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SEMICLEAN RINGS AND RINGS OF CONTINUOUS FUNCTIONS

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Dedicated to Professor Larry J. Gerstein

ABSTRACT. As defined by Ye [**12**], a ring is semiclean if every element is the sum of a unit and a periodic element. Ahn and Anderson [**1**] called a ring weakly clean if every element can be written as $u+e$ or $u-e$, where u is a unit and e an idempotent. A weakly clean ring is semiclean. We show the existence of semiclean rings that are not weakly clean. Every semiclean ring is 2-clean. New classes of semiclean subrings of $\mathbf{R}$ and $\mathbf{C}$ are introduced and conditions are given when these rings are clean. Cleanliness and related properties of $C(X,A)$ are studied when A is a dense semiclean subring of $\mathbf{R}$ or $\mathbf{C}$.

1. Introduction. All rings will be commutative with identity. As defined by Nicholson, an element a in a ring R is *clean* [**2, 7**] if a can be written as $a = u + e$, where $u \in U(R)$, the group of units of R, and $e \in \mathrm{Id}\,(R)$, the set of idempotents of R. R is a *clean ring* if every element is clean. Ye [**12**] called an element a in a ring R *semiclean* if a can be written as $a = u + p$, where $u \in U(R)$ and $p \in \mathrm{Per}\,(R)$, the set of periodic elements of R (that is, $p^k = p^l$ for $k \neq l$). R is called a *semiclean ring* if every element is semiclean. Several related notions have been studied in the literature, in particular, *weakly clean rings* [**1**], *n-clean rings* and *Σ-clean rings* [**11**].

Let X be a completely regular Hausdorff space. Let $1 \in A$ be a subring and subspace of $\mathbf{R}$ in the usual topology, and let $C(X,A)$ denote the set of all continuous A-valued functions on X. Under pointwise addition and multiplication $C(X,A)$ is a commutative ring with unity.

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The subset $C^*(X, A)$ of $C(X, A)$, consisting of all bounded functions in $C(X, A)$, is a subring of $C(X, A)$. When $A = \mathbf{R}$, these rings are simply denoted $C(X)$ (likewise $C^*(X)$) [**4**]. We are interested in studying cleanliness and related properties for the rings $C(X, A)$ and $C^*(X, A)$ which depend on the topological properties of X and algebraic properties of A. Azarpanah [**3**] and McGovern [**6**] independently proved that $C(X)$ and $C^*(X)$ are clean *if and only if* X is a strongly zero-dimensional space. If A is a proper subfield of $\mathbf{R}$, then $C(X, A)$ is always clean regardless of X. Hager and Kimber [**5**] considered the rings $C(X, A)$ when A is a dense clean subring of $\mathbf{R}$, with unity, which is not a field and X is zero-dimensional. In this case, $C(X, A)$ is clean *if and only if* X is a P-space [**4**, 4J], that is, every G_δ-set in X is open (or equivalently, every zero-set is open). We consider the case when $1 \in A$ is a dense semiclean subring of $\mathbf{R}$ or $\mathbf{C}$ which need not be clean.

In Section 2, we show the existence of semiclean rings that are not weakly clean and prove that every semiclean ring is 2-clean. In Section 3, we introduce classes of dense semiclean subrings of $\mathbf{R}$ and $\mathbf{C}$, and give conditions on when they will be clean. In Section 4, we go on to study cleanliness and related properties of rings of continuous functions, in particular we consider $C(X, A)$, where A is a dense semiclean subring of $\mathbf{R}$. In the final section we study these properties for rings of complex valued continuous functions.

2. Semiclean rings and other cleanliness related notions.

Example 2.1. For p prime, denote by $\mathbf{Z}_{(p)}$ the ring $\{(m/n) \in \mathbf{Q} \mid n \notin (p)\}$. It is easy to see that these rings are clean.

In fact, a ring having only trivial idempotents is clean if and only if it is a local ring. In particular, an integral domain is clean if and only if it is local. In [**1**] an element a in a ring R is called *weakly clean* if it can be written as $u + e$ or $u - e$, where $u \in U(R)$ and $e \in \mathrm{Id}\,(R)$. R is called *weakly clean* if every element is weakly clean. Clearly, every weakly clean ring is semiclean.

Example 2.2. For p, q distinct primes, the ring $\mathbf{Z}_{(p)} \cap \mathbf{Z}_{(q)} = \{(m/n) \in \mathbf{Q} \mid p \nmid n, q \nmid n\}$ is not clean. For example, $p/(p-q)$ is not a clean element. But these rings are always weakly clean. Suppose

m/n is not weakly clean. Then each of m/n, $(m/n) \pm 1$ is a non-unit. This means each of $m-n$, m and $m+n$ is a multiple of either p or q. Say, p divides two of them, then p divides n, a contradiction.

In [**11**] an element a is called *n-clean* if it can be written as the sum of n units and an idempotent, and it is *Σ-clean* if it is the sum of a finite number of units and an idempotent. Analogously, n-clean and Σ-clean rings are defined. The ring $\mathbf{Z}$ of integers is Σ-clean but not n-clean for any n.

