JOURNAL OF COMMUTATIVE ALGEBRA
Volume 5, Number 1, Spring 2013

GENERIC CIRCUITS SETS AND GENERAL INITIAL IDEALS WITH RESPECT TO WEIGHTS

GIULIO CAVIGLIA AND ENRICO SBARRA

We would like to dedicate this paper to Jürgen Herzog, teacher and collaborator, to his ability for sharing his passion for Commutative Algebra with so many students all over the world.

ABSTRACT. We study the set of circuits of a homogeneous ideal and that of its truncations, and introduce the notion of generic circuits set. We show how this is a well-defined invariant that can be used, in the case of initial ideals with respect to weights as a counterpart of the (usual) generic initial ideal with respect to monomial orders. As an application we recover the existence of the generic fan introduced by Römer and Schmitz for studying generic tropical varieties. We also consider general initial ideals with respect to weights and show, in analogy to the fact that generic initial ideals are Borel-fixed, that these are fixed under the action of certain Borel subgroups of the general linear group.

0. Introduction. In the study of homogeneous ideals in a polynomial ring it is a standard technique to pass to initial ideals. Also, in order to work with a monomial ideal more closely related to a given homogeneous ideal I, i.e., with a monomial ideal which shares with I important numerical invariants other than the Hilbert function, one can choose to work in generic coordinates or, in other words, to consider a generic initial ideal of I with respect to some monomial order. Even though some of the ideas underlying the notion of generic initial ideal were already present in the works of Hartshorne [**12**] and Grauert [**10**], a proper definition as well as the study of some of its main properties is to be found only later in the work of Galligo [**9**], where characteristic zero

2010 AMS *Mathematics subject classification.* Primary 13F10, 13P20, Secondary 13A50.
Keywords and phrases. Generic initial ideal, weights, circuits, generic tropical variety.
The work of the first author was supported by a grant from the Simons Foundation (209661).
Received by the editors on October 3, 2012, and in revised form on November 16, 2012.
DOI:10.1216/JCA-2013-5-1-49

is assumed and the subsequent paper of Bayer and Stillman [**3**], where the assumption on the characteristic is dropped. It is also shown there that generic initial ideals are invariant under the action of the Borel subgroup of the general linear group of coordinates changes $\mathrm{GL}_n(K)$; thus, they are endowed with interesting combinatorial properties which depend on the characteristic, but are well understood (see for instance [**16**], also for the study of other group actions). More can be said if one considers generic initial ideals with respect to some special monomial orders such as the lexicographic and the reverse-lexicographic orders. The generic initial ideal of I with respect to the revlex order has the same depth as I; therefore, it has the same projective dimension as I and its quotient ring is Cohen-Macaulay exactly when that of I is Cohen-Macaulay; furthermore, it shares with I the same Castelnuovo-Mumford regularity and, in general, the same positions and values of extremal Betti numbers, cf., [**2, 3**]. On the other hand, when one considers the lex order, the generic initial ideal of I captures other geometric invariants of the projective variety defined by I, see for instance [**11**, Section 6], [**1, 6**]. The interested reader is referred to the standard references [**7, 11**] and will also find the dedicated parts of Herzog and Hibi's book [**13**] useful to understand the connection with extremal Betti numbers and with shifting operations.

The main question we address in this paper is the following: *How can one define the generic initial ideal with respect to a weight*? The initial ideal with respect to a weight ω of gI is not necessarily constant on a non-empty Zariski open subset of $\mathrm{GL}_n(K)$ as, for instance, if $\omega = (1, \dots, 1)$ then $\mathrm{in}_\omega(gI) = gI$ for all coordinates changes g. We provide an answer to the above question by introducing some new invariants of I.

This paper is organized as follows. The first section is dedicated to introducing some notation and recalling some well-known properties of monomial orders, initial ideals with respect to weights, reduced and universal Gröbner bases. In the second section, Definitions 2.1 and 2.4, we introduce the notion of circuits set and generic circuits set, we explain their basic properties, cf. Lemma 2.2 and Theorem 2.9, and we relate the circuits set and the generic circuits set of a homogeneous ideal I to its reduced Gröbner bases and Gröbner fan, cf. Lemma 2.11, Corollaries 2.12 and 2.14. As an application, we recover in Corollary 2.15 one of the main results of [**18**, Corollary 3.2], where the *generic Gröbner fan* is introduced. In the third and last section we explain why what would be the natural definition of generic initial ideal with respect to weights does not return an invariant of a homogeneous ideal, and we suggest what provides, in our opinion, valid alternatives to generic initial ideals when working with weights: generic and general circuits set, their truncations, their initial circuits set and general initial ideals. Finally, in analogy with what is known in the case of monomial orders, we show that general initial ideals with respect to weights are stable under the action of certain subgroups of the general linear group.

