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# ON PRIME MODULES AND DENSE SUBMODULES

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ABSTRACT. Let $R$ be a commutative ring with identity, and let $M$ be a unital $R$-module. A submodule $N$ of $M$ is called a dense submodule, if $M = \sum_{\varphi} \varphi(N)$ where $\varphi$ runs over all the $R$-morphisms from $N$ into $M$. An $R$-module $M$ is called a $\pi$-module if every nonzero submodule is dense in $M$. This paper makes some observations concerning prime modules and $\pi$-modules over a commutative ring. It is shown that an $R$-module $M$ is a prime module if and only if every nonzero cyclic submodule of $M$ is a dense submodule of $M$. Moreover, for modules with nonzero socles and co-semisimple modules over any ring and for all finitely generated modules over a principal ideal domain (PID), the two concepts $\pi$ and prime are equivalent. Rings $R$, over which the two concepts $\pi$ and prime are equivalent for all $R$-modules, are characterized. Also, it is shown that, if $M$ is a $\pi$-module over a domain $R$ with $\dim(R) = 1$, then either $M$ is a homogeneous semisimple module or a torsion free module. In particular, if $M$ is a multiplication module over a domain $R$ with $\dim(R) = 1$, then $M$ is a $\pi$-module if and only if either $M$ is a simple module or $R$ is a Dedekind domain and $M$ is a faithful $R$-module.

**0. Introduction.** All rings in this article are commutative with identity and modules are unital. For a ring $R$ we denote by $\dim(R)$ the classical Krull dimension of $R$ and for a module $M$ we denote by $\mathrm{soc}(M)$ and $\mathrm{Ann}(M)$ the socle and the annihilator of $M$, respectively.

Let $R$ be a ring. We recall that a nonzero $R$-module $M$ is said to be a *prime module* if $\mathrm{Ann}(N) = \mathrm{Ann}(M)$ for each non-zero submodule $N$ of $M$, i.e., $rx = 0$ for $x \in M$, $r \in R$ implies that $x = 0$ or $rM = (0)$. We call a proper submodule $N$ of an $R$-module $M$ a *prime submodule* of $M$ if $M/N$ is a prime module, i.e., whenever $rm \in N$, then either $m \in N$ or $rM \subseteq N$ for any $r \in R$, $m \in M$. Thus, $N$ is a prime submodule

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of $M$ if and only if $P = \mathrm{Ann}\,(M/N)$ is a prime ideal of $R$ and $M/N$ is a torsion free $R/P$-module. This notion of prime submodule was first introduced and systematically studied in [**4, 6**] and recently has received a good deal of attention from several authors (see for example [**2, 3, 7–9, 12, 13**]).

Let $N$ be an $R$-module. An $R$-module $M$ is said to be generated by $N$ or $N$-generated if, for every non-zero homomorphism $f : M \to L$ where $L$ is a nonzero $R$-module, there is a homomorphism $h : N \to M$ with $foh \neq 0$. Also, for a submodule $N$ of $M$, the submodule

$$Tr(N, M) := \sum\{\varphi(N) : \varphi \in \mathrm{Hom}_R(N, M)\} \subseteq M$$

is called the trace of $N$ in $M$. By [**15**, 13.5], $\mathrm{Tr}\,(N, M)$ is the largest submodule of $M$ generated by $N$. Following [**11**], we say that a submodule $N$ of $M$ is dense if $M = \mathrm{Tr}\,(N, M)$. In fact, this concept of a dense submodule was introduced by [**11**] as a generalization of the concept of an invertible ideal. Also, an $R$-module $M$ is called a $\pi$-module if every non-zero submodule of $M$ is dense, i.e., the natural map $\mathrm{Hom}_R(N, M) \otimes_R N \to M$ is surjective for every non-zero submodule $N$ of $M$ (see also, [**1, 10, 12, 14**], for more information about dense submodules and $\pi$-modules).

