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REES ALGEBRAS OF DIAGONAL IDEALS

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ABSTRACT. There is a natural epimorphism from the symmetric algebra to the Rees algebra of an ideal. When this epimorphism is an isomorphism, we say that the ideal is of linear type. Given two determinantal rings over a field, we consider the diagonal ideal, kernel of the multiplication map. We prove in many cases that the diagonal ideal is of linear type and recover the defining ideal of the Rees algebra. In our cases, the special fiber rings of the diagonal ideals are the homogeneous coordinate rings of the join varieties.

1. Introduction. In this paper we address the problem of determining the equations that define the Rees algebra of an ideal. Besides encoding the asymptotic properties of the powers of an ideal, the Rees algebra realizes, algebraically, the blow-up of a variety along a subvariety. Though blowing up is a fundamental operation in the birational study of algebraic varieties and, in particular, in the process of desingularization, an explicit description of the resulting variety in terms of defining equations remains a difficult problem.

Let I be an ideal in a Noetherian ring R. The Rees algebra $\mathcal{R}(I)$ of I is the graded subalgebra $R[It] \cong \oplus_{n\geq 0} I^n$ of $R[t]$. When I is generated by $f_1, \dots, f_u$, there is a natural map ϕ from $R[t_1, \dots, t_u]$ to $\mathcal{R}(I)$ sending t_i to $f_i t$. The kernel of ϕ is the defining ideal of $\mathcal{R}(I)$ in the ring $R[t_1, \dots, t_u]$. There is another natural map ψ from $\mathrm{Sym}\,(R^u) = R[t_1, \dots, t_u]$ to $\mathrm{Sym}\,(I)$, the symmetric algebra of I, and the kernel of ψ is the defining ideal of $\mathrm{Sym}\,(I)$. This ideal is generated by the entries of the product of $(t_1, \dots, t_u)$ and the presentation matrix of I. The defining ideal of $\mathrm{Sym}\,(I)$ is contained in the kernel of ϕ;

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therefore, there is a surjective map from Sym (I) to $\mathcal{R}(I)$. The ideal I is said to be of *linear type* if Sym (I) is naturally isomorphic to $\mathcal{R}(I)$. Hence, we obtain the defining equations of $\mathcal{R}(I)$ for free in this case.

In this paper, we give a new class of ideals of linear type, namely, diagonal ideals of determinantal rings. In general, an ideal is not of linear type. The first known class of ideals of linear type are complete intersection ideals [**11**]. Ideals generated by d-sequences are another large class of ideals of linear type [**9, 15**]. These sequences play a role in the theory of approximation complexes similar to the role regular sequences play in the theory of Koszul complexes. Later Herzog, Simis and Vansconcelos and Herzog, Vansconcelos and Villarreal used strongly Cohen-Macaulay and sliding depth conditions to describe classes of ideals of linear type [**5, 6, 8**]. Huneke proved that, if X is a generic $n \times n$ matrix and I is the ideal of $n-1$ size minors of X in $R = \mathbf{Z}[x_{ij}]$, then I is of linear type [**10**]. Villarreal showed the edge ideals of a tree or a graph with a unique odd cycle are ideals of linear type [**16**].

Let k be a field, R a polynomial ring over the field k with variables $\{x_{ij}\}$, and X the generic $m \times n$ matrix (x_{ij}). Given two homogeneous R-ideals, I_1 and I_2, we consider the kernel of the multiplication map from $S = R/I_1 \otimes_k R/I_2$ to $R/(I_1 + I_2)$. The kernel is the *diagonal* ideal $\mathbf{D}$ of the ring S and $\mathbf{D}$ is generated by the images of $x_{ij} \otimes 1 - 1 \otimes x_{ij}$ in the ring S. The main result of this paper shows that the ideal $\mathbf{D}$ is of linear type if I_1, I_2 are the ideals of maximal minors of given submatrices of X. Notice I_1 and I_2 are in general not of linear type (see [**10**, 2.6]).

In this particular case, the *special fiber ring* of I, $\mathcal{F}(\mathbf{D}) = \mathcal{R}(\mathbf{D}) \otimes_S k$, is the homogeneous coordinate ring of the embedded join varieties of $V(I_1)$ and $V(I_2)$ in projective space $\mathbf{P}_k^{m\times n-1}$ [**13**]. Hence, when $\mathbf{D}$ is an ideal of linear type, the embedded join is the whole space. But it is not true in general that if the embedded join variety is the whole space, the diagonal ideal $\mathbf{D}$ is of linear type. See Example 2.2 in Section 2.

Basic aspects of the proof appear in Section 2. We now describe the idea of the proof. We use the defining ideals of Sym $(\mathbf{D})$ to understand the defining ideals of $\mathcal{R}(\mathbf{D})$. We identify some specific equations in the defining ideal $\mathcal{J}$ of Sym $(\mathbf{D})$ and consider the subideal $\mathcal{L}$ of $\mathcal{J}$ they generate.

Notice that $\mathcal{L} \subset \mathcal{J} \subset \mathcal{K}$, where $\mathcal{K}$ is the defining ideal of $\mathcal{R}(\mathbf{D})$; hence, the goal is to prove that $\mathcal{L} = \mathcal{K}$, which is accomplished in Section 4, after some preliminary results in linear algebra are established in Section 3. In Section 4, we use Buchberger's algorithm to find a Gröbner basis of the ideal $\mathcal{L}$ with respect to some monomial order. More precisely, we find a set of polynomials that are in the ideal $\mathcal{L}$ and show that all the remainders between elements in this set are zero. In this way, we find a Gröbner basis of the ideal $\mathcal{L}$. Once we have the Gröbner basis, we have the generating set for the initial ideal $\operatorname{in}(\mathcal{L})$ of $\mathcal{L}$. This way we find a non zero-divisor modulo $\mathcal{L}$ which we may invert, thereby reducing to the case of a smaller matrix. Thus, we show that $\mathcal{L} = \mathcal{K}$. As a consequence, the two algebras $\operatorname{Sym}(\mathbf{D})$ and $\mathcal{R}(\mathbf{D})$ are naturally isomorphic, and we obtain an explicit description of the defining equations of $\mathcal{R}(\mathbf{D})$.

2. Main results. Let k be a field, $2 \leq m \leq n$ integers, $X_{mn} = [x_{ij}]$, $Y_{mn} = [y_{ij}]$. $Z_{mn} = [z_{ij}]$, $m \times n$ matrices of variables over k. For $i = 1, 2$, let s_i, t_i be integers with $2 \leq s_i \leq t_i$ and $s_2 \leq s_1$, and let $X_{s_1 t_1}$, $Y_{s_2 t_2}$ be the submatrices of X and Y consisting of the first s_i rows and first t_i columns respectively. We write $I_1 = I_{s_1}(X_{s_1 t_1})$, $I_2 = I_{s_2}(X_{s_2 t_2})$ to denote the ideals of $k[X]$ generated by the maximal minors of $X_{s_1 t_1}$ and the maximal minors of $X_{s_2 t_2}$. Let $R_1 = k[X]/I_1$ and $R_2 = k[X]/I_2$ be the two determinantal rings. We consider the diagonal ideal $\mathbf{D}$ of $R_1 \otimes_k R_2$, defined via the exact sequence

$$0 \longrightarrow \mathbf{D} \longrightarrow R_1 \otimes_k R_2 \overset{\text{mult.}}{\longrightarrow} k[X]/(I_1 + I_2) \longrightarrow 0.$$

The ideal $\mathbf{D}$ is generated by the images of $x_{ij} \otimes 1 - 1 \otimes x_{ij}$ in $R_1 \otimes_k R_2$.

We write the diagonal ideal $\mathbf{D} = (\{x_{ij} - y_{ij}\})$ in

$$S = k[X_{mn}, Y_{mn}]/(I_{s_1}(X_{s_1 t_1}), I_{s_2}(Y_{s_2 t_2})) \cong R_1 \otimes_k R_2.$$

We have a presentation of $\mathbf{D}$,

$$S^l \overset{\phi}{\longrightarrow} S^{mn} \longrightarrow \mathbf{D} \longrightarrow 0.$$

From this, we obtain a presentation of the symmetric algebra of $\mathbf{D}$,

$$0 \longrightarrow (\operatorname{image}(\phi)) = J \longrightarrow \operatorname{Sym}(S^{mn}) = S[Z_{mn}] = T \longrightarrow \operatorname{Sym}(\mathbf{D}) \longrightarrow 0.$$

Here J is the ideal generated by the entries of the row vector $[z_{11}, z_{12}, \dots, z_{1n}, \dots, z_{mn}] \cdot \phi$. Hence

$$\mathrm{Sym}\,(\mathbf{D}) \cong T/J,$$

where J is generated by linear forms in the variables z_{ij}. We write $\mathcal{R}(\mathbf{D}) = T/K$, $J \subset K$. In general K is not generated by linear forms. We can rewrite $\mathrm{Sym}\,(\mathbf{D}) = T/J = k[X_{mn}, Y_{mn}, Z_{mn}]/\mathcal{J}$ and $\mathcal{R}(\mathbf{D}) = k[X_{mn}, Y_{mn}, Z_{mn}]/\mathcal{K}$. In this particular case, the *special fiber ring* of I, $\mathcal{F}(\mathbf{D}) = \mathcal{R}(\mathbf{D}) \otimes_S k$, is the homogeneous coordinate ring of the embedded join varieties of $V(I_1)$ and $V(I_2)$ in projective space $\mathbf{P}_k^{m\times n-1}$.

Theorem 2.1. *The ideal* $\mathbf{D}$ *is of linear type if* I_1 *and* I_2 *are generated by the maximal minors of given submatrices of* X*, respectively. So, in this case,* $\mathcal{R}(\mathbf{D}) \cong \mathrm{Sym}\,(\mathbf{D})$.

Notice that, if $s_1 < m$, then $\{x_{ij} - y_{ij}\}_{i>s_1}$ is a regular sequence of S. Hence, the defining ideal of $\mathcal{R}(\mathbf{D})$ in $S[Z]$ is generated by the defining equations of $\mathcal{R}(\mathbf{D}')$ in $S[Z']$ and the Koszul relationships of $\{x_{ij} - y_{ij}\}$ for all i, j in $S[Z]$ where $\mathbf{D}' = (\{x_{ij} - y_{ij}\}_{i\le s_1})$ and $Z' = \{z_{ij}\}_{i\le s_1}$. Therefore, it is sufficient to prove Theorem 2.1 for the case $s_1 = m$. From now on, we assume $s_1 = m$.

The embedded join is the whole space in the case of Theorem 2.1. We obtained the following example by Singular [**4**], and it shows that, in general, even if the fiber ring is the whole space, the ideal $\mathbf{D}$ may not be of linear type.

Example 2.2. Let X, Y and Z be 3×3 matrices and $I_1 = I_3(X)$, $I_2 = I_2(X)$ the ideal generated by 3×3 and 2×2 minors of X. Write

$$S = k[X, Y]/(I_3(X), I_2(Y)) \cong R_1 \otimes_k R_2,$$

$\mathrm{Sym}\,(\mathbf{D}) = S[Z]/J$ and $\mathcal{R}(\mathbf{D}) = S[Z]/K$. Then $J = (g_{ij,lk},\ f_{1,2,3})$ where $g_{ij,lk} = (\{(x_{ij} - y_{ij})z_{lk} - (x_{lk} - y_{lk})z_{ij}\})$ and

$$f_{1,2,3} = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ z_{21} & z_{22} & z_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix}.$$

$K = (J,\, h)$ where

$$h = \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix} + \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ y_{21} & y_{22} & y_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix}.$$

We can see each generator is in the ideal $(X, Y)S[Z]$. Hence, the special fiber ring has defining ideal as zero ideal in the ring $k[Z]$, which shows the secant variety is the whole space.

The remaining part of this section is devoted to proving basic aspects of Theorem 2.1. In the course of this, we also describe the defining equations of $\mathcal{R}(\mathbf{D})$. We identify some specific equations in the defining ideal $\mathcal{J}$ of $\mathrm{Sym}\,(\mathbf{D})$. To clarify the notations, we define matrices which will be used repeatedly.

Definition 2.3. Let $X = [x_{ij}]$, $Y = [y_{ij}]$, $1 \le i \le m$, $1 \le j \le n$, be $m \times n$ matrices, and $X^{l,k}_{a_1 \cdots a_s} = [x_{ia_i}]$, $Y^{l,k}_{a_1 \cdots a_s} = [y_{ia_i}]$, $l \le i \le k$, $1 \le a_1 < \cdots < a_s \le n$, $X^{l,k}_{1 \cdots \hat{s} \cdots n} = [x_{ij}]$, $l \le i \le k$, $1 \le j \le n$, $j \ne s$ be submatrices. For the convenience of notations, we write $\det M = |M|$ when M is a square matrix. We set the determinant of a 0×0 matrix equal to 1. We also write

$$\begin{bmatrix} X^{1,j} \\ Y^{j+1,m} \end{bmatrix}_{a_1 \cdots a_m} = \begin{bmatrix} x_{1a_1} & \cdots & x_{1a_m} \\ \vdots & & \vdots \\ x_{ja_1} & \cdots & x_{ja_m} \\ y_{j+1a_1} & \cdots & y_{j+1a_m} \\ \vdots & & \vdots \\ y_{ma_1} & \cdots & y_{ma_m} \end{bmatrix}$$

as a matrix with 'mixed' variables. We also use obvious extensions of this notation and allow vacuous block submatrices.

It is well known that we can write a matrix with variable y's as a matrix of variables x's and a combination of differences of x's and y's.

Lemma 2.4. *Let X and Y be $n \times n$ matrices. With notation as above,*

$$|Y| = |X| + \sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j} \begin{vmatrix} Y^{1,i-1}_{1\cdots\hat{j}\cdots n} \\ X^{i+1,n}_{1\cdots\hat{j}\cdots n} \end{vmatrix} (y_{ij} - x_{ij}).$$

Proof. This follows by a reasonably straightforward induction on n. □

In the following lemma, we define the special equations to be considered and we show that these equations are in the defining ideal of symmetric algebra of **D**.

Lemma 2.5. *Let $X_{a_1\cdots a_{s_1}}$ be the $s_1 \times s_1$ submatrix of $X_{s_1t_1}$ with columns $a_1, \ldots, a_{s_1}$, $Y_{b_1\cdots b_{s_2}}$ the $s_2 \times s_2$ submatrix of $Y_{s_2t_2}$ with columns $b_1, \ldots, b_{s_2}$, $X^{l,k}_{a_1\cdots a_{s_1}}$ the $k-l+1$ by s_1 submatrix of X with rows $l, l+1, \ldots, k$ and columns $a_1, \ldots, a_{s_1}$, and similarly for Y and Z.*

We define

$$g_{ij,lk} = \begin{vmatrix} z_{ij} & z_{lk} \\ x_{ij} - y_{ij} & x_{lk} - y_{lk} \end{vmatrix},$$

$$f_{a_1,\ldots,a_{s_1}} = \sum_{q=1}^{s_2}(-1)^{q+1} \left| \begin{bmatrix} Z^{q,q} \\ Y^{1,q-1} \\ X^{q+1,m} \end{bmatrix}_{a_1\cdots a_{s_1}} \right|,$$

where $1 \le a_1 < a_2 < \cdots < a_{s_1} \le \min(t_1, t_2)$ and $1 \le i \le m = s_1$, $1 \le l \le m = s_1$, $1 \le j \le n$, $1 \le k \le n$.

