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DETERMINANTS OF INCIDENCE AND HESSIAN MATRICES ARISING FROM THE VECTOR SPACE LATTICE

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ABSTRACT. Let $\mathcal{V} = \bigsqcup_{i=0}^{n} \mathcal{V}_i$ be the lattice of subspaces of the n-dimensional vector space over the finite field $\mathbb{F}_q$, and let $\mathcal{A}$ be the graded Gorenstein algebra defined over $\mathbb{Q}$ which has $\mathcal{V}$ as a $\mathbb{Q}$ basis. Let F be the Macaulay dual generator for $\mathcal{A}$. We explicitly compute the Hessian determinant $|\partial^2 F/\partial X_i \partial X_j|$, evaluated at the point $X_1 = X_2 = \cdots = X_N = 1$, and relate it to the determinant of the incidence matrix between $\mathcal{V}_1$ and $\mathcal{V}_{n-1}$. Our exploration is motivated by the fact that both of these matrices naturally arise in the study of the Sperner property of the lattice and the Lefschetz property for the graded Artinian Gorenstein algebra associated to it.

1. Introduction. Let P be a poset with a rank function $\rho : P \to \mathbb{N}$. Then, P decomposes into a disjoint union of the level sets, namely, $P = \bigsqcup_{i=0}^{c} P_i$, where $P_i = \{x \in P \mid \rho(x) = i\}$. We say that P has the *Sperner property* if the maximum size of antichains of P is equal to the maximum of the rank numbers $|P_i|$. Some of the basic examples of finite ranked posets known to have the Sperner property are the Boolean lattice, the divisor lattice, and the vector space lattice over a finite field. One way to show that the Sperner property holds for the vector space lattice is as a consequence of the fact that certain incidence matrices have full rank as illustrated in [**5,** Theorem 1.83]. We will say that a ranked poset with a symmetric sequence of rank numbers has the strong Lefschetz property if the incidence matrices

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between every pair of symmetric level sets are invertible. This implies the Sperner property for posets with symmetric sequence of rank numbers by [**5,** Lemmas 1.51, 1.52]. For the vector space lattice, the fact that it has the strong Lefschetz property follows from a result of Kantor [**7**]. There are several other ways to show that the vector space lattice has the Sperner property; the reader may consult [**3**] for details.

It is remarkable that some posets with a rank function can be vector space bases for some graded Artinian algebras over a field in such a way that the multiplication of the algebra is compatible with the incidence structure of the poset. For example, the Boolean lattice $2^{\{x_1,\dots,x_n\}}$ can be the basis for the algebra

$$K[x_1, x_2, \dots, x_n]/(x_1^2, x_2^2, \dots, x_n^2).$$

Recently Maeno and Numata [**9**] succeeded in constructing a family of algebras over a field for which vector space lattices are the bases. To briefly explain their construction, let $\mathbb{F}_q$ be the finite field with q elements, $V = \mathbb{F}_q^n$ the n-dimensional vector space and $\mathcal{V} = \bigsqcup_{i=0}^n \mathcal{V}_i$ the vector space lattice with rank decomposition. Introduce as many variables as the number of the one-dimensional subspaces of V, and then define the form

$$F = \sum x_{i_1} x_{i_2} \cdots x_{i_n},$$

where the indices run over the combinations such that $\operatorname{span}\langle x_{i_1}, x_{i_2}, \dots, x_{i_n}\rangle$ is the whole space V. (A variable like x_i represents a one-dimensional vector subspace of V, and distinct variables represent distinct spaces.) Let $R = K[x_1, \dots, x_N]$ be the polynomial ring in N variables, where N is the number of one-dimensional subspaces of V. (Note that K is any field and should not be confused with $\mathbb{F}_q$.) Set $\mathcal{A} = R/\operatorname{Ann}(F)$. The Artinian algebra $\mathcal{A}$ has the Hilbert function displayed below

$$\left(\begin{bmatrix} n \\ 0 \end{bmatrix}_q, \begin{bmatrix} n \\ 1 \end{bmatrix}_q, \dots, \begin{bmatrix} n \\ n \end{bmatrix}_q \right).$$

An explicit formula for $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_q$ is given at the beginning of Section 2. Every monomial m in $\mathcal{A}$ represents a vector subspace in V of the dimension which is equal to the degree of m.

We are interested in the Hessian determinant $|\partial^2 F/\partial x_i \partial x_j|$ of F, evaluated at $x_1 = \cdots = x_N = 1$. The motivation for it is as follows: it is proven in **[10]** that the non-vanishing of the Hessian, together with the non-vanishing of the higher Hessians of the Macaulay dual generator F, i.e.,

$$\left| \frac{\partial^{2k} F}{\partial x_{i_1} \cdots \partial x_{i_k} \partial x_{j_1} \cdots \partial x_{j_k}} \right|,$$

is equivalent to the strong Lefschetz property for the Gorenstein algebra (Definition 4.4), which ensures the Sperner property of the poset. This suggests that a connection exists between the higher Hessians evaluated at a certain point (x_i) and the determinants of the incidence matrices for the vector space lattice. (Recall that the first Hessian of F is the Hessian in the usual sense.) Our main result is Theorem 4.11, where we make explicit the relation between the Hessian matrix and the incidence matrix of the vector space lattice, and we derive from it a closed formula in Corollary 4.12 for the Hessian of F evaluated at $x_1 = \cdots = x_N = 1$.

In the literature, efforts have been made to obtain the Smith normal form of incidence matrices for various posets (**[12]**). In particular, the Smith normal form for the incidence matrix between the sets $\mathcal{V}_1$ and $\mathcal{V}_{n-1}$ was obtained by Xiang **[15]**. The determinant itself is much easier to obtain; it is sufficient to note that

$$A^{\,T} A = (N - \lambda) I + \lambda J,$$

where I is the $N \times N$ identity matrix and J is the matrix with all 1s as entries. This is due to Xiang **[15,** (1.1)**]**. In this paper, we reproduce a proof for it since this does not seem to be well known among the commutative algebraists (Theorem 3.6 (c)).

Computations similar in spirit have been performed for evaluating the determinants of all incidence matrices of the Boolean lattice in **[4, 11]**, obtaining explicit and recursive formulas, respectively. For a comprehensive survey of determinant evaluations and their many applications, see **[8]**.

Our paper is organized as follows. In Section 2, we gather useful properties of the vector space lattice, focusing on enumerative results.

