JOURNAL OF COMMUTATIVE ALGEBRA
Volume 2, Number 2, Summer 2010

NILPOTENT WEBS

OLIVIER RIPOLL AND JULIEN SEBAG

ABSTRACT. In this article, we introduce a new class of planar webs, the *nilpotent webs*. The study of these webs shows that they can be realized as a natural generalization of algebraic webs.

1. Introduction. A non singular planar d-web $\mathcal{W}$ is locally given by $d \geq 1$ holomorphic foliations in $(\mathbf{C}^2, 0)$ in general position. The data of a d-web corresponds to the data of a differential equation $0 = F(x, y, y') := \sum_{i=0}^{d} a_{d-i}(x, y) \cdot (y')^i \in \mathbf{C}\{x, y\}[y']$ with non vanishing y'-resultant of F and $\partial_{y'} F$. In this correspondence, the *leaves of the web* are given by the integral curves of the differential equation, locally, and out of the singular locus defined by this resultant. Such a differential polynomial F is called a *presentation equation* of the d-web (see for example [**1, 2, 6, 8**] for further details).

A d-web presented by a polynomial in $\mathbf{C}[x, y, y']$ can also be considered as an affine algebraic surface $X_F := \operatorname{Spec} \mathbf{C}[x, y, y']/(F)$, endowed with a derivation D_F (see subsection 3.2). We will essentially adopt these two last points of view in this article.

Among webs, *Algebraic webs* are entirely determined by an affine algebraic plane curve over $\mathbf{C}$ (see [**6**, Introduction). This property implies that the linearization polynomial associated with the web vanishes. But it also induces some other more or less classical properties as the existence of an essential singular solution (which is given by a local parametrization of the algebraic curve), or the fact that the hypersurface X_F of $\mathbf{A}^3_{\mathbf{C}}$ is an affine ruled surface.

A natural question is to determine whether or not such properties are exclusive to the class of algebraic webs (see also [**14**]).

The answer to this question is the main motivation of this work. In this article, we construct a strictly larger class of webs than algebraic

Received by the editors on May 18, 2009, and in revised form on November 15, 2009.

DOI:10.1216/JCA-2010-2-2-209

ones, with similar properties. We call them *nilpotent webs* (Definition 1). In a way, this class is maximal with such properties (see Theorems 1 and 2). The definition and the study of this class is made *algebraically*, by assuming that the derivation D_F is locally nilpotent (Theorem 1). But differential algebra allows us to obtain also a geometric characterization of them (Theorem 2).

In Sections 2 and 3, we set up notation and recall the basics of web theory. Sections 4 and 5 form the core of the paper, where we prove that nilpotent webs are algebrizable and explicitly compute their abelian relations (Theorem 2). We also prove that they are "held" by an affine ruled surface (Proposition 2).

2. Notations and conventions.

• Let $\mathbf{C}$ be the field of complex numbers, and let $\mathbf{C}\{x_1, \dots, x_n\}$ be the ring of convergent power series in the variables $x_1, \dots, x_n$. Let A be a $\mathbf{C}$-algebra. A $\mathbf{C}$-derivation D of A is a $\mathbf{C}$-linear map $D : A \to A$ which satisfies the Leibniz rule

$$\text{for all } a, b \in A, \quad D(ab) = a \cdot D(b) + a \cdot D(b).$$

We denote by $\mathrm{Der}_{\mathbf{C}}(A, A)$ the $\mathbf{C}$-vector space of the $\mathbf{C}$-derivations of A. We say that $D \in \mathrm{Der}_{\mathbf{C}}(A, A)$ is *locally nilpotent* if, for every $f \in A$, there exists an integer $n \in \mathbf{N}$ such that $D^n(f) = 0$, where $D^n = D \circ \cdots \circ D$ is the composition of D, with itself, n times, and $D^0(a) = a$ for every $a \in A$. All the results on locally nilpotent derivations that we used can be found, for example, in [**2**]. We denote by ∂_x, ∂_y, and ∂_p the classical derivations of $\mathbf{C}[x, y, p]$.

• Let $\mathbf{C}(x)\langle y\rangle$ be the polynomials ring $\mathbf{C}(x)[y_0, y_1, y_2, \dots]$ in an infinite number of variables. It is endowed with the derivation $\delta(y_i) := y_{i+1}$ and $\delta(x) = 1$. By convention, a polynomial $F \in \mathbf{C}[x, y, p]$ is seen as an element of $\mathbf{C}(x)\langle y\rangle$ by identifying y to y_0 and p to y_1. We will also denote y_1 by y' and y_2 by y''.

