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OPEN PROBLEMS ON SYZYGIES AND HILBERT FUNCTIONS

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1. Introduction.

In this paper we list a number of open problems and conjectures on Hilbert functions and syzygies. Some of the problems are closely related to Algebraic Geometry, Combinatorics, and Hyperplane Arrangements Theory.

Our aim is to stimulate interest, rather than to give a complete survey. When describing a problem, we sometimes state one or two related results, and give pointers to a few references, rather than giving an exhaustive list of references and what is known. A detailed survey of the covered topics would make the paper far longer than we (and perhaps, the readers) could handle.

Our list of problems is certainly not complete. We have focused on problems that we see as most exciting, or important, or popular. We present three types of problems: Conjectures, Problems, and Open-Ended Problems. Some of the problems and especially the Open-Ended problems are general problems which point to interesting directions for exploration.

The books [**35**] and [**95**] contain expository papers on some of the problems and related topics. Section 17 is a (probably non-complete) list of helpful books. [**3, 38, 63, 71, 72, 80, 116, 117**] provide lecture notes. A good way to get a feel of the recent research is to browse the web pages of the mathematicians working in this area.

2. Notation.

Throughout k stands for a field. For simplicity, we assume that k is algebraically closed and has characteristic 0. However, many of the open problems and conjectures make sense without these assumptions. In the paper, the polynomial ring $S = k[x_1, \ldots, x_n]$ is graded by

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$\deg(x_i) = 1$ for all i. A polynomial f is *homogeneous* if $f \in S_i$ for some i, that is, if all monomial terms of f have the same degree. An ideal I is *graded* (or homogeneous) if it has a system of homogeneous generators. Throughout the paper, I stands for a graded ideal in S and R stands for S/I. The quotient ring R inherits the grading by $(S/I)_i = S_i/I_i$ for all i.

Let T be a graded finitely generated R-module.

A very interesting and important numerical invariant of T is its *Hilbert function* $\mathrm{Hilb}_T(i) = \dim(T_i)$ for $i \in \mathbf{N}$.

The idea to associate a free resolution to T was introduced in Hilbert's famous 1890, 1893 papers [**76, 77**]. Let $\mathbf{U}$ be a graded minimal free resolution of T. The submodule $\mathrm{Ker}(d_{i-1}) = \mathrm{Im}(d_i)$ of U_{i-1} is called the *i'th syzygy module of* T, and its elements are called *i'th syzygies.* The rank of U_i is called the *i'th Betti number* of T and is denoted $b_i^R(T)$. The Betti numbers are among the most studied invariants of T.

The modules in the resolution $\mathbf{U}$ are graded and the differential has degree 0. For $p \in \mathbf{Z}$ denote by $R(-p)$ the free graded R-module such that $R(-p)_i = R_{i-p}$; the module $R(-p)$ is generated by one element in degree p. Since each module U_i is a free finitely generated R-module, we can write it as $U_i = \oplus_{p\in\mathbf{Z}} R(-p)^{b_{i,p}}$. Therefore,

$$\mathbf{U}:\ \dots \to U_i = \oplus_{p\in\mathbf{Z}}\, R(-p)^{b_{i,p}} \xrightarrow{d_i} U_{i-1} = \oplus_{p\in\mathbf{Z}} R(-p)^{b_{i-1,p}} \to \dots$$

The numbers $b_{i,p}^R(T)$ are called the *graded Betti numbers* of T. We say that $b_{i,p}^R(T)$ is the Betti number in homological degree i and (inner) degree p.

Recent computational methods have made it possible to compute graded free resolutions and Hilbert functions by computer. Algorithms for computation of syzygies and Hilbert functions are implemented in computer algebra systems as COCOA [**104**], MACAULAY [**15**], MACAULAY2 [**62**], and SINGULAR [**64**].

3. Regularity.

The Castelnuovo-Mumford regularity (or simply regularity) of S/I is

$$\mathrm{reg}(S/I) = \max\{j \mid b^S_{i,i+j}(S/I) \neq 0 \text{ for some } i\}$$

and $\mathrm{reg}(I) = \mathrm{reg}(S/I)+1$. [**25**] is an expository paper on the properties and open problems on regularity. The following conjecture has been open for about 25 years and is the most exciting (currently) open conjecture on syzygies.