In Section 3, we carry out our computation of the determinant of the incidence matrix between the first level set and the $(n-1)$st level set. In Section 4, we recall Maeno-Numata's construction of the graded Artinian Gorenstein algebra $\mathcal{A}$ associated with the vector space lattice, as introduced in [**9**]. We explicitly describe the Hessian matrix of the Macaulay dual generator of $\mathcal{A}$, and we compute the Hessian determinant. Furthermore, we show that the same method can be used to obtain the determinant for the multiplication map $\times L : \mathcal{A}_1 \to \mathcal{A}_{n-1}$, where $L := \sum_{j=1}^{N} x_j$, and the matrix is written with respect to the monomial bases.

2. The vector space lattice.

Throughout this paper, let $\mathbb{F}$ be the finite field with q elements, and let n be a positive integer.

Definition 2.1. The *vector space lattice* on $\mathbb{F}^n$, denoted $\mathcal{V}(n,q)$, is the set of all subspaces of $\mathbb{F}^n$ naturally ordered by inclusion. Note that $\mathcal{V}(n,q)$ is a poset with the *rank function* ρ defined by $\rho(W) = \dim_{\mathbb{F}}(W)$, for each $W \in \mathcal{V}(n,q)$. This gives rise to the *rank decomposition* $\mathcal{V}(n,q) = \bigsqcup_{j=0}^{n} \mathcal{V}_j$ into *level sets* $\mathcal{V}_j := \{W \in \mathcal{V}(n,q) \mid \dim_{\mathbb{F}}(W) = j\}$.

Using the notation $[i] = (q^i - 1)/(q-1)$ for the q-integers, we recall the formula for the sizes of the level sets in the vector space lattice (see [**5,** Proposition 1.81]):

$$\mathrm{card}(\mathcal{V}_j) = \begin{bmatrix} n \\ j \end{bmatrix}_q,$$

where

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \begin{cases} \frac{[n][n-1]\cdots[n-j+1]}{[j][j-1]\cdots[1]} & (0 \le j \le n), \\ 0 & (j < 0 \text{ or } j > n). \end{cases}$$

Let $G(n,m)$ denote the Grassmannian variety of m-dimensional subspaces of an n-dimensional vector space. One reason for studying the vector space lattice $\mathcal{V}$ is that each level set $\mathcal{V}_j$ may be regarded as the set of rational points of the Grassmannian variety $G(n,j)$ corresponding to a finite vector space. In our work, we routinely identify the set $\mathcal{V}_j$ as the collection of $n \times j$ matrices in echelon form with entries in $\mathbb{F}$. For example, for $n = 4$ and $j = 2$, the set $\mathcal{V}_j$ is in one-to-one correspondence with the set

$$\left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}, \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}, \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

where each echelon form corresponds to the subspace spanned by the rows of the respective matrix.

For $W \in \mathcal{V}(n, q)$, define the *dual subspace* $W^\perp \in \mathcal{V}(n, q)$ by

$$W^\perp = \left\{ w \in \mathbb{F}^n \mid \sum_{i=1}^{n} v_i w_i = 0 \text{ for all } v \in W \right\}.$$

The map $\mathcal{V}(n, q) \to \mathcal{V}(n, q)$, given by $W \mapsto W^\perp$, is an inclusion-reversing bijection meaning that it satisfies the condition: $U \subseteq W$ if and only if $W^\perp \subseteq U^\perp$.

Focusing on the level sets of elements of rank 1 and $n - 1$, respectively, the formula for the sizes of the level sets gives $\operatorname{card}(\mathcal{V}_1) = \operatorname{card}(\mathcal{V}_{n-1}) = \left[{n \atop 1} \right]_q$. Set $N = \operatorname{card}(\mathcal{V}_1)$, and fix the following notation for elements of the level set $\mathcal{V}_1$:

$$\mathcal{V}_1 = \{v_1, v_2, \ldots, v_N\}.$$

In particular, the set $\mathcal{V}_1$ is in one-to-one correspondence with the rational points of the projective space $\mathbb{P}_{\mathbb{F}}^{n-1}$. Thus, it will be convenient to regard $\mathcal{V}_1$ as the set of vectors $(a_1, \ldots, a_n)$ such that the first nonzero component is 1. These vectors are a special case of the echelon matrices described above. Since $\mathbb{P}_{\mathbb{F}}^{n-1} = \mathbb{P}_{\mathbb{F}}^{n-2} \sqcup \mathbb{A}_{\mathbb{F}}^{n-1}$, we have the identity $N = \left[{n \atop 1} \right]_q = \left[{n-1 \atop 1} \right]_q + q^{n-1}$.

We denote by $v_k^\perp$ the dual space of $\mathbb{F}v_k$, which allows us to identify the $(n-1)$st level set of the vector space lattice with the set of duals of elements of the first level set as follows:

$$\mathcal{V}_{n-1} = \{v_1^\perp, v_2^\perp, \ldots, v_N^\perp\}.$$

The following definition introduces the focal point of our attention in this work.

Definition 2.2. The *incidence matrix* $A = (a_{ij})$ for $\mathcal{V}_1$ and $\mathcal{V}_{n-1}$ is the $N \times N$ matrix, whose entries are

$$a_{ij} = \begin{cases} 1 & (v_i \in v_j^{\perp})^1 \\ 0 & (v_i \notin v_j^{\perp}). \end{cases}$$

The first goal of this note is to find a closed formula for the determinant of the incidence matrix A. While our vector space lattice is defined over a field of positive characteristic, all of our determinant computations will be performed in characteristic zero. This is to preserve the enumerative properties of the entries in our matrices. Note that the truly meaningful invariant of the incidence structure between $\mathcal{V}_1$ and $\mathcal{V}_{n-1}$ is, in fact, the absolute value of this determinant, denoted $|\det A|$, since this is preserved under permuting the order of the elements in $\mathcal{V}_1$ and $\mathcal{V}_{n-1}$.

We begin by describing the incidence matrix in a concrete example.

Example 2.3. Let $q = 2$ and $n = 3$. In this case, we have $N = 7$. Then, $\mathcal{V}_1 = \{v_1, v_2, \dots, v_7\}$, in which

$$v_1 = (0,0,1) \qquad v_2 = (0,1,0) \qquad v_3 = (0,1,1) \qquad v_4 = (1,0,0)$$
$$v_5 = (1,0,1) \qquad v_6 = (1,1,0) \qquad v_7 = (1,1,1).$$

Now, we have $\mathcal{V}_2 = \mathcal{V}_1^{\perp} = \{u_1, u_2, \dots, u_7\}$, where

$$u_1 := v_1^{\perp} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad u_2 := v_2^{\perp} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$u_3 := v_3^{\perp} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad u_4 := v_4^{\perp} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$u_5 := v_5^{\perp} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad u_6 := v_6^{\perp} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$u_7 := v_7^{\perp} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Therefore, we can compute the incidence matrix A as displayed below, which gives $\det(A) = -3 \cdot 2^3$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

For later use in our computations, we record a few enumerative invariants of the lattice $\mathcal{V}(n,q)$. We employ the notation card for the cardinality of a finite set.