1. Notation and preliminaries. Let $A = K[X_1, \dots, X_n]$ be a polynomial ring over a field K. Given a non-zero polynomial $f \in A$, we write it uniquely as a sum of monomials with non-zero coefficients, and we call the set of all such monomials, denoted by $\operatorname{supp}(f)$, the (*monomial*) *support* of f. When $S \subseteq A$ is a set, $\operatorname{supp}(S)$ will be the set of all monomial supports of the polynomials in S. When I is a homogeneous ideal of A and d an integer, by I_d and $I_{\leq d}$ we denote the degree d part of I and the *truncation* $\oplus_{j \leq d} I_j$ of I at (and below) d, respectively.

1.1. Monomial orders. We recall that a *monomial order* on A is a total order $\prec$ on the monomials of A which is also compatible with multiplication, i.e., for all $\mathbf{X}^{\mathbf{a}}, \mathbf{X}^{\mathbf{b}}, \mathbf{X}^{\mathbf{c}}$ monomials of A with $\mathbf{X}^{\mathbf{c}} \neq 1$ and $\mathbf{X}^{\mathbf{a}} \prec \mathbf{X}^{\mathbf{b}}$ one has $\mathbf{X}^{\mathbf{a}} \prec \mathbf{X}^{\mathbf{a}}\mathbf{X}^{\mathbf{c}} \prec \mathbf{X}^{\mathbf{b}}\mathbf{X}^{\mathbf{c}}$. Given a monomial order $\prec$, we denote by $\operatorname{in}_\prec(f)$ the greatest with respect to $\prec$ monomial in $\operatorname{supp}(f)$; we call it the *initial monomial* (or *leading monomial*) of f. Accordingly, given a homogeneous ideal I, we call *initial ideal* of I with respect to $\prec$ and denote it by $\operatorname{in}_\prec(I)$, the ideal generated by all the initial monomials of elements of I. A finite set $G = \{f_1, \dots, f_r\}$ of elements of I such that $\{\operatorname{in}_\prec(f_1), \dots, \operatorname{in}_\prec(f_r)\}$ is a set of generators for $\operatorname{in}_\prec(I)$ is called a *Gröbner basis* of I (with respect to $\prec$). Furthermore, if $f_1, \dots, f_r$ are monic (with respect to $\prec$) and $\operatorname{in}_\prec(f_i)$ does not divide any monomial in $\operatorname{supp}(f_j)$ for $i \neq j$, then we call G the *reduced Gröbner basis* of I (with respect to $\prec$). It is not difficult to see that such a basis always exists, and it is uniquely determined by $\prec$ and I; moreover, if $\prec$ and $\prec'$ are two monomial orders such that $\operatorname{in}_\prec(I) = \operatorname{in}_{\prec'}(I)$, then the reduced Gröbner bases of I with respect to $\prec$ and to $\prec'$ are the same. It is well known that a given homogeneous ideal has only a finite number of initial ideals; therefore, it has finitely many reduced Gröbner bases. A subset G of A is called a *universal Gröbner basis* of I if G is a Gröbner basis of I with respect to all monomial orders simultaneously. Such a basis can be obtained, for instance, as the union of all the reduced Gröbner bases of I (cf. [**19**, Corollary 1.3) in which case we call it the *canonical universal Gröbner basis* of I.

1.2. Initial ideals with respect to weights. We now consider the general case of initial ideals defined by using weights and summarize some of the basic properties and constructions; our main references are, as before, the books of Eisenbud [**7**] and Sturmfels [**19**]. We shall call a vector of $\mathbf{R}^n$ a *weight vector* or, simply, a *weight*. Given a polynomial $f = \sum_i \alpha_i \mathbf{X}^{\mathbf{a_i}}$, one lets the *initial form* of f with respect to ω be the sum of all terms $\alpha_j \mathbf{X}^{\mathbf{a_j}}$ of f which have maximal weight, i.e., such that the scalar product $\omega \cdot \mathbf{a_j}$ is maximal. Accordingly, one defines the *initial ideal with respect to* ω of a given ideal I as the ideal $\mathrm{in}_\omega(I)$ generated by all the initial forms of polynomials in I. This ideal will not be monomial in general. Similarly, one defines $\mathrm{in}_\omega(W)$ for a K-vector subspace W of A.

Let now $\prec$ be a monomial order; it is natural to define a new monomial order $\prec_\omega$ by refining ω by means of $\prec$, so that for all $f \in A$ one has $\mathrm{in}_{\prec_\omega}(f) = \mathrm{in}_\prec(\mathrm{in}_\omega(f))$ and, similarly, for all ideals $I \subseteq A$ one has $\mathrm{in}_{\prec_\omega}(I) = \mathrm{in}_\prec(\mathrm{in}_\omega(I))$, which also yields that I and $\mathrm{in}_\omega(I)$ share the same Hilbert function. Furthermore, if $G = \{f_1, \dots, f_r\}$ is a (reduced, universal) Gröbner basis of I with respect to $\prec_\omega$, then $\{\mathrm{in}_\omega(f_i) : i = 1, \dots, r\}$ is a (reduced, universal) Gröbner basis of $\mathrm{in}_\omega(I)$ with respect to $\prec$. The use of weights generalizes monomial orders also in the following sense: for any monomial order $\prec$ and any homogeneous ideal I, there exists a non-negative integral weight ω such that $\mathrm{in}_\prec(I) = \mathrm{in}_\omega(I)$, by [**19**, Proposition 1.11] (see also [**17**]). We also observe that, if $I \subseteq A$ is a homogeneous ideal, $\prec$ is a monomial order and $\omega, \omega' \in \mathbf{R}^n$ are such that $\mathrm{in}_\omega(\cdot)$ and $\mathrm{in}_{\omega'}(\cdot)$ coincide on all elements of a reduced Gröbner basis of I with respect to $\prec_\omega$, then the initial ideal of I with respect to ω and ω' coincide, since both have the same Hilbert function as I.