Let $M$ be a $\pi$-module. Then by [**11**, Proposition 1.3], $\mathrm{Ann}\,(N) = \mathrm{Ann}\,(M)$ for each non-zero submodule $N$ of $M$, i.e., $M$ is a prime module. In general, the converse is not true (even for $M = R$). In fact, a domain $R$ is a $\pi$-module if and only if $R$ is a Dedekind domain (see [**11**, Corollary 1.6]). In [**1**, Theorem 1] it is shown that the two concepts $\pi$ and prime are equivalent if the module is quasi-injective (see also [**12**, Theorem 1.2]). Also, the two concepts $\pi$ and prime are equivalent for Artinian modules (see [**12**, Theorem 1.3]). In Section 1, we show that an $R$-module $M$ is a prime module if and only if every nonzero cyclic submodule of $M$ is a dense submodule of $M$. Moreover, it is shown that, for modules with nonzero socles and for co-semisimple modules, the two concepts $\pi$ and prime are equivalent. This yields, for Artinian modules and semisimple modules, that the two concepts $\pi$ and prime are equivalent. In particular, we characterize rings $R$ over which the two concepts $\pi$ and prime are equivalent for all $R$-modules.

In [**11**], Naoum and Al-Alwani have investigated dense submodules of multiplication modules and in particular when such submodules are themselves multiplication modules. Recall that an $R$-module $M$ is called a multiplication module if, for every submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$ (see for example [**5**] for more details). In Section 2, we study $\pi$-modules over one-dimensional domains. First we show that if $M$ is a $\pi$-module over a domain $R$ with $\dim(R) = 1$, then either $M$ is a homogeneous semisimple module or a torsion free module. Then we show that a finitely generated module $M$ over a principal ideal domain (PID) $R$ is a $\pi$-module if and only if $M$ is a prime module, if and only if either $M$ is a homogeneous semisimple module or $M$ is a free module. Also, we characterize all multiplication $\pi$-modules over one-dimensional domains. In fact, it is shown that if $M$ is a multiplication $\pi$-module, then either $M$ is a simple module or $R$ is a Dedekind domain and $M$ is a faithful $R$-module. This yields that, if $R$ is a Dedekind domain, then a multiplication $R$-module $M$ is a $\pi$-module if and only if either $M$ is simple or $M$ is faithful.

**1. On prime modules and dense submodules.** We begin with the following useful result.

**Theorem 1.1.** *Let $M$ be an $R$-module. Then the following statements are equivalent*:

(1) *$M$ is a prime module.*

(2) *Every nonzero cyclic submodule of $M$ is a dense submodule of $M$.*

(3) *For every $m, m' \in M$, $Rm$ is a dense submodule of $Rm + Rm'$.*

*Proof.* (1) $\Rightarrow$ (2). Let $M$ be a prime module and $0 \neq m \in M$. Then, for each $m' \in M$, $\mathrm{Ann}(m) = \mathrm{Ann}(m')$. We define $f : Rm \to M$ by $f(rm) = rm'$. Then it is easy to check that $f \in \mathrm{Hom}(Rm, M)$ and $f(m) = m'$, and hence $M = \mathrm{Tr}(Rm, M)$, i.e., $Rm$ is a dense submodule of $M$.

(2) $\Rightarrow$ (1). Let $m \in M$, $r \in \mathrm{Ann}(m)$. Assume that $m' \in M$. Since $Rm$ is a dense submodule of $M$, $r_i m \in Rm$ and $\phi_i \in \mathrm{Hom}(Rm, M)$ exist where $r_i \in R$, $(1 \leq i \leq n, n \in \mathbf{N})$, such that $m' = \sum \phi_i(r_i m)$. Then $rm = \sum \phi_i(r r_i m) = 0$, and hence $\mathrm{Ann}(m) \subseteq \mathrm{Ann}(M)$. Thus $\mathrm{Ann}(Rm) = \mathrm{Ann}(M)$ for each $0 \neq m \in M$, i.e., $M$ is a prime module.

(1) $\Rightarrow$ (3). Let $M$ be a prime module. Then for every $m, m' \in M$, $\mathrm{Ann}(m) = \mathrm{Ann}(m')$, and hence $Rm \cong Rm'$. It follows that $Rm$ is a dense submodule of $Rm + Rm'$.

