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UPPER BOUNDS OF DEPTH OF MONOMIAL IDEALS

DORIN POPESCU

ABSTRACT. Let $J \subsetneq I$ be two ideals of a polynomial ring S over a field, generated by square free monomials. We show that some inequalities among the numbers of square free monomials of $I \setminus J$ of different degrees give upper bounds of $\mathrm{depth}_S I/J$.

Introduction. Let $S = K[x_1, \dots, x_n]$ be the polynomial algebra in n variables over a field K, $d \leq t$, two positive integers and $I \supsetneq J$, two square free monomial ideals of S such that I is generated in degrees $\geq d$, respectively J in degrees $\geq d+1$. By [**2**, Proposition 3.1] and [**4**, Lemma 1.1] $\mathrm{depth}_S I/J \geq d$. Let $\rho_t(I \setminus J)$ be the number of all square free monomials of degree t of $I \setminus J$.

Theorem 0.1 [**4**, Theorem 2.2]. *If $\rho_d(I) > \rho_{d+1}(I \setminus J)$, then $\mathrm{depth}_S I/J = d$, independently of the characteristic of K.*

The aim of this paper is to extend this theorem. Our Theorem 1.3 says that $\mathrm{depth}_S I/J = t$ if $\mathrm{depth}_S I/J \geq t$ and

$$\rho_{t+1}(I \setminus J) < \sum_{i=0}^{t-d} (-1)^{t-d+i} \rho_{d+i}(I \setminus J).$$

The proof of this theorem (similarly of [**4**, Theorem 2.2]) uses the Koszul homology (especially the rigidity property of the Koszul homology [**1**, Exercise 1.6.31]) which proves to be a very strong tool in this frame. If $t = d$, then our theorem is precisely Theorem 0.1 (a

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previous result is given in [**3**]). If $t = d+1$, then the above result says that $\operatorname{depth}_S I/J \leq d+1$ if

$$\rho_{d+1}(I \setminus J) > \rho_{d+2}(I \setminus J) + \rho_d(I \setminus J).$$

A particular case with I principal is given, with a different proof, in our Proposition 1.1. Theorem 0.1 is a small step in an attempt to show Stanley's conjecture for some classes of factors of square free monomial ideals (see our Remark 1.6 for some details), and we hope that our Theorem 1.3 will be useful in the same frame.

1. Upper bounds of depth. The aim of this section is to show the extension of Theorem 0.1 stated in the introduction. We start with a particular case.

Proposition 1.1. *Suppose that I is generated by a square free monomial f of degree d, and $s = \rho_{d+1}(I \setminus J) > \rho_{d+2}(I \setminus J) + 1$. Then* $\operatorname{depth}_S I/J = d+1$.

Proof. First suppose that $q = \rho_{d+2}(I \setminus J) > 0$. Let $g \in I \setminus J$ be a square free monomial of degree $d+2$. Renumbering the variable x, we may suppose that I is generated by $f = x_1 \cdots x_d$ and $g = f x_{d+1} x_{d+2}$. Since $g \notin J$, we see that $b_1 = f x_{d+1}$ and $b_2 = f x_{d+2}$ are not in J. Again, renumbering x, we may suppose that $b_i = f x_{d+i}$, $i \in [s]$, are all the square free monomials of degree $d+1$ from $I \setminus J$. Set $T = (b_3, \ldots, b_s)$ (by hypothesis $s \geq 3$). In the exact sequence

$$0 \longrightarrow T/T \cap J \longrightarrow I/J \longrightarrow I/(T+J) \longrightarrow 0,$$

we see that the left end has depth $d+1$ by Theorem 0.1 since $T \cap J$ is generated in degree $\geq d+2$ and $\rho_{d+1}(T) = s-2 > q-1 = \rho_{d+2}(T \setminus T \cap J)$. On the other hand, $(T+J) : f = (x_{d+3}, \ldots, x_n)$ because $b_1, b_2, g \notin T+J$. It follows that $\operatorname{depth}_S I/(T+J) = d+2$, and so the depth lemma says that $\operatorname{depth}_S I/J = d+1$.

Now suppose that $q = 0$. As above, we may assume that $b_i = f x_{d+i}$, $i \in [s]$, are the square free monomials of degree $d+1$ of $I \setminus J$. Then $J : f = (x_{d+s+1}, \ldots, x_n) + L$, where L is the Veronese ideal generated by all square free monomials of degree 2 in $x_{d+1}, \ldots, x_{d+s}$. It follows that $I/J \cong K[x_1, \ldots, x_{d+s}]/L$, which has depth $d+1$. $\square$

Next we present some details on the Koszul homology (see [**1**]) which we need for the proof of our main result. Let $\partial_i : K_i(x;I/J) \to K_{i-1}(x;I/J)$, $K_i(x;I/J) \cong (I/J)^{\binom{n}{i}}$, $i \in [n]$, be the Koszul derivation given by

$$\partial_i(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{k=1}^{i} (-1)^{k+1} x_{j_k} e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \wedge e_{j_{k+1}} \wedge \cdots \wedge e_{j_i}.$$