Clearly, every clean ring is weakly clean and every weakly clean ring is semiclean. Also every *n-clean* ring is *Σ-clean.* Below, we will see that every semiclean ring is 2-clean.

All the examples of semiclean rings that we have seen in literature happen to be weakly clean rings. For example, it was proven in [**12**, Theorem 3.1] that the group rings $\mathbf{Z}_{(p)}(G)$ where G is a cyclic group of order 3 are semiclean but need not be clean. One can adapt the same proof to prove the following result.

Theorem 2.3. *The group rings $\mathbf{Z}_{(p)}(G)$, where $G = \{1, a, a^2\}$ is a cyclic group of order 3, are weakly clean whenever p is a prime.*

Proof. Let us first consider the case $p > 3$. In this case the non-trivial idempotents in $\mathbf{Z}_{(p)}(G)$ are $(1+a+a^2)/3$ and $(2-a-a^2)/3$ [**12**, Proposition 3.1]. An element $(k+la+ma^2)/n$, $p \nmid n$ in $\mathbf{Z}_{(p)}(G)$ can be expressed as:

$$\begin{aligned}
0+\frac{k+la+ma^2}{n} &= 1+\frac{(k-n)+la+ma^2}{n} = -1+\frac{(k+n)+la+ma^2}{n}\\
&= \frac{1+a+a^2}{3}+\frac{(3k-n)+(3l-n)a+(3m-n)a^2}{3n}\\
&= \frac{-1-a-a^2}{3}+\frac{(3k+n)+(3l+n)a+(3m+n)a^2}{3n}\\
&= \frac{2-a-a^2}{3}+\frac{(3k-2n)+(3l+n)a+(3m+n)a^2}{3n}\\
&= \frac{-2+a+a^2}{3}+\frac{(3k+2n)+(3l-n)a+(3m-n)a^2}{3n}.
\end{aligned}$$

We need to show that at least one of the fractional terms to the right of the plus sign is a unit. Suppose not; then, by [**12**, Corollary 3.1], p

divides all of:

$$k^3 + l^3 + m^3 - 3klm \tag{2.1}$$

$$(k-n)^3 + l^3 + m^3 - 3(k-n)lm \tag{2.2}$$

$$(k+n)^3 + l^3 + m^3 - 3(k+n)lm \tag{2.3}$$

$$(3k-n)^3 + (3l-n)^3 + (3m-n)^3 - 3(3k-n)(3l-n)(3m-n) \tag{2.4}$$

$$(3k+n)^3 + (3l+n)^3 + (3m+n)^3 - 3(3k+n)(3l+n)(3m+n) \tag{2.5}$$

$$(3k-2n)^3 + (3l+n)^3 + (3m+n)^3 - 3(3k-2n)(3l+n)(3m+n) \tag{2.6}$$

$$(3k+2n)^3 + (3l-n)^3 + (3m-n)^3 - 3(3k+2n)(3l-n)(3m-n). \tag{2.7}$$

As in [**12**, Theorem 3.1], (2.1), (2.2) and (2.3) above give:

$$p \mid (-3k^2n + 3kn^2 - n^3 + 3nlm) \tag{2.8}$$

$$p \mid 3k. \tag{2.9}$$

Using (2.9) and the fact that $p \nmid n$, (2.8) further gives:

$$p \mid (n^2 - 3lm). \tag{2.10}$$

Subtracting (2.4) from (2.7) and using (2.9), we get that p divides

$$3n(4n^2 + n^2 - 2n^2) - 9n(3l-n)(3m-n) = -27n(3lm - ln - mn).$$

Using (2.10), $p > 3$ and $p \nmid n$ gives:

$$p \mid (n - l - m). \tag{2.11}$$

Similarly, subtracting (2.6) from (2.5) and using (2.9), we get that p divides

$$3n(3n^2) - 9n(3l+n)(3m+n) = -27n(3lm + ln + mn).$$

Using (2.10), $p > 3$ and $p \nmid n$ gives:

$$p \mid (n + l + m). \tag{2.12}$$

Adding (2.11) and (2.12) gives $p \mid 2n$, a contradiction. Hence, p does not divide at least one of (2.1)–(2.7). Subsequently, every element of $\mathbf{Z}_{(p)}(G)$ is weakly clean.

When $p = 3$, (2.1) and (2.2) give (2.8) which implies $p \mid n^3$, a contradiction. So $\mathbf{Z}_{(3)}(G)$ is clean. When $p = 2$, we can argue similar to [**12**, Theorem 3.1] to show that $\mathbf{Z}_{(2)}(G)$ is clean. □

Is there a semiclean ring which is not weakly clean? The answer is affirmative. But, in order to have this affirmative answer, we need the next lemma.