We write $\mathcal{L} = (I_{s_1}(X_{s_1t_1}),\ I_{s_2}(Y_{s_2t_2}),\ g_{ij,lk},\ f_{a_1,\ldots,a_{s_1}})$, an ideal of $k[X_{mn}, Y_{mn}, Z_{mn}]$. Then $\mathcal{L} \subset \mathcal{J}$.

Proof. We can see that the $|X_{a_1\cdots a_{s_1}}|$, $|Y_{b_1\cdots b_{s_2}}|$, $g_{ij,lk}$'s are in $\mathcal{J}$. Notice that, when $t_2 < s_1$, by the way we define $f_{a_1,\ldots,a_{s_1}}$, this is an empty condition, because in this case we have $1 \le a_1 < a_2 < \cdots < a_{s_1} \le t_2 < s_1$. When $t_2 \ge s_1$, we substitute z_{ij} via $x_{ij} - y_{ij}$ and use Lemma 2.4, we can see f's are in $\mathcal{J}$. □

Instead of proving Theorem 2.1, we will prove $\mathcal{L}$ is the defining ideal of $\mathcal{R}(\mathbf{D})$ in the following theorem. Hence, Theorem 2.1 immediately follows from the theorem.

Theorem 2.6. *The ideal $\mathcal{L}$ is the defining ideal of $\mathcal{R}(\mathbf{D})$ and $\mathbf{D}$ is of linear type.*

In order to use the induction hypothesis, we need to find a non zero-divisor of $k[X,Y,Z]/\mathcal{L}$. The following lemma gives us one. Its proof is given in Section 4. It involves finding a Gröbner basis of the ideal.

Lemma 2.7. *The variable x_{11} is a non zero-divisor of the quotient ring $k[X,Y,Z]/\mathcal{L}$.*

Proof of Theorem 2.6. From Lemma 2.5, we have $\mathcal{L} \subset \mathcal{J} \subset \mathcal{K}$, where $\mathcal{J}$ is the defining ideal of $\operatorname{Sym}(\mathbf{D})$. We would like to show $\mathcal{L} = \mathcal{K}$ and, as a consequence, $\mathcal{L} = \mathcal{J} = \mathcal{K}$, i.e., $\mathbf{D}$ is an ideal of linear type. By Lemma 2.7, x_{11} is a non zero-divisor on $k[X,Y,Z]/\mathcal{L}$. Changing the roles of X and Y, we also obtain that y_{11} is a non zero-divisor on $k[X,Y,Z]/\mathcal{L}$. Since $\mathcal{K}$ is the defining ideal of the Rees algebra, it is a prime ideal. Hence, it suffices to show that $\mathcal{L}_{x_{11}y_{11}} = \mathcal{K}_{x_{11}y_{11}}$. The latter holds by inducting on the size of the matrix X.

To explain this, we consider the $(m-1)\times(n-1)$ matrices of variables $X' = [x'_{ij}]$, $Y' = [y'_{ij}]$, $Z' = [z'_{ij}]$, $2 \le i \le m = s_1$, $2 \le j \le n$. We define a natural isomorphism ϕ from $k[\{x_{ij}\}_{i=1 \text{ or } j=1}, X']_{x_{11}}$ to $k[X]_{x_{11}}$ via $\phi(x_{ij}) = x_{ij}$ when $i = 1$ or $j = 1$, and $\phi(x'_{ij}) = x_{ij} - x_{i1}x_{1j}/x_{11}$ when $i \neq 1$ and $j \neq 1$. Let

$$R'_1 = k[\{x_{ij}\}_{i=1 \text{ or } j=1}, X']_{x_{11}}/I'_1 \cong (R_1)_{x_{11}},$$
$$R'_2 = k[\{x_{ij}\}_{i=1 \text{ or } j=1}, X']_{x_{11}}/I'_2 \cong (R_2)_{x_{11}},$$

where $I'_1 = I_{s_1-1}(X'_{s_1-1,t_1-1})$, $I'_2 = I_{s_2-1}(X'_{s_2-1,t_2-1})$ and

$$S' := k[\{x_{ij}\}_{i=1 \text{ or } j=1},\ X',\ \{y_{ij}\}_{i=1 \text{ or } j=1}, Y']_{x_{11}y_{11}}/(I'_1, I'_2)$$
$$\cong R'_1 \otimes R'_2 \cong (R_1)_{x_{11}} \otimes (R_2)_{x_{11}}.$$

Then we have

$$\widetilde{\mathbf{D}}' = (\{x_{ij} - y_{ij}\}_{i=1 \text{ or } j=1}, \{x'_{ij} - y'_{ij}\}_{2\le i\le m,\, 2\le j\le n})$$
$$\cong \mathbf{D} = (\{x_{ij} - y_{ij}\}_{1\le i\le m,\, 1\le j\le n}),$$

and $T' = S'[\{z_{ij}\}_{i=1 \text{ or } j=1}, Z'] \cong T_{x_{11}y_{11}}$ by the map $\overline{\phi}$ defined as follows: $\overline{\phi}(x_{ij}) = x_{ij}$, $\overline{\phi}(y_{ij}) = y_{ij}$ and $\overline{\phi}(z_{ij}) = z_{ij}$ when $i = 1$ or $j = 1$, and $\overline{\phi}(x'_{ij}) = x_{ij} - x_{i1}x_{1j}/x_{11}$, $\overline{\phi}(y'_{ij}) = y_{ij} - y_{i1}y_{1j}/y_{11}$, and $\overline{\phi}(z'_{ij}) = z_{ij} - x_{i1}z_{1j}/y_{11} - y_{1j}z_{i1}/y_{11} + x_{i1}x_{1j}z_{11}/x_{11}y_{11}$ when $i \neq 1$ and $j \neq 1$. Let ϕ' denote the induced map of $\overline{\phi}$ from $\mathcal{R}_{S'}(\widetilde{\mathbf{D}}')$ to $\mathcal{R}_{S_{x_{11}y_{11}}}(\mathbf{D})$. Let ψ and ψ' denote the map from $T_{x_{11}y_{11}}$ to $\mathcal{R}_{S_{x_{11}y_{11}}}(\mathbf{D})$ and T' to $\mathcal{R}_{S'}(\widetilde{\mathbf{D}}')$. We obtain the following diagram:

$$\begin{array}{ccc} T' & \longrightarrow & T_{x_{11}y_{11}} \\ {\scriptstyle \psi'}\downarrow & & \downarrow{\scriptstyle \overline{\phi}} \\ \mathcal{R}_{S'}(\widetilde{\mathbf{D}}') & \underset{\phi'}{\longrightarrow} & \mathcal{R}_{S_{x_{11}y_{11}}}(\mathbf{D}) \end{array}$$

$\overline{\phi}(z'_{ij})$ is defined to ensure the commutativity of the diagram. It is sufficient to show $\phi'(\psi'(z'_{ij})) = \psi(\overline{\phi}(z'_{ij}))$, which is straightforward by the following equations.

$$\begin{aligned} \psi(\overline{\phi}(z'_{ij})) &= \psi(z_{ij} - x_{i1}z_{1j}/y_{11} - y_{1j}z_{i1}/y_{11} + x_{i1}x_{1j}z_{11}/x_{11}y_{11}) \\ &= x_{ij} - y_{ij} - x_{i1}(x_{1j} - y_{1j})/y_{11} - y_{1j}(x_{i1} - y_{i1})/y_{11} \\ &\quad + x_{i1}x_{1j}(x_{11} - y_{11})/x_{11}y_{11}, \end{aligned}$$

and

$$\begin{aligned} \phi'(\psi'(z'_{ij})) &= \phi'(x'_{ij} - y'_{ij}) \\ &= x_{ij} - y_{ij} - x_{i1}x_{1j}/x_{11} + y_{i1}y_{1j}/y_{11} \\ &= x_{ij} - y_{ij} - x_{i1}(x_{1j} - y_{1j})/y_{11} - y_{1j}(x_{i1} - y_{i1})/y_{11} \\ &\quad + x_{i1}x_{1j}(x_{11} - y_{11})/x_{11}y_{11}. \end{aligned}$$

Hence, ϕ' is an isomorphism. Let $\mathbf{D}' = (\{x'_{ij} - y'_{ij}\}_{2\leq i\leq m,\ 2\leq j\leq n})$. Then, by the induction hypothesis, the defining ideal of $\mathcal{R}_{S'}(\mathbf{D}')$ in T' is of the form $\mathcal{L}' = \{I_{s_1-1}(X'_{s_1-1,t_1-1}), I_{s_2-1}(Y'_{s_2-1,t_2-1}), g'_{ij,lk}, f'_{a_2,\dots,a_{s_1}}\}$, where

$$g'_{ij,lk} = \begin{vmatrix} z'_{ij} & z'_{lk} \\ x'_{ij} - y'_{ij} & x'_{lk} - y'_{lk} \end{vmatrix}$$

$$f'_{a_2,\dots,a_{s_1}} = \sum_{q=2}^{s_2} (-1)^{q+1} \left| \begin{bmatrix} Z'^{q,q} \\ Y'^{2,q-1} \\ X'^{q+1,m} \end{bmatrix}_{a_2\cdots a_{s_1}} \right|$$

with $2 \leq a_2 < \cdots < a_{s_1} \leq \min\{t_1, t_2\}$ and $2 \leq i \leq m = s_1$, $2 \leq l \leq m = s_1$, $2 \leq j \leq n$, $2 \leq k \leq n$. Let W denote the set of Koszul relations:

$$g^1{}_{ij,lk} = \begin{vmatrix} z_{ij} & z'_{lk} \\ x_{ij} - y_{ij} & x'_{lk} - y'_{lk} \end{vmatrix}$$

with $i = 1$ or $j = 1$ and

$$g^2{}_{ij,lk} = \begin{vmatrix} z_{ij} & z_{lk} \\ x_{ij} - y_{ij} & x_{lk} - y_{lk} \end{vmatrix}$$

with $i = 1$ or $j = 1$ and $l = 1$ or $k = 1$. Then $(\mathcal{L}', W)$ is the defining ideal of $\mathcal{R}_{S'}(\widetilde{\mathbf{D}}') \cong \mathcal{R}_{S_{x_{11}y_{11}}}(\mathbf{D})$. Once we show that $\overline{\phi}(\mathcal{L}', W) \subset \mathcal{L}_{x_{11}y_{11}}$, then $\mathcal{L}_{x_{11}y_{11}} = \mathcal{K}_{x_{11}y_{11}}$.

From the way we define the map $\overline{\phi}$, we have

$$\overline{\phi}(I_{s_1-1}(X'_{s_1-1,t_1-1}),\ I_{s_2-1}(Y'_{s_2-1,t_2-1}), g'_{ij,lk}, W) \subset \mathcal{L}_{x_{11}y_{11}}.$$

Notice the following equality:

$$\overline{\phi}(f'_{a_2,\dots,a_{s_1}}) = \frac{1}{y_{11}} f_{1,a_2,\dots,a_{s_1}} - \frac{z_{11}}{x_{11}y_{11}} |X_{1a_2\cdots a_{s_1}}|;$$

hence, $\overline{\phi}(f'_{a_2,\dots,a_{s_1}}) \in \mathcal{L}_{x_{11}y_{11}}$. $\square$

3. Some linear algebra. This section details some determinantal identities to be used in the proof of Section 4.

The following lemma writes the determinant of a certain matrix in x and y variables in terms of y variables and differences of $x_{ij} - y_{ij}$.

Lemma 3.1. *With notation as in Definition 2.3, for fixed i,j, $1 \leq i \leq j \leq n$,*

$$\begin{vmatrix} & & Y^{1,i-1}_{1,\dots,n} & & & \\ y_{i,1} & \cdots & y_{i,j} & x_{i,j+1} & \cdots & x_{i,n} \\ & & X^{i+1,n}_{1,\dots,n} & & & \end{vmatrix}$$
$$= |Y| + \sum_{k=j+1}^{n} (-1)^{i+k} \begin{vmatrix} Y^{1,i-1}_{1\cdots\hat{k}\cdots n} \\ X^{i+1,n}_{1\cdots\hat{k}\cdots n} \end{vmatrix} (x_{ik} - y_{ik})$$
$$+ \sum_{l=i+1}^{n} \sum_{k=1}^{n} (-1)^{l+k} \begin{vmatrix} Y^{1,l-1}_{1\cdots\hat{k}\cdots n} \\ X^{l+1,n}_{1\cdots\hat{k}\cdots n} \end{vmatrix} (x_{lk} - y_{lk}).$$

Proof. This lemma can be proved by using Lemma 2.4 and induction on i. □

In order to show the remainders of the S-pairs between elements of the ideal $\mathcal{L}$ are zero, we often use the Koszul relations repeatedly. The following two lemmas give conditions when we can use the Koszul relations. We omit the proofs which are basic linear algebra.

Lemma 3.2. *Let* $1 \leq i,\ l \leq m,\ 1 \leq j,\ k \leq n,\ a_1 < a_2 < a_3$. *Let*

$$g_{ij,lk} = \begin{vmatrix} z_{ij} & z_{lk} \\ x_{ij} - y_{ij} & x_{lk} - y_{lk} \end{vmatrix}, \qquad M = \begin{vmatrix} z_{1a_1} & z_{1a_2} & z_{1a_3} \\ x_{1a_1} & x_{1a_2} & x_{1a_3} \\ y_{1a_1} & y_{1a_2} & y_{1a_3} \end{vmatrix}.$$

Then

$$M = y_{1a_1} g_{1a_2,1a_3} - y_{1a_2} g_{1a_1,1a_3} + y_{1a_3} g_{1a_1,1a_2}.$$

Notice that, in Lemma 3.2, the matrix M can be replaced by any matrix containing three rows of z_i's, x_i's and y_i's and yield a similar result; the determinant is in the ideal generated by $\{g_{ij,lk}\}$'s.