We say that an ideal I of $\mathbf{C}(x)\langle y\rangle$ is a differential ideal if $\delta(I) \subset I$. Let S be a part of $\mathbf{C}(x)\langle y\rangle$. Then $[S]$ is the differential ideal generated by S, and $\{S\}$ is the radical of $[S]$.

• Let $F \in \mathbf{C}[x, y, p]$ be a polynomial with coefficients in $\mathbf{C}$. We denote by $R_F := \mathrm{Result}_p(F, \partial_p F)$ the resultant (in p) of F and $\partial_p F$. We say that a polynomial $F \in \mathbf{C}[x,y,p]$ is a *$\mathcal{W}$-polynomial*, if the degree (in p) of F is greater than 3, and if $R_F \in \mathbf{C}[x,y]\backslash\{0\}$. Setting $p = y'$, polynomial planar d-webs are thus presented by $\mathcal{W}$-polynomials. The choice we make to consider d-webs for $d \geq 3$ comes from the fact that 1 and 2 webs are locally trivial from this point of view.

• By convention, all the webs that we consider in this article are supposed to be non singular, polynomial and planar webs.

3. Web theory.

3.1. Geometric web theory. We refer to [**6**] for further developments and proofs of the following properties. If $F_i : \mathbf{C}^2 \longrightarrow \mathbf{C}$ for $1 \leq i \leq d$, are local submersions defining the foliations of a d-web $\mathcal{W}$, a classical theorem in web geometry asserts that the $\mathbf{C}$-vector space of its *abelian relations*, defined by

$$\mathcal{A}(d) = \Big\{(g_1(F_1),\dots,g_d(F_d)) \in \mathbf{C}\{x,y\} \text{ with } g_i \in \mathbf{C}\{t\} \\ \text{and } \sum_{i=1}^{d} g_i(F_i)dF_i = 0\Big\},$$

is of finite dimension. This dimension is called the *rank of the web*. It is bounded by the integer $\pi_d := (d-1)(d-2)/2$, and invariant by local analytic isomorphisms.

Linearizable webs are webs equivalent (by local analytic isomorphism) to webs whose leaves are locally given by straight lines (called *linear webs*).

3.2. Web algebra. Let $F(x,y,p) = \sum_{i=0}^{d} a_{d-i}(x,y)\cdot p^i \in \mathbf{C}[x,y,p]$ be a $\mathcal{W}$-polynomial. For all $0 \leq i \leq d-3$, one can define (see [**10**, Theorem 3.1]) two *associated polynomials* U_F^i and V_F^i (of rank i) respectively of degree at most $d-2$ and $d-1$, verifying the following identity:

$$p^i \cdot R_F \cdot (\partial_x F + p\cdot\partial_y F) = U_F^i \cdot F + V_F^i \cdot \partial_p F.$$

Because of the conditions on the degrees in p, one can remark that such a couple (U_F^i, V_F^i) is unique. The polynomials V_F^i are called the *higher linearization polynomials* of F, and $V_F := V_F^0$ is the *linearization polynomial* of F. This last polynomial gives differential conditions for the web to be linearizable (see [**10**, Proposition 3.3] in this setting).

One can show (see [**10**, Theorem 3.6]) that there exists $c_i \in \mathbf{C}(x,y)$ for $0 \leq i \leq d-4$, verifying

$$V_F^{i+1} = -c_i \cdot F + p \cdot V_F^i \quad \text{and} \quad U_F^{i+1} = c_i \cdot \partial_p F + p \cdot U_F^i$$

in $\mathbf{C}[x,y][p]$, with $U_F^0 := U_F$. The term c_i is the coefficient (possibly zero) corresponding to the term of degree $d-1$ in V_F^i, divided by a_0; we can check that, for $0 \leq i \leq d-4$, the coefficients c_i are, in fact, in $\mathbf{C}[x,y]$.