(3) $\Rightarrow$ (1). Let $m, m' \in M$. Since $Rm$ is dense in $Rm+Rm'$, it is easy to check that $\mathrm{Ann}\,(Rm) = \mathrm{Ann}\,(Rm + Rm')$. Also, $Rm'$ is dense in $Rm+Rm'$. This follows that $\mathrm{Ann}\,(m) = \mathrm{Ann}\,(m')$ for each $m, m' \in M$ i.e., $M$ is a prime module. $\square$

**Corollary 1.2.** *Let $M$ be an $R$-module. A proper submodule $N$ of $M$ is a prime submodule if and only if every nonzero cyclic submodule of $M/N$ is a dense submodule of $M/N$.*

*Proof.* Clearly, $N$ is a prime submodule of $M$ if and only if $M/N$ is a prime module. Now apply Theorem 1.1. $\square$

**Corollary 1.3.** *A ring $R$ is a domain if and only if every nonzero cyclic ideal of $R$ is dense in $R$.*

*Proof.* By Theorem 1.1, the proof is clear. $\square$

**Corollary 1.4.** *Let $M$ be a $\pi$-module. Then $M$ is a prime module.*

*Proof.* By Theorem 1.1, the proof is clear. $\square$

*Remark* 1.5. Corollary 1.4 was first proved in [**11**]. In general, the converse of Corollary 1.4 is not true. In fact, a domain $R$ is a $\pi$-module if and only if $R$ is a Dedekind domain (see [**11**, Corollary 1.6]). Even if $R$ is a Dedekind domain, the converse of Corollary 1.4 is not true in general. In fact, it is easily seen that the $\mathbf{Z}$-module $\mathbf{Z} \oplus \mathbf{Q}$ is a prime module but not a $\pi$-module (see [**11**, page 417]). In [**1**, Theorem 1] it is shown that the two concepts $\pi$ and prime are equivalent if the module is quasi-injective (also see [**12**, Theorem 1.2]). Recall that module $M$ is quasi-injective if for each submodule $N$ of $M$ and each homomorphism $g : N \to M$, $g$ has an extension to all of $M$. Also, the two concepts $\pi$ and prime are equivalent for Artinian modules (see [**12**, Theorem 1.3]).

Here we give a class of modules which is properly larger than the class of Artinian modules where the two concepts $\pi$ and prime are equivalent. Recall that, for a module $M$, the socle of $M$ (denoted by $\mathrm{soc}\,(M)$) is the sum of all simple (minimal) submodules of $M$. If there are no minimal submodules in $M$, we put $\operatorname{soc}(M) = (0)$. Thus, $M$ is a semisimple module if $\operatorname{soc}(M) = M$. Also, a semisimple module $M$ is called homogeneous if any two simple submodules of $M$ are isomorphic (see, for example, [**15**]).

**Proposition 1.6.** *Let $R$ be a ring, and let $M$ be an $R$-module with* $\operatorname{soc}(M) \neq (0)$. *Then, $M$ is prime if and only if $M$ is a $\pi$-module.*

*Proof.* ($\Leftarrow$), by Corollary 1.4.

($\Rightarrow$). Let $M$ be a prime module with nonzero socle, and let $Rm$ be a simple submodule of $M$. Then $\operatorname{Ann}(m) = \operatorname{Ann}(M) = P$, and hence $P$ is a maximal ideal of $R$. Since $\operatorname{Ann}(m) = \operatorname{Ann}(m')$ for each $0 \neq m' \in M$, $M$ is a homogeneous semisimple $R$-module. Hence, we get $M = \sum_{\lambda\in\Lambda} \oplus Rm_\lambda$, where $\Lambda$ is an index set and $Rm_{\lambda_1} \cong Rm_{\lambda_2}$ $(\lambda_1, \lambda_2 \in \Lambda)$. Note that every nonzero submodule of $M$ is also a homogeneous semisimple $R$-module. Thus, for every nonzero submodule $N$ of $M$, we have $N = \sum_{\gamma\in\Gamma} \oplus Rx_\gamma$, where $\Gamma$ is an index set and $x_\gamma \in N$ for every $\gamma \in \Gamma$. Now let $\gamma \in \Gamma$, and we consider the projection $\varphi_\lambda : N \to Rm_\gamma \cong Rm_\lambda$, $(\lambda \in \Lambda)$. Hence, we obtain $\sum_{\lambda\in\Lambda} \varphi_\lambda(N) = M$. This completes the proof. $\square$

The following corollary is now immediate.