1.3. A flat family argument. We would like to conclude this section with a technical observation we shall need later when we use a classical flat family argument, as of [**7**, Theorem 15.17]. Let $\omega \in \mathbf{Z}^n$, I a given ideal of A and $A[t]$ a polynomial ring over A. Let $\mathbf{e_i}$ denote the ith element of the standard basis of $\mathbf{Z}^n$; for all $f = \sum_i \alpha_i \mathbf{X}^{\mathbf{a_i}} \in I$, we denote the homogenization $t^{\max_i\{\omega\cdot\mathbf{a_i}\}} f(t^{-\omega\cdot\mathbf{e_1}}X_1, \dots, t^{-\omega\cdot\mathbf{e_n}}X_n)$ of f with respect to ω by $\widetilde{f}$; also, $\widetilde{I}$ will denote the ideal of $A[t]$ generated by all $\widetilde{f}$ with $f \in I$. The ideal $\widetilde{I}$ is the homogenization of I with respect to ω. One can thus build a family $\{I_a : a \in K\}$ of ideals of A, where $I_a = \widetilde{I}_{t=a}$ is the ideal $\widetilde{I}$ evaluated at $t = a$. It is important to notice that $I_1 = I$, $I_0 = \mathrm{in}_\omega(I)$, and that for all $a \neq 0$ the ideal I_a is the image of I under the diagonal change of coordinates D_a which maps X_i to $a^{-\omega\cdot\mathbf{e_i}}X_i$. This family is flat because the Hilbert function is constant on its elements, see also Definition 1.17 and Theorem 1.18 in [**11**].

2. Generic circuits set. In this section we define the notions of circuits set and generic circuits set of homogeneous ideals and show how to use these definitions to compute reduced and universal Gröbner bases, and Gröbner fans.

Definition 2.1. Let I be a subset of A. We define the *circuits set of* I, denoted by $\mathbf{cs}(I)$, to be the set of all minimal (with respect to inclusion) elements of $\mathrm{supp}(I)$. We say that a set T is a *circuits set* if $T = \mathbf{cs}(I)$ for some subset I of A. In particular, T is a collection of finite sets of monomials of A.

The name we chose in the above definition comes from Matroid Theory, see [**15**] or any other standard reference: a circuit in a matroid is a minimal dependent subset, i.e., a dependent set whose proper subsets are all independent. When I is a K-vector subspace of A, as it is for instance when I is a homogeneous ideal, one can define a matroid by considering the set S of all monomials of A and declaring a subset of S independent (respectively dependent) if its image in A/I consists of linearly independent (respectively dependent) elements. The support of a polynomial $f \in I$ is minimal among all the supports of elements of I if and only if it is a circuit in the above matroid. It is immediately seen that, if I is finite, then so is $\mathbf{cs}(I)$. Moreover, if I is a homogeneous ideal of A, then $\mathbf{cs}(I) = \sqcup_d \mathbf{cs}(I_d)$ and $\mathbf{cs}(I_{\leq d}) = \sqcup_{h\leq d}\mathbf{cs}(I_h)$. Also, we notice that, if I is a monomial ideal, then $\mathbf{cs}(I)$ is just the set of all monomials in I. Clearly, when I and J are monomial ideals, then $\mathbf{cs}\,(I) = \mathbf{cs}\,(J)$ if and only if $I = J$, a fact which is false in general, e.g., in $A = K[X_1, X_2]$, where $\mathrm{char}\,(K) \neq 2$, the ideals $(X_1+X_2)+(X_1, X_2)^2$, $(X_1 - X_2) + (X_1, X_2)^2$ are distinct and have same circuits sets.

It is useful to point out that, if $\{f_1, \dots, f_r\}$ is a Gröbner basis with respect to a monomial order $\prec$, then $\mathrm{supp}\,(f_i)$ is not necessarily an element of $\mathbf{cs}\,(I)$, take for instance $A = K[X_1, X_2]$ and $I = (X_1 + X_2, X_2)$. In fact, if $\{f_1, \dots, f_r\}$ is a reduced Gröbner basis, then $\mathrm{supp}\,(f_i) \in \mathbf{cs}\,(I)$ for all $i = 1, \dots, r$: If $\mathrm{supp}\,(f_h) \notin \mathbf{cs}\,(I)$ for some h, then there would exist a $g \in I$ with $\mathrm{supp}\,(g) \subsetneq \mathrm{supp}\,(f_h)$; it is easily seen that this would contradict the fact that $\{f_1, \dots, f_r\}$ is reduced, whether $\mathrm{in}_\prec(g) = \mathrm{in}_\prec(f_h)$ or not. We have thus proven the following lemma.