The Regularity Conjecture 3.1. (Eisenbud-Goto) [**14, 44**] If $P \subset (x_1, \dots, x_n)^2$ is a prime graded ideal, then

$$\mathrm{reg}(P) \;\leq\; \deg(S/P) - \mathrm{codim}(S/P) + 1\,.$$

It is known to hold for irreducible curves by [**65**], and for irreducible smooth surfaces and 3-folds by [**85, 101**]. The following particular case is very interesting; it is open for toric ideals.

Conjecture 3.2. If $P \subset (x_1, \dots, x_n)^2$ is a prime graded ideal, then the maximal degree of an element in a minimal system of homogeneous generators is $\leq \deg(S/P)$.

A number of examples show that Conjecture 3.1 is sharp. For example, the equality holds for the defining ideal of the twisted cubic curve. It also holds for a rational curve in $\mathbf{P}^3$ with a $(q-1)$-secant line; this provides an example for every degree $\deg(S/P)$. It will be interesting to explore when the equality holds.

Open-Ended Problem 3.3. Find classes of graded prime ideals so that for every ideal $P \subset (x_1, \dots, x_n)^2$ in this class we have $\mathrm{reg}(P) = \deg(S/P) - \mathrm{codim}(S/P) + 1$.

There is only one known family of ideals – the Mayr-Meyer's examples – where the regularity is doubly exponential in the number of variables, while the maximum degree of an element in a minimal system of homogeneous generators of the ideal is fixed (it is 4) [**16, 83, 88, 113**]. Eisenbud has pointed recently that it is of interest to construct and study more such examples.

Problem 3.4. Find families of graded ideals with large regularity (doubly exponential, or exponential, or polynomial) in the number of variables, while the maximum degree of an element in a minimal system of homogeneous generators of an ideal is bounded (by a constant).

In the spirit of works by Bertram-Ein-Lazarsfeld [**18**] and Chardin-Ulrich [**28**], we consider:

Open-Ended Problem 3.5. Let $a_1 \geq \cdots \geq a_p \geq 2$ be the degrees of the elements in a minimal system of homogeneous generators of I. Set $r = \operatorname{codim}(S/I)$. Find nice sufficient conditions on I so that

$$\operatorname{reg}(S/I) \leq a_1 + \cdots + a_r - r\,.$$

One can also consider a multiple of $a_1 + \cdots + a_r - r$ as a possible bound.

A general problem, which has inspired a lot of work is:

Open-Ended Problem 3.6. Assuming the ideal I satisfies some special conditions, find a sharp upper bound for $\operatorname{reg}(I)$ in terms of the maximum degree of an element in a minimal system of homogeneous generators of I.

For a generic linear form f, we have that $\operatorname{reg}(I + (f)) \leq \operatorname{reg}(I)$. However, it is not known what happens if f is not generic.

Problem 3.7. (Caviglia) Let f be a linear form. Is $\operatorname{reg}(I + (f))$ bounded by a polynomial (possibly quadratic) function of $\operatorname{reg}(I)$?

The following two problems stem form a result of Ravi, who proved that $\operatorname{reg}(\operatorname{rad}(I)) \leq \operatorname{reg}(I)$ if I is a monomial ideal. Chardin-D'Cruz [**26**] constructed examples where $\operatorname{reg}(\operatorname{rad}(I))$ is the cube of $\operatorname{reg}(I)$.

Problem 3.8. (Ravi) Find classes of ideals for which $\operatorname{reg}(\operatorname{rad}(I)) \leq \operatorname{reg}(I)$.

Problem 3.9. Is $\operatorname{reg}(\operatorname{rad}(I))$ bounded by a (possibly polynomial) function of $\operatorname{reg}(I)$?

Results of Eisenbud-Huneke-Ulrich [**48**] in the case $\dim \operatorname{Tor}_1(M, N) \leq 1$ give rise to the following problem.

Problem 3.10. Let M and N be finitely generated graded S-modules. Is $\mathrm{reg}(\mathrm{Tor}_i^S(M,N))$ bounded in terms of $\mathrm{reg}(M)$ and $\mathrm{reg}(N)$ (possibly under some conditions on M and N)?

Furthermore, one can study the regularity of intersections, sums, or powers.

Open-Ended Problem 3.11. Find classes of ideals for which you can obtain a nice upper bound on the regularity of intersections.