Proposition 2.4. *The following enumerative identities hold*:

(a) $\mathrm{card}(\mathrm{GL}(n,\mathbb{F})) = (q^n - 1)(q^n - q^1)(q^n - q^2) \cdots (q^n - q^{n-1})$.

(b) *The number of ordered n-tuple subsets of $\mathcal{V}_1$ which form bases for $\mathbb{F}^n$ is*

$$t_{n,q} = \frac{\mathrm{card}(\mathrm{GL}(n,\mathbb{F}))}{(q-1)^n} = (q^{n(n-1)/2})\left(\prod_{k=1}^{n} \begin{bmatrix} k \\ 1 \end{bmatrix}_q\right).$$

(c) *The number of n-tuple subsets of $\mathcal{V}_1$ which form bases for $\mathbb{F}^n$ is*

$$s_{n,q} = \frac{\mathrm{card}(\mathrm{GL}(n,\mathbb{F}))}{n!(q-1)^n} = \left(\frac{q^{n(n-1)/2}}{n!}\right)\left(\prod_{k=1}^{n} \begin{bmatrix} k \\ 1 \end{bmatrix}_q\right).$$

(d) *The number of ordered n-tuple subsets of $\mathcal{V}_1$ which form bases for $\mathbb{F}^n$ and contain a fixed linearly independent ordered subset of size j is*

$$t_{n,j,q} = (q^{(n(n-1)-j(j-1))/2})\left(\prod_{k=1}^{n-j} \begin{bmatrix} k \\ 1 \end{bmatrix}_q\right).$$

(e) *The number of n-tuple subsets of $\mathcal{V}_1$ which form bases for $\mathbb{F}^n$ and contain a fixed linearly independent subset of size j is*

$$s_{n,j,q} = \left(\frac{q^{(n(n-1)-j(j-1))/2}}{(n-j)!}\right)\left(\prod_{k=1}^{n-j} \begin{bmatrix} k \\ 1 \end{bmatrix}_q\right).$$

(f) *The number of paths in $\mathcal{V}(n,q)$ from the minimum element to the maximum element in the vector space lattice of $\mathbb{F}^n$ is equal to*

$$p_{n,q} = \prod_{k=1}^{n} \begin{bmatrix} k \\ 1 \end{bmatrix}_q .$$

Proof.

(a) Any nonzero vector can be the first row of an $n \times n$ invertible matrix. If the first k rows $u_1, \dots, u_k \in \mathbb{F}^n$ of an invertible matrix are chosen, then any vector in $\mathbb{F}^n \setminus \sum_{i=1}^{k} \mathbb{F}u_i$ can be the $(k+1)$st row for such a matrix. This inductively proves the formula for the number of elements in $\mathrm{GL}(n, \mathbb{F})$.

(b) We regard such an ordered n-tuple of vectors as a matrix $U \in \mathrm{GL}(n, \mathbb{F})$, and we let u_i be the ith row. Then, for each integer i, we may find a unique vector $v_{k_i} \in \mathcal{V}_1$ such that $\mathbb{F}v_{k_i} = \mathbb{F}u_i$. The correspondence $U \mapsto (v_{k_1}, \dots, v_{k_n})$ is $(q-1)^n : 1$, where, by $(v_{k_1}, \dots, v_{k_n})$, we mean the ordered n-tuple. This proves that the number of ordered n-tuple subsets of $\mathcal{V}_1$ which form bases for $\mathbb{F}^n$ is equal to

$$\frac{(q^n-1)(q^n-q^1)(q^n-q^2)\dots(q^n-q^{n-1})}{(q-1)^n}.$$

Noting that

$$\frac{q^n-q^k}{q-1} = q^k \frac{q^{n-k}-1}{q-1} = q^k \begin{bmatrix} n-k \\ 1 \end{bmatrix}_q,$$

we may rewrite the above expression as the claimed formula.

(c) This is easily deduced by observing that the correspondence between the ordered tuples of part (b) and the unordered ones is $n! : 1$.

(d) We regard such an ordered n-tuple of vectors as a matrix $U \in \mathrm{GL}(n, \mathbb{F})$, where the first j rows are fixed. Similar reasoning as in part (b) yields the following count

$$\frac{(q^n - q^j)(q^n - q^{j+1}) \cdots (q^n - q^{n-1})}{(q-1)^{n-j}}$$
$$= (q^{(n(n-1)-j(j-1))/2})\Bigg(\prod_{k=1}^{n-j} \begin{bmatrix} k \\ 1 \end{bmatrix}_q\Bigg).$$

(e) The statement follows from (d) because the correspondence between the ordered tuples of part (d) and the unordered ones is $(n-j)! : 1$.

(f) A path from the minimum element to the maximum element in the lattice $\mathcal{V}(n,q)$ is a chain of vector subspaces in $\mathbb{F}^n$

$$W_0 = \mathbb{F}^0 \subset W_1 \subset W_2 \subset \cdots \subset W_n = \mathbb{F}^n,$$

with $W_k \in \mathcal{V}_k$. Let $W \in \mathcal{V}_k$. The number of $(k+1)$-dimensional subspaces in $\mathbb{F}^n$ which contains W is $\left[{n-k \atop 1}\right]_q$, since this number is the same as the number of linearly independent vectors in $\mathbb{F}^n/W$, which is $(n-k)$-dimensional. Hence, the assertion follows. □

3. The determinant of the incidence matrix between $\mathcal{V}_1$ and $\mathcal{V}_{n-1}$. We use the notation fixed in Section 2. A recurring theme in our work will be the occurrence of matrices of a special form, for which determinants are relatively easily computed. We find it useful to introduce a uniform notation for these matrices.

Notation 3.1. Let $\Phi(\nu, \alpha, \beta)$ denote the matrix of size $\nu \times \nu$ with entries

$$\phi_{ij} = \begin{cases} \alpha & (i = j), \\ \beta & (i \neq j). \end{cases}$$

Lemma 3.2.