Fix $0 \leq i < n-d$. Let $f_1, \ldots, f_r$, $r = \rho_{d+i}(I \setminus J)$, be all square free monomials of degree $d+i$ from $I \setminus J$ and $b_1, \ldots, b_s$, $s = \rho_{d+i+1}(I \setminus J)$ be all square free monomials of degree $d+i+1$ from $I \setminus J$. Let $\operatorname{supp} f_i = \{j \in [n] : x_j \mid f_i\}$, $e_{\sigma_i} = \wedge_{j \in ([n] \setminus \operatorname{supp} f_i)} e_j$ and $\operatorname{supp} b_k = \{j \in [n] : x_j \mid b_k\}$, $e_{\tau_k} = \wedge_{j \in ([n] \setminus \operatorname{supp} b_k)} e_j$. We consider the element $z = \sum_{q=1}^{r} y_q f_q e_{\sigma_q}$ of $K_{n-d-i}(x;I/J)$, where $y_q \in K$. Then

$$\partial_{n-d-i}(z) = \sum_{k=1}^{s} \left(\sum_{q \in [r]} \varepsilon_{kq} y_q \right) b_k e_{\tau_k},$$

where $\varepsilon_{kq} \in \{1,-1\}$ if $f_q \mid b_k$, otherwise $\varepsilon_{kq} = 0$. Thus, $\partial_{n-d-i}(z) = 0$ if and only if $\sum_{q \in [r]} \varepsilon_{kq} y_q = 0$ for all $k \in [s]$, that is, $y = (y_1, \ldots, y_r)$ is in the kernel of the linear map $h_{n-d-i} : K^r \to K^s$ given by the matrix ε_{kq}.

Now we will see when $z \in \operatorname{Im} \partial_{n-d-i+1}$. Since the Koszul derivation is a graded map we note that $z \in \operatorname{Im} \partial_{n-d-i+1}$ if and only if $z = \partial_{n-d-i+1}(w)$ for a $w = \sum_{p=1}^{c} u_p g_p e_{\nu_p}$, where $c = \rho_{d+i-1}(I \setminus J)$, $u_p \in K$, $g_1, \ldots, g_c$ are all square free monomials of degree $d+i-1$ from $I \setminus J$ and $e_{\nu_p} = \wedge_{j \in ([n] \setminus \operatorname{supp} g_p)} e_j$. It follows that

$$z = \sum_{q=1}^{r} \left(\sum_{p \in [c]} \gamma_{qp} u_p \right) f_q e_{\sigma_q},$$

where $\gamma_{qp} \in \{1,-1\}$ if $g_p \mid f_q$, otherwise $\gamma_{qp} = 0$. Thus, $z \in \operatorname{Im} \partial_{n-d-i+1}$ if and only if y belongs to the image of the linear map $h_{n-d-i+1} : K^c \to K^r$ given by the matrix γ_{qp}. When $i = 0$, we have $h_{n-d-i+1} = 0$.

Note that $\operatorname{Im} h_{n-d-i+1} \subset \operatorname{Ker} h_{n-d-i}$, and the inclusion is strict if and only if there exists $y \in K^r$ such that z induces a nonzero element in $H_{n-d-i}(x; I/J)$. This implies $\operatorname{depth}_S I/J \leq d+i$ by [**1**, Theorem 1.6.17]. In fact, this is based on the rigidity property of the Koszul homology [**1**, Exercise 1.6.31]. If $\operatorname{depth} I/J > d+i$, then $\operatorname{Im} \partial_{n-d-i+1} = \operatorname{Ker} \partial_{n-d-i}$, and it follows that $\operatorname{Im} h_{n-d-i+1} = \operatorname{Ker} h_{n-d-i}$.

Lemma 1.2. *Let $0 \leq i < n-d$. Then the following statements hold independently of the characteristic of K:*

(1) *the complex $K^c \xrightarrow{h_{n-d-i+1}} K^r \xrightarrow{h_{n-d-i}} K^s$ is exact if* $\operatorname{depth} I/J > d+i$,

(2) *if* $\operatorname{depth}_S I/J > d+i$ *then* $r = \operatorname{rank} h_{n-d-i+1} + \operatorname{rank} h_{n-d-i}$,

(3) *if* $r > \operatorname{rank} h_{n-d-i+1} + \operatorname{rank} h_{n-d-i}$ *then* $\operatorname{depth}_S I/J \leq d+i$.