We define a derivation D_F of $\mathrm{Der}_{\mathbf{C}}(\mathbf{C}[x,y,p], \mathbf{C}[x,y,p])$ by

$$D_F := R_F \cdot (\partial_x + p \cdot \partial_y) - V_F \cdot \partial_p.$$

Thus, $D_F(F) = U_F \cdot F$. The derivation D_F is said to be the *derivation associated with* F. If R_F divides V_F, we adopt the following notation:

$$d_F := D_F / R_F = \partial_x + p \cdot \partial_y - (V_F/R_F) \cdot \partial_p$$

and $v_F := V_F/R_F$.

Remark 1. The data of such a derivation comes naturally from the setting of the "implicit" theory of webs. Furthermore, the condition R_F divides V_F is related to the existence of (essential) singular solutions in the first order differential equation $F(x,y,y') = 0$ (see [**11**]). This condition is also linked to the fact that the differential equation $F(x,y,y') = 0$ verifies the weak Painlevé-Fuchs property (see [**12**]).

Lemma 1. *Let F be a $\mathcal{W}$-polynomial of degree d. We have the following properties*:

(1) *$F \in \operatorname{Ker} D_F$ if and only if $U_F = 0$ and R_F divides V_F.*

(2) *Let $v \in \mathbf{C}[x,y,p]$ be a polynomial of degree at most $d-1$ in p such that $(\partial_x + p \cdot \partial_y - v \cdot \partial_p)(F) = 0$. Then $R_F \cdot v = V_F$ and $U_F = 0$. Moreover, $F \in \operatorname{Ker} d_F$. Conversely, if R_F divides V_F and $U_F = 0$, then $F \in \operatorname{Ker} d_F$.*

Proof. The proof is straightforward, using the unicity in the property satisfied by U_F and V_F, and the definition of R_F as the resultant of F and $\partial_p F$. $\square$

4. Characterization of nilpotent webs. Let $F \in \mathbf{C}[x, y, p]$ be a $\mathcal{W}$-polynomial and let $V_F \in \mathbf{C}[x, y, p]$ be its linearization polynomial. We denote by (N) the following hypothesis:

(N): *the resultant R_F divides V_F and the derivation $d_F \in \operatorname{Der}(\mathbf{C}[x, y, p], \mathbf{C}[x, y, p])$ is locally nilpotent.*

Example 1. If a (planar) polynomial d-web $\mathcal{W}$ is algebraic, there exists a polynomial $G \in \mathbf{C}[s, t]$ such that $G(y - px, p)$ is a presentation of the web $\mathcal{W}$. This condition is equivalent to the fact that $V_F = 0$ (see [**11**, Proposition 10]). In particular, R_F divides V_F, and the derivation $d_F = \partial_x + p \cdot \partial_y$ is clearly locally nilpotent.

Lemma 2. *Let $F \in \mathbf{C}[x, y, p]$ be a $\mathcal{W}$-polynomial satisfying the hypothesis (N). Then $\partial_p v_F = 0$, $U_F = 0$ and $F \in \operatorname{Ker} d_F$.*

Proof. Since d_F is locally nilpotent, by using [**15**, Proposition 1.3.51], we obtain that $\partial_p v_F = 0$. As R_F divides V_F, it follows, from the relation

$$R_F \cdot (\partial_x F + p \cdot \partial_y F) - V_F \cdot \partial_p F = U_F \cdot F,$$

that R_F divides U_F, and that $U_F = 0$, by [**7**, Lemma 1.10]. $\square$

Lemma 3. *Let F be a $\mathcal{W}$-polynomial. Assume that $f := y - px - s$ and $g := p + t$ are in $\operatorname{Ker} D_F$, with s, $t \in \mathbf{C}[x]$ satisfying $\partial_x s = x \cdot \partial_x t$. Then $V_F = R_F \cdot \partial_x t$, and we have the equality $d_F = \Delta_{f,g}$ where $\Delta_{f,g}$ is the Jacobian derivation associated with (f, g) defined by $\Delta_{f,g}(h) = \det(\partial(f, g, h)/\partial(x, y, p))$ for all $h \in \mathbf{C}[x, y, p]$.*

Proof. Since $d_F(g) = 0$, we have $V_F = \partial_x t \cdot R_F$. We have $\Delta_{f,g}(h) = -x \cdot \partial_x t \cdot \partial_y h + \partial_x h - \partial_x t \cdot \partial_p h + \partial_y h \cdot (p + \partial_x s)$. Hence $\Delta_{f,g} = \partial_x + p \cdot \partial_y - \partial_x t \cdot \partial_p = d_F$. $\square$