(a) *The determinant of* $\Phi(\nu, \alpha, \beta)$ *is given by*

$$\det \Phi(\nu, \alpha, \beta) = (\alpha - \beta)^{\nu-1}(\nu\beta + \alpha - \beta).$$

(b) *If* $\alpha - \beta = \alpha' - \beta'$, *then*

$$\frac{\det \Phi(\nu, \alpha, \beta)}{\det \Phi(\nu, \alpha', \beta')} = \frac{\nu\beta + \alpha - \beta}{\nu\beta' + \alpha' - \beta'}.$$

Proof. Part (a) follows after performing convenient row and column operations on $\Phi(\nu, \alpha, \beta)$ to transform the matrix to an almost diagonal form. Part (b) then follows from (a). □

Definition 3.3. In addition to the incidence matrix A of Definition 2.2, we consider the $N \times N$ matrix $B = (b_{ij})$ whose entries are

$$b_{ij} = 1 - a_{ij} = \begin{cases} 0 & (v_i \in v_j^{\perp}) \\ 1 & (v_i \notin v_j^{\perp}). \end{cases}$$

As it will turn out, the determinant of B is easier to compute than that of A, and we use the relation between A and B to complete our computation. Furthermore, both of these matrices carry deeper algebraic meaning, as shall be seen in Section 3.

We begin with a few structural observations regarding the matrices A and B.

Lemma 3.4.

(a) *Matrices A and B are symmetric*;
(b) $\sum_{i=1}^{N} a_{ij} = \sum_{j=1}^{N} a_{ij} = \left[{n-1 \atop 1} \right]_q$;
(c) $\sum_{i=1}^{N} b_{ij} = \sum_{j=1}^{N} b_{ij} = \left[{n \atop 1} \right]_q - \left[{n-1 \atop 1} \right]_q = q^{n-1}$.

Proof. Note that $v_i \in v_j^{\perp}$ if and only if $v_j \in v_i^{\perp}$, which follows from the inclusion-reversing property of dual spaces. This implies part (a). The row sum of A is equal to the number of codimension 1 subspaces in $\mathbb{F}^{n-1}$ which contain v_1, and this is equal to the number of the one-dimensional subspaces in $v_1^{\perp} \cong \mathbb{F}^{n-1}$. Hence, part (b) follows. Finally, since $N = \left[{n \atop 1} \right]_q$, part (c) follows as a consequence of the relations $b_{ij} = 1 - a_{ij}$. □

The following result shows the role played by the matrices $\Phi(\nu, \alpha, \beta)$ in relation to A and B.

Lemma 3.5. *The following hold*:

(a) $A^2 = \Phi\left(N, \left[\begin{smallmatrix} n-1 \\ n-2 \end{smallmatrix}\right]_q, \left[\begin{smallmatrix} n-2 \\ n-3 \end{smallmatrix}\right]_q\right)$;

(b) $B^2 = \Phi(N, q^{n-1}, q^{n-2}(q-1))$;

(c) $AB = \Phi(N, 0, q^{n-2})$.

Proof.

(a) The (i,j)th entry of A^2 is $\sum_{k=1}^N a_{ik}a_{kj}$. Note that

$$a_{ik}a_{kj} = \begin{cases} 1 & (v_i,\ v_j \in v_k^{\perp}) \\ 0 & (\text{otherwise}). \end{cases}$$

Hence, if $i = j$, the sum $\sum_{k=1}^N a_{ik}a_{kj}$ is equal to the number of codimension 1 subspaces in $\mathbb{F}^n$ which contain v_1, and this number is $\left[\begin{smallmatrix} n-1 \\ n-2 \end{smallmatrix}\right]_q$ since these subspaces are in bijection with codimension one subspaces of $v_1^{\perp} \simeq \mathbb{F}^{n-1}$. If $i \neq j$, the sum $\sum_{k=1}^N a_{ik}a_{kj}$ is equal to the number of codimension 1 subspaces in $\mathbb{F}^n$ which contain both v_1 and v_2. This number is $\left[\begin{smallmatrix} n-2 \\ n-3 \end{smallmatrix}\right]_q$ due to the fact that the codimension 1 subspaces in $\mathbb{F}^n$ which contain both v_1 and v_2 are in bijection with codimension 1 subspaces in $\{v_1, v_2\}^{\perp} \simeq \mathbb{F}^{n-2}$. This proves the assertion for A^2.

(b) The (i,j)th entry of B^2 is $\sum_{k=1}^N b_{ik}b_{kj}$. For the diagonal entry of B^2, we must count the number of the codimension 1 subspaces of $\mathbb{F}^n$ which do not contain v_1. This number is q^{n-1} since we have $\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right]_q - \left[\begin{smallmatrix} n-1 \\ 1 \end{smallmatrix}\right]_q = q^{n-1}$. To compute the off-diagonal entry of B^2, we use the inclusion-exclusion formula, since we must count the number of the subspaces of $\mathbb{F}^n$ of codimension 1 which contain neither v_1 nor v_2. The number of the subspaces in $\mathbb{F}^n$ of codimension 1 is $\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right]_q$, and the number of the subspaces of codimension 1 which contain v_1 is $\left[\begin{smallmatrix} n-1 \\ 1 \end{smallmatrix}\right]_q$, and the same is true for v_2. The number of the subspaces of codimension 1 which contain both v_1 and v_2 is $\left[\begin{smallmatrix} n-2 \\ 1 \end{smallmatrix}\right]_q$. Hence,

$$\sum_{k=1}^N b_{ik}b_{kj} = \begin{bmatrix} n \\ 1 \end{bmatrix}_q - 2\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q = q^2(q-1).$$

(c) By the definition, $A+B=\Phi(N,1,1)$, which is the $N\times N$ matrix with 1 for all entries. Hence, $(A+B)B$ is the matrix which has the row sum of B for all entries. By Lemma 3.4, this row sum is q^{n-1}. Thus, the diagonal entries of AB are 0, and the off-diagonal entries are equal to $q^{n-1}-q^{n-2}(q-1)=q^{n-2}$. □

At this point, part (a) of Lemma 3.5, together with the formula in Lemma 3.2, would allow us to complete the computation of $|\det(A)|$. It turns out, however, that it is easier to find $|\det(B)|$ first and utilize the relationship between the two determinants than to simplify the expression resulting from a direct approach. The following is the main result of this section.

Theorem 3.6. *For the matrices A and $B=\Phi(N,1,1)-A$, we have*

(a) $\det(B^2)=q^{(n-2)N+n}$;
(b) $|\det B|=q^{((n-2)N+n)/2}$;
(c) $|\det A|=(q^{(n-2)(N-1)/2})\left[\begin{smallmatrix}n-1\\1\end{smallmatrix}\right]_q$.

Proof. Lemmas 3.2 (a) and 3.5 (b) imply that $\det(B^2)=q^{(n-2)N}(N(q-1)+1)$. Now, part (a) follows from the formula $N=(q^n-1)/(q-1)$, and part (b) immediately follows from (a).