Open-Ended Problem 3.12. Find classes of ideals for which you can obtain a nice upper bound on the regularity of products.

For example, the following results are of this type: Let $I_1, \dots, I_r$ be ideals in S generated by linear forms. By [**31**], $\mathrm{reg}(I_1 \cdots I_r) = r$. By [**32**], $\mathrm{reg}(I_1 \cap \dots \cap I_r) = r$.

Open-Ended Problem 3.13. Find classes of ideals for which you can obtain a nice upper bound on the regularity of powers. For example, question [**25**, Question **7.3**] is asking for a bound on the regularity of a square of an ideal.

Caviglia proved that the following problem is equivalent to Problem 3.11 on regularity.

Problem 3.14. (Stillman) Fix a sequence of natural numbers $a_1, \dots, a_r$. Does there exist a number p, such that

$$\mathrm{pd}\,(W/J) \leq p$$

if W is a polynomial ring (over k) and J is a graded ideal with a minimal system of homogeneous generators of degrees $a_1, \dots, a_r$? Note that the number of variables in the polynomial ring W is not fixed.

Problem 3.15. Fix a sequence of natural numbers $a_1, \dots, a_r$. Does there exist a number q, such that

$$\mathrm{reg}\,(W/J) \leq q$$

if W is a polynomial ring (over k) and J is a graded ideal with a minimal system of homogeneous generators of degrees $a_1, \dots, a_r$? Note that the number of variables in the polynomial ring W is not fixed.

The following problem is related to regularity, since it yields an upper bound on it. For problems of this type in the toric case, cf. [**21**].

Open-Ended Problem 3.16. Fix a certain class of graded ideals. Obtain a nice upper bound on the maximal degree of an element in a minimal homogeneous Gröbner basis.

By [**9, 12**] we have that every graded finitely generated S/I-module has finite regularity if the quotient ring S/I is Koszul (see Section 13 for definition of Koszulness). This leads to a problem on infinite free resolutions:

Open-Ended Problem 3.17. Study the properties of regularity over a Koszul (non-polynomial) ring.

In several cases of interest, we study multigraded rings, ideals, and modules. For Hilbert functions and regularity in that setting see [**67, 87, 108, 114**].

Open-Ended Problem 3.18. Study the properties of multigraded regularity.

4. Characterization of Hilbert Functions.

The following question is very natural and important: "What sequences of numbers are Hilbert functions of ideals (subject to some property)?".

The characterization of all Hilbert functions of graded ideals in S was discovered by Macaulay [**86**]. The characterization of all Hilbert functions of graded ideals containing $x_1^2, \dots, x_n^2$ was obtained by Kruskal-Katona [**82, 84**]; such Hilbert functions are often studied by counting faces of simplicial complexes via the Stanley-Reisner theory. Furthermore, Clements-Lindström [**29**] (cf. [**90**]) generalized Macaulay's idea and provided a characterization of all Hilbert functions of graded ideals containing $x_1^{a_1}, \dots, x_n^{a_n}$ for $a_1 \leq \cdots \leq a_n \leq \infty$.

The following very challenging conjecture aims to answer the question "What sequences of numbers are Hilbert functions of ideals containing a regular sequence?".

The Eisenbud-Green-Harris Conjecture 4.1. (Eisenbud-Green-Harris) [**45**] If I contains a regular sequence of homogeneous elements of degrees $a_1, \dots, a_j$, then there exists a monomial ideal containing $x_1^{a_1}, \dots, x_j^{a_j}$ with the same Hilbert function.

Using Clements-Lindström's Theorem, it is easy to see that if $j = n$ then Conjecture 4.1 is equivalent to the numerical criterion conjectured in [**45**]. The following special case is open, and is the main case of interest.

Conjecture 4.2. [**45**] If I contains a regular sequence of n quadrics, then there exists a monomial ideal containing $x_1^2, \dots, x_n^2$ with the same Hilbert function.