Proposition 1. *Let $F \in \mathbf{C}[x, y, p]$ be a $\mathcal{W}$-polynomial, for which the hypothesis (N) is satisfied. The kernel of d_F is equal to the polynomial ring $\mathbf{C}[f_F, g_F]$ with $f_F := y - px - s_F$ and $g_F := p + t_F$, where s_F, $t_F \in \mathbf{C}[x]$ satisfy $\partial_x s_F = x \cdot \partial_x t_F$ and $\partial_x t_F = v_F$.*

Proof. Remark that $d_F(f_F) = d_F(g_F) = 0$. So, $\mathbf{C}[f_F, g_F]$ is included in the kernel of d_F. Conversely, let $h \in \mathbf{C}[x, y, p]$ be an element of the kernel of d_F. Using the relations $y = f_F + x \cdot g_F + s_F - x \cdot t_F$ and $p = g_F - t_F$, we can write

$$h = \sum_{i,j=0}^{n} a_{i,j}(x) \cdot f_F^i \cdot g_F^j,$$

where $a_{i,j} \in \mathbf{C}[x]$, for each $0 \leq i \leq n$ and $0 \leq j \leq n$. Note that the data $(x, y, p) \mapsto (x, f_F, g_F)$ induces an automorphism of $\mathbf{C}^3$. So x, f_F, and g_F are algebraically independent (over $\mathbf{C}$). By assumption on h, we have $0 = d_F(h) = \sum_{i,j=0}^{n} (\partial_x a_{i,j})(x) \cdot f_F^i \cdot g_F^j$. Thus $a_{i,j} \in \mathbf{C}$ for all i and j. It shows that $\operatorname{Ker} d_F$ is equal to $\mathbf{C}[f_F, g_F]$. □

Remark 2. By Miyanishi's theorem (see [**15**, Theorem 1.3.41]), we know that $\operatorname{Ker} d_F$ is a polynomial ring in two variables. It follows that Proposition 1 can be interpreted as a specialization of this general and main theorem in the case of our derivation d_F.

In the following parts of this article, we will define and study a strictly larger class of webs than algebraic ones which satisfy the hypothesis (N), and share some notable properties with algebraic webs.

Definition 1. Let $\mathcal{N}$ be a (non singular, planar) polynomial d-web, presented by a $\mathcal{W}$-polynomial $F \in \mathbf{C}[x, y, p]$. We say that $\mathcal{N}$ is nilpotent if the hypothesis (N) is satisfied, i.e., if R_F divides V_F and d_F is a locally nilpotent derivation.

Remark 3. This definition is legitimated by the fact that V_F/R_F does not depend on the chosen presentation of the web (see [**10**, Proposition 3.2]).

Example 2. Consider the $\mathcal{W}$-polynomial given by:

$$F := (-x^3+1)p^3 + \left(3\left(y-\frac{x^2}{2}\right)x^2+3x\right)p^2 + \left(-3\left(y-\frac{x^2}{2}\right)^2 x+3\,x^2\right)p + \left(y-\frac{x^2}{2}\right)^3 + x^3.$$

Its resultant is $R_F = -(27/64)\,(x-1)(x^2+x+1)(2\,y+x^2)^6$. We can compute V_F and find that $V_F = R_F$. One can check that this web is nilpotent, as announced in the previous proposition, but not algebraic, since $V_F \neq 0$.

Theorem 1. *Let $\mathcal{N}$ be a (non singular, planar) polynomial web presented by a $\mathcal{W}$-polynomial $F \in \mathbf{C}[x,y,p]$. Then the following assertions are equivalent:*

(1) *$\mathcal{N}$ is nilpotent (i.e., the hypothesis (N) is satisfied);*

(2) *R_F divides V_F and $v_F := V_F/R_F \in \mathbf{C}[x]$;*

(3) *There exist a polynomial $G \in \mathbf{C}[X,Y]$ of total degree d such that*

$$F(x,y,p) = G(f,g),$$

where $f := y - px - s$ and $g := p + t$, with $s,\ t \in \mathbf{C}[x]$, such that $\partial_x s = x \cdot \partial_x t$. In this case, $V_F = \partial_x t \cdot R_F$.