Recall from Lemma 3.5 the identities $A^2=\Phi\left(N,\left[\begin{smallmatrix}n-1\\n-2\end{smallmatrix}\right]_q,\left[\begin{smallmatrix}n-2\\n-3\end{smallmatrix}\right]_q\right)$ and $B^2=\Phi(N,q^{n-1},q^{n-2}(q-1))$. Since $N=\left[\begin{smallmatrix}n\\1\end{smallmatrix}\right]_q$, $\left[\begin{smallmatrix}n-1\\n-2\end{smallmatrix}\right]_q-\left[\begin{smallmatrix}n-2\\n-3\end{smallmatrix}\right]_q=\left[\begin{smallmatrix}n-1\\1\end{smallmatrix}\right]_q-\left[\begin{smallmatrix}n-2\\1\end{smallmatrix}\right]_q=q^{n-2}$ and $q^{n-1}-q^{n-2}(q-1)=q^{n-2}$, it follows from Lemma 3.2 (b) that

$$\det(A^2):\det(B^2)=\left(N\cdot\frac{q^{n-2}-1}{q-1}+q^{n-2}\right):\left(Nq^{n-2}(q-1)+q^{n-2}\right)$$
$$=\left(\frac{(q^n-1)(q^{n-2}-1)}{(q-1)^2}+q^{n-2}\right):\left((q^n-1)q^{n-2}+q^{n-2}\right)$$
$$=\frac{(q^n-1)^2}{(q-1)^2}:q^{2n-2}=\left(\left[\begin{smallmatrix}n-1\\1\end{smallmatrix}\right]_q\right)^2:(q^{n-1})^2.$$

Taking the square root gives $|\det A|:|\det B|=\left[\begin{smallmatrix}n-1\\1\end{smallmatrix}\right]_q:q^{n-1}$. Together with part (b), this implies part (c) of the theorem. □

Remark 3.7. We can also compute the determinant for A using the description of AB. From Lemmas 3.2 and 3.5, noting that $N-1=\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right]_q - 1 = q\left[\begin{smallmatrix} n-1 \\ 1 \end{smallmatrix}\right]_q = |\det \Phi(N,0,1)|$, we get $|\det(AB)| = (N-1)q^{N(n-2)}$, whence $|\det A| = (N-1)q^{(N(n-2)-n)/2}$. This description for $|\det A|$ is slightly different from Theorem 3.6. Of course, they are, in fact, the same since we have

$$N-1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q - 1 = q\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q .$$

Remark 3.8. The result in Theorem 3.6 (c) recovers the non-vanishing of one of the determinants involved in the definition of the strong Lefschetz property for ranked posets, given in the introduction. We recall that the vector space lattice has been proved to have the strong Lefschetz property in **[7]**.

Example 3.9. The determinant computations below were directly obtained using `Mathematica` **[14]**, independent of Theorem 3.6 for $q = 2, 3, 5$.

$q = 2$:

n	3	4	5	6	7	8
N	7	15	31	63	127	255
$\det A$	$2^3 \cdot 3$	$2^{14} \cdot 7$	$2^{45} \cdot 15$	$2^{124} \cdot 31$	$2^{315} \cdot 63$	$2^{762} \cdot 127$
$\det B$	$2^3 \cdot 2^2$	$2^{14} \cdot 2^3$	$2^{45} \cdot 2^4$	$2^{124} \cdot 2^5$	$2^{315} \cdot 2^6$	$2^{762} \cdot 2^7$

$q = 3$:

n	3	4	5	6
N	13	40	121	364
$\det A$	$3^6 \cdot 2^2$	$3^{39} \cdot 13$	$3^{180} \cdot 2^3 \cdot 5$	$3^{726} \cdot 11^2$
$\det B$	$3^6 \cdot 3^2$	$2^{39} \cdot 3^3$	$3^{180} \cdot 3^4$	$3^{726} \cdot 3^5$

$q = 5$:

n	3	4
N	31	156
$\det A$	$5^{15} \cdot 2 \cdot 3$	$5^{155} \cdot 31$
$\det B$	$5^{15} \cdot 5^2$	$5^{155} \cdot 5^3$

4. The Hessian of the Macaulay dual generator for the Gorenstein algebra associated to the vector space lattice. In this section, we relate the combinatorial data of Section 2 to algebraic invariants arising from a graded ring associated to the vector space lattice.

Recall that $N = \left[{n \atop 1} \right]_q$. Consider the polynomial rings $R = K[X_1, \ldots, X_N]$ and $Q = K[x_1, \ldots, x_N]$, where K is a field of characteristic zero. Setting $x_i = \partial/\partial X_i$ allows us to view R as a Q-module via the partial differentiation action of Q on R, given by $x_i \circ f = \partial f/\partial X_i$, for $f \in R$.

A bijection can be established between the set of variables in R and the set $\mathbb{P}_{\mathbb{F}}^{n-1}$ of vectors of length n with entries in the field $\mathbb{F}$ in which the first non-zero entry is 1. We fix this bijection once and for all, so that the variable X_i corresponds to the vector $v_i \in \mathbb{P}_{\mathbb{F}}^{n-1}$.

We now outline the construction given in **[9]** of a graded Artinian Gorenstein algebra associated to the vector space lattice. This uses the theory of Macaulay inverse systems, which provides a correspondence between homogeneous polynomials in the ring R and graded Artinian Gorenstein quotient algebras of Q. For more details on Macaulay inverse systems, the reader may consult **[1, 6]**.

Definition 4.1. For a homogeneous polynomial $F \in R$, the annihilator of F in Q is the ideal $I \subset Q$, defined by

$$\operatorname{Ann}_Q(F) := \{f \in Q \mid f \circ F = 0\}.$$

If I is an ideal of Q the following set is the annihilator of I in R:

$$\operatorname{Ann}_R(I) := \{F \in R \mid f \circ F = 0, \text{ for all } f \in I\}.$$

Let $I \subset Q$ be a homogeneous ideal of finite colength. It is well known that, if Q/I is Gorenstein, then there exists a homogeneous form $F \in R$ such that $I = \operatorname{Ann}_Q(F)$. On the other hand, if $F \in R$ is homogeneous, then $I = \operatorname{Ann}_Q(F)$ is a homogeneous ideal and $Q/\operatorname{Ann}_Q(F)$ is an Artinian Gorenstein algebra.

The idea of constructing a Gorenstein algebra associated to the vector space lattice is that its combinatorial structure can be encoded in a homogeneous polynomial of R and then the graded Gorenstein quotient of Q corresponding to it can be considered.