**Corollary 1.7.** *Let $R$ be a ring, and let $M$ be an $R$-module. If $M$ is Artinian or $M$ is semisimple, then, $M$ is prime if and only if $M$ is a $\pi$-module.*

Now we give another class of modules where the two concepts $\pi$ and prime are equivalent.

We recall that, if $U$, $M$ are $R-$modules, then following Azumaya, $U$ is called $M-$injective if, for any submodule $N$ of $M$, each homomorphism $N \to U$ can be extended to $M \to U$; see [**15**, 16.3], where it is shown that if $U$ is $N-$injective for each cyclic submodule $N$ of $M$, then $U$ is $M-$injective. Also, an $R$-module $M$ is called co-semisimple if every simple module is $M-$injective. This is a class of modules properly larger than the class of semisimple modules, and also, a ring $R$ is co-semisimple as $R$-module if and only if $R$ is a regular ring (see [**15**, 23.1] for these facts and several characterizations of co-semisimple modules).

**Proposition 1.8.** *Let $R$ be a ring, and let $M$ be a co-semisimple $R$-module. Then, $M$ is prime if and only if $M$ is a $\pi$-module.*

*Proof.* ($\Leftarrow$), by Corollary 1.4.

($\Leftarrow$). Let $M$ be a prime module. Then, for each $0 \neq m \in M$, $\mathrm{Ann}\,(m) = \mathrm{Ann}\,(M)$, and it is a prime ideal of $R$. On the other hand, since $M$ is co-semisimple, $Rm$ is also a co-semisimple $R$-module (see [**15**, 23.1]). It follows that the ring $R/\mathrm{Ann}\,(m)$ is a (von Neumann) regular ring (see [**15**, 23.5]). Since $\mathrm{Ann}\,(m) = \mathrm{Ann}\,(M)$ is a prime ideal, $R/\mathrm{Ann}\,(M)$ is a field, i.e., $\mathrm{Ann}\,(M)$ is a maximal ideal of $R$. Thus, $M$ is a vector space over the field $R/\mathrm{Ann}\,(M)$. Clearly, every nonzero $R/\mathrm{Ann}\,(M)$-submodule of $M$ is a dense $R/\mathrm{Ann}\,(M)$-submodule of $M$. It follows that every nonzero $R$-submodule of $M$ is a dense $R$-submodule of $M$, i.e., $M$ is a $\pi$-module. $\square$

The following result shows that rings $R$ for which the two concepts $\pi$ and prime are equivalent for all $R$-modules are abundant.

**Theorem 1.9.** *Let $R$ be any ring. Then the two concepts $\pi$ and prime are equivalent for all $R$-modules if and only if* $\dim\,(R) = 0$.

*Proof.* ($\Rightarrow$). Let $P$ be a prime ideal of $R$ which is not a maximal ideal. Assume that $M = \overline{R} \oplus \overline{Q}$ where $\overline{R} := R/P$ and $\overline{Q}$ is the field of fraction of $\overline{R}$. Clearly, $\mathrm{Ann}\,(M) = P$, and it is easy to check that $M$ is a prime module as $\overline{R}$-module. It follows that $M$ is a prime $R$-module. Thus, $M$ must be a $\pi$-module. On the other hand, since $\overline{R}$ is not a field, $\mathrm{Hom}_{\overline{R}}(\overline{Q}, \overline{R}) = 0$, and hence $\mathrm{Hom}_R(\overline{Q}, \overline{R}) = 0$. Now it is easily checked that the submodule $(0) \oplus \overline{Q}$ is not a dense $R$-submodule of $M$, a contradiction. Thus, every prime ideal of $R$ is maximal i.e., $\dim\,(R) = 0$.