Proof. It is clear that (2) implies (1), because, in this case, d_F is triangular, or, more directly, because $d_F(p) \in \mathbf{C}[x]$. Conversely, assume that d_F is locally nilpotent. By Lemma 2 we have $\partial_p v_F = 0$ and $d_F(F) = 0$ in $\mathbf{C}[x,y,p]$. It follows, by deriving this equation, that $d_F(\partial_p F) + \partial_y F = 0$ and $d_F(\partial_y F) - \partial_y v_F \cdot \partial_p F = 0$. Thus,

$$d_F^2(\partial_p F) - (\partial_y v_F) \cdot \partial_p F = 0.$$

By Miyanishi's theorem (see [**15**, Theorem 1.3.41]) and by the slice theorem (see [**15**, Proposition 1.3.21]), we are reduced to study, in the polynomial ring $\mathbf{C}[X,Y][x]$, the following differential equation

$$\partial_x^2(T) = u \cdot T.$$

By comparing the degrees in x of the two members, we see that, if this equation has a non zero solution, then $u = 0$. So, $\partial_y v_F = 0$ (because $\partial_p F$ is, by assumption on F, a non zero solution).

If $\mathcal{N}$ is nilpotent, Lemma 2 asserts that F is in $\operatorname{Ker} d_F$. So (1) implies (3) follows from Proposition 1. Now if $F = G(f, g)$, one can check that $(\partial_x + p \cdot \partial_y - \partial_x t \cdot \partial_p)(F) = 0$. So, by Lemma 2, (3) implies (2). □

Remark 4. We see that nilpotent webs are particular Clairaut web in the sense of [**4**, subsection 2.4].

Example 3. If $v = 0$, Proposition 1 gives the polynomials of the form $G(y - px, p)$, with $G \in \mathbf{C}[X, Y]$. Again, nilpotent webs with $v = 0$ are algebraic ones.

Let $F \in \mathbf{C}[x, y, p]$ be a polynomial. Let us set $X_F := \operatorname{Spec} \mathbf{C}[x, y, p]/(F) \subset \mathbf{A}^3_{\mathbf{C}}$.

Proposition 2. *Let $\mathcal{N}$ be a nilpotent web, presented by a $\mathcal{W}$-polynomial $F \in \mathbf{C}[x, y, p]$. Then X_F is an affine ruled surface.*

Proof. Since $d_F \in \operatorname{Der}_{\mathbf{C}}(A, A)$, with $A := \mathbf{C}[x, y, p]/(F)$, and since $d_F(x) = 1$, this property is just a translation of Miyanishi's theorem (see [**15**, 1.3.41]), and the slice theorem (see [**15**, Proposition 1.3.21]). □

Example 4. • Assume $\mathcal{N}$ is presented by $F(x, y, p) = \prod_{i=1}^{d}(p - p_i)$ where the slopes p_i of the leaves of $\mathcal{N}$ are given by $p_i := -\partial_x u + c_i$, with $c_i \in \mathbf{C}$ and $u \in \mathbf{C}[x]$, then $\mathcal{N}$ is nilpotent and the linearization polynomial is $V_F = \partial_x^2 u \cdot R_F$. Using the relation of subsection 3.2, we deduce that

$$(V_F - R_F \cdot \partial_x^2 u) \sum_{1 \le i \le d} \prod_{j \ne i} (p - p_j) = -U_F \cdot \prod_{1 \le i \le d} (p - p_i).$$

By comparing the degree in p, we see that $U_F = 0$ and $V_F - R_F \cdot \partial_x^2 u = 0$.

• Consider the 4-web given by the leaves $\{F_i = cste\}$ where $F_1 = y + x^2$, $F_2 = y + x^2 - x$, $F_3 = y + x^2 + x$, and $F_4 = y + x^2 - 2x$. Computing the slopes of the leaves, we find that they are of the form

$p_i := -\partial_x t + c_i$, for $1 \leq i \leq 4$, where $t := x^2$ and $c_1 = 0, c_2 = 1, c_3 = -1$ and $c_4 = 2$. This gives the following presentation of the web

$$F := p^4 + (8x-2)p^3 + (24x^2 - 12x - 1)p^2 + (32x^3 - 24x^2 - 4x + 2)p \\ + 16x^4 - 16x^3 - 4x^2 + 4x.$$

Here, $R_F = 144$ and $V_F = 2R_F$ as it was expected. Moreover, one can see that $F = G(f, g)$, where $G(X, Y) := Y^4 - 2Y^3 - Y^2 + 2Y$, and $g = p + \partial_x t$, as announced in Theorem 1.