Definition 4.2. We define the *Macaulay dual generator* for the vector space lattice to be the following degree n homogeneous polynomial in R:

$$F_{\mathcal{V}(n,q)} = \sum_{X_{i_1}X_{i_2}\cdots X_{i_n}\in\mathcal{B}} X_{i_1}X_{i_2}\cdots X_{i_n}.$$

In the sum, the sets of indices of the variables appearing in each monomial represent the subsets of $\mathcal{V}_1$ that form bases for $\mathbb{F}^n$, namely:

$$\mathcal{B} = \{X_{i_1}X_{i_2}\cdots X_{i_n} \mid 1 \le i_1 < i_2 < \cdots i_n \le N \text{ and } \det[v_{i_1}v_{i_2}\cdots v_{i_n}] \ne 0 \text{ in } \mathbb{F}\}.$$

The cardinality of the set $\mathcal{B}$ above is according to Proposition 2.4:

$$\operatorname{card}(\mathcal{B}) = s_{n,q} = \left(\frac{q^{n(n-1)/2}}{n!}\right)\left(\prod_{k=1}^{n} \begin{bmatrix} k \\ 1 \end{bmatrix}_q\right).$$

Definition 4.3. Setting $I = \operatorname{Ann}_Q(F_{\mathcal{V}(n,q)})$ yields a graded Artinian Gorenstein quotient ring $\mathcal{A}_{\mathcal{V}(n,q)} = Q/I$, which we call the *Gorenstein algebra associated to the vector space lattice.* For simplicity, we write $\mathcal{A}$ for $\mathcal{A}_{\mathcal{V}(n,q)}$ henceforth, unless otherwise specified.

This graded ring decomposes into homogeneous components as follows:

$$\mathcal{A} = Q/I = \bigoplus_{i=0}^{n}(Q/I)_i = \bigoplus_{i=0}^{n}\mathcal{A}_i.$$

We note the similarity between the homogeneous decomposition of $\mathcal{A}$ and the rank decomposition of $\mathcal{V}(n,q)$. It is shown in [**5**, Lemma 1.48, Proof of Theorem 1.83, step 4] and [**9**, Lemma 4.1, Theorem 4.2] that the non-zero monomials in $\mathcal{A}$ are in bijective correspondence with the elements of $\mathcal{V}(n,q)$ in such a way that the level set $\mathcal{V}_i$ corresponds to the monomials in the graded component $\mathcal{A}_i$. In particular, we have the following correspondences:

$$\mathcal{A}_0 \ni 1 \longleftrightarrow \mathbb{F}^0 \in \mathcal{V}_0$$
$$\mathcal{A}_n \ni g \longleftrightarrow \mathbb{F}^n \in \mathcal{V}_n,$$

where g is a monomial of degree n, called a socle generator for $\mathcal{A}$. The socle of $\mathcal{A}$ is a one-dimensional vector space; thus, in $\mathcal{A}$, g is unique up to scalar. However, any product of variables of Q whose indices correspond to a basis of V can be chosen to be a representative for g.

Next, we recall the algebraic counterpart of the Lefschetz properties, defined for ranked posets in the introduction, with the end goal of explicitly relating the incidence matrices of Section 2 with certain matrices arising from the Macaulay dual generator in Definition 4.2.

Consider, for some scalar values $a_1, \dots, a_N \in K$, the linear form

$$L = a_1x_1 + \cdots + a_Nx_N \in Q,$$

and let $0 \le j \le \lfloor n/2 \rfloor$. We set $\times L^{n-2j} : \mathcal{A} \to \mathcal{A}$ to be the Q-module homomorphism given by $x \mapsto L^{n-2j}x$. Restricting to the degree j and $n-j$ homogeneous components of $\mathcal{A}$, we obtain the K-linear maps

$$\times L^{n-2j} : \mathcal{A}_j \longrightarrow \mathcal{A}_{n-j}.$$

The motivation for considering such a map originally arises from the study of cohomology rings of compact Kähler manifolds, where we can regard such a map as taking a class in cohomology and intersecting it with hyperplanes (represented by L) $n-2j$ times.

Fixing the sets of monomials corresponding to elements of $\mathcal{V}_j$ and $\mathcal{V}_{n-j}$, respectively, as bases for $\mathcal{A}_j$ and $\mathcal{A}_{n-j}$, we can express the linear transformations $\times L^{n-2}$ as matrices M_j. Note that $\dim_K \mathcal{A}_j = \dim_K \mathcal{A}_{n-j}$, since the bases for these vector spaces correspond to symmetric level sets $\mathcal{V}_j$ and $\mathcal{V}_{n-j}$ of $\mathcal{V}(n,q)$ which have the same size. Thus, it is logical to consider $\det M_j$.

Definition 4.4. Let $\mathcal{A}$ be any graded Gorenstein Artinian algebra. If there exist scalars $a_1, \dots, a_N \in K$ such that the matrices M_j representing the K-linear maps $\times L^{n-2j} : \mathcal{A}_j \to \mathcal{A}_{n-j}$ for $L = a_1x_1 + \cdots + a_Nx_N$ have $\det M_j \neq 0$ for all $0 \le j \le \lfloor n/2 \rfloor$, the algebra $\mathcal{A}$ is said to have the *strong Lefschetz property.*

We turn to our case of interest, $\mathcal{A} = \mathcal{A}_{\mathcal{V}(n,q)}$, and focus on a particular choice of linear form, $\ell = x_1 + x_2 + \cdots + x_N$. We shall be particularly concerned with computing the determinant of the matrix that represents the map $\times\ell^{n-2}$. Setting $x_i^\perp = \prod_{v_j \in v_i^\perp} x_j$, consider the bases $\mathcal{B}_1 = \{x_1, \ldots, x_N\}$ for $\mathcal{A}_1$ and $\mathcal{B}_{n-1} = \{x_1^\perp, \ldots, x_N^\perp\}$ for $\mathcal{A}_{n-1}$, which we shall call canonical bases, and let M be the matrix that represents the linear transformation ℓ^{n-2} with respect to these fixed bases.

Example 4.5. Let $q = 2$ and $n = 3$, which yield $N = 7$. We use the notation of Example 2.3. A computation with `Macaulay2` [**2**] yields that the matrix representing $\times\ell : \mathcal{A}_1 \to \mathcal{A}_2$ with respect to the bases

$$\begin{aligned}\mathcal{B}_1 &= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}\\ \mathcal{B}_2 &= \{x_2x_4, x_1x_4, x_3x_4, x_1x_2, x_2x_5, x_1x_6, x_3x_5\}\end{aligned}$$

is the matrix M below, related to the incidence matrix A computed in Example 2.3, as follows:

$$M = \begin{pmatrix} 0 & 2 & 0 & 2 & 0 & 2 & 0\\ 2 & 0 & 0 & 2 & 2 & 0 & 0\\ 0 & 0 & 2 & 2 & 0 & 0 & 2\\ 2 & 2 & 2 & 0 & 0 & 0 & 0\\ 0 & 2 & 0 & 0 & 2 & 0 & 2\\ 2 & 0 & 0 & 0 & 0 & 2 & 2\\ 0 & 0 & 2 & 0 & 2 & 2 & 0 \end{pmatrix} = 2A.$$

The next theorem describes the precise relation between the incidence matrix A of Definition 2.2 and the matrix describing multiplication by ℓ^{n-2}.