($\Leftarrow$). Let $\dim\,(R) = 0$ and $M$ be a prime $R$-module. Then, for each $0 \neq m \in M$, $\mathrm{Ann}\,(m) = \mathrm{Ann}\,(M) = P$ is a maximal ideal of $R$, and hence $M$ is a (homogenous) semisimple $R$-module. Now, by Corollary 1.7, $M$ is a $\pi$-module. $\square$

The following corollary is now immediate.

**Corollary 1.10.** *Let $R$ be a domain. Then the two concepts $\pi$ and prime are equivalent for all $R$-modules if and only if $R$ is a field.*

We conclude this section with the following interesting result.

**Proposition 1.11.** *Let $R$ be a domain which is not a field. Then the two concepts $\pi$ and prime are equivalent for all torsion $R$-modules if and only if $\dim(R) = 1$.*

*Proof.* ($\Rightarrow$). Let $P$ be a nonzero prime ideal of $R$, and let $M = \overline{R} \oplus \overline{Q}$ where $\overline{R} := R/P$ and $\overline{Q}$ is the field of fraction of $\overline{R}$. Since $\operatorname{Ann}(M) = P \neq (0)$, $M$ is a torsion $R$-module. Now, by the proof of Theorem 1.9, $P$ is a maximal ideal of $R$. It follows that $\dim(R) = 1$.

($\Leftarrow$). Let $M$ be a torsion prime $R$-module. Then, for each $0 \neq m \in M$, $\operatorname{Ann}(m)$ is a nonzero prime ideal of $R$. Since $\dim(R) = 1$, $P$ is a maximal ideal, and hence $M$ is a (homogenous) semisimple $R$-module. Now, by Corollary 1.7, $M$ is a $\pi$-module. □

**2. On $\pi$-modules over one-dimensional domains.** This section makes some observations concerning prime modules and $\pi$-modules over a commutative domain $R$ with $\dim(R) = 1$.

**Proposition 2.1.** *Let $R$ be domain with $\dim(R) = 1$, and let $M$ be a $\pi$-module. Then, either $M$ is a homogeneous semisimple module or a torsion free module.*

*Proof.* Let $M$ be a $\pi$-module. Then, by Corollary 1.4, $M$ is a prime module. Thus for each $0 \neq m \in M$, $\operatorname{Ann}(m) = \operatorname{Ann}(M)$ and $P = \operatorname{Ann}(M)$ is a prime ideal of $R$. Since $R$ is a domain with $\dim(R) = 1$, either $P = (0)$ or $P$ is a maximal ideal. If $P = (0)$, then $M$ is a torsion free $R$-module. If $P$ is a maximal ideal, then $M$ is a homogeneous semisimple $R$-module. □

**Theorem 2.2.** *Let $M$ be a finitely generated module over a principal ideal domain* (PID) *$R$. Then the following statements are equivalent*:

(1) *$M$ is a $\pi$-module.*

(2) *$M$ is a prime module.*

(3) *Either $M$ is a homogeneous semisimple module or $M$ is a free module.*

*Proof.* (1) $\Rightarrow$ (2), by Corollary 1.4.

(2) $\Rightarrow$ (3). Let $M$ be a prime module. Thus $\mathrm{Ann}\,(M) = \mathcal{P}$ is a prime ideal of $R$ and $\mathrm{Ann}\,(m) = \mathrm{Ann}\,(M)$ for each $0 \neq m \in M$, i.e., $Rm \cong Rm'$ for every nonzero element $m$, $m' \in M$. On the other hand, each finitely generated module over a PID is a direct sum of cyclic submodules with prime power annihilators. It follows that $M$ is a direct sum of isomorphic cyclic submodules. Since $\dim\,(R) \leq 0$, either $\mathcal{P} = (0)$ or $\mathcal{P}$ is a maximal ideal of $R$. Now it is easy to check that, if $\mathcal{P} = (0)$, then $M$ is a free module and, if $\mathcal{P}$ is a maximal ideal, then $M$ is a homogeneous semisimple module.