Lemma 2.4. *Let p_i be a periodic element in ring $R_i : 1 \leq i \leq N$. Then $p = (p_1, p_2, \ldots, p_N)$ is a periodic element in $R = R_1 \times R_2 \times \cdots \times R_N$.*

Proof. If $p_i^{m_i} = p_i^{n_i}$ $(m_i > n_i)$, then $p_i^{n_i} = p_i^{m_i} = p_i^{m_i - n_i + n_i} = p_i^{2(m_i - n_i) + n_i} = \cdots = p_i^{k(m_i - n_i) + n_i}$ for any positive integer k. Let $K = (m_1 - n_1)(m_2 - n_2) \cdots (m_N - n_N)$, $L = \max(n_1, n_2, \ldots, n_N)$. Then $p^{K+L} = (p_1^{K+n_1+(L-n_1)}, p_2^{K+n_2+(L-n_2)}, \ldots, p_N^{K+n_N+(L-n_N)}) = (p_1^{n_1+(L-n_1)}, p_2^{n_2+(L-n_2)}, \ldots, p_N^{n_N+(L-n_N)}) = p^L$. □

Theorem 2.5. *Let $\{R_i : 1 \leq i \leq N\}$ be commutative rings. Then the direct product $R = R_1 \times R_2 \times \cdots \times R_N$ is semiclean if and only if each R_i is semiclean.*

Proof. $(\Rightarrow)$. This is clear since every homomorphic image of a semiclean ring is semiclean [**12**, Lemma 2.1].

$(\Leftarrow)$. Suppose each R_i is semiclean. Let $x = (x_i) \in R$. For each i, let $x_i = u_i + p_i$, where u_i is a unit and p_i a periodic element in R_i. Then $x = u + p$, where $u = (u_i)$ is a unit and $p = (p_i)$ is a periodic element in R by Lemma 2.4. Hence, R is semiclean. □

In particular, a direct product $R = R_1 \times R_2$ of two weakly clean rings is semiclean. However, by [**1**, Theorem 1.7], R will be weakly clean if and only if at least one of R_1 or R_2 is clean. This proves that a semiclean ring need not be weakly clean.

Example 2.6. Let $R = \mathbf{Z}_{(3)} \cap \mathbf{Z}_{(5)}$. $((3/2),(5/2)) \in R \times R$ is not weakly clean. But, subtracting $(1,-1)$ from this gives a unit. By Theorem 2.5, $R \times R$ is a semiclean ring.

An element x of a ring R is *strongly* π*-regular* if there exists an $n \in \mathbf{N}$ and $b \in R$ such that $x^n = x^{n+1}b$.

Theorem 2.7. *Every semiclean ring is* 2*-clean.*

Proof. Clearly, every periodic element is *strongly* π*-regular* and, by [**8**, Theorem 1], every *strongly* π*-regular* element is strongly clean. In particular, every periodic element is clean. Hence, every semiclean element can be written as the sum of two units and an idempotent. □

Again, the subset relation is proper. For example, we will see that $C(X)$ is semiclean *if and only if* X is strongly zero-dimensional. But every $f \in C(X)$ can be written as the sum of two units (irrespective of X) as $f = (f+|f|+1)/2 + (f-|f|-1)/2$. So $C(X)$ is 2-clean irrespective of the topology of X. For more details on the rings generated by the units, see [**9, 10**].

Hence, the family of all weakly clean rings is properly contained in the family of all semiclean rings, and the latter family is properly contained in the family of all 2-clean rings.

3. Semiclean subrings of R and C. Note that the only idempotents in $\mathbf{R}$ are $\{0,1\}$, and the only periodic elements are $\{0,1,-1\}$. Hence, the notions of semiclean and weakly clean coincide for subrings of $\mathbf{R}$. Our conjecture is that this may not be the case with $\mathbf{C}$ which has infinitely many periodic elements.

Below we give more examples of semiclean subrings of $\mathbf{R}$ and $\mathbf{C}$ which need not be clean. We also give the condition when they will be clean.

By $\mathbf{Z}_{(p)}[\sqrt{q}]$, where $p \in \mathbf{N}$ is a prime and $q \in \mathbf{N}$ is not a square, denote the set $\{a_0 + a_1\sqrt{q} \mid a_0, a_1 \in \mathbf{Z}_{(p)}\}$. Clearly these are subrings of $\mathbf{R}$.

Also note that an element in these rings can be represented as $(k_0 + k_1\sqrt{q})/m$, $k_0, k_1, m \in \mathbf{Z}, p \nmid m$.

Lemma 3.1. *The element* $(k_0 + k_1\sqrt{q})/m$ *in* $\mathbf{Z}_{(p)}[\sqrt{q}]$ *is a unit* $\Leftrightarrow p$ *does not divide* $k_0^2 - qk_1^2$.

Proof. The inverse of $(k_0 + k_1\sqrt{q})/m$ in $\mathbf{R}$ is $(m(k_0 - k_1\sqrt{q}))/(k_0^2 - qk_1^2)$. This is in $\mathbf{Z}_{(p)}[\sqrt{q}]$ if and only if p does not divide the denominator. (Note that the representation $a_0 + a_1\sqrt{q}$ for a real number with $a_0, a_1 \in \mathbf{Q}$ is unique; otherwise, we can represent $\sqrt{q}$ as a rational.) $\square$

Now, $\mathbf{Z}_{(p)}[\sqrt{q}]$ may or may not be clean as shown by the examples below.