Lemma 3.3. *Let* $g_{ij,lk}$ *be as in Lemma* 3.2. *Then*

$$\begin{aligned} M' &= \begin{vmatrix} z_{1a_1} & z_{1a_2} \\ x_{2a_1} - y_{2a_1} & x_{2a_2} - y_{2a_2} \end{vmatrix} \\ &= g_{1a_1,2a_2} - g_{1a_2,2a_1} + \begin{vmatrix} x_{1a_1} - y_{1a_1} & x_{1a_2} - y_{1a_2} \\ z_{2a_1} & z_{2a_2} \end{vmatrix}. \end{aligned}$$

Notice that, in Lemma 3.3, the matrix M' can be replaced by any matrix containing two rows of z_{la_j}'s and $x_{ia_j} - y_{ia_j}$ with $l < i$ and yield the similar result: the determinant is the sum of elements in the ideal generated by $\{g_{la_j,ia_k}\}$ and the determinant of a matrix containing two rows of z_{ia_j}'s and $x_{la_j} - y_{la_j}$.

4. Gröbner basis. This section is devoted to proving Lemma 2.7. We will recall Buchberger's criterion and give several lemmas that will help us reduce the computations of S-pairs between elements of $\mathcal{L}$. We define several polynomials and show those polynomials sit inside the ideal $\mathcal{L}$. Those polynomials are quite complicated; hence we will write each class of polynomials as one definition. And the remarks and lemmas following those definitions show those polynomials indeed sit inside the ideal $\mathcal{L}$. Theorem 4.18 will show the collection of those classes of polynomials is a Gröbner basis of $\mathcal{L}$ via a particular term ordering. The proof of Theorem 4.18 will be broken down as several lemmas computing the S-pairs of the elements and showing all of the remainders of S-pairs are zero. Each lemma will show the remainders of S-pairs between two classes of polynomials are zero.

Let $I = (g_1, \dots, g_s)$ be an ideal in a polynomial ring with a fixed term ordering. Define

$$\mathrm{in}\,(g_j)/\gcd\,(\mathrm{in}\,(g_i), \mathrm{in}\,(g_j)) = m_{ji},$$
$$\mathrm{in}\,(g_i)/\gcd\,(\mathrm{in}\,(g_i), \mathrm{in}\,(g_j)) = m_{ij},$$

and

$$m_{ji}g_i - m_{ij}g_j = \sum f_u^{(ij)} g_u + h_{g_i g_j},$$

where $\mathrm{in}\,(m_{ji}g_i) > \mathrm{in}\,(f_u^{(ij)} g_u)$ for all u.

Theorem 4.1 (Buchberger's criterion). *The elements $g_1, \dots, g_s$ form a Gröbner basis if and only if $h_{g_i g_j} = 0$ for all i and j.*

The polynomial $m_{ji}g_i - m_{ij}g_j$ is commonly referred to as the S-pair between g_i and g_j, and $h_{g_i g_j}$ is called the remainder.

Using Buchberger's criterion, we obtain a Gröbner basis of $\mathcal{L}$. Since we focus on determinantal rings, the computation of S-pairs between elements involves the values of matrix determinants. For computational purposes, we provide the following definition.

Definition 4.2. Let $k[X]$ be a polynomial ring with a fixed term ordering where X is an $m \times n$ matrix. Given two polynomials p_1 and p_2 in $k[X]$, we define $m_{12} = \mathrm{in}\,(p_1)/\gcd\,(\mathrm{in}\,(p_1), \mathrm{in}\,(p_2))$ and $m_{21} = \mathrm{in}\,(p_2)/\gcd\,(\mathrm{in}\,(p_1), \mathrm{in}\,(p_2))$. Assume $m_{12} = x_{u_1 a_1} \cdots x_{u_r a_r}$ and

$m_{21} = x_{v_1 b_1} \cdots x_{v_w b_w}$. Then define the matrix

$$M_{12} := \begin{bmatrix} x_{u_1 a_1} & \cdots & x_{u_1 a_r} \\ x_{u_2 a_1} & \cdots & x_{u_2 a_r} \\ \vdots & & \vdots \\ x_{u_r a_1} & & x_{u_r a_r} \end{bmatrix}$$

and the matrix

$$M_{21} := \begin{bmatrix} x_{v_1 b_1} & \cdots & x_{v_1 b_w} \\ x_{v_2 b_1} & \cdots & x_{v_2 b_w} \\ \vdots & & \vdots \\ x_{v_w b_1} & & x_{v_w b_w} \end{bmatrix}.$$

The following lemma helps us replace a polynomial with a leading term involving $x_{i,j}$'s by a polynomial with a leading term not involving $x_{i,j}$'s.

Lemma 4.3. *Let* $1 \le a_1 < \cdots < a_{s_1} \le n$, $1 \le r \le s_1$, *and let* $g_{i_1 j_1, i_2 j_2}$ *be as in Lemma* 2.5. *Then*:

$$\left| \begin{bmatrix} Y^{1,r-1} \\ Z^{r,r} \\ X^{r+1,s_1} \end{bmatrix}_{a_1,\dots,a_{s_1}} \right| = \left| \begin{bmatrix} Y^{1,r-1} \\ Z^{r,r} \\ Y^{r+1,s_1} \end{bmatrix}_{a_1,\dots,a_{s_1}} \right| + \sum_{u=r+1}^{s_1} \left| \begin{bmatrix} Y^{1,r-1} \\ X^{r,r} - Y^{r,r} \\ Y^{r+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix}_{a_1,\dots,a_{s_1}} \right| + \sum_{u=r+1}^{s_1} \sum_{\{c_1,c_2,d_1,\dots,d_{s_1-2}\}=\{a_1,\dots,a_{s_1}\}} \pm \left(g_{rc_1,uc_2} - g_{rc_2,uc_1}\right) \left| \begin{bmatrix} Y^{1,r-1} \\ Y^{r+1,u-1} \\ X^{u+1,s_1} \end{bmatrix}_{d_1,\dots,d_{s_1-2}} \right|.$$

Proof. Column indices can be dropped in this proof. Lemma 3.1 implies

$$\left|\left[\begin{matrix} Y^{1,r-1} \\ Z^{r,r} \\ X^{r+1,s_1} \end{matrix}\right]\right| = \left|\left[\begin{matrix} Y^{1,r-1} \\ Z^{r,r} \\ Y^{r+1,s_1} \end{matrix}\right]\right| + \sum_{u=r+1}^{s_1} \left|\left[\begin{matrix} Y^{1,r-1} \\ Z^{r,r} \\ Y^{r+1,u-1} \\ X^{u,u} - Y^{u,u} \\ X^{u+1,s_1} \end{matrix}\right]\right|.$$

Notice that

$$\begin{aligned}
&\left|\left[\begin{matrix} Y^{1,r-1} \\ Z^{r,r} \\ Y^{r+1,u-1} \\ X^{u,u} - Y^{u,u} \\ X^{u+1,s_1} \end{matrix}\right]\right| \\
&= \sum_{\{c_1,c_2,d_1,\dots,d_{s_1-2}\}=\{a_1,\dots,a_{s_1}\}} \\
&\quad \pm \left|\left[\begin{matrix} z_{rc_1} & z_{rc_2} \\ x_{uc_1} - y_{uc_1} & x_{uc_2} - y_{uc_2} \end{matrix}\right]\right| \left|\left[\begin{matrix} Y^{1,r-1} \\ Y^{r+1,u-1} \\ X^{u+1,s_1} \end{matrix}\right]_{d_1,\dots,d_{s_1-2}}\right| \\
&= \sum_{\{c_1,c_2,d_1,\dots,d_{s_1-2}\}=\{a_1,\dots,a_{s_1}\}} \\
&\quad \pm \left(g_{rc_1,uc_2} - g_{rc_2,uc_1} + \left|\left[\begin{matrix} x_{rc_1} - y_{rc_1} & x_{rc_2} - y_{rc_2} \\ z_{uc_1} & z_{uc_2} \end{matrix}\right]\right|\right) \\
&\quad \times \left|\left[\begin{matrix} Y^{1,r-1} \\ Y^{r+1,u-1} \\ X^{u+1,s_1} \end{matrix}\right]_{d_1,\dots,d_{s_1-2}}\right| \\
&= \sum_{\{c_1,c_2,d_1,\dots,d_{s_1-2}\}=\{a_1,\dots,a_{s_1}\}} \\
&\quad \pm (g_{rc_1,uc_2} - g_{rc_2,uc_1}) \left|\left[\begin{matrix} Y^{1,r-1} \\ Y^{r+1,u-1} \\ X^{u+1,s_1} \end{matrix}\right]_{d_1,\dots,d_{s_1-2}}\right| + \left|\left[\begin{matrix} Y^{1,r-1} \\ X^{r,r} - Y^{r,r} \\ Y^{r+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{matrix}\right]\right|. \quad \square
\end{aligned}$$

While computing the Gröbner bases, we encounter determinants which are very similar to those in the following lemma. We reduce this kind of case here, i.e., the lemma enables the determinant to be written as a combination of elements of $I_{s_2}(Y)$ and various $g_{i_1j_1,i_2j_2}$.

Lemma 4.4. *Let* $a_1 < \cdots < a_{s_1+1}$ *and* $1 \le r \le s_1$. *One has*

$$\sum_{u=r}^{s_2} \left| \begin{bmatrix} X^{r,r} \\ Y^{1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix}_{a_1,\dots,a_{s_1+1}} \right| \in I_{s_2}(Y) + (g_{i_1j_1,i_2j_2} |\ 1 \le i_v \le m,\ 1 \le j_v \le n,\ v = 1,2).$$

Proof. The column indices are omitted again. First we write

$$\sum_{u=r}^{s_2} \left| \begin{bmatrix} X^{r,r} \\ Y^{1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix} \right| = \left| \begin{bmatrix} X^{r,r} \\ Y^{1,r-1} \\ Z^{r,r} \\ X^{r+1,s_1} \end{bmatrix} \right| + \sum_{u=r+1}^{s_2} \left| \begin{bmatrix} X^{r,r} \\ Y^{1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix} \right| = \alpha + \beta.$$

Then, using Lemmas 3.2 and 4.3, we obtain

$$\begin{aligned} \alpha &= \left| \begin{bmatrix} X^{r,r} \\ Y^{1,r-1} \\ Z^{r,r} \\ X^{r+1,s_1} \end{bmatrix} \right| = \left| \begin{bmatrix} X^{r,r} - Y^{r,r} \\ Y^{1,r-1} \\ Z^{r,r} \\ X^{r+1,s_1} \end{bmatrix} \right| + \left| \begin{bmatrix} Y^{r,r} \\ Y^{1,r-1} \\ Z^{r,r} \\ X^{r+1,s_1} \end{bmatrix} \right| \\ &= \sum_{\{c_1,c_2,d_1,\dots,d_{s_1-1}\}=\{a_1,\dots,a_{s_1+1}\}} \\ &\quad \pm g_{rc_1,rc_2} \left| \begin{bmatrix} Y^{1,r-1} \\ X^{r+1,s_1} \end{bmatrix}_{d_1,\dots,d_{s_1-1}} \right| + \left| \begin{bmatrix} Y^{r,r} \\ Y^{1,r-1} \\ Z^{r,r} \\ Y^{r+1,s_1} \end{bmatrix} \right| \\ &\quad + \sum_{u=r+1}^{s_1} \sum_{\{c_1,c_2,d_1,\dots,d_{s_1-1}\}=\{a_1,\dots,a_{s_1+1}\}} \\ &\quad \pm (g_{rc_1,uc_2} - g_{rc_2,uc_1}) \left| \begin{bmatrix} Y^{r,r} \\ Y^{1,r-1} \\ Y^{r+1,u-1} \\ X^{u+1,s_1} \end{bmatrix}_{d_1,\dots,d_{s_1-1}} \right| \end{aligned}$$

$$+\sum_{u=r+1}^{s_2}\left|\begin{bmatrix} Y^{r,r}\\ Y^{1,r-1}\\ X^{r,r}-Y^{r,r}\\ Y^{r+1,u-1}\\ Z^{u,u}\\ X^{u+1,s_1}\end{bmatrix}\right|$$

$$+\sum_{u=s_2+1}^{s_1}\left|\begin{bmatrix} Y^{r,r}\\ Z^{r,r}\\ Y^{1,r-1}\\ Y^{r+1,u-1}\\ X^{u,u}-Y^{u,u}\\ X^{u+1,s_1}\end{bmatrix}\right| = \alpha_1+\alpha_2,$$

where

$$\alpha_1 = \sum_{\{c_1,c_2,d_1,\dots,d_{s_1-1}\}=\{a_1,\dots,a_{s_1+1}\}}$$

$$\pm\, g_{rc_1,rc_2}\left|\begin{bmatrix} Y^{1,r-1}\\ X^{r+1,s_1}\end{bmatrix}_{d_1,\dots,d_{s_1-1}}\right| + \left|\begin{bmatrix} Y^{r,r}\\ Y^{1,r-1}\\ Z^{r,r}\\ Y^{r+1,s_1}\end{bmatrix}\right|$$

$$+\sum_{u=r+1}^{s_2}\ \sum_{\{c_1,c_2,d_1,\dots,d_{s_1-1}\}=\{a_1,\dots,a_{s_1+1}\}}$$

$$\pm\,(g_{rc_1,uc_2}-g_{rc_2,uc_1})\left|\begin{bmatrix} Y^{r,r}\\ Y^{1,r-1}\\ Y^{r+1,u-1}\\ X^{u+1,s_1}\end{bmatrix}_{d_1,\dots,d_{s_1-1}}\right|$$

$$+\sum_{u=s_2+1}^{s_1}\left|\begin{bmatrix} Y^{r,r}\\ Z^{r,r}\\ Y^{1,r-1}\\ Y^{r+1,u-1}\\ X^{u,u}-Y^{u,u}\\ X^{u+1,s_1}\end{bmatrix}\right|,$$

and

$$\alpha_2 = \sum_{u=r+1}^{s_2} \left| \begin{bmatrix} Y^{r,r} \\ Y^{1,r-1} \\ X^{r,r} - Y^{r,r} \\ Y^{r+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix} \right| .$$

After removing the repeated row y_r in α_2, we have:

$$\alpha_2 = \sum_{u=r+1}^{s_2} \left| \begin{bmatrix} Y^{r,r} \\ Y^{1,r-1} \\ X^{r,r} - Y^{r,r} \\ Y^{r+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix} \right| = \sum_{u=r+1}^{s_2} \left| \begin{bmatrix} Y^{r,r} \\ Y^{1,r-1} \\ X^{r,r} \\ Y^{r+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix} \right|$$

$$= - \sum_{u=r+1}^{s_2} \left| \begin{bmatrix} X^{r,r} \\ Y^{1,r-1} \\ Y^{r,r} \\ Y^{r+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix} \right|$$

$$= - \sum_{u=r+1}^{s_2} \left| \begin{bmatrix} X^{r,r} \\ Y^{1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix} \right| = -\beta.$$

Therefore $\alpha + \beta = \alpha_1 + \alpha_2 + \beta = \alpha_1$ and the element α_1 is in $I_{s_2}(Y_{s_2 t_2}) + (g_{i_1 j_1, i_2 j_2})$. ◻

Since, in our case, the polynomials are sums of products of determinants coming from the same column indices, instead of writing each polynomial as a sum of monomials, we write each polynomial as a sum of determinants. This way, we can simplify the notations and computations. The following definition will be a key point to help us reduce the computations of S-pairs between elements.