5. The study of the abelian relations. Nilpotent webs are algebrizable. This property can be easily deduced from our Theorem 1, by considering an appropriate change of coordinates in $\mathbf{C}^2$. In the present paragraph, we describe precisely the "abelian relations" of nilpotent webs. In particular, we again proved that such webs are algebrizable, by other arguments. We show that nilpotent webs are also characterized by their abelian relations.

Let $\mathcal{W}$ be a (planar) polynomial d-web presented by a $\mathcal{W}$-polynomial $F \in \mathbf{C}[x, y, p]$. We simply denote by $(U_i, V_i)_{0 \leq i \leq d-3}$ the family of associated polynomials of F.

Definition 2. We say that $r = \sum_{i=3}^{d} b_i \cdot p^{d-i} \in \mathbf{C}\{x, y\}[p]$ is an abelian polynomial associated with the d-web $\mathcal{W}$, if the degree of r is at most $d-3$ and if r satisfies the following differential equation

$$R_F \cdot (\partial_x r + p \cdot \partial_y r) = U_r + \partial_p V_r,$$

where $U_r = \sum_{i=3}^{d} b_i \cdot U_F^{d-i}$ and $V_r = \sum_{i=3}^{d} b_i \cdot V_F^{d-i}$. Let us denote by $\mathcal{R}$ the $\mathbf{c}$-vector space of the abelian polynomials of $\mathcal{W}$.

Our interest in abelian polynomials of $\mathcal{W}$ comes from the fact that they are in correspondence with abelian relations of $\mathcal{W}$ (see [**10**, Lemma 1]). Thus, the $\mathbf{C}$-vector space $\mathcal{R}$ identified to $\mathcal{A}(d)$, is of dimension at most π_d.

Let r be an abelian polynomial. Using the relations $V_F^{i+1} = -c_i \cdot F + p \cdot V_F^i$ and $U_F^{i+1} = c_i \cdot \partial_p F + p \cdot U_F^i$ (see subsection 3.2), it is easy to show that there exists $w \in \mathbf{C}\{x, y\}[p]$ (which can be expressed in terms of the c_i) such that:

$$\begin{cases} V_r = r \cdot V_F - w \cdot F \\ U_r = r \cdot U_F + w \cdot \partial_p F. \end{cases}$$

In particular, this implies that $U_r + \partial_p V_r = r \cdot (U_F + \partial_p V_F) + V_F \cdot \partial_p r - F \cdot \partial_p w$. One can remark that if we suppose $r \in \mathbf{C}[x,y,p]$, then $w \in \mathbf{C}[x,y,p]$. Making this last assumption corresponds to the study of *algebraic* abelian relations. Remark also that if $\mathcal{W}$ is nilpotent web, one can prove that an abelian polynomial $r \in \mathbf{C}\,[x,y,p]$ belongs to $\operatorname{Ker} d_F$.

Theorem 2. *Let f and g be two algebraically independent polynomials in $\mathbf{C}\,[x,y,p]$. Let $\mathbf{C}_{d-3}[f,g]$ be the $\mathbf{C}$-vector space of polynomials in $\mathbf{C}\,[f,g]$ of degree at most $d-3$ in p. We have the following properties*:

(1) *A nilpotent web $\mathcal{N}$, presented by a $\mathcal{W}$-polynomial F, is of maximal rank. More, we have $\mathcal{R} = \mathbf{C}_{d-3}[f_F, g_F]$, with $f_F := y - px - s_F$ and $g_F := p + t_F$, where s_F, $t_F \in \mathbf{C}\,[x]$ satisfy $\partial_x s_F = x \cdot \partial_x t_F$ and $\partial_x t_F = v_F$.*

(2) *Let $\mathcal{W}$ be a d-web with $d \geq 4$, presented by a $\mathcal{W}$-polynomial F. Then $\mathcal{W}$ is a nilpotent web if and only if $\mathcal{R} = \mathbf{C}_{d-3}[f,g]$, with $f := y - px - s$ and $g := p + t$, where s, $t \in \mathbf{C}[x]$ satisfy $\partial_x s = x \cdot \partial_x t$.*

(3) *A nilpotent web $\mathcal{N}$ is algebrizable.*

(4) *The residue of a nilpotent web is defined and equal to* 0 *on each irreducible components of the singular locus of the web.*