Example 3.2. Consider $R = \mathbf{Z}_{(2)}[\sqrt{q}]$. This is a clean ring for all q. Suppose $a = (k_0 + k_1\sqrt{q})/n \in R$ (n odd) is such that a and $a - 1$ are not units. By Lemma 3.1, $2 \mid k_0^2 - k_1^2 q$ and $2 \mid (k_0 - n)^2 - k_1^2 q$. Subtracting, we get $2 \mid n(2k_0 - n)$. Since n is odd, this is impossible.

Example 3.3. Consider $S = \mathbf{Z}_{(7)}[\sqrt{2}]$. Let $a = (2 + 3\sqrt{2})/4 \in S$. Clearly, $7 \mid 2^2 - 2 \cdot 3^2 = -14$ as well as $(-2)^2 - 2 \cdot 3^2 = -14$. Hence, a and $a - 1$ are both non-units. S is not a clean ring.

Theorem 3.4. *Let* $p \in \mathbf{N}$ *be an odd prime,* $q \in \mathbf{N}$ *is not a square. Let* R *be the ring* $\mathbf{Z}_{(p)}[\sqrt{q}]$. *If* $p \mid q$, *then the ring* R *is clean. Otherwise,* R *is clean* $\Leftrightarrow q$ *is a quadratic nonresidue of* p.

Proof. Case 1. $p \mid q$. Let $a = (k_0 + k_1\sqrt{q})/n$, $p \nmid n$. Suppose a and $a - 1$ are not units. So $p \mid k_0^2 - k_1^2 q \Rightarrow p \mid k_0$ and $p \mid (k_0 - n)^2 - k_1^2 q \Rightarrow p \mid k_0 - n$. Subtracting, we get $p \mid n$, a contradiction. Hence, R is a clean ring.

Case 2. $p \nmid q$. ($\Rightarrow$). Suppose q is a quadratic residue of p, that is, there exists a $k \in [1, p-1]$ such that $k^2 \equiv q \pmod{p}$. Take $a = (k + \sqrt{q})/(2k) \in R$. Clearly a and $a - 1$ are non-units, hence R cannot be clean.

($\Leftarrow$). Suppose R is not clean. Then there exists an $a = (k_0 + k_1\sqrt{q})/n$, $p \nmid n$, such that both a and $a-1$ are non-units. So,

$$p \mid k_0^2 - qk_1^2 \tag{3.1}$$

$$p \mid (k_0 - n)^2 - qk_1^2. \tag{3.2}$$

Subtracting these two, we get $p \mid n(2k_0 - n)$. Since $p \nmid n$, $p \nmid k_0$. Using (3.1), $p \nmid k_1$. Hence, there exists a multiplicative inverse $\pmod{p}$ of k_1. Now, from (3.1), $(k_1^{-1}k_0)^2 \equiv q \pmod{p}$. So, q is a quadratic residue of p. □

Theorem 3.5. *The rings* $\mathbf{Z}_{(p)}[\sqrt{q}]$*, for* p *prime, are dense weakly clean subrings of* $\mathbf{R}$ *which need not be clean.*

Proof. We only have to prove that these rings are weakly clean. Suppose, on the contrary, there exists $a = (k_0 + k_1\sqrt{q})/n$, $p \nmid n$, such that none of a, $a-1$, $a+1$ is a unit. This means

$$p \mid k_0^2 - qk_1^2 \tag{3.3}$$

$$p \mid (k_0 - n)^2 - qk_1^2 \tag{3.4}$$

$$p \mid (k_0 + n)^2 - qk_1^2 \tag{3.5}$$

Subtracting (3.4) from (3.3) and (3.3) from (3.5) gives

$$p \mid n(2k_0 - n) \Longrightarrow p \mid 2k_0 - n \tag{3.6}$$

$$p \mid n(2k_0 + n) \Longrightarrow p \mid 2k_0 + n. \tag{3.7}$$

Subtracting these two gives $p \mid 2n$. Since $p \nmid n$, this can happen only if p is 2. But then (3.6) implies $p \mid n$. Again, a contradiction.

Thus, every element a is weakly clean, and hence the ring $\mathbf{Z}_{(p)}[\sqrt{q}]$ is weakly clean. □

Therefore, there are ample examples of dense weakly clean (hence, semiclean) subrings of $\mathbf{R}$ which are not clean. Also, in this case we can find units u in these rings such that $u-1$ and $u+1$ are both non-units. This is because Theorem 3.4 implies we can find $a \in [1, p-1]$ such that $a^2 \equiv q \pmod{p}$. Then $(p + \sqrt{q})/a$ is one such unit, because $p \nmid p^2 - q$, but $p \mid (p-a)^2 - q = p(p-2a) + a^2 - q$ and $p \mid (p+a)^2 - q = p(p+2a) + a^2 - q$.