Definition 4.5. Let $k[X, Y, Z]$ be a polynomial ring with X, Y, Z as $m \times n$ matrices of variables over the field k, and we fix a term ordering in $k[X,Y,Z]$. Let G be a collection of polynomials in the ring $k[X,Y,Z]$. Let $P^u_{a_1,\dots,a_{q_u}}$, $u \in I$, be an element of G such that each $P^u_{a_1,\dots,a_{q_u}}$ is the sum of determinants P^u_i of $m \times m$ matrices with the same column indices, $a_1,\dots,a_{q_u}$, in variables X, Y and Z. Denote $P^u_{a_1,\dots,a_{q_u}} = \sum_{i=1}^{p_u} P^u_i$ with P^u_1 containing the leading term of $P^u_{a_1,\dots,a_{q_u}}$. For example, the element $f_{a_1,\dots,a_{s_1}}$ in Lemma 2.5 is written as $f_{a_1,\dots,a_{s_1}} = \sum_{i=1}^{s_2} f_i$.

Given $P^u_{a_1,\dots,a_{q_u}}$ and $P^v_{b_1,\dots,b_{q_v}}$ in G, we can define m_{12}, m_{21}, M_{12} and M_{21} as in Definition 4.2 by setting $p_1 = P^u_{a_1,\dots,a_{q_u}}$ and $p_2 = P^v_{b_1,\dots,b_{q_v}}$. Assume M_{12} has column indices $c_1,\dots,c_{p_{12}}$ and M_{21} has column indices $d_1,\dots,d_{p_{21}}$. Define $\overline{P^u_{a_1,\dots,a_{q_u},d_1,\dots,d_{p_{21}}}} = \sum_{i=1}^{p_u} \overline{P^u_i}$ and $\overline{P^v_{b_1,\dots,b_{q_v},c_1,\dots,c_{p_{12}}}} = \sum_{i=1}^{p_v} \overline{P^v_i}$ as follows: $\overline{P^u_i}$ is the determinant of the matrix having rows of M_{21} and rows of the matrix of P^u_i with column indices $a_1,\dots,a_{q_u},d_1,\dots,d_{p_{21}}$, and $\overline{P^v_i}$ is the determinant of the matrix having rows of M_{12} and rows of the matrix of P^v_i with column indices $b_1,\dots,b_{q_v},c_1,\dots,c_{p_{12}}$. Note that each P^u_i and P^v_i comes from the determinant of square matrices, and the way we define those new matrices will insure each matrix as a square matrix again. Also the leading term of $\overline{P^u_{a_1,\dots,a_{q_u},d_1,\dots,d_{p_{21}}}}$ and $\overline{P^v_{b_1,\dots,b_{q_v},c_1,\dots,c_{p_{12}}}}$ will be contained in $\overline{P^u_1}$ and $\overline{P^v_1}$. Moreover, the leading term of $\overline{P^u_{a_1,\dots,a_{q_u},d_1,\dots,d_{p_{21}}}}$ and $\overline{P^v_{b_1,\dots,b_{q_v},c_1,\dots,c_{p_{12}}}}$ will be equal up to sign difference. For example in Lemma 2.5, we have $f_{a_1,\dots,a_{s_1}} = f^1_{a_1,\dots,a_{s_1}}$ and $f_{b_1,a_2,\dots,a_{s_1}} = f^2_{b_1 a_2,\dots,a_{s_1}}$. Then $m_{12} = M_{12} = z_{1,a_1}$ and $m_{21} = M_{21} = z_{1,b_1}$, which implies

$$\overline{f^1_{b_1,a_1,a_2,\dots,a_{s_1}}} = \sum_{q=1}^{s_2}(-1)^{q+1}\left|\begin{bmatrix} Z^{1,1} \\ Z^{q,q} \\ Y^{1,q-1} \\ X^{q+1,m}\end{bmatrix}_{b_1,a_1,a_2,\dots,a_{s_1}}\right|$$
$$= \sum_{i=1}^{s_2}\overline{f^1_i} = -\overline{f^2_{a_1,b_1,a_2,\dots,a_{s_1}}} = -\sum_{i=1}^{s_2}\overline{f^2_i}.$$

When we compute the S-pair between two polynomials, we consider the initial monomials of those two monomials. Since, in our cases, the polynomials are sums of determinants, we consider the determinants of submatrices that contain the initial monomials. We are going to use a similar computation technique repeatedly, and the following lemma shows why this technique proves the reminders of S-pairs are zero.

Lemma 4.6. *With the above notation, assume*

$$\begin{aligned} \mathrm{in}\,(m_{21}P^u_{a_1,\dots,a_{q_u}}) &= \mathrm{in}\,(|M_{21}|P^u_{a_1,\dots,a_{q_u}}) \\ &= \mathrm{in}\,(\overline{P^u_{a_1,\dots,a_{q_u},d_1,\dots,d_{p_{21}}}}) = \mathrm{in}\,(\overline{P^u_1}) \end{aligned}$$

and

$$\begin{aligned} \mathrm{in}\,(m_{12}P^v_{b_1,\dots,b_{q_v}}) &= \mathrm{in}\,(|M_{12}|P^v_{b_1,\dots,b_{q_v}}) = \mathrm{in}\,(\overline{P^v_{b_1,\dots,b_{q_v},c_1,\dots,c_{p_{12}}}}) \\ &= \mathrm{in}\,(\overline{P^v_1}). \end{aligned}$$

Furthermore, $\sum_{i=2}^{p_u}\overline{P^u_i}$ and $\sum_{i=2}^{p_v}\overline{P^v_i}$ can be written as a combination of elements of G with the leading term smaller than in$(\overline{P^u_1})$. Then the S-pairs of $P^u_{a_1,\dots,a_{q_u}}$ and $P^v_{b_1,\dots,b_{q_v}}$ have zero remainder.

Proof. From the definition of M_{12} and M_{21}, we have $\overline{P^u_1} = \overline{P^v_1}$. Hence, the following equation holds

$$\overline{P^u_{a_1,\dots,a_{q_u},d_1,\dots,d_{p_{21}}}} - \overline{P^v_{b_1,\dots,b_{q_v},c_1,\dots,c_{p_{12}}}} = \sum_{i=2}^{u}\overline{P^u_i} - \sum_{i=2}^{v}\overline{P^v_i}. \tag{4.1}$$

$\overline{P^u_{a_1,\dots,a_{q_u},d_1,\dots,d_{p_{21}}}}$ can be written as

$$\sum_{\{\alpha_1,\dots,\alpha_{p_{21}}\}\cup\{\beta_1,\dots,\beta_{q_u}\}=\{a_1,\dots,a_{q_u},d_1,\dots,d_{p_{21}}\}} |M^{21}_{\alpha_1,\dots,\alpha_{p_{21}}}|P^u_{\beta_1,\dots,\beta_{q_u}},$$

where $M^{21}_{\alpha_1,\dots,\alpha_{p_{21}}}$ has the same rows as M_{21} with columns, $\alpha_1,\dots,\alpha_{p_{21}}$ and $P^u_{\beta_1,\dots,\beta_{q_u}}$ is in G with columns, $\beta_1,\dots,\beta_{q_u}$. Similarly, $\overline{P^v_{b_1,\dots,b_{q_v},c_1,\dots,c_{p_{12}}}}$ can be written as

$$\sum_{\{\alpha_1,\dots,\alpha_{p_{21}}\}\cup\{\beta_1,\dots,\beta_{q_u}\}=\{a_1,\dots,a_{q_u},d_1,\dots,d_{p_{21}}\}} |M^{12}_{\alpha_1,\dots,\alpha_{p_{21}}}|P^v_{\beta_1,\dots,\beta_{q_u}}.$$

$|M_{21}|P^u_{a_1,\dots,a_{q_u}}$ and $|M_{12}|P^v_{b_1,\dots,b_{q_v}}$ are one of the summands, and their initial terms are the initial terms of each sum. After moving everything other than $m_{21}P^u_{a_1,\dots,a_{q_u}}$ and $m_{12}P^v_{b_1,\dots,b_{q_v}}$ from the left-hand side of (4.1) to the right-hand side, we obtain the equality:

$$m_{21}P^u_{a_1,\dots,a_{q_u}} - m_{12}P^v_{b_1,\dots,b_{q_v}} = \sum r_i g_i$$

with $g_i \in G$ and $\mathrm{in}\,(r_i g_i) < \mathrm{in}\,(m_{12}P^v_{b_1,\dots,b_{q_v}})$. □

Instead of computing S-pairs between elements of $\mathcal{L}$ and adding the remainder of those S-pairs, we will define some polynomials that are in the ideal $\mathcal{L}$ and show they are indeed in the ideal $\mathcal{L}$. Those polynomials will be part of the Gröbner basis of $\mathcal{L}$. Since it is a huge collection of classes of polynomials, and each class of polynomials is complicated, we will define each class of polynomials one by one. The lemmas or remarks following the definitions show those polynomials are indeed in the ideal $\mathcal{L}$. Once there is a new class of remainders from the S-pairs, we add it as a new class of polynomials. When we compute the S-pairs between elements of $f_{a_1,\dots,a_s}$'s, the remainders produce new polynomials, defined below.

Definition 4.7. Letting $1 \le a_1 < a_2 < \cdots < a_{s_1+k-1} \le \min\{t_1, t_2\}$, and $1 \le l \le k \le s_2$, we define $f^{l,k}_{a_1,\dots,a_{s_1+k-1}}$ as follows:

$$f^{l,k}_{a_1,\dots,a_{s_1+k-1}} := \sum_{r=k}^{s_2} (-1)^{r+1} \left| \begin{bmatrix} Z^{l,k-1} \\ Z^{r,r} \\ X^{1,l-1} \\ Y^{1,r-1} \\ Y^{r+1,s_1} \end{bmatrix}_{a_1,\dots,a_{s_1+k-1}} \right| + \sum_{r=k}^{s_2} (-1)^{r+1} \sum_{u=r+1}^{s_1} \left| \begin{bmatrix} Z^{l,k-1} \\ X^{r,r} - Y^{r,r} \\ X^{1,l-1} \\ Y^{1,r-1} \\ Y^{r+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix}_{a_1,\dots,a_{s_1+k-1}} \right|.$$

Lemma 4.8. *$f^{l,k}_{a_1,\dots,a_{s_1+k-1}} \in \mathcal{L}$.*

Proof. We first define $p^{l,k}_{a_1,\dots,a_{s_1+k-1}}$ as follows.

$$p^{l,k}_{a_1,\dots,a_{s_1+k-1}} = \sum_{r=k}^{s_2}(-1)^{r+1}\left|\begin{bmatrix} Z^{l,k-1} \\ Z^{r,r} \\ X^{1,l-1} \\ Y^{1,r-1} \\ X^{r+1,s_1} \end{bmatrix}_{a_1,\dots,a_{s_1+k-1}}\right|,$$

where $1 \le a_1 < a_2 < \cdots < a_{s_1+k-1} \le \min\{t_1, t_2\}$, and $1 \le l \le k \le s_2$. We notice that $p^{1,1}_{a_1,\dots,a_{s_1}} = f_{a_1,\dots,a_{s_1}}$. We will show $p^{l,k}_{a_1,\dots,a_{s_1+k-1}} \in \mathcal{L}$. Since $p^{l,k+1}_{a_1\cdots a_{s_1+k}} = \sum_{i=1}^{s_1+k}(-1)^{i+1} z_{ka_i} f^{l,k}_{a_1\cdots\hat{a_i}\cdots a_{s_1+k}}$ and $p^{l+1,l+1}_{a_1\cdots a_{s_1+l}} = \sum_{i=1}^{s_1+l}(-1)^{i+1} x_{la_i} p^{l,l}_{a_1\cdots\hat{a_i}\cdots a_{s_1+l}}$, we have that the $p^{l,k}_{a_1\cdots a_{s_1+k-1}}$'s are all in $\mathcal{L} \subset \mathcal{J}$. By Lemma 4.3, we have

$$\begin{aligned} p^{l,k}_{a_1,\dots,a_{s_1+k-1}} &= \sum_{r=k}^{s_2}(-1)^{r+1}\left|\begin{bmatrix} Z^{l,k-1} \\ Z^{r,r} \\ X^{1,l-1} \\ Y^{1,r-1} \\ X^{r+1,s_1} \end{bmatrix}_{a_1,\dots,a_{s_1+k-1}}\right| \\ &= \sum_{r=k}^{s_2}(-1)^{r+1}\left|\begin{bmatrix} Z^{l,k-1} \\ Z^{r,r} \\ X^{1,l-1} \\ Y^{1,r-1} \\ Y^{r+1,s_1} \end{bmatrix}_{a_1,\dots,a_{s_1+k-1}}\right| \\ &\quad + \sum_{r=k}^{s_2}(-1)^{r+1}\sum_{u=r+1}^{s_1}\left|\begin{bmatrix} Z^{l,k-1} \\ X^{r,r} - Y^{r,r} \\ X^{1,l-1} \\ Y^{1,r-1} \\ Y^{r+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix}_{a_1,\dots,a_{s_1+k-1}}\right| \\ &\quad + \sum_{r=k}^{s_2}(-1)^{r+1}\sum_{u=r+1}^{s_1}\sum_{\{c_1,c_2,d_1,\dots,d_{s_1+k-3}\}=\{a_1,\dots,a_{s_1+k-1}\}} \end{aligned}$$

$$\left(\pm(g_{rc_1,uc_2} - g_{rc_2,uc_1}) \left| \begin{bmatrix} Z^{l,k-1} \\ X^{1,l-1} \\ Y^{1,r-1} \\ Y^{r+1,u-1} \\ X^{u+1,s_1} \end{bmatrix}_{d_1,\dots,d_{s_1+k-3}} \right| \right).$$

Since $p^{l,k}_{a_1,\dots,a_{s_1+k-1}} \in \mathcal{L}$, and

$$\sum_{r=k}^{s_2}(-1)^{r+1} \sum_{u=r+1}^{s_1} \sum_{\{c_1,c_2,d_1,\dots,d_{s_1+k-3}\}=\{a_1,\dots,a_{s_1+k-1}\}} \pm(g_{rc_1,uc_2} - g_{rc_2,uc_1}) \left| \begin{bmatrix} Z^{l,k-1} \\ X^{1,l-1} \\ Y^{1,r-1} \\ Y^{r+1,u-1} \\ X^{u+1,s_1} \end{bmatrix}_{d_1,\dots,d_{s_1+k-3}} \right| \in \mathcal{L},$$

we have

$$f^{l,k}_{a_1,\dots,a_{s_1+k-1}} = \sum_{r=k}^{s_2}(-1)^{r+1} \left| \begin{bmatrix} Z^{l,k-1} \\ Z^{r,r} \\ X^{1,l-1} \\ Y^{1,r-1} \\ Y^{r+1,s_1} \end{bmatrix}_{a_1,\dots,a_{s_1+k-1}} \right| + \sum_{r=k}^{s_2}(-1)^{r+1} \sum_{u=r+1}^{s_1} \left| \begin{bmatrix} Z^{l,k-1} \\ X^{r,r} - Y^{r,r} \\ X^{1,l-1} \\ Y^{1,r-1} \\ Y^{r+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix}_{a_1,\dots,a_{s_1+k-1}} \right| \in \mathcal{L}. \quad \square$$

We now define couple of new polynomials that come from the remainders of the S-pairs of the generators of $\mathcal{L}$. We define those polynomials in the following order: we start with the generators of $\mathcal{L}$ and compute the S-pairs between those generators, then add the remainder if necessary. The newly defined polynomial is always coming from the

S-pairs between the polynomials defined earlier. The first encounter remainders are coming from the S-pairs of $X_{a_1,\dots,a_{s_1}}$ and $g_{ij,lk}$ as in Lemma 2.5, and those remainders are defined as new polynomials below.

