[21] S.S. Chern & J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. (2) **99** (1974) 48–69, MR 0353327, Zbl 0283.53036.

[22] E.B. Curtis, *Simplicial homotopy theory*, Advances in Math. **6** (1971) 107–209, MR 0279808, Zbl 0225.55002.

[23] P. Deligne, *Le déterminant de la cohomologie*, Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), Amer. Math. Soc., Providence, RI, 1987, 93–177, MR 0902592, Zbl 0629.14008.

[24] ———, *Théorie de Hodge*, II, Inst. Hautes Études Sci. Publ. Math. **40** (1971) 5–57, MR 0498551, Zbl 0219.14007.

[25] L. Dolan & Ch.R. Nappi, *A modular invariant partition function for the fivebrane*, Nuclear Phys. B **530**(**3**) (1998) 683–700, MR 1653488, Zbl 0961.81082.

[26] A. Douady, *Variétés à bord anguleux et voisinages tubulaires*, Séminaire Henri Cartan, 1961/62, Exp. 1, Secrétariat mathématique, Paris, 1961/1962, MR 0160221, Zbl 0116.40304.

[27] A.D. Elmendorf, I. Kříž, M.A. Mandell, & J.P. May, *Modern foundations for stable homotopy theory*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, 213–253, MR 1361891, Zbl 0865.55007.

[28] A.D. Elmendorf, I. Kriz, M.A. Mandell, & J.P. May, *Rings, modules, and algebras in stable homotopy theory*, American Mathematical Society, Providence, RI, 1997, With an appendix by M. Cole, MR 1417719, Zbl 0894.55001.

[29] D.S. Freed, *Dirac charge quantization and generalized differential cohomology*, Surveys in differential geometry, 129–194, Surv. Differ. Geom., **VII**, Int. Press, Somerville, MA, 2000, MR 1919425, Zbl 1058.81058.

[30] D.S. Freed & M. Hopkins, *On Ramond-Ramond fields and* K*-theory*, J. High Energy Phys. **5** (2000) Paper 44, 14 pp., MR 1769477, Zbl 0990.81624.

[31] P.G. Goerss & J.F. Jardine, *Simplicial homotopy theory*, Birkhäuser Verlag, Basel, 1999, MR 1711612, Zbl 0949.55001.

[32] J. Harris, *Theta-characteristics on algebraic curves*, Trans. Amer. Math. Soc. **271**(**2**) (1982) 611–638, MR 0654853, Zbl 0513.14025.

[33] R. Harvey & H. Blaine Lawson, Jr, *From sparks to grundles–differential characters*, arXiv:math.DG/0306193.

[34] M.W. Hirsch, *Differential topology*, Springer-Verlag, New York, 1994, Corrected reprint of the 1976 original, MR 1336822.

[35] M.J. Hopkins & P. Goerss, *Multiplicative stable homotopy theory*, book in preparation.

[36] M.J. Hopkins, *Topological modular forms, the Witten genus, and the theorem of the cube*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, 554–565, MR 1403956, Zbl 0848.55002.

[37] M. Kervaire, *La méthode de Pontryagin pour la classification des applications sur une sphère*, Topologia Differenziale (Centro Internaz. Mat. Estivo, deg. 1 Ciclo, Urbino, 1962), Lezione 3, Edizioni Cremonese, Rome, 1962, MR 0164353, Zbl 0203.56403.

[38] M.A. Kervaire, *A manifold which does not admit any differentiable structure*, Comment. Math. Helv. **34** (1960) 257–270, MR 0139172, Zbl 0145.20304.

[39] G. Laures, *On cobordism of manifolds with corners*, Trans. Amer. Math. Soc. **352**(**12**) (2000) 5667–5688 (electronic), MR 1781277, Zbl 0954.55008.

[40] L.G. Lewis, J.P. May, & M. Steinberger, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, **1213**, Springer–Verlag, New York, 1986, MR 0866482, Zbl 0611.55001.

[41] J. Lott, *Higher-degree analogs of the determinant line bundle*, http://www.math.lsa.umich.edu/∼lott.

[42] ———, **r/z** *index theory*, Comm. Anal. Geom. **2**(**2**) (1994) 279–311, MR 1312690, Zbl 0840.58044.

[43] V. Mathai & D. Quillen, *Superconnections, Thom classes, and equivariant differential forms*, Topology **25**(**1**) (1986) 85–110, MR 0836726, Zbl 0592.55015.

[44] J.P. May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin, 1972, Lectures Notes in Mathematics, **271**, MR 0420610, Zbl 0244.55009.