Proof. 1) It is sufficient to show that $\mathbf{C}_{d-3}[f_F, g_F] \subseteq \mathcal{R}$, since the $\mathbf{C}$-vector space $\mathbf{C}_{d-3}[f_F, g_F]$ is of dimension π_d and $\mathcal{R}$ is of dimension at most π_d. By Proposition 1, remark also that $\mathbf{C}_{d-3}[f_F, g_F]$ is the set of polynomials in $\operatorname{Ker} d_F$ of degree at most $d-3$ in p. Let us write $r = \sum_{i=3}^{d} b_i \cdot p^{d-i} \in \mathbf{C}[x,y][p]$ and assume that $r \in \operatorname{Ker} d_F$. Then $U_r + \partial_p V_r = V_F \cdot \partial_p r - F \cdot \partial_p w$. But $d_F(r) = 0$ so we have $U_r + \partial_p V_r = R_F \cdot (\partial_x r + p \cdot \partial_y r) - F \cdot \partial_p w$. We have seen that $V_F^{i+1} = -c_i \cdot F + p \cdot V_F^i$. But V_F^{i+1} and V_F^i are of degree at most $d-1$ in p, by definition (see subsection 3.2). If $a_0 \in \mathbf{C}\,[x,y] \backslash \{0\}$ denotes the coefficient of p^d in F, we see that $a_0 \cdot c_i$ is the coefficient of p^{d-1} in V_F^i, which can be equal to zero. Since V_F is of degree 0 in p, it follows by induction that $c_i = 0$ for $0 \leq i \leq d-3$ and so, $w = 0$. This proves that r is an abelian polynomial, and the requested equality.

2) The first implication follows from (1). Conversely, since 1 is in $\mathcal{R}$, we have $U_F + \partial_p V_F = 0$. But $g = p + t$ is also in $\mathcal{R}$ since $d \geq 4$, so, as an abelian polynomial, we have $U_g + \partial_p V_g = g \cdot (U_F + \partial_p V_F) + V_F = R_F \cdot \partial_x t$ and so, $V_F = R_F \cdot \partial_x t$, which proves that $\mathcal{W}$ is nilpotent by Theorem 1.

3) Since $\mathcal{N}$ is of maximal rank, it is enough to check that $\mathcal{N}$ is linearizable. Actually, we know (see [**6**, Introduction] for details) that a linearizable maximal rank web is algebrizable. We can check that the web $\mathcal{N}$ is linearizable, since v_F, which is here in $\mathbf{C}[x]$, satisfies the differential conditions given in [**10**, Proposition 3.3] for instance.

4) Nilpotent webs are algebrizable, so the residue of the web is defined (see [**5**, part 3] for details on the residue of a web). By Lemma 2 and Theorem 1, we have $U_F = 0$ and $V_F / R_F \in \mathbf{C}[x]$. Thus, Proposition 3.5 in [**10**] says that the trace of the connection associated with the nilpotent web is equal to zero, hence by Proposition 1 in [**5**], the residue is equal to zero. $\square$

Remark 5. If $\mathcal{N}$ is a nilpotent web, we have described precisely its abelian relations, which are polynomials, and shown that the residue of nilpotent web is equal to 0, as in the case of algebraic webs.

By considering abelian polynomials, many other characterizations of nilpotent webs can be given, such as the following propositions.

Proposition 3. *Let $\mathcal{W}$ be a d-web presented by a $\mathcal{W}$-polynomial F, with $d \geq 4$. The following properties are equivalent*:

(1) *$\mathcal{W}$ is nilpotent*;

(2) *F is in $\operatorname{Ker} D_F$, and there exists $t \in \mathbf{C}\,[x]$ such that $g := p + t$ is in $\mathcal{R}$.*

Proof. The first implication is a consequence of Lemma 2 and Theorem 1. Conversely, since $F \in \operatorname{Ker} D_F$, Lemma 1 asserts that $U_F = 0$ and R_F divides V_F. Since $g = p + t$ is in $\mathcal{R}$, we have $U_g + \partial_p V_g = R_F \cdot \partial_x t$ and so $V_F + g \cdot \partial_p V_F - R_F \cdot \partial_x t = 0$. If the degree in p of V_F is strictly positive, this equality leads to a contradiction by comparing the degrees in p. So $\partial_p V_F = 0$ and $V_F = R_F \cdot \partial_x t$, which proves that $\mathcal{W}$ is nilpotent by Theorem 1. $\square$