Definition 4.9. Let $1 \le p_1 \le m$, $1 \le q_1 \le n$, $a_{s_1} < \cdots < a_{j+1} \le q_1 < a_{j-1} < \cdots < a_1$. We define $U_{p_1,q_1,a_1,\dots,a_{s_1}}$ as follows:

$$U_{p_1,q_1,a_{s_1},\dots,a_1} := z_{p_1q_1}\left|\begin{bmatrix} & & X^{1,p_1-1} & & & \\ x_{p_1a_{s_1}} & \cdots & x_{p_1a_j} & y_{p_1a_{j-1}} & \cdots & y_{p_1a_1} \\ & & Y^{p_1+1,s_1} & & & \end{bmatrix}\right|$$
$$+ \sum_{k=j+1}^{m} (x_{p_1q_1} - y_{p_1q_1})(-1)^{k+p_1} z_{p_1a_k}|X^{1,\dots,\hat{p_1},\dots,m}_{a_m,\dots,\hat{a_k},\dots,a_1}|$$
$$+ \sum_{u=p_1+1}^{s_1} (x_{p_1q_1} - y_{p_1q_1})$$
$$\times \left|\begin{bmatrix} & & X^{1,p_1-1} & & & \\ x_{p_1a_{s_1}} & \cdots & x_{p_1a_j} & y_{p_1a_{j-1}} & \cdots & y_{p_1a_1} \\ & & Y^{p_1+1,u-1} & & & \\ & & Z^{u,u} & & & \\ & & X^{u+1,s_1} & & & \end{bmatrix}_{a_{s_1},\dots,a_1}\right|.$$

Lemma 4.10. $U_{p_1q_1a_{s_1},\dots,a_1} \in \mathcal{L}$.

Proof. We use Lemma 4.3 on $|X^{1,s_1}_{a_{s_1},\dots,a_1}|$. Then

$$\begin{aligned}\alpha &= z_{p_1,q_1}|X^{1,s_1}_{a_{s_1},\dots,a_1}| \\ &= z_{p_1q_1}\Bigg(\left|\begin{bmatrix} & & X^{1,p_1-1} & & & \\ x_{p_1a_{s_1}} & \cdots & x_{p_1a_j} & y_{p_1a_{j-1}} & \cdots & y_{p_1a_1} \\ & & Y^{p_1+1,s_1} & & & \end{bmatrix}\right| \\ &\quad + \sum_{k=j+1}^{m} (-1)^{k+p_1}(x_{p_1a_k} - y_{p_1a_k})|X^{1,\dots,\hat{p_1},\dots,m}_{a_m,\dots,\hat{a_k},\dots,a_1}| \\ &\quad + \sum_{u=p_1+1}^{s_1}\sum_{k=1}^{s_1}(-1)^{u+k}(x_{ua_k} - y_{ua_k})\end{aligned}$$

$$\times \left| \begin{bmatrix} & X^{1,p_1-1} & \\ x_{p_1a_{s_1}} \cdots & x_{p_1a_j}y_{p_1a_{j-1}} \cdots & y_{p_1a_1} \\ & X^{p_1+1,u-1} & \\ & X^{u+1,s_1} & \end{bmatrix}_{a_m,\dots,\widehat{a_k},\dots,a_1} \right| \Bigg).$$

We substitute all the monomials that are the leading terms of $\{g_{ij,lk}\}$. The above expression becomes:

$$\begin{aligned} & z_{p_1q_1} \left| \begin{bmatrix} & X^{1,p_1-1} & \\ x_{p_1a_{s_1}} \cdots & x_{p_1a_j}y_{p_1a_{j-1}} \cdots & y_{p_1a_1} \\ & Y^{p_1+1,s_1} & \end{bmatrix} \right| \\ & + \sum_{k=j+1}^{s_1} (-1)^{p_1+k} g_{p_1q_1,p_1a_k} |X^{1,\dots,\hat{p_1},\dots,m}_{a_m,\dots,\hat{a_k},\dots,a_{s_1}}| \\ & + \sum_{k=j+1}^{m} (x_{p_1q_1} - y_{p_1q_1})(-1)^{k+p_1} z_{p_1a_k} |X^{1,\dots,\hat{p_1},\dots,m}_{a_m,\dots,\hat{a_k},\dots,a_1}| \\ & + \sum_{u=p_1+1}^{s_1} \sum_{k=1}^{m} (-1)^{k+u} (g_{p_1q_1,ua_k} - g_{p_1a_k,uq_1}) \\ & \times \left(\left| \begin{bmatrix} & X^{1,p_1-1} & \\ x_{p_1a_{s_1}} \cdots & x_{p_1a_j}y_{p_1a_{j-1}} \cdots & y_{p_1a_1} \\ & Y^{p_1+1,u-1} & \\ & X^{u+1,s_1} & \end{bmatrix}_{a_m,\dots,\widehat{a_k},\dots,a_1} \right| \right) \\ & + \sum_{u=p_1+1}^{s_1} (x_{p_1q_1} - y_{p_1q_1}) \\ & \times \left| \begin{bmatrix} & X^{1,p_1-1} & \\ x_{p_1a_{s_1}} \cdots & x_{p_1a_j}y_{p_1a_{j-1}} \cdots & y_{p_1a_1} \\ & Y^{p_1+1,u-1} & \\ & Z^{u,u} & \\ & X^{u+1,s_1} & \end{bmatrix}_{a_{s_1},\dots,a_1} \right|. \end{aligned}$$

Let β be

$$\beta = \sum_{k=j+1}^{s_1} (-1)^{p_1+k} g_{p_1q_1,p_1a_k} \left| X^{1,\dots,\hat{p_1},\dots,m}_{a_m,\dots,\hat{a_k},\dots,a_1} \right|$$

$$+\sum_{u=p_1+1}^{s_1}\sum_{k=1}^{m}(-1)^{k+u}(g_{p_1q_1,ua_k}-g_{p_1a_k,uq_1})$$
$$\times\left(\left|\begin{bmatrix} X^{1,p_1-1} \\ x_{p_1a_m}\cdots x_{p_1a_j}y_{p_1a_{j-1}}\cdots y_{p_1a_1} \\ Y^{p_1+1,u-1} \\ X^{u+1,s_1}\end{bmatrix}_{a_m,\dots,\widehat{a_k},\dots,a_1}\right|\right).$$

β is in $\mathcal{L}$ and α is in $\mathcal{L}$; hence, $U_{p_1q_1a_{s_1},\dots,a_1}=\alpha-\beta$ is in $\mathcal{L}$. □

While computing the S-pairs of $U_{p,q,a_{s_1},\dots,a_1}$ as in Definition 4.9 and $Y_{a_1,\dots,a_{s_2}}$ as in Lemma 2.5, the remainders produce new polynomials, defined below.

Definition 4.11. Let $1\le b_{s_2}<\cdots<b_1\le n$, $1\le p_1\le m$, $1\le q_1\le n$, $a_{s_1}<\cdots<a_{s_2+1}<a_{p_1}<\cdots<a_1$ and $a_{p_1}\le q_1$. Let i be an integer so that $1\le i\le p$ and $a_{s_2+1}<b_{s_2}<\cdots<b_{i+1}<a_{p_1-1}\le b_i$, and let $b_l\ne a_{p_1}$ for $l\ge i+1$.

We define M_{12} as follows:

$$M_{12}=z_{p_1q_1}x_{p_1a_{p_1}}\left|\begin{bmatrix} X^{1,p_1-1} \\ Y^{s_2+1,s_1}\end{bmatrix}_{a_{s_1},\dots,a_{s_2+1},a_{p_1-1},\dots,a_1}\right|.$$

We define

$$\begin{aligned}W_{p_1,q_1,a_{s_1},\dots,a_{s_2+1},a_{p_1},\dots a_1,b_{s_2},\dots,b_1} &:= M_{12}|Y^{1,s_2}_{b_{s_2},\dots,b_1}| \\ &\quad-\left|Y^{1,i}_{b_i,\dots,b_1}\right|\sum_{\{c_{p_1},\dots,c_{i+1},d_{s_2},\dots,d_{p_1+1}\}=\{b_{s_2},\dots,b_{i+1}\}} \\ &\quad\times\left|Y^{i+1,p_1}_{c_{p_1},\dots,c_{i+1}}\right|U_{p_1q_1,a_{s_1},\dots,a_{s_2+1},d_{s_2},\dots,d_{p_1+1},a_{p_1},\dots,a_1}.\end{aligned}$$

Remark. From the way we define $W_{p_1,q_1,a_{s_1},\dots,a_{s_2+1},a_{p_1},\dots a_1,b_{s_2},\dots,b_1}$, it is in $\mathcal{L}$. Notice that all the submatrices $|Y^{p_1,s_2}_{d_{s_2},\dots,d_{p_1}}|$ of $|Y^{1,s_2}_{b_{s_2},\dots,b_1}|$

such that $a_{s_2+1} < d_{s_2} < \cdots < d_{p_1+1} < a_{p_1-1}$ are canceled. Hence, the leading term is

$$\text{in}\,(M_{12}|Y^{1,i-1}_{b_{i-1},\ldots,b_1}||Y^{i,p_1}_{b_{p_1+1},\ldots,b_{i+1}}||Y^{p_1+1,s_2}_{b_{s_2},\ldots,b_{p_1+2},b_i}|).$$

While computing the S-pairs of $W_{p_1,q_1,a_{s_1},\ldots,a_{s_2+1},a_{p_1},\ldots a_1,b_{s_2},\ldots,b_1}$ as in Definition 4.11 and $U_{p,q,a_{s_1},\ldots,a_1}$ as in Definition 4.9, the remainders produce new polynomials, defined below.

Definition 4.12. Let $1 \leq p_1 \leq m = s_1$, $1 \leq q_1 \leq n$, $v = p_1+1,\ldots,s_2-1$, $1 \leq a_{s_1} < \cdots < a_{s_2+1} < a_{p_1} < \cdots < a_1 \leq t_1$ and $a_{p_1} \leq q_1$. Let i be an integer so that $1 \leq i \leq p$, and let $a_{s_2+1} < b_{s_2} < \cdots < b_{v+2} < b'_v < \cdots < b'_{p_1+1} < b_{p_1} < \cdots < b_{i+1} < a_{p_1-1} \leq b_{p_1+1}$ and $b'_l \neq a_{p_1}$ for $l \geq i+1$ and $b'_{v-1} \leq b_{v+1}$. Let $a_{s_1} < \cdots < a_{s_2+1} < b_{s_2} < \cdots < b_{v+2} < b_{v+1} < b_v < b_{v-1} < \cdots < b_{p_1+2} < a_{p_1} < a_{p_1-1} \leq b_{p_1+1}$, and $b'_r \leq b_{r+2} < b_{r+1}$ for $r = p_1,\ldots,v-2$.

We define

$$\begin{aligned} &W^{p_1+1,v}_{p_1,q_1,a_{s_1},\ldots,a_{s_2+1},a_{p_1},\ldots a_1,b_{s_2},\ldots,b_1,b'_v,\ldots,b'_{p_1+1}} \\ &\quad := y_{v-1,b'_{v-1}} y_{v,b_v} \\ &\qquad \times W^{p_1+1,v-2}_{p_1,q_1,a_{s_1},\ldots,a_{s_2+1},a_{p_1},\ldots,a_1,b_{s_2},\ldots,b_{v+1},b'_v,b_{v-1}b_1,b'_{v-2},\ldots,b'_{p_1+1}} \\ &\qquad - y_{v,b'_v} W^{p_1+1,v-1}_{p_1,q_1,a_{s_1},\ldots,a_{s_2+1},a_{p_1},\ldots a_1,b_{s_2},\ldots,b_1,b'_{v-1},\ldots,b'_{p_1+1}}. \end{aligned}$$

Here,

$$W^{p_1+1,p_1-1}_{p_1,q_1,a_{s_1},\ldots,a_{s_2+1},a_{p_1},\ldots a_1,b_{s_2},\ldots,b_1,} = U_{p_1,q_1,a_{s_1},\ldots,a_1}$$

and

$$W^{p_1+1,p_1}_{p_1 q_1,a_{s_1},\ldots,a_{s_2+1},a_{p_1},\ldots a_1,b_{s_2},\ldots,b_1,} = W_{p_1 q_1,a_{s_1},\ldots,a_{s_2+1},a_{p_1},\ldots,a_1,b_{s_2},\ldots,b_1,}.$$

Remark. From the way we define

$$W^{p_1+1,v}_{p_1,q_1,a_{s_1},\ldots,a_{s_2+1},a_{p_1},\ldots a_1,b_{s_2},\ldots,b_1,b'_v,\ldots,b'_{p_1+1}},$$

it is clear that it belongs to $\mathcal{L}$. Notice that it has the leading term

$$\mathrm{in}\,(z_{p_1q_1}\left|\begin{bmatrix} X^{1,p_1} \\ Y^{s_2+1,s_1}\end{bmatrix}_{a_{s_1},\dots,a_{s_2+1},a_{p_1},\dots,a_1}\right||Y^{1,i-1}_{b_{i-1},\dots,b_1}||Y^{i,p_1}_{b_{p_1+1},\dots,b_{i+1}}|$$
$$y_{p_1+1,b_i}y_{p_1+1,b_{p_1+1}}y_{p_1+2,b_{p_1+2}}y_{p_1+2,b'_{p_1+2}}\cdots y_{v-1,b'_{v-1}}y_{v-1,b'_{v-1}}y_{vb_v}y_{v,b'_v}$$
$$|Y^{l+1,s_2}_{b_{s_2},\dots,b_{v+2},b_{v+1}}|).$$

While computing the S-pairs of $f^{l,k}_{a_1,\dots,a_{s_1+k-1}}$ as in Definition 4.7 and $Y_{a_1,\dots,a_{s_2}}$ as in Definition 2.5, the remainders produce new polynomials, defined below.