For $p \in \mathbf{N}$, a prime, let us now consider subrings of $\mathbf{C}$: $\mathbf{Z}_{(p)}[i] = \{a + bi \mid a, b \in \mathbf{Z}_{(p)}\}$.

We will see that again these rings are weakly clean but need not be clean. Note that there are many more periodic elements in $\mathbf{C}$. So it may be possible to have semiclean subrings of $\mathbf{C}$ which are not weakly clean. For example, one might want to consider rings $\mathbf{Z}_{(p)}[\alpha]$ where α is a kth root of unity.

Lemma 3.6. *An element $(k_0 + k_1 i)/m \in \mathbf{Z}_{(p)}[i]$ is a unit $\Leftrightarrow p \nmid k_0^2 + k_1^2$.*

Proof. The inverse in $\mathbf{C}$ of the above element is $m(k_0 - k_1 i)/(k_0^2 + k_1^2)$. □

Theorem 3.7. *For any odd prime p, $\mathbf{Z}_{(p)}[i]$ is a dense weakly clean (hence, semiclean) subring of $\mathbf{C}$, which is clean if and only if $p \equiv 3 \pmod 4$.*

Proof. Suppose $\mathbf{Z}_{(p)}[i]$ is not weakly clean. Then there exists an element $a = (k_0 + k_1 i)/n$, $p \nmid n$ such that a, $a-1$, $a+1$ are all non-units. Therefore, by Lemma 3.6,

$$p \mid k_0^2 + k_1^2 \tag{3.8}$$

$$p \mid (k_0 - n)^2 + k_1^2 \tag{3.9}$$

$$p \mid (k_0 + n)^2 + k_1^2. \tag{3.10}$$

Subtracting (3.9) from (3.10), $p \mid 2k_0 \cdot 2n \Rightarrow p \mid 2k_0$. Subtracting (3.8) from (3.10), $p \mid n(2k_0 + n) \Rightarrow p \mid 2k_0 + n$. These two give $p \mid n$, a contradiction.

Hence, every element of $\mathbf{Z}_{(p)}[i]$ has to be weakly clean. However, it need not be clean. For example, consider the element $b = (7 + 6i)/4 \in \mathbf{Z}_{(5)}[i]$. Both b and $b-1$ are non-units in $\mathbf{Z}_{(5)}[i]$ since $3^2 + 6^2$ and $7^2 + 6^2$ are both multiples of 5.

As in Theorem 3.4, we can get that $\mathbf{Z}_{(p)}[i]$ is clean $\Leftrightarrow -1$ is a quadratic non-residue of p. By Euler's criterion, this will happen whenever $(-1)^{(p-1)/2} \not\equiv 1 \pmod p$, that is, $p \equiv 3 \pmod 4$. □

The rings $\mathbf{Z}_{(2)}[i]$ are clean. The proof is similar to that for the rings $\mathbf{Z}_{(2)}[\sqrt{q}]$.

4. Cleanliness in $C(X, A)$.

A non-empty T_1 space is *zero-dimensional* if it has a base of clopen sets. The rings $C(X)$ are known [**3, 6**] to be clean *if and only if* X is strongly zero dimensional, i.e., the Stone-Čech compactification βX is zero-dimensional.

Theorem 4.1. *The following are equivalent.*

(1) $C(X)$ *is clean.*

(2) $C(X)$ *is weakly clean.*

(3) $C(X)$ *is semiclean.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (1). This follows from [**6**, Theorem 13 (iv)] since the periodic elements in $C(X)$ overlap with roots of idempotents. Here we give a direct proof since [**6**] had omitted one. Suppose $C(X)$ is semiclean. Let f be any function in $C(X)$. Then $g = 2f - 1$ is 1 on $Z(1 - f)$ and -1 on $Z(f)$. Since $C(X)$ is semiclean, there exists a periodic function p, which assumes one of the values -1, 0, 1, such that $g - p$ is a unit u. It is clear that $u < 0$ on $Z(f)$ and $u > 0$ on $Z(1 - f)$. Hence, if we let $K = \{x : u(x) < 0\}$, a clopen set, and χ_K be the characteristic function of K, then $f - \chi_K$ is nowhere zero and hence a unit. So f is clean. □

Let A be a dense clean subring of $\mathbf{R}$, with unity, which is not a field. Let $u(A)$ denote the set of its units and $c(A) = \{a \in A \setminus u(A) : a - 1 \in u(A)\}$. By [**5**, Lemma 2.2], $u(A)$ and $c(A)$ are dense in $\mathbf{R}$. Further, in [**5**] it was proved that the rings $C(X, A)$, with X zero-dimensional, are clean *if and only if* X is a P-space.

Is it again the case that $C(X, A)$ is semiclean if and only if it is clean? It certainly is the case when 2 is not a unit in A. As in [**5**], we assume, without loss of generality, that the space X is zero-dimensional. First, we need a couple of lemmas.