Theorem 4.6. *The matrix M representing $\times\ell^{n-2}$, with respect to the standard bases for $\mathcal{A}_1$ and $\mathcal{A}_{n-1}$, is $M = t_{n-1,1,q}A$. Hence, $|\det M| = t_{n-1,1,q}^N |\det A|$.*

Proof. To find the entry of M in the position indexed by the variable $x_i \in \mathcal{A}_1$ corresponding to v_i and the basis element $x_j^\perp = \prod_{v_k \in v_j^\perp} x_k \in \mathcal{A}_{n-1}$ corresponding to the element $v_j^\perp \in \nu_{n-1}$, we need to count the number of monomials $x_{k_1}x_{k_2}\cdots x_{k_{n-1}}$ in the expansion of ℓ^{n-1} in the polynomial ring, which satisfy the following conditions:

(a) one of $x_{k_1}, x_{k_2}, \ldots, x_{k_{n-1}}$ is x_i;
(b) $\mathrm{Span}\langle v_{k_1}, v_{k_2}, v_{k_3} \ldots, v_{k_{n-1}}\rangle = v_j^{\perp}$.

If $v_i \notin v_j^{\perp}$, then, clearly, this number is zero. If $v_i \in v_j^{\perp}$, then we need to count the number of ordered $(n-1)$-tuples which form bases for $v_j^{\perp}$ and contain v_1. By Proposition 2.4, this number is

$$t_{n-1,1,q} = \left(q^{(n-1)(n-2)/2}\right)\left(\prod_{k=1}^{n-2}\begin{bmatrix}k\\1\end{bmatrix}_q\right).$$

Hence, it follows from Definition 2.2 that the matrix for $\times\ell^{n-2}$ is $t_{n-1,1,q}A$. ☐

It is shown in **[10]** that there is a close connection between the matrices representing $\times L^{n-2j}$ for $L = a_1x_1 + \cdots + a_Nx_N$ and the determinants of higher analogues of the classical Hessian matrix of the Macaulay dual generator $F_{\mathcal{V}(q,n)}$, evaluated at $X_1 = a_1, \ldots, X_N = a_N$. For our purposes, it suffices to consider the classical Hessian, as this corresponds to $\times L^{n-2}$, which we have been able to relate to the incidence matrix in Theorem 4.6.

Definition 4.7. The *Hessian matrix* of a polynomial $F \in R = K[X_1, \ldots, X_N]$ is the matrix of partial derivatives

$$H(F) = \left(\frac{\partial^2 F}{\partial X_i \partial X_j}\right)_{1\le i\le N, 1\le j\le N}.$$

We begin by describing the Hessian matrix in our running example.

Example 4.8. Let $q = 2$ and $n = 3$. We use the notation of Examples 2.3 and 4.5. The Macaulay dual generator, as introduced in Definition 4.2, is:

$$\begin{aligned} F_{\mathcal{V}(3,2)} = {} & X_1X_2X_4 + X_1X_3X_4 + X_2X_3X_4 + X_1X_2X_5 + X_1X_3X_5 \\ & + X_2X_3X_5 + X_2X_4X_5 + X_3X_4X_5 + X_1X_2X_6 + X_1X_3X_6 \\ & + X_2X_3X_6 + X_1X_4X_6 + X_3X_4X_6 + X_1X_5X_6 + X_2X_5X_6 \\ & + X_4X_5X_6 + X_1X_2X_7 + X_1X_3X_7 + X_2X_3X_7 + X_1X_4X_7 \\ & + X_2X_4X_7 + X_1X_5X_7 + X_3X_5X_7 + X_4X_5X_7 + X_2X_6X_7 \\ & + X_3X_6X_7 + X_4X_6X_7 + X_5X_6X_7. \end{aligned}$$

Note that this polynomial has $s_{3,2} = 28$ terms, in accordance to the formula in Proposition 2.4. A computation with `Macaulay2` [**2**] yields that, after evaluating at $X_1 = \cdots = X_7 = 1$, the Hessian matrix is:

$$H(F_{\mathcal{V}(3,2)})|_{X_1=X_2=\cdots=X_7=1} = \begin{pmatrix} 0 & 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 0 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 0 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 0 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 0 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 0 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 0 \end{pmatrix} = \Phi(7,0,4).$$

In the following, we aim to understand this especially nice form of the Hessian matrix by describing the relation between the Hessian of the Macaulay dual generator of $\mathcal{A}$ and the matrices introduced in Section 2.

Lemma 4.9. *Let $F \in R = K[X_1, \ldots, X_N]$ be a homogeneous polynomial of degree n, let $a_1, \ldots, a_N \in K$, and consider the linear form*

$$L = a_1 \frac{\partial}{\partial X_1} + \cdots + a_N \frac{\partial}{\partial X_N} \in Q.$$

Then, there is a commutative diagram

$$\begin{tikzcd} \mathcal{A}_1 \otimes_K \mathcal{A}_1 \arrow[r, "\mathbf{1}_{\mathcal{A}_1} \otimes_K (\times L^{n-2})"] \arrow[rrr, bend right, "(n-2)! H(F)|_{X_1=a_1,\ldots,X_N=a_N}"'] & \mathcal{A}_1 \otimes_K \mathcal{A}_{n-1} \arrow[r, "\mu"] & \mathcal{A}_n \arrow[r, "\circ F"] & K \end{tikzcd}$$

where

(a) μ *denotes the internal multiplication on* $\mathcal{A}$;

(b) *the map* $\circ F$ *maps* $f \in \mathcal{A}_n \mapsto f \circ F \in K$; *and*

(c) $H(F)|_{X_1=a_1,\ldots,X_N=a_N}$ *denotes the* K*-bilinear form* $\mathcal{A}_1 \otimes_K \mathcal{A}_1 \to K$, *represented with respect to the basis* $\{\partial/\partial X_1, \partial/\partial X_2, \ldots, \partial/\partial X_N\}$ *of* $\mathcal{A}_1$ *by the matrix in Definition* 4.7, *evaluated at* $X_1 = a_1, \ldots, X_N = a_N$.