Definition 4.13. Let $b_{s_1} < \cdots < b_1$, and $1 \le p_l < \cdots < p_k < b_{s_1} < \cdots < b_{s_2+1} < c_{s_2} < \cdots < c_{k+1} < b_{k-1} < \cdots < b_1 < a_{k-1} < \cdots < a_1 \le t_1$.

Let

$$M_{12} = \left|\begin{bmatrix} Z^{l,k} \\ X^{1,k-1} \\ Y^{s_2+1,s_1}\end{bmatrix}_{p_l,\dots,p_k,a_{k-1},\dots,a_1,b_{s_2},\dots,b_{s_2+1}}\right|.$$

We define

$$V_{p_l,\dots,p_k,a_{k-1},\dots,a_1,b_{s_1},\dots,b_1}$$
$$:= M_{12}|Y^{1,s_2}_{b_{s_2},\dots,b_1}| - \sum_{\{e_k,c_{s_2},\dots,c_{k+1}\}=\{b_{s_2},\dots,b_k\}}$$
$$\pm\, y_{ke_k} f^{l,k}_{p_l,\dots,p_k,a_{k-1},\dots,a_1,b_{s_1},\dots,b_{s_2+1},c_{s_2},\dots,c_{k+1},b_{k-1},\dots,b_1}.$$

Remark. From the way we define $V_{p_l,\dots,p_k,a_{k-1},\dots,a_1,b_{s_1},\dots,b_1}$, it is in $\mathcal{L}$. Notice that the submatrices $|Y^{k+1,s_2}_{c_{s_2},\dots,c_{k+1}}|$ of $|Y^{1,s_2}_{b_{s_2},\dots,b_1}|$ such that $b_{s_2+1} < c_{s_2} < \cdots < c_{k+1} < b_{k-1}$ are canceled. Hence, the leading term of $V_{p_l,\dots,p_k,a_{k-1},\dots,a_1,b_{s_1},\dots,b_1}$ is

$$\mathrm{in}\,(M_{12}|Y^{1,k-2}_{b_{k-1},\dots,b_1}|y_{k-1,b_k}|Y^{k,s_2}_{b_{s_2},\dots,b_{k+1},b_{k-1}}|).$$

While computing the S-pairs of elements in $\{V_{p_l,\ldots,p_k,a_{k-1},\ldots,a_1,b_{s_1},\ldots,b_1}\}$ as in Definition 4.13, the remainders produce new polynomials, defined below.

Definition 4.14. Let $1 \leq l \leq k \leq s_2$ and $1 \leq p_l < \cdots < p_k < b_{s_1} < \cdots < b_{s_2+1} < \cdots < b_{k+1} < b_{k-1} < b_k < b_{k-2} < \cdots < b_1 < a_{l-1} < \cdots < a_1 \leq t_1$. Let $w = k, \ldots, s_2 - 1$, $1 \leq b_{s_2} < \cdots < b_{w+2} < b'_w < b'_{w-1} < \cdots < b'_k < b_{k-1} < b_{k-2} \cdots < b_1 \leq t_2$, $b'_{w-1} \leq b_{w+1}$ and $b'_r \leq b_{r+2} < b_{r+1}$ for $r = k, \ldots, l-2$.

We define

$$\begin{aligned} &V^{k,w}_{p_l,\ldots,p_k,a_{k-1},\ldots,a_1,b_{s_1},\ldots,b_1,b'_w,\ldots,b'_k} \\ &\qquad := y_{w-1,b_{w-1}} y_{w,b_w} V^{k,w-2}_{p_l,\ldots,p_k,a_{k-1},\ldots,a_1,b_{s_1},\ldots,b'_w,\ldots,b_1,b'_{w-1},\ldots,b'_k} \\ &\qquad\quad - y_{wb'_w} V^{k,w-1}_{p_l,\ldots,p_k,a_{k-1},\ldots,a_1,b_{s_1},\ldots,b_1,b'_{w-1},\ldots,b'_k}. \end{aligned}$$

Here,

$$\begin{aligned} V^{k,k-2}_{p_l,\ldots,p_k,a_{k-1},\ldots,a_1,b_{s_1},\ldots,b_1} &= V^{k,k-1}_{p_l,\ldots,p_k,a_{k-1},\ldots,a_1,b_{s_1},\ldots,b_1} \\ &= V_{p_l,\ldots,p_k,a_{k-1},\ldots,a_1,b_{s_1},\ldots,b_1}. \end{aligned}$$

Remark. From the way we define $V^{k,w}_{p_l,\ldots,p_k,a_{k-1},\ldots,a_1,b_{s_1},\ldots,b_1,b'_w,\ldots,b'_k}$, it is in $\mathcal{L}$. It has the leading term

$$\operatorname{in}(M_{12}|Y^{1,k-2}_{b_{k-2},\ldots,b_1}|y_{k-1,b_{k-1}} y_{kb_k} y_{kb'_k} \cdots y_{lb_w} y_{lb'_w}|Y^{l+1,s_2}_{b_{s_2},\ldots,b_{w+1}}|).$$

While computing the S-pairs of $g_{ij,lk}$ as in Definition 2.5 and $f^{l,k}_{a_1,\ldots,a_{s_1+k-1}}$ as in Definition 4.7, the remainders produce new polynomials, defined below.

Definition 4.15. Let $1 \leq l \leq k \leq s_2$, $1 \leq q \leq n$, $1 \leq a_{s_1+k-1} < \cdots < a_1 \leq t_1$, $a_{s_1+k-1} < q$ and $a_{j+1} \leq q < a_j$ for some $j = l-1, \ldots, s_1+k-3$. Let $\overline{f^{l,k,x_{l-1}}_{a_{s_1+k-1},\ldots,\hat{a_c},\ldots,a_1}}$ be the determinant of matrices that come from deleting row x_{l-1} and column a_c. We define $H^{l,k,q}_{a_{s_1+k-1},\dots,a_1}$ as follows:

$$H^{l,k,q}_{a_{s_1+k-1},\dots,a_1} = z_{l-1,q} f^{l,k}_{a_{s_1+k-1},\dots,a_1} - \sum_{c=k}^{j} (-1)^{k+c} g_{l-1,q,l-1,a_c} \overline{f^{l,k,x_{l-1}}_{a_{s_1+k-1},\dots,\hat{a_c},\dots,a_1}}.$$

Remark. It is clear that $H^{l,k,q}_{a_1,\dots,a_{s_1+k-1}}$ is in $\mathcal{L}$ from the way we define it. Notice in the row x_l of $f^{l,k}_{a_{s_1+k-1},\dots,a_1}$, the x_{l,a_c} are canceled by the $g_{l-1,q,l-1a_c}$. Hence, the leading term of $H^{l,k,q}_{a_{s_1+k-1},\dots,a_1}$ is

$$z_{l-1,q}\text{in}\left(\left|Z^{l,k}_{a_k,\dots,a_l}\right| x_{l-1,a_{j+1}} \left|X^{1,l-2}_{a_{l-1},\dots,a_1}\right| \left|\begin{bmatrix} Y^{1,k-1} \\ Y^{k+1,s_1} \end{bmatrix}_{b_{s_1},\dots,b_{k+1},b_{k-1},\dots,b_1}\right|\right).$$

Here, $a_i \neq a_{j+1}$, $b_i \neq a_{j+1}$ for all i.

While computing the S-pairs of $H^{l,k,q}_{a_{s_1+k-1},\dots,a_1}$ as in Definition 4.15 and $Y_{a_{s_2},\dots,a_1}$ as in Definition 2.5, the remainders produce new polynomials, defined below.

Definition 4.16. Let $1 \leq l \leq k \leq s_2$, $1 \leq q \leq n$, $1 \leq a_{s_1+k-1} < \cdots < a_1 \leq t_1$, $a_{s_1+k-1} < q$ and $a_{j+1} \leq q < a_j$ for some $j = l-1,\dots,s_1+k-3$. Let $a_{l+s_2-1} < b_{s_2} < \cdots < b_k < a_{l-1+k-1} = b_{k-1} < \cdots < a_{l-1+1} = b_1$.

Let

$$M = z_{l-1,q} x_{l-1,a_{j+1}} \left|\begin{bmatrix} Z^{l,k} \\ X^{1,l-2} \\ Y^{s_2+1,s_1} \end{bmatrix}_{a_{s_1+k-1},\dots,a_{l+s_2-1},a_{l-1},\dots,a_1}\right|.$$

We define

$$I^{l,k,q}_{a_{s_1+k-1},\dots,a_{l+s_2-1},a_{l-1},\dots,a_1,b_{s_2},\dots,b_1} := M\left|Y^{1,s_2}_{b_{s_2},\dots,b_1}\right| - \sum_{\{e_k,c_{k+1},\dots,c_{s_2}\}=\{b_k,\dots,b_{s_2}\}} \pm y_{ke_k} H^{l,k,q}_{a_{s_1+k-1},\dots,a_{l+s_2-1},c_{s_2},\dots,c_{k+1},b_{k-1},\dots,b_1,a_{l-1},\dots,a_1}.$$

Remark. It is clear that $I^{l,k,q}_{a_{s_1+k-1},\dots,a_{l+s_2-1},a_{l-1},\dots,a_1,b_{s_2},\dots,b_1}$ is in $\mathcal{L}$ from the way we define it. Notice that the submatrices $|Y^{k+1,s_2}_{c_{s_2},\dots,c_{k+1}}|$ of $|Y^{1,s_2}_{b_{s_2},\dots,b_1}|$ with $a_{l+s_2-1} < c_{s_2} < \cdots < c_{k+1} < b_{k-1}$ are canceled by $H^{l,k,q}$'s; hence, the leading term of $I^{l,k,q}_{a_{s_1+k-1},\dots,a_{l+s_2-1},a_{l-1},\dots,a_1,b_{s_2},\dots,b_1}$ is:

$$\text{in}\left(z_{l-1,q}x_{l-1,a_{j+1}} \left| \begin{bmatrix} Z^{l,k} \\ X^{1,l-2} \\ Y^{s_2+1,s_1} \end{bmatrix}_{a_{s_1+k-1}\cdots a_{l+s_2-1}a_{l-1}\cdots a_1} \right| y_{k-1,b_k} \left| \begin{bmatrix} Y^{1,k-2} \\ Y^{k,s_2} \end{bmatrix}_{b_{s_2}\cdots b_{k+1}b_{k-1}b_{k-2}\cdots b_1} \right| \right).$$

While computing the S-pairs of elements of

$$\{I^{l,k,q}_{q_l,\dots,q_k,b_{s_1},\dots,b_{s_2},\dots,b_1,a_{l-1},\dots,a_1}\},$$

the remainders produce new polynomials, defined below.

Definition 4.17. Let $1 \le l \le k \le s_2$, $1 \le q \le n$, $k \le w \le s_2 - 1$, $1 \le q_l < \cdots < q_k < b_{s_1} < \cdots < b_{s_2} < \cdots < b_{k+1} < b_{k-1} < b_k < b_{k-2} < \cdots < b_1 < a_{l-2} < \cdots < a_1 \le t_1$, $q_l < q$, $q_l < a_{l-1} \le q$. Let $w = k,\dots,s_2-1$, $1 \le b_{s_2} < \cdots < b_{w+2} < b'_w < b'_{w-1} < \cdots < b'_k < b_{k-1} < b_{k-2} \cdots < b_1 \le t_2$, $b'_{w-1} \le b_{w+1}$ and $b'_r \le b_{r+2} < b_{r+1}$ for $r = k,\dots,l-2$.

We define

$$\begin{aligned} &{}^{k,w}I^{l,k,q}_{q_l,\dots,q_k,b_{s_1},\dots,b_{s_2},\dots,b_1,a_{l-1},\dots,a_1,b'_w,\dots,b'_k} \\ &:= y_{w-1,b'_{w-1}}y_{wb_w}\,{}^{k,w-2}I^{l,k,q}_{q_l,\dots,q_k,b_{s_1},\dots,b_{s_2},\dots,b'_w,\dots,b_1,a_{l-1},\dots,a_1,b'_{w-2},\dots,b'_k} \\ &\quad - y_{wb'_w}\,{}^{k,w-1}I^{l,k,q}_{q_l,\dots,q_k,b_{s_1},\dots,b_{s_2},\dots,b_1,a_{l-1},\dots,a_1,b'_{w-},\dots,b'_k}. \end{aligned}$$

Here,

$$\begin{aligned} &{}^{k,k-2}I^{l,k,q}_{q_l,\dots,q_k,b_{s_1},\dots,b_{s_2},\dots,b_1,a_{l-1},\dots,a_1} \\ &\qquad = {}^{k,k-1}I^{l,k,q}_{q_l,\dots,q_k,b_{s_1},\dots,b_{s_2},\dots,b_1,a_{l-1},\dots,a_1} \\ &\qquad = I^{l,k,q}_{q_l,\dots,q_k,b_{s_1},\dots,b_{s_2},\dots,b_1,a_{l-1},\dots,a_1}. \end{aligned}$$

Remark. From the way we define ${}^{k,w}I^{l,k,q}_{q_l,..,q_k,b_{s_1},..,b_{s_2},..,b_1,a_{l-1},..,a_1,b'_w,..,b'_k}$, it is in $\mathcal{L}$. The leading term of ${}^{k,w}I^{l,k,q}_{q_l,\dots,q_k,b_{s_1},\dots,b_{s_2},\dots,b_1,a_{l-1},\dots,a_1,b'_w,\dots,b'_k}$ is:

$$\operatorname{in}\left(z_{l-1,q}x_{l-1,a_{j+1}} \left| \begin{bmatrix} Z^{l,k} \\ X^{1,l-2} \\ Y^{s_2+1,s_1} \end{bmatrix}_{a_{s_1+k-1},\dots,a_{l+s_2-1},a_{l-1},\dots,a_1} \right| \right.$$
$$\left. \times\; y_{k-1,b_{k-1}}y_{kb_k}y_{kb'_k}\cdots y_{wb_w}y_{wb_w} \left| \begin{bmatrix} Y^{1,k-2} \\ Y^{w+1,s_2} \end{bmatrix}_{b_{s_2},\dots,b_{w+1},b_{k-2}\cdots b_1} \right| \right).$$

We are now ready to show the collection of polynomials that defined the above and the generators of $\mathcal{L}$ form a Gröbner basis of $\mathcal{L}$.