Lemma 4.2. *Let A be a clean (local) ring with trivial idempotents. Then A has a unit u such that both $u \pm 1$ are non-units* if and only if 2 *is not a unit in A.*

Proof. ($\Leftarrow$). Take $u = 1$.

($\Rightarrow$). Let u be such a unit and suppose 2 is a unit. Then $(u \pm 1)/2$ are both non-units. So $(u + 1)/2$ is not clean, a contradiction. $\square$

Lemma 4.3 [**5**, Lemma 3.6]. *Let X be a completely regular Hausdorff space. The following are equivalent.*

(1) *X is a P-space.*

(2) *X is zero-dimensional and every countable union of clopen sets is closed.*

(3) *$f^{-1}(T) \in \operatorname{clop}(X)$ for every $f \in C(X)$ and for every $T \subseteq \mathbf{R}$.*

Theorem 4.4. *Let A be a dense clean subring of $\mathbf{R}$, with unity, which is not a field, and let X be a zero-dimensional space. If 2 is not a unit in A, the following are equivalent.*

(1) *X is a P-space.*

(2) *$C(X, A)$ is a pm-ring.*

(3) *$C(X, A)$ is clean.*

(4) *$C(X, A)$ is semiclean.*

Proof. In view of [**5**, Theorem 3.7], it suffices to prove that X is a P-space whenever $C(X, A)$ is semiclean. Let K_n be a sequence of pairwise disjoint clopen sets in X. Let u be a unit such that $u \pm 1$ are both non-units. Since $c(A)$ is dense in $\mathbf{R}$, let (b_n) be a sequence in $c(A)$ that converges to u. Define $f \in C(X, A)$ such that $f \mid K_n = b_n$ and $f(x) = u$ otherwise. Let U be an open set in A. If $u \in U$, then U contains all but finitely many b_n's; thus, $X \setminus f^{-1}(U)$ is closed being a finite union of clopen sets. If $u \notin U$, then $f^{-1}(U)$ is a union of clopen sets and so open. Thus, $f \in C(X, A)$. Since $C(X, A)$ is semiclean, there exist disjoint clopen sets U and V such that $f - \chi_U + \chi_V$ is a unit of $C(X, A)$. f is already a unit ($= u$) outside $\cup_n K_n$ and adding ± 1 will make it a non-unit. In $\cup_n K_n$ it is a non-unit so its value needs to be changed. Hence, $\cup_n K_n$ has to be $U \cup V$, a clopen set. By Lemma 4.3 (2), it follows that X is a P-space. $\square$

Now the question arises as to what happens if $2 \in u(A)$.

Theorem 4.5. *Let A be a dense clean subring of $\mathbf{R}$, with unity, which is not a field and such that $2 \in u(A)$. If X is a P-space, then $C^*(X, A)$ is a semiclean ring.*

Proof. Let $f \in C^*(X, A)$. Let $c^+(f) = \{x \in X \mid f(x) \geq 0 \in c(A)\}$ and $c^-(f) = \{x \in X \mid f(x) < 0 \in c(A)\}$. Further, let $u^+(A) = \{a \in u(A) \mid a + 1 \in u(A)\}$ and $u^-(A) = \{a \in u(A) \mid a - 1 \in u(A)\}$. Since $2 \in u(A)$, by Lemma 4.2, $u^+(A) \cup u^-(A) = u(A)$. Let $u^+(f) = \{x \in X \mid f(x) \in (-1/2, 1/2) \cap u^+(A)\}$ and $u^-(f) = \{x \in X \mid f(x) \in (-1/2, 1/2) \cap u^-(A)\}$. By Lemma 4.3 (3) (and looking at f as a member of $C(X)$), the sets $c^+(f), c^-(f), u^+(f)$ and $u^-(f)$ are all clopen sets and $f - \chi_{c^-(f) \cup u^-(f)} + \chi_{c^+(f) \cup u^+(f)}$ is a unit in $C^*(X, A)$. □

Theorem 4.6. *Let A be a dense clean subring of $\mathbf{R}$, with unity, which is not a field. If X is an infinite zero-dimensional space, then $C^*(X, A)$ is not a weakly clean ring.*

Proof. We can find a sequence $\{U_n\}_{n \in \mathbf{N}}$ of disjoint non-empty clopen sets in X, since X is infinite, Hausdorff and zero-dimensional. Let $\{a_n\}_{n \in \mathbf{N}} \subseteq c(A)$ be a sequence such that $a_n \to -1$ and $\{b_n\}_{n \in \mathbf{N}} \subseteq c(A)$ such that $b_n \to 1$. Let $f \in C^*(X, A)$ be such that $f |_{U_{2n}} = a_n + 1$, $f |_{U_{2n-1}} = b_n - 1$ and $f(x) = 0$ otherwise. If $f = u + e$, with u a unit and e an idempotent, then $u |_{U_{2n}} = a_n + 1$. Since $a_n + 1 \to 0$, this makes u^{-1} unbounded, a contradiction. Similarly, $f = u - e$ implies $u |_{U_{2n-1}} = b_n - 1$, again a contradiction. □

Every P-space is zero-dimensional [**4**]. Hence, by Theorems 4.5 and 4.6, we have a semiclean ring of continuous functions which is not weakly clean.