Another challenging conjecture aims to answer the question "What sequences of numbers are Hilbert functions of ideals generated by generic forms?". The conjecture is that if I is generated by generic forms, then I_i is expected to generate in degree $i+1$ as much as possible; the numerical form of the conjecture is:

Conjecture 4.3. (Fröberg) [**54**] Let $f_1, \dots, f_r$ be generic forms of degrees $a_1, \dots, a_r$. Set $I = (f_1, \dots, f_r)$. The Hilbert series of S/I is

$$\mathrm{Hilb}_{S/I}(t) = \left| \frac{\prod_{1 \le i \le r} (1 - t^{a_i})}{(1-t)^n} \right|,$$

where $|\,|$ means that a term $c_i t^i$ in the series is omitted if there exists an earlier term $c_j t^j$ with $j \le i$ and negative coefficient $c_j \le 0$.

The conjecture holds for $r \le n$ since the generic forms form a regular sequence in this case. A solution of the following problem could lead to a solution of 4.3 or at least will shed light on it.

Problem 4.4. (cf. [**116**, Problem **4.4**]) What is the generic initial ideal with respect to revlex order of the ideal generated by generic forms $f_1, \dots, f_r$ of degrees $a_1, \dots, a_r$?

Another open problem of this type is:

Open-Ended Problem 4.5. (cf. [**116**, Problem **2.16**]) Does there exist a nice characterization of the Hilbert functions of artinian Gorenstein graded algebras?

5. Lex ideals.

The key idea in Macaulay's Theorem, which provides a characterization of the Hilbert functions of graded ideals in S, is that every Hilbert function is attained by a lex ideal. If I is a monomial or toric ideal, then we can define the notion of a lex ideal in the quotient ring $R = S/I$, see [**55**].

Open-Ended Problem 5.1. (Mermin-Peeva) [**90**] Find classes of either monomial or projective toric ideals I so that Macaulay's Theorem holds over R, that is, every Hilbert function over R is attained by a lex ideal.

Toric varieties are an important class of varieties which occur at the intersection of Algebraic Geometry, Commutative Algebra, and Combinatorics. They might provide several examples of interesting rings in which all Hilbert functions are attained by lex ideals.

It is easy to find rings over which Macaulay's Theorem does not hold. Sometimes, the trouble is in the degrees of the minimal generators of I. Thus, it makes sense to relax the problem as follows.

Open-Ended Problem 5.2. (Mermin-Peeva) [**90**] Let p be the maximal degree of an element in a minimal homogeneous system of generators of I. Find classes of (either monomial or projective toric) ideals I so that every Hilbert function over R of a graded ideal generated in degrees $> p$ is attained by a lex ideal.

Furthermore, in view of Hartshorne's Theorem [**69**] that every graded ideal in S is connected by a sequence of deformations to a lex ideal, it is natural to ask:

Problem 5.3. Let J be a graded ideal in R, where I is either a monomial or a projective toric ideal, and let L be a lex ideal with the same Hilbert function. When is J connected to L by a sequence of deformations? What can be said about the structure of the Hilbert scheme that parametrizes all graded ideals in R with the same Hilbert function as L?

A consequence of the proof of Hartshorne's Theorem is that the lex ideal attains maximal graded Betti numbers among all graded ideals in S with the same Hilbert function; it should be noted that there are examples where no ideal attains minimal Betti numbers. In the same spirit we can consider:

Open-Ended Problem 5.4. (Mermin-Peeva) Let J be a graded ideal in R, where I is either a monomial or a toric ideal. Suppose that L is a lex ideal with the same Hilbert function in R. Find conditions on R or J so that some of the following hold.

(1) The Betti numbers of J over R are less than or equal to those of L.

(2) The Betti numbers of $J+I$ over S are less than or equal to those of $L+I$.

Problem 5.4(2) was inspired by work of G. Evans and his conjecture 5.5 , cf. the expository paper [**53**].

The Lex-plus-powers Conjecture 5.5. (Evans) [**53**] If a graded ideal J in S contains a regular sequence of graded elements of degrees $a_1, \dots, a_j$, and if there exists a lex-plus-$(x_1^{a_1}, \dots, x_j^{a_j})$ ideal L with the same Hilbert function as J, then the Betti numbers of L are greater or equal to those of J.

Conjecture 5.5 was inspired by the Eisenbud-Green-Harris Conjecture 3.1. It is quite challenging. It is proved for ideals containing powers of the variables by Mermin-Murai [**89**].