Theorem 4.18. *Use the notations of Definitions* 4.7*,* 4.9*,* 4.11–4.17*, and let*

$$\begin{aligned}\mathcal{G} := \{&|X^{1,s_1}_{a_1,\dots,a_{s_1}}|, |Y^{1,s_2}_{b_1,\dots,b_{s_2}}|, g_{p_1q_1,p_2q_2},\\ &f^{l,k}_{a_1,\dots,a_{s_1+k-1}}, U_{p_1,q_1,a_{s_1},\dots,a_1}, W_{p_1,q_1,a_{s_1},\dots,a_{s_2+1},a_{p_1},\dots a_1,b_{s_2},\dots,b_1},\\ &W^{p_1+1,v}_{p_1,q_1,a_{s_1},\dots,a_{s_2+1},a_{p_1},\dots a_1,b_{s_2},\dots,b_1,b'_v,\dots,b'_{p_1+1}},\\ &V_{p_l,\dots,p_k,a_{k-1},\dots,a_1,b_{s_1},\dots,b_1},\\ &V^{k,w}_{p_l,\dots,p_k,a_{k-1},\dots,a_1,b_{s_1},\dots,b_1,b'_w,\dots,b'_k}, H^{l,k,q}_{a_{s_1+k-1},\dots,a_1},\\ &I^{l,k,q}_{a_{s_1+k-1},\dots,a_{l+s_2-1},a_{l-1},\dots,a_1,b_{s_2},\dots,b_1},\\ &\qquad\qquad {}^{k,w}I^{l,k,q}_{q_l,\dots,q_k,b_{s_1},\dots,b_{s_2},\dots,b_1,a_{l-1},\dots,a_1,b'_w,\dots,b'_k}\}.\end{aligned}$$

$\mathcal{G}$ is a Gröbner basis of $\mathcal{L}$ with respect to the lexicographic term order and the variables ordered by $z_{ij} > x_{lk} > y_{pq}$ for any i, j, l, k, p, q and $x_{ij} < x_{lk}$, $y_{ij} < y_{lk}$ if $i > l$ or $i = l$ and $j < k$ and $z_{ij} < z_{lk}$ if $i > l$ or if $i = l$ and $j > k$.

The proof of the above theorem is divided into a sequence of lemmas when we treat S-pairs between elements of $\mathcal{G}$. We only have to compute the S-pairs of elements whose leading terms are not relatively prime. In each lemma, we show $h_{P,Q} = 0$ for some P, Q in $\mathcal{G}$.

Lemma 4.19. *$h_{P,Q} = 0$ when P and Q are in the same group of $\mathcal{G}$. Here the same group means summands of P and Q come from matrices with the same row variables but different column indices. For example, $f^{l,k}_{a_1,\dots,a_{s_1+k-1}} = P$ and $f^{l,k}_{b_1,\dots,b_{s_1+k-1}} = Q$.*

Proof. We use the notation in the Definition 4.5 with $P = P^u_{a_1,\dots,a_{q_u}} = p_1$ and $Q = Q^v_{b_1,\dots,b_{q_v}} = p_2$. Notice that m_{12} and m_{21} have the same row variables, $P^u_{a_1,\dots,a_{q_u}}$ and $Q^v_{b_1,\dots,b_{q_v}}$ have the same number of columns, i.e., $q_u = q_v$; hence,

$$\overline{P^u_{a_1,\dots,a_{q_u},d_1,\dots,d_{p_{21}}}} = \overline{Q^v_{b_1,\dots,b_{q_v},c_1,\dots,c_{p_{12}}}}.$$

Also, $\operatorname{in}(m_{12}P^u_{a_1,\dots,a_{q_u}}) = \operatorname{in}(m_{21}Q^v_{b_1,\dots,b_{q_u}})$ are indeed the leading terms of $\overline{P^u_{a_1,\dots,a_{q_u},d_1,\dots,d_{p_{21}}}}$. The first matrix of $\overline{P^u_{a_1,\dots,a_{q_u},d_1,\dots,d_{p_{21}}}}$ has determinant zero because it has repeated rows. Hence, we have $m_{12}P^u_{a_1,\dots,a_{q_u}}$ and $m_{21}Q^v_{b_1,\dots,b_{q_u}}$ with different signs in the sum. Except $P^u_{a_1,\dots,a_{q_u}} = f^{l,k}_{a_1,\dots,a_{q_u}}$ and $Q^v_{b_1,\dots,b_{q_v}} = f^{l,k}_{a_1,\dots,b_{q_u}}$ with $a_i = b_i$ for $i \neq k-l+1$ and $a_{k-l+1} \neq b_{k-l+1}$, each summand of all possible cases have either repeated row, or all the rows, $y_1, \dots, y_{s_2}$ or rows of Lemma 4.4. Hence, they give

$$\sum_{i=2}^{u} \overline{P^u_i} \in \mathcal{G}, : \sum_{i=2}^{u} \overline{Q^v_i} \in \mathcal{G}.$$

For the remaining case,

$$\sum_{i=2}^{u} \overline{P^u_i} = f^{l,k+1}_{a_1,\dots,a_{k-l+1},b_{k-l+1},a_{k-l+2},\dots,a_{q_u}}$$

from the proof of Lemma 4.8. Similarly for $\sum_{i=2}^{u} \overline{Q^v_i}$. Hence,

$$\overline{P^u_{a_1,\dots,a_{q_u},d_1,\dots,d_{p_{21}}}} = \sum_{i=2}^{u} \overline{P^u_i} \in \mathcal{G} \cdot \overline{Q^v_{a_1,\dots,a_{q_u},d_1,\dots,d_{p_{21}}}} = \sum_{i=2}^{u} \overline{Q^v_i} \in \mathcal{G}.$$

Now we can apply Lemma 4.6. □

Lemma 4.20. *$h_{P,Q} = 0$ when $P \in \{|X^{1,s_1}_{a_1,\dots,a_{s_1}}|\}$ in $\mathcal{G}$.*

Proof. As in the notation of Definition 4.5 with $P = p_1$ and $Q = p_2$, we look at $\sum_{i=2}^{u} \overline{P_i}$. For most of the cases, $\sum_{i=2}^{u} \overline{P_i} = 0$ because each summand has repeated rows x_j for some $j = 1, \dots, s_1$. In some other cases, we have rows, either y_i or x_i and z_i, in each matrix. Then Lemma 3.2 can be applied. Or the part of the sum has sum as Lemma 4.4. Then deduce that it is in $(\{|Y^{1,s_2}_{b_1,\dots,b_{s_2}}|\}) + (\{g_{ij,lk}\})$. Similarly, $\sum_{i=2}^{u} \overline{Q_i} \in (\{|Y^{1,s_2}_{b_1,\dots,b_{s_2}}|\}) + (\{g_{ij,lk}\})$, hence Lemma 4.6 applies. □

Lemma 4.21. *$h_{P,Q} = 0$ when $P \in \{|Y^{1,s_2}_{b_1,\dots,b_{s_2}}|\}$ in $\mathcal{G}$.*

Proof. The computation of S-pairs between $f^{l,k}_{a_1,\dots,a_{s_1+k-1}}$ and $|Y^{1,s_2}_{b_1,\dots,b_{s_2}}|$ gives us $V_{p_l,\dots,p_k,a_{k-1},\dots,a_1,b_{s_1},\dots,b_1}$ as in Definition 4.13. The S-pairs between

$$V_{p_l,\dots,p_k,a_{k-1},\dots,a_1,b_{s_1},\dots,b_1} \quad \text{and} \quad |Y^{1,s_2}_{b_1,\dots,b_{s_2}}|$$

give $V^{k,w}_{p_l,\dots,p_k,a_{k-1},\dots,a_1,b_{s_1},\dots,b_1,b'_w,\dots,b'_k}$. The S-pairs between

$$V^{k,w}_{p_l,\dots,p_k,a_{k-1},\dots,a_1,b_{s_1},\dots,b_1,b'_w,\dots,b'_k} \quad \text{and} \quad |Y^{1,s_2}_{b_1,\dots,b_{s_2}}|$$

give $V^{k,w+1}_{p_l,\dots,p_k,a_{k-1},\dots,a_1,b_{s_1},\dots,b_1,b'_{w+1},\dots,b'_k}$. Similarly, the computation of S-pairs between $H^{l,k,q}_{a_1,\dots,a_{s_1+k-1}}$ and $|Y^{1,s_2}_{b_1,\dots,b_{s_2}}|$ gives

$$I^{l,k,q}_{a_{s_1+k-1},\dots,a_{l+s_2-1},a_{l-1},\dots,a_1,b_{s_2},\dots,b_1}$$

as Definition 4.16, and the S-pairs between

$$I^{l,k,q}_{a_{s_1+k-1},\dots,a_{l+s_2-1},a_{l-1},\dots,a_1,b_{s_2},\dots,b_1} \quad \text{and} \quad |Y^{1,s_2}_{b_1,\dots,b_{s_2}}|$$

give

$${}^{k,w}I^{l,k,q}_{q_l,\dots,q_k,b_{s_1},\dots,b_{s_2},\dots,b_1,a_{l-1},\dots,a_1,b'_w,\dots,b'_k}.$$

Also, the S-pairs between $U_{p_1,q_1,a_{s_1},\dots,a_1}$ and $|Y^{1,s_2}_{b_1,\dots,b_{s_2}}|$ give

$$W_{p_1,q_1,a_{s_1},\dots,a_{s_2+1},a_{p_1},\dots a_1,b_{s_2},\dots,b_1}$$

as in Definition 4.11, and the S-pairs between

$$W_{p_1,q_1,a_{s_1},\dots,a_{s_2+1},a_{p_1},\dots a_1,b_{s_2},\dots,b_1} \quad \text{and} \quad |Y^{1,s_2}_{b_1,\dots,b_{s_2}}|$$

give $W^{p_1+1,v}_{p_1,q_1,a_{s_1},\dots,a_{s_2+1},a_{p_1},\dots a_1,b_{s_2},\dots,b_1,b'_v,\dots,b'_{p_1+1}}$. □

Lemma 4.22. *$h_{P,Q} = 0$ when $P \in \{g_{ij,lk}\}$ in $\mathcal{G}$.*

Proof. If $Q \in \{g_{ij,lk}\}$, we have $Q = z_{p_1q_1}(x_{p_2q_2} - y_{p_2q_2}) - z_{p_2q_2}(x_{p_1q_1} - y_{p_1q_1})$ and $P = g_{ij,lk} = z_{ij}(x_{lk} - y_{lk}) - z_{lk}(x_{ij} - y_{ij})$. It's sufficient to consider either $(p_1, q_1) = (i, j)$ or $(p_2, q_2) = (l, k)$. For the first case,

$$(x_{lk} - y_{lk})Q - (x_{p_2q_2} - y_{p_2q_2})P = (x_{ij} - y_{ij})g_{lk,p_2q_2}.$$

For the second case,

$$z_{ij}Q - z_{p_1q_1}P = z_{p_2q_2}g_{p_1q_1,ij}.$$

Notice that $P = g_{ij,lk} = z_{i,j}(x_{l,k} - y_{l,k}) - z_{l,k}(x_{i,j} - y_{i,j})$ with $z_{i,j} > z_{l,k}$. If $Q \in \{|X^{1,s_1}_{a_1,\dots,a_{s_1}}|\}$, the computing of S-pairs of P and Q is similar to Lemma 4.10. And it gives $U_{ija_1,\dots,a_{s_1}}$ as in Definition 4.9. If $Q \in \{f^{l,k}_{a_1,\dots,a_{s_{1+k-1}}}\}$, the computing of S-pairs of P and Q, gives us $H^{l,k,j}_{a_1,\dots,a_{s_1+k-1}}$ as in Definition 4.15 when $P = g_{ij,lk}$ and $i = l - 1$. Otherwise $\gcd(\mathrm{in}(P),\ \mathrm{in}(Q)) = z_{i,j}$ with $i \in \{l, l+1, \dots, k\}$ or $\gcd(\mathrm{in}(P),\ \mathrm{in}(Q)) = x_{l,k}$ with $i < l - 1$. For $\gcd(\mathrm{in}(P),\ \mathrm{in}(Q)) = z_{i,j}$ with $i \in \{l, l+1, \dots, k\}$, the computation of S-pairs gives us $f^{l+1,k}_{a_1,\dots,a_{s_1+k-1}}$. For $\gcd(\mathrm{in}(P),\ \mathrm{in}(Q)) = x_{l,k}$, the computation of S-pairs gives us repeated row, y_l, in every matrix of Q, and this makes the determinant zero. For all other cases, $Q \in \mathcal{G}$, which come from the S-pairs of $P \in \{g_{ij,lk}\}$ and $|X^{1,s_1}_{a_1,\dots,a_{s_1}}|$ or $f^{l,k}_{a_1,\dots,a_{s_1+k-1}}$. Hence, the computations of S-pairs are very similar to the cases above. □

The following lemma is the main computation of S-pairs between the elements of $\mathcal{G}$, and it is the most difficult case. Actually, this is the only case where the new remainders of S-pairs are not straightforward. After this particular case is proven, all other cases are trivial.

Lemma 4.23. *$h_{P,Q} = 0$ when $P = f^{l_1,k_1}_{a_1,\dots,a_{s_1+k_1-1}}$ and $Q = f^{l_2,k_2}_{b_1,\dots,b_{s_1+k_2-1}}$ and $l_1 \neq l_2$ or $k_1 \neq k_2$ in $\mathcal{G}$.*

Proof. We prove this part in two cases: (a) $k_1 \neq k_2$, (b) $l_1 \neq l_2$.

In case (a), without loss of generality, let $k_1 > k_2$. Then the matrix that appears in the first summand of $f^{l_1 k_1}_{a_1,\dots,a_{s_1+k_1-1}}$ has row y_{k_2} without row y_{k_1}, and the matrix that appears in the first summand of $f^{l_2 k_2}_{b_1,\dots,b_{s_1+k_2-1}}$ has row y_{k_1} without y_{k_2}. Consider m_{12}, m_{21}, M_{12} and M_{21} as defined in Definition 4.2 with $P = p_1$ and $Q = p_2$. Assume M_{12} has columns $c_1,\dots,c_r$ and M_{21} has columns $d_1,\dots,d_w$. Define $\overline{f^{l_1,k_1}_{a_1,\dots,a_{s_1+k_1-1},d_1,\dots,d_w}}$ and $\overline{f^{l_2,k_2}_{b_1,\dots,b_{s_1+k_2-1},c_1,\dots,c_r}}$ as in Definition 4.5. Let $\{c_1,\dots,c_r,b_1,\dots,b_{s_1+k_2-1}\} = \{d_1,\dots,d_w,a_1,\dots,a_{s_1+k_1-1}\} = \mathcal{I}$; from the way we define M_{12} and M_{21}, we have the initial term of $\overline{f^{l_1,k_1}_{\mathcal{I}}}$ is in $(M_{12} f^{l_1,k_1}_{a_1,\dots,a_{s_1+k_1-1}})$ and similarly for $\overline{f^{l_2,k_2}_{\mathcal{I}}}$. We would like to apply Lemma 4.6 to this case.