Lemma 2.2. *Let I be a homogeneous ideal, $\prec$ a fixed monomial order and G the reduced Gröbner basis of I with respect to $\prec$. Then,* $\mathrm{supp}\,(G) \subseteq \mathbf{cs}\,(I)$.

Let now $\mathbf{y} = (y_{ij})_{i,j=1,\dots,n}$ be a matrix of indeterminates and $K(\mathbf{y})$ an extension field of K. In the following we shall denote by γ the K-algebra homomorphism

$$\gamma : K(\mathbf{y})[X_1, \dots, X_n] \longrightarrow K(\mathbf{y})[X_1, \dots, X_n],$$

$$\gamma X_i \longmapsto \sum_{j=1}^{n} y_{ij} X_j \quad \text{for all } i = 1, \dots, n. \tag{2.3}$$

Definition 2.4. Let I be a homogeneous ideal of A. We define the *generic circuits set* of I as $\mathbf{cs}\,(\gamma I)$, and we denote it by $\mathbf{gcs}\,(I)$. Given a non-negative integer d, we let the *generic circuits set of I truncated at d*, denoted by $\mathbf{gcs}\,(I_{\leq d})$, be the circuits set $\mathbf{cs}\,(\gamma I_{\leq d})$.

It is easy to see that $\mathbf{gcs}\,(I_{\leq d}) = \mathbf{cs}\,((\gamma I)_{\leq d}) = \mathbf{cs}\,(\gamma I)_{\leq d} = \mathbf{gcs}\,(I)_{\leq d}$.

Remark 2.5. The generic circuits set of a homogeneous ideal I is invariant under coordinates changes, i.e., for all $h \in \mathrm{GL}_n(K)$, one has $\mathbf{gcs}\,(hI) = \mathbf{gcs}\,(I)$. To this end, observe that, if $\mathbf{z}$ is the matrix associated with γh, then $K(\mathbf{y})$ and $K(\mathbf{z})$ are the same field; in particular, the entries of $\mathbf{z}$ are algebraically independent over K. Moreover, γhI is the image of I under the map $K(\mathbf{z})[X_1,\dots,X_n] \to K(\mathbf{z})[X_1,\dots,X_n]$, with $X_i \mapsto \sum_{j=1}^n z_{ij}X_j$ for all $i=1,\dots,n$. Hence, $\operatorname{supp}(\gamma f) = \operatorname{supp}(\gamma h f)$ for all $f \in A$.

Notation 2.6. Let S be a finite set of monomials of A and W a K-vector subspace of A. We set

$$\operatorname{rank}_S W := \dim_K(W + \langle S\rangle)/W, \qquad \operatorname{rank}^S W := \dim_K(W + \langle S\rangle)/\langle S\rangle.$$

Evidently, $\operatorname{rank}^S W = \operatorname{rank}_S(W) + \dim_K W - \dim_K\langle S\rangle$.

Remark 2.7. It is an easy observation completing the discussion before Lemma 2.2 that the circuits set of a homogeneous vector space can be determined using ranks: Given a K-vector space $W \subseteq A_d$, a set S is an element of $\mathbf{cs}\,(W)$ if and only if $\operatorname{rank}_S W < |S|$ and $\operatorname{rank}_{S'}(W) = |S'|$ for all $\varnothing \neq S' \subsetneq S$.

Let W be a K-vector subspace of A_d with basis B. Consider an ordered monomial basis of A_d, and let M_W be the $\dim_K W \times \dim_K A_d$ matrix whose (i,j)th entry is the coefficient of the jth-monomial in the ith basis element of W. Clearly, a minor of M_W is an element of K.

Now we consider γW together with its basis γB, where γ is as in (2.3). The minors of the matrix $M_{\gamma W}$ are polynomials in $K[\mathbf{y}]$ which specialize to the minors of M_W when all of the y_{ij} are evaluated at 1 if $i=j$ and at 0 otherwise.

When S is a set of monomials in A_d, we may thus conclude that $\operatorname{rank}_S W \leq \operatorname{rank}_S \gamma W$; if K is infinite, then there exists a non-empty Zariski open set $U \subseteq \mathrm{GL}_n(K) \subset K^{n^2}$ such that if $g \in U$, then

$$\operatorname{rank}_S gW = \operatorname{rank}_S \gamma W = \max\{\operatorname{rank}_S hW : h \in \mathrm{GL}_n(K)\}, \tag{2.8}$$

for all $S \subseteq A_d$.

Theorem 2.9. *Let K be an infinite field, d a positive integer and $I \subseteq A$ a homogeneous ideal. Then, there exists a non-empty Zariski open set $U \subseteq \mathrm{GL}_n(K) \subset K^{n^2}$ such that $\mathbf{gcs}\,(I_{\leq d}) = \mathbf{cs}\,(gI_{\leq d})$ for all $g \in U$.*

Proof. For any integer i, a set S belongs to $\mathbf{cs}\,(\gamma I_i)$ if and only if the condition on ranks of Remark 2.7 holds for the K-vector space γI_i or for gI_i, where g belongs to a non-empty Zariski open set, say U_i, for which (2.8) holds. The desired open set U can be taken simply as the intersection of all U_i, $i = 0, \dots, d$. □

We do not know at the present time whether there exists a non-empty Zariski open set U such that $\mathbf{gcs}\,(I) = \mathbf{cs}\,(gI)$ for all $g \in U$, or whether there exists any such g at all.