The above results suggest that if we can find a suitable topological space with lots of clopen sets which is not a P-space, we may also be able to demonstrate:

Conjecture 4.7. *If 2 is invertible in A, a dense clean subring of $\mathbf{R}$, then $C(X, A)$ can be semiclean without being clean.*

Let us now consider dense semiclean subrings of $\mathbf{R}$ that are not fields and are not necessarily clean (recall the semiclean rings introduced in previous section). Let S denote one such semiclean ring, and consider the ring $C(X,S)$. If S is not clean, then there exists an element s such that both s and $s-1$ are not units. Then the constant function s on any space is not clean. On the other hand, we will see that $C(X,S)$ is semiclean whenever X is a P-space. This gives another trivial example of a semiclean ring of continuous functions which is not clean.

Lemma 4.8. *Let S be a commutative semiclean ring with unity and only periodic elements $\{-1,0,1\}$. If S is not clean, then S has a unit u such that $u-1$ and $u+1$ are both non-units. If $2 \in u(S)$, then the converse is also true.*

Proof. If S is not clean, there exists an $s \in S$ such that s and $s-1$ are both non-units. Hence, $2s$ and $2s-2$ are also non-units. Then, $2s-1$ is the required unit by semicleanliness, since the only periodic elements are $\{-1,0,1\}$.

If $2 \in u(S)$, then by Lemma 4.2, it follows that S is not clean. $\square$

Lemma 4.9. *If S is a dense semiclean subring of $\mathbf{R}$ that is not a field, then the set of all non-units in S is also dense in $\mathbf{R}$.*

Proof. Let b be a nonzero non-unit in S. For every $s \in S$, $\alpha_s = sb$ is also a non-unit. For $r \in \mathbf{R}$, let (s_n) be a sequence in S such that $s_n \to r/b$. Then $\alpha_{s_n} \to r$ shows that the set of non-units is dense in $\mathbf{R}$. $\square$

Theorem 4.10. *Let S be a dense semiclean subring of $\mathbf{R}$ with unity that is not a field. Let X be a zero-dimensional space.*

(1) *If X is a P-space, then $C(X,S)$ is semiclean.*

(2) *If S is not clean, then the converse of the above also holds true.*

(3) *$C(X,S)$ is clean $\Leftrightarrow$ X is a P-space and S is clean.*

Proof. (1) Let $f \in C(X,S)$. Since we are considering the subspace topology on S, f is also in $C(X)$. Let $c^+(f) = \{x|f(x) \notin u(S), f(x)+1 \in u(S)\}$, $c^-(f) = \{x|f(x) \notin u(S), f(x)-1 \in u(S)\}$. Since X is a P-space, by Lemma 4.3 (3) it follows that $c^+(f)$ and $c^-(f)$ are clopen sets and $f+\chi_{c^+(f)}-\chi_{c^-(f)\setminus c^+(f)}$ is a unit. Hence, $C(X,S)$ is semiclean.

(2) If S is not clean, then by Lemma 4.8, there exists a unit u such that $u-1$ and $u+1$ are both non-units. By arguing similarly as in Theorem 4.4 and observing that the set of non-units in S is dense in $\mathbf{R}$, we can now show that the converse holds as well.

(3) If X is a P-space and S is clean, then by [**5**, Theorem 3.7], $C(X,S)$ is clean. Conversely, if $C(X,S)$ is clean then S, being a homomorphic image of $C(X,S)$, will be clean. Once again by [**5**, Theorem 3.7], X has to be a P-space. □

5. Complex valued continuous functions. In this section we would like to consider the rings of complex valued continuous functions.

For $\mathcal{S}$ a non-empty set of periodic elements of R, we say R is $\mathcal{S}$-semiclean if each $x \in R$ can be written as $x = u + p$ where u is a unit and $p \in \mathcal{S}$.

We begin with proving the following theorem.

Theorem 5.1. *The following are equivalent.*

(1) $C(X)$ *is clean.*

(2) $C(X)$ *is weakly clean.*

(3) $C(X)$ *is semiclean.*

(4) $C(X,\mathbf{C})$ *is clean.*

(5) $C(X,\mathbf{C})$ *is weakly clean.*

(6) $C(X,\mathbf{C})$ *is $\mathcal{S}$-semiclean, where $\mathcal{S}$ is the family of all continuous $\{-1,0,1\}$-valued functions.*

(7) X *is strongly zero-dimensional.*

Proof. (1) ⇔ (2) ⇔ (3). Theorem 4.1.

(4) ⇒ (5) ⇒ (6). Clear.

(6) ⇒ (4) Similar to (3) ⇒ (1) of Theorem 4.1.