Rewrite $\overline{f^{l_2,k_2}_{\mathcal{I}}}$ as α_1:

$$\alpha_1 := \sum_{r=k_2}^{s_2} (-1)^{r+1} \left| \begin{bmatrix} \overline{M_{12}}' \\ Y^{k_2,k_2} \\ Z^{l_2,k_2-1} \\ Z^{r,r} \\ X^{1,l_2-1} \\ Y^{1,r-1} \\ Y^{r+1} \end{bmatrix}_{\mathcal{I}} \right| + \sum_{r=k_2}^{s_2} (-1)^{r+1} \sum_{u=r+1}^{s_1} \left| \begin{bmatrix} \overline{M_{12}}' \\ Y^{k_2,k_2} \\ Z^{l_2,k_2-1} \\ X^{r,r} - Y^{r,r} \\ X^{1,l_2-1} \\ Y^{1,r-1} \\ Y^{r+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix}_{\mathcal{I}} \right| .$$

Notice that, in the first sum of α_1, when $r > k_2$, the matrices have repeated row y_{k_2}. Hence, the determinants are zero. The first sum becomes α_{11}:

$$\alpha_{11} := \left| \begin{bmatrix} \overline{M_{12}}' \\ Y^{k_2,k_2} \\ Z^{l_2,k_2-1} \\ Z^{k_2,k_2} \\ X^{1,l_2-1} \\ Y^{1,k_2-1} \\ Y^{k_2+1} \end{bmatrix}_{\mathcal{I}} \right|.$$

We notice the leading term of α_{11} is $m_{12}(\operatorname{in} f^{l_2k_2}_{b_1,\dots,b_{s_1+k-1}})$. Let the second sum of α_1 be α_{12}:

$$\alpha_{12} := \sum_{r=k_2}^{s_2} (-1)^{r+1} \sum_{u=r+1}^{s_1} \left| \begin{bmatrix} \overline{M_{12}}' \\ Y^{k_2,k_2} \\ Z^{l_2,k_2-1} \\ X^{r,r} - Y^{r,r} \\ X^{1,l_2-1} \\ Y^{1,r-1} \\ Y^{r+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix}_{\mathcal{I}} \right|.$$

In order to apply Lemma 4.6, we have to show α_{12} is a combination of elements of $\mathcal{G}$ such that the leading term of each summand is smaller than $m_{12} f^{l_2k_2}_{b_1,\dots,b_{s_1+k_2-1}}$. Observe that, in the sum of α_{12}, when $r > k_2$, the matrices have repeated row y_{k_2}. Hence, their determinants are zero.

We are only left with $r = k_2$, and α_{12} becomes:

$$\alpha_{13} := (-1)^{k_2+1} \sum_{u=k_2+1}^{s_1} \left| \begin{bmatrix} \overline{M_{12}}' \\ Y^{k_2,k_2} \\ Z^{l_2,k_2-1} \\ X^{k_2,k_2} - Y^{k_2,k_2} \\ X^{1,l_2-1} \\ Y^{1,k_2-1} \\ Y^{k_2+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix}_{\mathcal{I}} \right|.$$

Applying Lemma 3.3 on α_{13}, then α_{13} becomes α_{14}:

$$\alpha_{14} := (-1)^{k_2+1} \sum_{u=k_2+1}^{s_1} \left| \begin{bmatrix} \overline{M_{12}}' \\ Y^{k_2,k_2} \\ X^{l_2,l_2} - Y^{l_2,l_2} \\ Z^{l_2+1,k_2-1} \\ Z^{k_2,k_2} \\ X^{1,l_2-1} \\ Y^{1,k_2-1} \\ Y^{k_2+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix}_{\mathcal{I}} \right|$$

$$+ (-1)^{k_2+1} \sum_{u=k_2+1}^{s_1} \sum_{\{p_1,p_2,q_1,\dots,q_{s_1+k_2-1}\}=\mathcal{I}}$$

$$\times \left(\pm(g_{l_2p_1,k_2,p_2} - g_{l_2p_2,k_2p_1}) \left| \begin{bmatrix} \overline{M_{12}}' \\ Y^{k_2,k_2} \\ Z^{l_2+1,k_2-1} \\ X^{1,l_2-1} \\ Y^{1,k_2-1} \\ Y^{k_2+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix}_{q_1,\dots,q_{s_1+k-2}} \right| \right).$$

After removing the repeated row y_{l_2} in the first sum in the above expression for α_{14}, let this sum be α_{15}:

$$\alpha_{15} := (-1)^{k_2+1} \sum_{u=k_2+1}^{s_1} \left| \begin{bmatrix} \overline{M_{12}}' \\ Y^{k_2,k_2} \\ X^{l_2,l_2} \\ Z^{l_2+1,k_2-1} \\ Z^{k_2,k_2} \\ X^{1,l_2-1} \\ Y^{1,k_2-1} \\ Y^{k_2+1,u-1} \\ Z^{u,u} \\ X^{u+1,s_1} \end{bmatrix}_{\mathcal{I}} \right|$$

$$= \pm \sum_{u=k_2+1}^{s_1} (-1)^{u+1} \left| \begin{bmatrix} \overline{M_{12}}' \\ Z^{l_2+1,k_2} \\ Z^{u,u} \\ X^{1,l_2} \\ Y^{1,u-1} \\ X^{u+1,s_1} \end{bmatrix}_{\mathcal{I}} \right| .$$

Now α_{15} becomes

$$\begin{aligned} \alpha_{16} := & \sum_{\{p_1,\dots,p_{v-1},q_1,\dots,q_{s_{1+k_2}}\}=\mathcal{I}} \\ & \pm |M^{12}_{p_1,\dots,p_{v-1}}| \left(\sum_{u=k_2+1}^{s_1} (-1)^{u+1} \left| \begin{bmatrix} Z^{l_2+1,k_2} \\ Z^{u,u} \\ X^{1,l_2} \\ Y^{1,u-1} \\ X^{u+1,s_1} \end{bmatrix}_{\mathcal{I}} \right| \right) \\ = & \sum_{\{p_1,\dots,p_{v-1},q_1,\dots,q_{s_{1+k_2}}\}=\mathcal{I}} \pm |M^{12}_{p_1,\dots,p_{v-1}}| p^{l_2+1,k_2+1}_{q_1,\dots,q_{s_1+k_2}} . \end{aligned}$$

Here $\{p^{l_2+1,k_2+1}_{q_1,\dots,q_{s_1+k_1}}\}$ are as in Lemma 4.7, and the proof of Lemma 4.7 shows that they are in $\mathcal{L}$. This shows that α_{12} is a combination of elements of $\mathcal{G}$ such that the leading term of each summand is smaller than $m_{12} f^{l_2 k_2}_{b_1,\dots,b_{s_1+k_2-1}}$.

We can apply a similar technique to $\overline{f^{l_1,k_1}_{\mathcal{I}}}$ and show the second part of the sum of $\overline{f^{l_1,k_1}_{\mathcal{I}}}$ is a combination of elements of $\mathcal{G}$ such that the leading term of each summand is smaller than in $(m_{21} f^{l_1 k_1}_{a_1,\dots,a_{s_1+k_1-1}})$.

In case (b), assume $k_1 = k_2$ and $l_1 < l_2 \le k_2 = k_1$. The proof of the technique is very similar to case (a). Notice that the first matrix appearing in the expression for $f^{l_1 k_1}_{a_1,\dots,a_{s_1+k_1-1}}$ has row z_{l_1} without row x_{l_1}, and the first matrix appearing in the expression for $f^{l_2 k_2}_{b_1,\dots,b_{s_1+k_1-1}}$ has row x_{l_1} without row z_{l_1}. Since $l_1 \le l_2 - 1$ and $l_1 \le k_2 - 1$, each matrix of $\overline{f^{l_2,k_2}_{b_1,\dots,b_{s_1+k_2-1},c_1,\dots,c_r}}$ has the rows x_{l_1} and y_{l1}. They also all have row z_{l_1}. Lemma 3.2 gives that all the determinants of those matrices are in $(\{g_{l_1 i, l_1 j}\})$. ◻

Lemma 4.24. $h_{PQ} = 0$ *if*

$$P,\ Q \in \Big\{ f^{l,k}_{a_1,\dots,a_{s_1+k-1}}, U_{p_1,q_1,a_{s_1},\dots,a_1},\\
W_{p_1,q_1,a_{s_1},\dots,a_{s_2+1},a_{p_1},\dots a_1,b_{s_2},\dots,b_1},\\
W^{p_1+1,v}_{p_1,q_1,a_{s_1},\dots,a_{s_2+1},a_{p_1},\dots a_1,b_{s_2},\dots,b_1,b'_v,\dots,b'_{p_1+1}},\\
V_{p_l,\dots,p_k,a_{k-1},\dots,a_1,b_{s_1},\dots,b_1},\\
V^{k,w}_{p_l,\dots,p_k,a_{k-1},\dots,a_1,b_{s_1},\dots,b_1,b'_w,\dots,b'_k},\\
H^{l,k,q}_{a_{s_1+k-1},\dots,a_1}, I^{l,k,q}_{a_{s_1+k-1},\dots,a_{l+s_2-1},a_{l-1},\dots,a_1,b_{s_2},\dots,b_1},\\
{}^{k,w}I^{l,k,q}_{q_l,\dots,q_k,b_{s_1},\dots,b_{s_2},\dots,b_1,a_{l-1},\dots,a_1,b'_w,\dots,b'_k} \Big\}.$$

Proof. In this proof, column indices are dropped for convenience. The remainder of the S-pairs of $f^{l,k}$ and U are in the ideal generated by $(\{V\},\{W\},\{g\},\{Y\})$. Similarly, the remainders of S-pairs of $f^{l,k}$ and W are in $(\{V^{k,w}\},\{W^{p_1+1,v}\},\{g\},\{Y\})$, and the remainders of S-pairs of $f^{l,k}$ and $W^{p_1+1,v}$ are in $(\{V^{k,v+1}\},\{W^{p_1+1,v+1}\},\{g\},\{Y\})$. The remainder of S-pairs of $f^{l,k}$ and V are $V^{k,w}$, and the remainder of S-pairs of $f^{l,k}$ and $V^{k,w}$ are $V^{k,w+1}$. The remainder of S-pairs of $f^{l,k}$ and $H^{l,k,q}$ are in $(\{I^{l,k,q}\},\{g\},\{Y\})$, and the remainder of S-pairs of $f^{l,k}$ and $I^{l,k,q}$ are in $(\{{}^{k,w}I^{l,k,q}\},\{g\},\{Y\})$. Finally, the remainder of S-pairs of $f^{l,k}$ and ${}^{k,w}I^{l,k,q}$ are in $(\{{}^{k,w+1}I^{l,k,q}\},\{g\},\{Y\})$. All the other S-pairs of elements have a similar relationship as above.

The proof of Theorem 4.18 is completed. □

Proof of Lemma 2.7. Notice that

$$U_{1,1,1,a_{s_1},\dots,a_2},\\
W_{1,1,1,a_{s_1},\dots,a_{s_2+1},b_{s_2},\dots,b_1}$$

and

$$W^{2,v}_{1,1,1,a_{s_1},\dots,a_{s_2+1},b_{s_2},\dots,b_1,b'_v,\dots,b'_2}$$

are the only possible polynomials whose leading monomials are divisible by x_{11}. But those monomials are divisible by the leading monomials of $f^{1,1}_{1,a_2,\dots,a_{s_1}}$, $V_{1,a_{s_1},\dots,a_{s_2+1},b_{s_2},\dots,b_1}$ and $V^{2,v}_{1,a_{s_1},\dots,a_{s_2+1},b_{s_2},\dots,b_1,b'_v,\dots,b'_2}$. Therefore, x_{11} is a non zero-divisor in the ring $k[X,Y,Z]/\mathrm{in}(\mathcal{L})$. Hence, x_{11} is also a non zero-divisor in $k[X,Y,Z]/\mathcal{L}$. ◻

To get an idea of what the initial ideal looks like, we compute the following example in `Singular` [**5**].

Example 4.25. Let X, Y and Z be a 3×4 matrix, $X_{3,4}$, $Y_{2,4}$ are 3×4 and 2×4 submatrices of X, and let Y then be the defining ideal of the $\mathcal{R}(\mathbf{D})$ generated by $I_3(X_{3,4})$, $I_2(Y_{2,4})$ and $g_{ij,lk}$ where $1\le i,\, l\le 3$, $1\le l,k\le 4$ and

$$f_{a_1,\dots,a_3}=\begin{vmatrix} z_{1a_1} & z_{1a_2} & z_{1a_3}\\ x_{2a_1} & x_{2a_2} & x_{2a_3}\\ x_{3a_1} & x_{3a_2} & x_{3a_3}\end{vmatrix}+\begin{vmatrix} y_{1a_1} & y_{1a_2} & y_{1a_3}\\ z_{2a_1} & z_{2a_2} & z_{2a_3}\\ x_{3a_1} & x_{3a_2} & x_{3a_3}\end{vmatrix},$$

where $1\le a_1<a_2<a_3\le 4$. The initial ideal of $\mathcal{L}$ via the term order defined in Theorem 4.18 is generated by:

$$\begin{aligned}
&\{x_{1a_3}x_{2a_2}x_{3a_1}\}_{1\le a_1<a_2<a_3\le 4}, \qquad \{y_{1b_2}y_{2b_1}\}_{1\le b_1<b_2\le 4},\\
&\{z_{ij}x_{lk}\}_{i<l \text{ or } i=l \text{ and } j<k}, \qquad \{z_{1a_1}y_{2a_2}y_{3a_3}\}_{1\le a_1<a_3<a_2\le 4},\\
&z_{11}z_{22}y_{34}y_{33}, \qquad z_{21}x_{14}y_{13}y_{32},\\
&z_{12}z_{21}x_{12}y_{14}y_{33}, \qquad z_{13}z_{21}x_{13}y_{14}y_{32}, \qquad z_{31}x_{14}x_{23}y_{32},\\
&\{z_{2j}x_{1a_3}x_{2a_2}y_{3a_1}\}_{1\le a_1<a_2\le j<a_3\le 4, \text{ or } 1\le a_2\le j<a_1<a_3\le 4},\\
&\{z_{2j}x_{1a_3}y_{2a_2}y_{1a_1}\}_{1\le a_1<a_2<a_3, a_2<j},\\
&\{z_{1j}x_{1a_3}y_{2a_2}y_{3a_1}\}_{1\le a_1<a_2<a_3\le j\le 4, \text{ or } 1\le a_1<a_3\le j<a_2\le 4},\\
&\{z_{1j}x_{1a_3}y_{1b_1}y_{2b_2}y_{3b_3}\}_{1\le b_2<b_3<a_3\le j\le 4 \text{ or } 1\le a_1<a_3\le j<a_2\le 4}.
\end{aligned}$$

We can see the variable x_{11} is not in the generating set of the initial ideal of $\mathcal{L}$.

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