Proposition 4. *Let $\mathcal{W}$ be a d-web presented by a $\mathcal{W}$-polynomial F. The following properties are equivalent*:

(1) *$\mathcal{W}$ is nilpotent*;

(2) *F and $\partial_y F$ are in* $\operatorname{Ker} D_F$, *and there exists an r in $\mathcal{R}\backslash\{0\}$ such that $r \in \mathbf{C}$.*

Proof. If $\mathcal{W}$ is nilpotent, $D_F(F) = 0$ and d_F is well defined by Lemma 2. But $0 = \partial_y(d_F(F)) = d_F(\partial_y F) - \partial_y v_F \cdot \partial_p F$ and v_F belongs to $\mathbf{C}[x]$, so $d_F(\partial_y F) = 0 = D_F(\partial_y F)$. Theorem 3 gives that 1 is an abelian polynomial, which proves the first implication. Conversely, if $r \in \mathbf{C}\backslash\{0\}$ is an abelian polynomial, we have $0 = U_r + \partial_p V_r = r \cdot U_F + r \cdot \partial_p V_F = r \cdot \partial_p V_F$ by Lemma 1, and more, d_F is defined. It follows that $\partial_p V_F = 0$. Thus $0 = \partial_y(d_F(F)) = d_F(\partial_y F) - \partial_y v_F \cdot \partial_p F$. Since $D_F(\partial_y F) = d_F(\partial_y F) = 0$ and $\partial_p F \neq 0 \in \mathbf{C}[x,y,p]$ by assumption on R_F, we see that v_F is in $\mathbf{C}[x]$, and so the web is nilpotent by Theorem 1. $\square$

REFERENCES

1. A. Beauville, *Géométries des tissus* (d'après S.S. Chern et P.A. Griffiths), Séminaire Bourbaki **531**, février 1976, Lecture Notes Math **770**, Springer, Berlin, 1980, 103–119.

2. W. Blaschke and G. Bol, *Geometrie der Gewebe*, Springer, Berlin, 1938.

3. A. Buium, *Differential algebra and diophantine geometry*, Hermann, Paris, 1994.

4. L. Dara, *Singularités génériques des équations différentielles multiformes*, Bol. Soc. Bras. Mat. **6** (1975), 95–128.

5. A. Hénaut, *Planar web geometry through abelian relations and singularities*, in *Inspired by Chern, A memorial volume in honor of a great mathematician*, World Scientific Publishing Co., River Edge, NJ, 2006.

6. ———, *On planar web geometry through abelian relations and connections*, Ann. Math. **159** (2004), 425–445.

7. M. Miyanishi, *Vector fields on factorial schemes*, J. Algebra **173** (1995), 144–165.

8. J.V. Pereira, *Algebrization of codimension one webs* [after Trépreau, Hénaut, Pirio, Robert, ...], Séminaire Bourbaki **974**, Mars 2007.

9. O. Ripoll, *Détermination du rang des tissus du plan et autres invariants géométriques*, C.R. Acad. Sci. Paris **341** (2005), 247–252.

10. ———, *Properties of the connection associated with planar webs and applications*, Arxiv math.DG/0702321, Manuscr. Math., to appear.

11. O. Ripoll and J. Sebag, *Solutions singulières des tissus polynomiaux du plan*, J. Algebra **310** (2007), 351–370.

12. ———, *The Cartan-Tresse linearization polynomial and applications*, J. Algebra **320** (2008), 1914–1932.

13. J. Ritt, *Differential algebra*, Coll. Publ. **33**, American Mathematical Society, Dover, 1966.

14. J. Sebag, *On a special class of non complete webs*, Annal. Fac. Sci. Toulouse, to appear.

15. A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Prog. Math. **190**, Birkhäuser Verlag, Berlin.

Université Bordeaux I, Institut Mathématique de Bordeaux, labag, 351 cours de la Libération, 33405 Talence, France
Email address: Olivier.Ripoll@gmail.com

Université Rennes 1, UFR Mathématiques, irmar, 263 Avenue du General Leclerc, CS 74205, 35042 Rennes cedex, France
Email address: Julien.Sebag@univ-rennes1.fr