[45] ———, *Simplicial objects in algebraic topology*, University of Chicago Press, Chicago, IL, 1992, Reprint of the 1967 original, MR 1206474, Zbl 0769.55001.

[46] R. Melrose, *Differential analysis on manifolds with corners*, http://www-math.mit.edu/∼rbm/book.html, in preparation.

[47] R.J. Milgram, *Surgery with coefficients*, Ann. of Math. (2) **100** (1974) 194–248, MR 0362359, Zbl 0324.57022.

[48] J. Milnor, *Spin structures on manifolds*, Enseignement Math. (2) **9** (1963) 198–203, MR 0157388, Zbl 0116.40403.

[49] ———, *A procedure for killing homotopy groups of differentiable manifolds.*, Proc. Sympos. Pure Math., **III**, American Mathematical Society, Providence, RI, 1961, 39–55, MR 0130696, Zbl 0118.18601.

[50] J. Milnor & D. Husemoller, *Symmetric bilinear forms*, Springer-Verlag, New York, 1973, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73, MR 0506372, Zbl 0292.10016.

[51] J.W. Morgan & D.P. Sullivan, *The transversality characteristic class and linking cycles in surgery theory* , Ann. of Math. (2) **99** (1974) 463–544, MR 0350748, Zbl 0295.57008.

[52] D. Mumford, *Theta characteristics of an algebraic curve*, Ann. Sci. École Norm. Sup. (4) **4** (1971) 181–192, MR 0292836, Zbl 0216.05904.

[53] M.S. Narasimhan & S. Ramanan, *Existence of universal connections*, Amer. J. Math. **83** (1961) 563–572, MR 0133772, Zbl 0114.38203.

[54] ———, *Existence of universal connections*, II, Amer. J. Math. **85** (1963) 223–231, MR 0151923, Zbl 0117.39002.

[55] L.S. Pontryagin, *A classification of continuous transformations of a complex into a sphere*, 1, Doklady Akad. Nauk SSSR (N.S.) **19** (1938) 361–363, Zbl 0019.23802.

[56] ———, *A classification of continuous transformations of a complex into a sphere*, 2, Doklady Akad. Nauk SSSR (N.S.) **19** (1938) 147–149, Zbl 0019.08803.

[57] ———, *Homotopy classification of mappings of an $(n+2)$-dimensional sphere on an n-dimensional one*, Doklady Akad. Nauk SSSR (N.S.) **19** (1950) 957–959, Russian, MR 0042121.

[58] ———, *Smooth manifolds and their applications in homotopy theory*, American Mathematical Society Translations, Ser. 2, Vol. 11, American Mathematical Society, Providence, RI, 1959, 1–114, MR 0115178, Zbl 0084.19002.

[59] D. Quillen, *Superconnections and the Chern character*, Topology **24**(**1**) (1985) 89–95, MR 0790678, Zbl 0569.58030.

[60] K. Reidemeister, *Einführung in die kombinatorische Topologie*, Wissenschaftliche Buchgesellschaft, Darmstadt, 1972, Unveränderter reprografischer Nachdruck der Ausgabe Braunschweig 1951, MR 0345088.

[61] E.H. Spanier, *Algebraic topology*, McGraw–Hill, New York, 1966, MR 0210112, Zbl 0145.43303.

[62] ———, *Algebraic topology*, Springer-Verlag, New York, 1981, Corrected reprint of the 1966 original. MR 0666554, Zbl 0477.55001.

[63] R. Thom, *Quelques propriétés globales des variétés differentiables*, Commentarii Mathematici Helvetici **28** (1954) 17–86, MR 0061823, Zbl 0057.15502.

[64] G.W. Whitehead, *The $(n+2)^{\text{nd}}$ homotopy group of the n-sphere*, Ann. of Math. (2) **52** (1950) 245–247, MR 0037507, Zbl 0037.39703.

[65] E. Witten, *Five-brane effective action in M-theory*, J. Geom. Phys. **22**(**2**) (1997) 103–133, MR 1451550, Zbl 0878.58063.

[66] ———, *On flux quantization in M-theory and the effective action*, J. Geom. Phys. **22**(**1**) (1997) 1–13, MR 1441697, Zbl 0908.53065.

[67] ———, *Overview of K-theory applied to strings*, Internat. J. Modern Phys. A **16**(**5**) (2001) 693–706, MR 1827946, Zbl 0980.81049.

[68] Zen-ichi Yosimura, *Universal coefficient sequences for cohomology theories of* cw-*spectra*, Osaka J. Math. **12**(**2**) (1975) 305–323, MR 0388375, Zbl 0309.55008.

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