Let I be a homogeneous ideal of A and consider now the following equivalence relation on $\mathbf{R}^n$: two weights ω and ω' are said to be equivalent if and only if $\mathrm{in}_\omega(I) = \mathrm{in}_{\omega'}(I)$. The closures with respect to the Euclidean topology of such equivalence classes are convex polyhedral cones, and the collection of all such cones form a fan, which is called the *Gröbner fan of I*, see [**14, 19**].

Notation 2.10. Let S be a finite set of monomials of A and ω a weight. We denote by $\mathrm{in}_\omega(S)$ the set of all elements of S with maximal weight. Similarly, for a collection T of finite sets of monomials, we denote by $\mathrm{in}_\omega(T)$ the set of all $\mathrm{in}_\omega(S)$ for S in T. When I is a subset of A we will refer to $\mathrm{in}_\omega(\mathbf{cs}\,(I))$ as the *initial circuits set of I with respect to* ω. It is not hard to see that, when I is a homogeneous ideal $\mathrm{in}_\omega(\mathbf{cs}\,(I)) = \mathbf{cs}\,(\mathrm{in}_\omega(I))$, and also that $\mathrm{in}_\omega(\mathbf{cs}\,(I_{\leq d})) = \mathbf{cs}\,((\mathrm{in}_\omega(I))_{\leq d})$.

The following technical result will be useful in the remaining part of the section. See also [**19**, Proposition 2.3] and [**8**, Proposition 2.6].

Lemma 2.11. *Let I be a homogeneous ideal of A, $\omega, \omega' \in \mathbf{R}^n$ two weights and $\prec$ a given monomial order. If $\{f_1, \dots, f_r\}$ is a reduced Gröbner basis of I with respect to $\prec_\omega$, then*

$$\mathrm{in}_\omega(I) = \mathrm{in}_{\omega'}(I)$$

if and only if

$$\mathrm{in}_\omega(\mathrm{supp}\,(f_i)) = \mathrm{in}_{\omega'}(\mathrm{supp}\,(f_i)) \quad \textit{for } i = 1, \dots, r.$$

Proof. We first assume that $\mathrm{in}_\omega(I) = \mathrm{in}_{\omega'}(I)$. Since $\mathrm{in}_\prec(\mathrm{in}_\omega(I)) = \mathrm{in}_{\prec_\omega}(I)$, we immediately have that $\mathrm{in}_{\prec_\omega}(I) = \mathrm{in}_{\prec_{\omega'}}(I)$, hence $\mathrm{in}_{\prec_{\omega'}}(f_i) \in$

$\mathrm{in}_{\prec_\omega}(I)$. Since $\mathrm{in}_{\prec_{\omega'}}(f_i)$ is a monomial of f_i and $\{f_1, \ldots, f_r\}$ is a reduced Gröbner basis with respect to $\prec_\omega$, this implies that $\mathrm{in}_{\prec_\omega}(f_i) = \mathrm{in}_{\prec_{\omega'}}(f_i)$ for $i = 1, \ldots, r$. Now, we assume by contradiction that $\mathrm{in}_\omega(\mathrm{supp}\,(f_i)) \neq \mathrm{in}_{\omega'}(\mathrm{supp}\,(f_i))$ for some i, and accordingly $\mathrm{in}_\omega(f_i) - \mathrm{in}_{\omega'}(f_i)$ is a non-zero element of $\mathrm{in}_\omega(I)$ which does not contain $\mathrm{in}_{\prec_\omega}(f_i)$ in its support. Thus, $\mathrm{in}_\prec(\mathrm{in}_\omega(f_i) - \mathrm{in}_{\omega'}(f_i)) \in \mathrm{in}_{\prec_\omega}(I)$ contradicts the fact that $\{f_1, \ldots, f_r\}$ is a reduced Gröbner basis of I with respect to $\prec_\omega$.

Vice versa, when $\mathrm{in}_\omega(\mathrm{supp}\,(f_i)) = \mathrm{in}_{\omega'}(\mathrm{supp}\,(f_i))$, then $\mathrm{in}_\omega(I) = (\mathrm{in}_\omega(f_i) : i = 1, \ldots, r) = (\mathrm{in}_{\omega'}(f_i) : i = 1, \ldots, r) \subseteq \mathrm{in}_{\omega'}(I)$, and equality is forced by the Hilbert function. □

From the previous result it follows as a corollary that the set of all supports of all reduced Gröbner bases of a homogeneous ideal I determines the equivalence relation on weights that defines the Gröbner fan of I.