In a different direction: very little is known on when (that is, in what rings) the Gotzmann's Persistence Theorem holds. For example:

Problem 5.6. Does the Gotzmann's Persistence Theorem hold over a Clements-Lindström ring $C = S/(x_1^{a_1}, \dots, x_n^{a_n})$, where $a_1 \leq a_2 \leq \cdots \leq a_n \leq \infty$?

A result of Peeva shows that the Gotzmann's Persistence Theorem holds for a Borel ideal in C. In order to solve Problem 5.6, it remains to make a reduction to a Borel ideal.

6. Extremal Betti numbers and Lower Bounds.

In the spirit of the above conjectures that the lex ideal attains maximal Betti numbers in various settings, there are several other difficult problems on minimal/maximal Betti numbers for specific classes of ideals.

Problem 6.1. (Geramita-Harima-Shin) [**58**] Does there exist an ideal which has greatest graded Betti numbers among all Gorenstein artinian graded ideals with a fixed Hilbert function?

The problem is solved under the additional hypothesis that the weak Lefschetz property holds in [**92**].

Conjecture 6.2. (Herzog-Hibi) cf. [**74**]. Let M be a square-free monomial ideal in S. Let P be the square-free monomial ideal in S such that P is the generic initial ideal of M over the exterior algebra (on the same variables as S). The Betti numbers of M over S are less than or equal to those of P.

This problem is motivated by the technique of algebraic and combinatorial shifting, cf. [**74**].

It is proved by Bigatti, Hulett, Pardue that a lex ideal in S attains the greatest Betti numbers among all graded ideals with the same Hilbert function, cf. [**27**]. However, there exist examples of Hilbert functions for which no ideal has smallest (total or graded) Betti numbers [**33, 103**]. Furthermore, they provide examples where no monomial ideal attains smallest (total or graded) Betti numbers among all monomial ideals with a fixed Hilbert function. In view of these examples, it is interesting to obtain constructions on how to get smallest Betti numbers. The next two problems propose such ideas.

Open-Ended Problem 6.3. (Nagel-Reiner) [**93**] Let M be a monomial ideal generated by q monomials of degree p. Let W be the monomial ideal generated by the first q square-free monomials (in a bigger polynomial ring if needed) of degree p in the reverse lex order. Find conditions on M that imply

$$b_i^S(S/W) \leq b_i^S(S/M) \quad \text{for every } i \geq 0\,.$$

Note that S/W and S/M may not have the same Hilbert function. The expectation is that many monomial ideals have the property in 6.3.

Problem 6.4. (Peeva-Stillman) [**99**] Let P be a projective toric ideal. Is it true that S/P has the smallest Betti numbers among all ideals with the same multigraded Hilbert function as P?

Problems 6.3 and 6.4 yield lower bounds on the Betti numbers in the cases that are considered. Obtaining lower bounds on the Betti numbers is usually a very hard problem. The following conjecture has been open for a long time, cf. the expository paper [**24**] .

Conjecture 6.5. (Buchsbaum-Eisenbud, Horrocks) If M is an artinian graded finitely generated S-module, then

$$b_i^S \geq \binom{n}{i} \qquad \text{for } i \geq 0\,.$$

Note that the lower bounds are given by the ranks of the free modules in the Koszul complex that is the minimal free resolution of k over S. A more general version of the conjecture is open:

Problem 6.6. (cf. [**24**]) Let $I \subseteq (x_1, \ldots, x_n)^2$, and let T be an artinian graded finitely generated R-module. Is it true that

$$b_i^R(M) \geq b_i^R(k) \qquad \text{for } i \geq 0\,.$$

The following weaker conjecture is also open, cf. [**7**].

Conjecture 6.7. If M is an artinian graded finitely generated S-module, then

$$\sum_{i\geq 0} b_i^S(M) \geq 2^n\,.$$

7. The linear strand.

Let **F** be the graded minimal free resolution of S/I over the polynomial ring S. The subcomplex

$$\cdots \to S(-i-1)^{b_{i,i+1}} \xrightarrow{d_i} S(-i)^{b_{i-1,i}} \to \cdots \to S(-2)^{b_{1,2}}$$

of **F** is called the 2-linear strand of S/I. All entries in the differential matrices in the 2-linear strand are linear forms. The length of the 2-linear strand is $\max\{i \mid b_{i,i+1} \neq 0\}$.