(3) $\Rightarrow$ (1). Let $M$ be a homogeneous semisimple module. Clearly, $M$ is a prime module. Thus, by Corollary 1.7, $M$ is a $\pi$-module. Now let $M$ be a free module. We note that each nonzero submodule of a free module with a finite base over a PID is free. It follows that every nonzero submodule of $M$ is dense, i.e., $M$ is a $\pi$-module. $\square$

Before stating the next lemma, we introduce some notation. If $\{M_\lambda\}_{\lambda\in\Lambda}$ is a non-empty collection of $R$-modules and $\lambda \in \Lambda$, let $\widehat{M_\lambda}$ denote the submodule $\oplus_{\mu\neq\lambda} M_\mu$ of $M = \oplus_{\lambda\in\Lambda} M_\lambda$. Note that $M = M_\lambda \oplus \widehat{M_\lambda}$ for each $\lambda \in \Lambda$.

**Lemma 2.3** [**5**, Corollary 2.3]. *Let $M_\lambda$ ($\lambda \in \Lambda$) be a collection of finitely generated $R$-modules and $M = \oplus_{\lambda\in\Lambda} M_\lambda$. Then $M$ is a multiplication module if and only if $M_\lambda$ is a multiplication module and $\mathrm{Ann}\,(M_\lambda) + \mathrm{Ann}\,(\widehat{M_\lambda}) = R$ for each $\lambda \in \Lambda$.*

Now we are in a position to characterize multiplication $\pi$-modules over a one-dimensional domain.

**Theorem 2.4.** *Let $R$ be a domain with $\dim\,(R) = 1$, and let $M$ be a multiplication $R$-module. Then $M$ is a $\pi$-module if and only if either $M$ is a simple module or $R$ is a Dedekind domain and $M$ is a faithful $R$-module.*

*Proof.* ($\Rightarrow$). Assume $M$ is a $\pi$-module. By Proposition 2.1, either $M$ is a homogeneous semisimple module or a torsion free module. Let $M$ be a homogeneous semisimple module. Hence, we get $M = \sum_{\lambda\in\Lambda} \oplus Rm_\lambda$, where $\Lambda$ is an index set and $\{Rm_\lambda\}_{\lambda\in\Lambda}$ is a non-empty collection of isomorphic simple $R$-modules. It follows that, for each $\lambda \in \Lambda$, $\operatorname{Ann}(Rm_\lambda) = \operatorname{Ann}(\widehat{Rm_\lambda}) = \mathcal{P}$ where $\mathcal{P}$ is a maximal ideal of $R$. Hence, $\operatorname{Ann}(Rm_\lambda) + \operatorname{Ann}(\widehat{Rm_\lambda}) = \mathcal{P} \neq R$ for each $\lambda \in \Lambda$. If $|\Lambda| > 1$, then by Lemma 2.3, $\operatorname{Ann}(Rm_\lambda) + \operatorname{Ann}(\widehat{Rm_\lambda}) = R$ for each $\lambda \in \Lambda$, a contradiction. Thus, $|\Lambda| = 1$, i.e., $M$ is a simple module. Now let $M$ be a torsion free $R$-module. Then $M$ is a faithful multiplication $R$-module. Thus, by [**11**, Theorem 3.1] or [**1**, Theorem 19], $R$ is a Dedekind domain.

($\Leftarrow$). Clearly every simple $R$-module (over any ring $R$) is a $\pi$-module. Now let $R$ be a Dedekind domain and $M$ a faithful multiplication $R$-module. Then, by [**11**, Corollary 3.7], $M$ is a $\pi$-module. $\square$

We conclude this paper with the following corollary.

**Corollary 2.5.** *Let $R$ be a Dedekind domain, and let $M$ be a multiplication $R$-module. Then $M$ is a $\pi$-module if and only if either $M$ is simple or $M$ is faithful.*

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