Corollary 2.12. *Let I and J be homogeneous ideals of A with canonical universal Gröbner bases G_1 and G_2, respectively. If* $\mathrm{supp}\,(G_1) = \mathrm{supp}\,(G_2)$, *then* $\mathrm{Gf}(I) = \mathrm{Gf}(J)$.

Proposition 2.13. *Let I and J be homogeneous ideals with the same Hilbert function, $\prec$ a fixed monomial order, G_1 and G_2 the reduced Gröbner bases (with respect to $\prec$) of I and J, respectively. If d is an integer greater than or equal to the largest degree of an element of G_1 and* $\mathbf{cs}\,(I_{\leq d}) = \mathbf{cs}\,(J_{\leq d})$, *then* $\mathrm{supp}\,(G_1) = \mathrm{supp}\,(G_2)$.

Proof. Let $G_1 = \{f_1, \ldots, f_r\}$. We know that $G_1 \subseteq I_{\leq d}$, and by Lemma 2.2, $\mathrm{supp}\,(G_1) \subseteq \mathbf{cs}\,(I_{\leq d}) = \mathbf{cs}\,(J_{\leq d})$. Thus, there exists a subset $H_2 = \{h_1, \ldots, h_r\}$ of J with $\mathrm{supp}\,(h_i) = \mathrm{supp}\,(f_i)$ for $i = 1, \ldots, r$, and we may assume that the initial monomials of $h_1, \ldots, h_r$ with respect to $\prec$ have coefficients equal to 1. Now, $(\mathrm{in}_\prec(h_i) : i = 1, \ldots, r)$ has the same Hilbert function as I and, thus, as J; consequently, H_2 is a Gröbner basis of J, and it is clearly reduced. Since such a basis is unique, $H_2 = G_2$ and we are done. □

Corollary 2.14. *Let I and J be homogeneous ideals with the same Hilbert function, and let d be the largest degree of a minimal generator of the lex-segment ideal with same Hilbert function as I and J. If* $\mathbf{cs}\,(I_{\leq d}) = \mathbf{cs}\,(J_{\leq d})$, *then* $\mathrm{Gf}(I) = \mathrm{Gf}(J)$.

As a special case of the above corollary, we now recover one of the main results of [**18**], namely Corollary 3.2, where the existence of generic Gröbner fans is proven.

Corollary 2.15. *Let I be a homogeneous ideal of A. Then, there exists a non-empty Zariski open set $U \subseteq \mathrm{GL}_n(K)$ such that $\mathrm{Gf}(gI) = \mathrm{Gf}(hI)$ for all $g, h \in U$.*

Proof. It is a direct consequence of the previous corollary and Theorem 2.9. □

3. General initial ideals with respect to weights. As we have explained in the introduction, the generic initial ideal of I with respect to a given monomial order $\prec$ is an important invariant of I, and one would like to have a similar invariant when using weights. There are two naive definitions of what a generic initial ideal with respect to a given weight could be. One might set $\mathrm{gin}_\omega(I)$ to be $\mathrm{in}_\omega(\gamma I) \subseteq K(\mathbf{y})[X_1, \ldots, X_n]$, where γ is as in (2.3) and Definition 2.4. The disadvantage here is that, to define $\mathrm{gin}_\omega(I)$ in this way would not provide a coordinate-independent invariant, as the following easy example shows. If we take $I = (X_1) \subseteq K[X_1, X_2]$, a coordinates change such that $X_1 \mapsto X_1 + X_2$ and $\omega = (1, 1)$ then we have $\mathrm{in}_\omega(X) = (y_{11}X_1 + y_{12}X_2) \neq ((y_{11} + y_{21})X_1 + (y_{12} + y_{22})X_2) = \mathrm{in}_\omega(\gamma(X_1 + X_2))$. Moreover, with this definition, the resulting generic initial ideal could not be viewed as an ideal of $K[X_1, \ldots, X_n]$. Alternatively, when K is infinite, one might be tempted to let $\mathrm{gin}_\omega(I)$ be $\mathrm{in}_\omega(gI)$ for a general $g \in \mathrm{GL}_n(K)$, but this is in some sense meaningless because it relies on the existence of a non-empty Zariski open set U where $\mathrm{in}_\omega(hI)$ is constant for all $h \in U$. In the above example, clearly such an open set does not exist. Therefore, we would like to emphasize that the expression *a general initial ideal with respect to* ω should be used in the same way as *a general linear form* or *a general hyperplane section* is used: always in combination with a specific and well-defined property $\mathcal{P}$ of $\mathrm{in}_\omega(gI)$ which is constant on a Zariski open set of $\mathrm{GL}_n(K)$. Keeping this in mind, it is correct to phrase Theorem 3.2 in the following way: *a general initial ideal with respect to ω is fixed under the action of the group B_ω.*

We have seen in the previous section how the generic circuits set of an ideal I can be used as an invariant of I. Also, rather than defining the generic initial ideal of I with respect to a weight, one can consider the generic initial circuits set $\mathbf{gcs}\,(\mathrm{in}_\omega(I)) = \mathrm{in}_\omega(\mathbf{gcs}\,(I))$ and its truncations $\mathbf{gcs}\,(\mathrm{in}_\omega(I)_{\le d})$ at some $d \in \mathbf{N}$, see Notation 2.10. Remark 2.5 yields that any such $\mathbf{gcs}\,(\mathrm{in}_\omega(I)_{\le d})$ is an invariant of I. Moreover, when K is infinite, a truncated generic circuits set of I can be defined using general changes of coordinates: One can let $\mathbf{gcs}\,(\mathrm{in}_\omega(I_{\le d}))$ be $\mathrm{in}_\omega(\mathbf{cs}\,(gI_{\le d}))$ for a general change of coordinates g, and this makes sense because of Theorem 2.9.