Proof. From the proof of [**5,** Theorem 3.76], [**10,** Theorem 3.1] or [**13,** Theorem 4], we have the following identity:

$$L^{n-2}\frac{\partial}{\partial X_i}\frac{\partial}{\partial X_j}F(X) = (n-2)!\frac{\partial}{\partial X_i}\frac{\partial}{\partial X_j}F(X)|_{X_1=a_1,\ldots,X_N=a_N}.$$

The left side of the above expression can be viewed as the composition of the three maps in the top line of the diagram, applied to the element $\partial/\partial X_i \otimes \partial/\partial X_j \in \mathcal{A}_1 \otimes_K \mathcal{A}_1$. The right side of the displayed equality is the bottom map in the diagram evaluated at the same element. The commutative diagram represents this equality in visual form. $\square$

To exploit the relations illustrated in the above diagram, we prove the following.

Proposition 4.10. *The matrix describing the natural* (*bilinear*) *multiplication map*

$$\mathcal{A}_1 \otimes_K \mathcal{A}_{n-1} \xrightarrow{\mu} \mathcal{A}_n$$

with respect to the canonical bases of $\mathcal{A}_1$, $\mathcal{A}_{n-1}$ *and* $\mathcal{A}_n$, *respectively, is the matrix* B *introduced in Definition* 3.3.

Proof. Since the squares of variables are in the ideal I, by [**9,** Proposition 3.1], we have that the action of μ on the pairs of basis elements is the following:

$$\mu(x_i, x_j^{\perp}) = \begin{cases} 0 & (x_i \mid x_j^{\perp}, \text{ equivalently } v_i \in v_j^{\perp}) \\ g & (x_i \nmid x_j^{\perp}, \text{ equivalently } v_i \notin v_j^{\perp}). \end{cases}$$

Clearly, then, μ is represented as a bilinear form by B with respect to the bases $\mathcal{B}_1$ and $\mathcal{B}_{n-1}$ of $\mathcal{A}_1$ and $\mathcal{A}_{n-1}$, and the basis $\{g\}$ for $\mathcal{A}_n$, where g is a monomial generator of $\mathcal{A}_n$. $\square$

We are now ready to see how the Hessian relates to matrices A and B.

Theorem 4.11. *The Hessian matrix of $F_{\mathcal{V}(q,n)}$, evaluated at $X_1 = \cdots = X_n = 1$, is*

$$H(F_{\mathcal{V}(q,n)})|_{X_1=\cdots=X_n=1} = \frac{t_{n-1,1,q}}{(n-2)!}AB.$$

Proof. It follows from Lemma 4.9 that the matrix of the Hessian is $1/(n-2)!$ times the product of the matrices of μ and $\times\ell^{n-2}$. Proposition 4.10 and Theorem 4.6, which give that the matrix representing μ is B and the matrix representing $\times\ell^{n-2}$ is $t_{n-1,1,q}A$, respectively, now finish the proof. $\square$

Corollary 4.12. *The Hessian matrix of the dual socle generator $F_{\mathcal{V}(n,q)}$, evaluated at $X_1 = X_2 = \cdots = X_N = 1$, is given by*

$$H(F_{\mathcal{V}(q,n)})|_{X_1=\cdots=X_n=1} = \Phi(N, 0, t_{n,2,q}).$$

Hence, the absolute value of the determinant for this matrix is

$$\left| \det H(F_{\mathcal{V}(q,n)})|_{X_1=X_2=\cdots=X_N=1} \right| = (N-1)t_{n,2,q}^N.$$

Proof. This follows from Theorem 4.11 and Proposition 3.5 (c), after we note that

$$\left(\frac{q^{n-2}}{(n-2)!}\right)(t_{n-1,1,q}) = \left(\frac{q^{(n^2-n-2)/2}}{(n-2)!}\right)\left(\prod_{k=1}^{n-2}\begin{bmatrix} k \\ 1 \end{bmatrix}_q\right) = t_{n,2,q}.$$

The determinantal formula in Lemma 3.2 finishes the proof. $\square$

We conclude the paper with a description of the zeroth Hessian of $F_{\mathcal{V}(n,q)}$ evaluated at $X_1 = X_2 = \cdots = X_N = 1$ which is by definition $F_{\mathcal{V}(n,q)}(1, 1, \ldots, 1)$ and its implications on the map $\ell^n : \mathcal{A}_0 \to \mathcal{A}_n$.

Proposition 4.13. *Recall that $\ell = \partial/\partial X_1 + \partial/\partial X_2 + \cdots + \partial/\partial X_N \in Q$. Then*:

(a) *the K-linear homomorphism $KF_{\mathcal{V}(n,q)} \to K$ mapping $F_{\mathcal{V}(n,q)} \mapsto \ell^n F_{\mathcal{V}(n,q)}$ is given by the formula*

$$\ell^n F_{\mathcal{V}(n,q)} = q^{(1/2)n(n-1)} \prod_{k=1}^{n} \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix} \right]_q;$$

(b) *the homomorphism* $\times \ell^n : \mathcal{A}_0 \to \mathcal{A}_n$ *is given with respect to the bases* $\mathcal{B}_0 = \{1\}$ *and* $\mathcal{B}_n = \{g\}$ *(where* g *is any monomial in* $\mathcal{A}_n$*) by multiplication by the integer*

$$q^{(1/2)n(n-1)} \prod_{k=1}^{n} \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix} \right]_q.$$

Proof.

(a) The coefficient of a square-free monomial in ℓ^n is $n!$, so, acting by partial differentiation, $\ell^n F_{\mathcal{V}(n,q)} = n! F_{\mathcal{V}(n,q)}(1,1,\ldots,1)$. Since the number of monomials in $F_{\mathcal{V}(n,q)}$ is $F_{\mathcal{V}(n,q)}(1,1,\ldots,1) = s_{n,q}$, Proposition 2.4 (c) proves the first assertion.

(b) Since the maps in (a) and (b) are dual to each other by the theory of inverse systems, it follows that $\times \ell^n : \mathcal{A}_0 \to \mathcal{A}_n$ is given by multiplication by the same integer as the map in (a). $\square$

Remark 4.14. Our results in Proposition 4.13 and Corollary 4.12 recover, via Lemma 4.9, the non-vanishing of two of the determinants involved in the definition of the strong Lefschetz property (Definition 4.4) of the Gorenstein algebra $\mathcal{A}$. This algebra has been proved to have the strong Lefschetz property in [**9**].

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ENDNOTES

1. Throughout this article, we write $v_i \in v_j^{\perp}$ rather than $v_i \subset v_j^{\perp}$ because we prefer to think of v_i as vectors rather than subspaces of V, via a canonical identification explained previously.

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