Proof. The first statement follows from the above and the second one is only a consequence. If $r > \operatorname{rank} h_{n-d-i+1} + \operatorname{rank} h_{n-d-i}$, then $\operatorname{Im} h_{n-d-i+1} \subsetneq \operatorname{Ker} h_{n-d-i}$, and the last statement also follows from the above. $\square$

Theorem 1.3. *Let $d \leq t \leq n$ be two integers, and set*

$$\alpha_j = \sum_{i=0}^{j-d} (-1)^{j-d+i} \rho_{d+i}(I \setminus J),$$

for $d \leq j \leq t$. Suppose that $\operatorname{depth}_S I/J \geq t$ *and $\rho_{t+1}(I \setminus J) < \alpha_t$. Then* $\operatorname{depth}_S I/J = t$ *independently of the characteristic of K.*

Proof. We have $\alpha_j = \rho_j(I \setminus J) - \alpha_{j-1}$ for $d < j \leq t$. By Lemma 1.2 (2), we get h_{n-d} injective and $\rho_{d+i}(I \setminus J) = \operatorname{rank} h_{n-d-i+1} + \operatorname{rank} h_{n-d-i}$ for $0 < i < t-d$. It follows that $\rho_d(I \setminus J) = \operatorname{rank} h_{n-d} = \alpha_d$, $\rho_{d+1}(I \setminus J) = \operatorname{rank} h_{n-d} + \operatorname{rank} h_{n-d-1} = \rho_d(I \setminus J) + \operatorname{rank} h_{n-d-1}$, and so $\operatorname{rank} h_{n-d-1} = \alpha_{d+1}$. By recurrence, we get $\operatorname{rank} h_{n-t+1} = \alpha_{t-1}$. Clearly, $\operatorname{rank} h_{n-t} \leq \rho_{t+1}(I \setminus J)$. By hypothesis, $\rho_{t+1}(I \setminus J) < \alpha_t = \rho_t(I \setminus J) - \alpha_{t-1}$. It follows that $\operatorname{rank} h_{n-t} < \rho_t(I \setminus J) - \alpha_{t-1} = \rho_t(I \setminus J) - \operatorname{rank} h_{n-t+1}$, which gives $\operatorname{depth}_S I/J = t$ by Lemma 1.2 (3). $\square$

The next example shows that the above theorem is tight.

Example 1.4. Let $n = 4$, $I = (x_1, x_3)$ and $J = (x_1x_4)$. Note that $x_1x_2, x_1x_3, x_2x_3, x_3x_4$ are all square free monomials of degree 2 from $I \setminus J$ and $x_1x_2x_3, x_2x_3x_4$ are all square free monomials of degree 3 from $I \setminus J$. Thus, $\rho_2(I \setminus J) = 4 = \rho_1(I) + \rho_3(I \setminus J)$, but $\operatorname{depth}_S I/J = 3$. On the other hand, taking $J' = J + (x_2x_3x_4)$ we see that $\operatorname{depth}_S I/J' = 2$ which is also given by Theorem 1.3 since $\rho_3(I \setminus J') = 1$, and we have $\rho_2(I \setminus J') = 4 > 3 = \rho_1(I) + \rho_3(I \setminus J')$.

Corollary 1.5. *Suppose that* $\operatorname{depth}_S I/J \geq d + 2$. *Then* $\rho_d(I) \leq \rho_{d+1}(I \setminus J) \leq \rho_d(I) + \rho_{d+2}(I \setminus J)$. *Moreover, if* $\rho_{d+2}(I \setminus J) = 0$, *then* $\rho_d(I) = \rho_{d+1}(I \setminus J)$.

Remark 1.6. Consider the poset $P_{I\setminus J}$ of all square free monomials of $I \setminus J$ (a finite set) with the order given by the divisibility. Let $\mathcal{P}$ be a partition of $P_{I\setminus J}$ in intervals $[u, v] = \{w \in P_{I\setminus J} : u \mid w, w \mid v\}$, let us say $P_{I\setminus J} = \cup_i [u_i, v_i]$, the union being disjoint. Define $\operatorname{sdepth} \mathcal{P} = \min_i \deg v_i$ and $\operatorname{sdepth}_S I/J = \max_{\mathcal{P}} \operatorname{sdepth} \mathcal{P}$, where $\mathcal{P}$ runs in the set of all partitions of $P_{I\setminus J}$. This is the Stanley depth of I/J, in the idea of [**2**] (also see [**5**]). Stanley's conjecture says that $\operatorname{sdepth}_S I/J \geq \operatorname{depth}_S I/J$. In the above corollary, $\rho_{d+2}(I \setminus J) = 0$ implies $\operatorname{sdepth}_S I/J \leq d + 1$ and so $\operatorname{depth}_S I/J \leq d + 1$ if Stanley's conjecture holds. This shows the weakness of the above corollary, which accepts the possibility of having $\operatorname{depth}_S I/J \geq d + 2$ when $\rho_d(I) = \rho_{d+1}(I \setminus J)$.

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"Simion Stoilow" Institute of Mathematics of Romanian Academy, Research Unit 5, P.O. Box 1-764, Bucharest 014700, Romania
Email address: dorin.popescu@imar.ro