3.1. The subgroup $\mathcal{B}_\omega$ of the Borel group. In the usual setting, generic initial ideals in any characteristic have the property of being fixed under the action of the Borel subgroup $\mathcal{B}$ of $\mathrm{GL}_n(K)$ consisting of invertible $n \times n$ upper-triangular matrices. Let $\omega = (\omega_1, \dots, \omega_n) \in \mathbf{Z}^n$ be a fixed weight. By relabeling the variables, if necessary, we will further assume that $\omega_1 \ge \cdots \ge \omega_n$. We define a subset of $\mathcal{B}$ by taking all $n \times n$ matrices $M = (m_{ij})$ in $\mathcal{B}$ such that $m_{ii} = 1$ for $i = 1, \dots, n$ and $m_{ij} = 0$ if $\omega_i = \omega_j$, and denote it by $\mathcal{B}_\omega$. Obviously, the identity matrix belongs to $\mathcal{B}_\omega$. If $M, N \in \mathcal{B}_\omega$ and $\omega_i = \omega_j$ for $i \ne j$, then the (i,j)th entry of MN is zero; it is also easy to verify that every $M \in \mathcal{B}_\omega$ has an inverse in $\mathcal{B}_\omega$ by computing the row-echelon form of M augmented with the identity. We have thus verified that $\mathcal{B}_\omega$ is a subgroup of $\mathcal{B}$. The main result we want to prove next, and we do in Theorem 3.2, is that general initial ideals with respect to a non-negative weight ω are fixed under the action of $\mathcal{B}_\omega$.

We now let d be a positive integer and, as before, let A_d be the degree d part of the polynomial ring A. The largest weight of a monomial in A_d is $\omega_1 d$. For all $0 \le a \le \omega_1 d + 1$, we let S_a be the set of all monomials of A_d of weight strictly less then a.

Given a vector subspace W of A_d, we let $\alpha_\omega(W)$ denote the vector

$$\alpha_\omega(W) := (\mathrm{rank}^{S_{\omega_1 d}} W, \mathrm{rank}^{S_{\omega_1 d - 1}} W, \dots, \mathrm{rank}^{S_1} W).$$

Clearly, when V, W are K-vector subspaces of A_d and $\mathbf{cs}\,(V) = \mathbf{cs}\,(W)$, then $\alpha_\omega(V) = \alpha_\omega(W)$, by Remark 2.7. Next, we write $\alpha_\omega(V) \ge \alpha_\omega(W)$ when the inequality holds pointwise; when, in addition, $\alpha_\omega(V) \ne \alpha_\omega(W)$, we write $\alpha_\omega(V) > \alpha_\omega(W)$.

Recall that a K-vector subspace W of A_d is homogeneous with respect to ω if it is spanned by polynomials which are homogeneous with respect to ω.

Proposition 3.1. *Let W be a K-vector subspace of A_d and $\omega = (\omega_1, \dots, \omega_n)$ a weight with $\omega_1 \geq \cdots \geq \omega_n \geq 0$. Then, for every integer $0 \leq a \leq \omega_1 d$, the dimension of the homogeneous component of* $\mathrm{in}_\omega(W)$ *of weight a is* $\mathrm{rank}^{S_a} W - \mathrm{rank}^{S_{a+1}} W$. *Furthermore, if W is homogeneous with respect to ω and $b \in \mathcal{B}$ is an upper-triangular change of coordinates, then $\alpha_\omega(bW) \geq \alpha_\omega(W)$.*

Proof. Let $\prec$ be a monomial order. We consider, as we did to prove (2.8), the matrix M_W associated with W after having ordered a monomial basis of A_d by means of $\prec_\omega$. The first part of the statement can be verified by computing the row-echelon form of M_W. The desired inequality follows from the definition of α_ω, since the image under b of a monomial is the sum of that monomial and a linear combination of other monomials of equal or greater weight. □

Theorem 3.2. *Let I be a homogeneous ideal of $A = K[X_1, \dots, X_n]$, with $|K| = \infty$. Let also $\omega = (\omega_1, \dots, \omega_n)$ be a weight with $\omega_1 \geq \cdots \geq \omega_n$. Then, a general initial ideal of I with respect to ω is $\mathcal{B}_\omega$-fixed, i.e., there exists a non-empty Zariski open set U such that $b(\mathrm{in}_\omega(g(I))) = \mathrm{in}_\omega(g(I))$, for all $g \in U$ and all $b \in \mathcal{B}_\omega$.*