(4) ⇒ (1). Given $C(X,\mathbf{C})$ is clean. For $f \in C(X,\mathbf{R}) \subseteq C(X,\mathbf{C})$, let $f = u + e$ where $u = u_1 + iu_2$ is a unit in $C(X,\mathbf{C})$ and e is an idempotent in $C(X,\mathbf{C})$. Now the only idempotents in $\mathbf{C}$ are 0 and 1, since for a complex c, $c = c^2$ implies $c = 0$ or $c = 1$. So e is real valued, implying u is real valued. Therefore, f is clean in $C(X,\mathbf{R})$ as well. Hence, $C(X,\mathbf{R})$ is clean.

(1) $\Rightarrow$ (4). Given $C(X, \mathbf{R})$ is clean. Let $f \in C(X, \mathbf{C})$ and $f = f_1 + if_2$. It is known that $f \in C(X, \mathbf{C}) \Leftrightarrow f_1, f_2 \in C(X, \mathbf{R})$.

From the hypothesis, there exist units u_1, u_2 and idempotents e_1, e_2 in $C(X, \mathbf{R})$, such that $f_1 = u_1 + e_1$ and $f_2 = u_2 + e_2$. So $f = u_1 + e_1 + i(u_2 + e_2)$. Now e_1 is also an idempotent in $C(X, \mathbf{C})$ and u_1 being non-zero, $u_1 + i(u_2 + e_2)$ is a unit in $C(X, \mathbf{C})$. Hence, f is clean and so is $C(X, \mathbf{C})$.

(1) $\Leftrightarrow$ (7). Proved in [**3, 6**]. $\square$

Now we would like to generalize some of the results of [**5**] for $C(X, A)$ where A is a subring and subspace of $\mathbf{C}$.

Let A be a dense clean subring of $\mathbf{C}$ with unity, which is not a field. We can generalize Lemma 2.2 of [**5**] to show that the set $u(A)$ of units in A and $c(A)$, non-units in A, are both dense sets in $\mathbf{C}$. Further, Lemma 4.3 (3) can be generalized so that $f^{-1}(T) \in \operatorname{clop}(X)$ for every $f \in C(X, A)$ and every $T \subseteq \mathbf{C}$. Using these we can show, as in [**5**, Theorem 3.7] that $C(X, A)$ is clean *if and only if* X is a P-space.

Now, let us consider the case when S is a dense semiclean subring of $\mathbf{C}$ with unity, which is not a field.

Theorem 5.2. *Let S be a dense semiclean subring of $\mathbf{C}$ with unity that is not a field. Let X be a zero-dimensional space.*

(1) *If X is a P-space and $|\operatorname{Per}(S)| < \infty$, then $C(X, S)$ is semiclean.*

(2) *$C(X, S)$ is clean $\Leftrightarrow$ X is a P-space and S is clean.*

Proof. (1) As in Lemma 4.3 (3), we can show that $f^{-1}(T) \in \operatorname{clop}(X)$ for every $f \in C(X, S)$ and every $T \subseteq \mathcal{C}$ whenever X is a P-space. Let $\{p_1, p_2, \dots, p_n\}$ be the set of periodics in S. Let $c_0(S) = u(S)$ and, for $1 \leq i \leq n$, $c_i(S) = \{a \in S \mid a - p_i \in u(S), a \notin c_j(S) \text{ for } j < i\}$. Then $c_i(f) = \{x \in X \mid f(x) \in c_i(S)\}$ are clopen sets. Let $p = \Sigma p_i \chi_{c_i(f)}$. It is easy to show that p is periodic. Then $f - p$ is a unit. Hence $C(X, S)$ is semiclean.

(2) Clear. $\square$

REFERENCES

1. Myung-Sook Ahn and D.D. Anderson, *Weakly clean rings and almost clean rings*, Rocky Mountain J. Math. **36** (2006), 783–798.

2. D.D. Anderson and V.P. Camillo, *Commutative rings whose elements are a sum of a unit and idempotent*, Comm. Algebra **30** (2002), 3327–3336.

3. F. Azarpanah, *When is $C(X)$ a clean ring?* Acta Math. Hung. **94** (2002), 53–58.

4. L. Gillman and M. Jerison, *Rings of continuous functions*, D. Van Nostrand Publ. Co., New York, 1960.

5. A. Hager and C. Kimber, *Clean rings of continuous functions*, Alg. Univ. **56** (2007), 77–92.

6. W.Wm. McGovern, *Clean semiprime f-rings with bounded inversion*, Comm. Alg. **31** (2003), 3295–3304.

7. W.K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269–278.

8. ———, *Strongly clean rings and fitting's lemma*, Comm. Alg. **27** (1999), 3583–3592.

9. R. Raphael, *Rings which are generated by their units*, J. Alg. **28** (1974), 199–205.

10. Peter Vámos, 2-*good rings*, Q.J. Math. **56** (2005), 417–430.

11. G. Xiao and W. Tong, *n-clean rings and weakly unit stable range rings*, Comm. Alg. **33** (2005), 1501–1517.

12. Y. Ye, *Semiclean rings*, Comm. Alg. **31** (2003), 5609–5625.

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