Proof. Observe that, if f is a homogeneous polynomial, ω a weight and we let $\omega' := \omega + (1, 1, \dots, 1)$, then $\mathrm{in}_\omega(f) = \mathrm{in}_{\omega'}(f)$; therefore we may assume that $\omega \in \mathbf{R}^n_{\geq 0}$. Now notice that there exists an upper bound D for the generating degrees of all the ideals with the same Hilbert function as I. Thus, if ω and ω' induce the same partial order on all monomials of degree less than or equal to D, then $\mathrm{in}_\omega(J) = \mathrm{in}_{\omega'}(J)$ for every homogeneous ideal J with such Hilbert function. Hence, we may further assume that $\omega \in \mathbf{Z}^n$, with $\omega_1 \geq \cdots \geq \omega_n \geq 0$.

By Theorem 2.9, we let U be a non-empty Zariski open set such that $\mathbf{gcs}\,(I_{\leq D}) = \mathbf{cs}\,(gI_{\leq D})$ for all $g \in U$. Clearly, it is enough to prove the equality degree by degree up to degree D. Let W be the degree d component of $\mathrm{in}_\omega(gI)$ with $0 \leq d \leq D$. First, we decompose A_d as a direct sum of vector spaces $V_p \oplus V_{p-1} \oplus \cdots \oplus V_0$, where V_i is generated by all polynomials in A_d which are homogeneous with respect to ω and of weight i. Accordingly, $p = d\omega_1$, $V_p = K[X_1, \dots, X_j]_d$ with $j = \max\{i : \omega_i = \omega_1\}$, and b acts as the identity on V_p. Now, if we decompose W in an analogous manner as $\oplus_{i=p}^0 W_i$, we immediately have that $W_p \subseteq V_p$ and W_p is fixed under the action of b. We may thus assume by induction that $b(\oplus_{i<j} W_{p-i}) = \oplus_{i<j} W_{p-i}$, and we need to prove that $b(\oplus_{i\leq j} W_{p-i})$ is equal to $\oplus_{i\leq j} W_{p-i}$. In order to see this, it is enough to prove a containment. Assume $W_j \neq 0$, otherwise the result follows from the inductive hypothesis. When $0 \neq f \in W_j$, we can write $b(f)$ as $f+q$, where q is a sum of polynomials of weight larger than j, i.e., $q \in V_p \oplus \cdots \oplus V_{j+1}$, and it is left to show that $q \in W_p \oplus \cdots \oplus W_{j+1}$. If this would not be the case, then $\mathrm{in}_\omega(f) = \mathrm{in}_\omega(q) \notin W_p \oplus \cdots \oplus W_{j+1}$. Thus, by Proposition 3.1, we would have $\mathrm{rank}^{S_{j+1}}(b(W)) > \mathrm{rank}^{S_{j+1}}(W)$ and, in particular, $\alpha_\omega(bW) > \alpha_\omega(W)$.

On the other hand, by definition of W and subsection 1.3, we have that

$$\alpha_\omega(bW) = \alpha_\omega(b(\mathrm{in}_\omega(gI)_d)) = \alpha_\omega\left(b((\widetilde{gI})_{t=0})_d\right),$$

where $\widetilde{gI}$ denotes the homogenization of the ideal gI with respect to ω. By arguing as before (2.8) we see that this quantity is determined by the ranks of the non-zero minors of the matrix $M_{b((\widetilde{gI})_{t=0})_d}$, each such minor corresponding to a non-zero minor of the matrix $M_{b((\widetilde{gI}))_d}$, which has entries in $K[t]$. In particular, we can find an $a \in K$ such that all the non-zero minors of $M_{b((\widetilde{gI}))_d}$ do not vanish after applying the substitution $t = a$. Hence, $\alpha_\omega(bW) \leq \alpha_\omega(b(((\widetilde{gI})_{t=a})_d) = \alpha_\omega(b(D_a(gI))_d)$. Accordingly,

$$\alpha_\omega(bW) \leq \max\{\alpha_\omega(hI_d) : h \in \mathrm{GL}_n(K)\} = \alpha_\omega(gI_d),$$

where the last equality follows from the choice of $g \in U$ and (2.8). Proposition 3.1 implies that $\alpha_\omega(gI_d) = \alpha_\omega(\mathrm{in}_\omega((gI_d))$, which is by definition $\alpha_\omega(W)$. We have thus obtained that $\alpha_\omega(bW) \leq \alpha_\omega(W)$ and the desired contradiction. $\square$

If we consider a homogeneous ideal I of A and take, for instance, ω to be the weight $(1, 1, \ldots, 1, 0)$, then by the previous theorem a general initial ideal of I with respect to ω is fixed under any coordinates change which is the identity on $X_1, \ldots, X_{n-1}$; this fact can be useful in applications, see for instance [**4**, Section 4] and [**5**, Proposition 1.6].

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Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907
Email address: gcavigli@math.purdue.edu

Dipartimento di Matematica, Università degli Studi di Pisa, Largo Bruno Pontecorvo 5-56127 Pisa, Italy
Email address: sbarra@dm.unipi.it