The basic idea is that the existence of a long 2-linear strand imposes strong geometric or combinatorial constraints. The following is a generalization of a conjecture of Green (see [**36**] for a more complete discussion):

Open-Ended Problem 7.1. (Eisenbud) [**36**] Let P be a prime graded ideal containing no linear form and whose quadratic part is spanned by quadrics of rank ≤ 4. Suppose that the 2-linear strand of S/P has length p. Find nice sufficient conditions on P so that P contains the 2×2-minors of a $v \times w$-matrix A satisfying the following conditions:

(1) $v + w - 3 = p$

(2) A has linear entries

(3) no entry is zero, and no entry can be made zero by row and column operations.

The need of extra conditions on P in the above problem, is shown to be necessary by the examples constructed by Schenck-Stillman [**106**]. Green's conjecture covers a special case of 7.1 when the ideal P satisfies the following additional conditions:

(1) S/I is normal (that is, it is integrally closed)

(2) $\dim(S/I) = 2$ (that is, P defines a projective curve)

(3) S/I is Koszul (see Section 13 for definition of Koszulness)

(4) S/I is Gorenstein

(5) $\deg(S/I) = 2(n-1)$.

Open-Ended Problem 7.2. How is the length of the linear strand related to the other invariants of S/I?

Next, we focus on the question for how long does the linear strand coincide with the minimal free resolution; this is captured in the property N_p defined as follows. Let $p \geq 1$. A graded ideal $I \subseteq (x_1, \dots, x_n)^2$ *satisfies the property* N_p if the graded Betti numbers $b^S_{i,i+j}(S/I)$ vanish for $j \geq 3$, $i \leq p$. Note that N_1 is equivalent to the property that I is generated by quadrics. In geometric situations, the property N_p typically includes also the property N_0, which states that S/I is projectively normal. We have the following general problem.

Open-Ended Problem 7.3. Fix a certain class of graded ideals. Find a geometric or combinatorial criteria for ideals in the considered class to satisfy N_p.

For example, [**46**] provides a criterion for monomial ideals to satisfy N_p.

Fix integer numbers $r, q \geq 1$. Set $n = \binom{r+q-1}{r-1}$. Let V be the q'th Veronese ring in r variables which defines the q'th Veronese embedding of $\mathbf{P}^{r-1}$. Set $T = k[t_1, \dots, t_r]$, graded by $\deg(t_j) = 1$ for $1 \leq j \leq r$. Then,

$$V \cong \oplus_{j=0}^{\infty} T_{jq} = k[\text{ all monomials of degree } q \text{ in } T\ ]\,.$$

Ottaviani-Paoletti [**95**] conjecture a criterion for V to satisfy N_p. They prove that the criterion gives a necessary condition. Surprisingly, the following part of their conjecture is not solved yet.

Conjecture 7.4. [**95**] If q and r are ≥ 3, and $p \leq 3q - 3$, then V satisfies N_p.

The general toric case is considered in [**70**].

8. Betti diagrams.

The graded Betti numbers of a graded finitely generated S-module form a diagram. A recent breakthrough in understanding such diagrams was generated by the conjectures of Boij-Söderberg [**19**], which were solved in [**20, 43, 51**]. Clearly, the diagrams form a semigroup $\mathcal{B}$. There are many open problems emerging involving the structure of this semigroup. We list two of them.

Problem 8.1. Is there an algorithm (or criterion) which takes a given diagram, and determines whether there exists a graded S-module with these graded Betti numbers?

The semigroup of *virtual* (or potential) Betti diagrams is the semigroup of lattice points in the positive rational cone generated by $\mathcal{B}$. The solution of the Boij-Soderberg conjectures allow us to check whether a given diagram is a virtual Betti diagram.

Erman [**34**] proved that the semigroup is finitely generated (if we restrict which Betti numbers can be non-zero); this raises the next problem.

Problem 8.2. Describe the generators of the semigroup (when we restrict which Betti numbers can be non-zero).

9. Free resolutions and Hilbert functions of points in $\mathbf{P}^{n-1}$.

This section focuses on the possible Hilbert functions and graded Betti numbers that can occur for ideals representing points in $\mathbf{P}^{n-1}$. A survey is given in Migliore's paper [**91**]. See [**38, 39, 63, 115**] for helpful background on this topic.

One can investigate "ordinary" points, which are reduced zero-dimensional subschemes of $\mathbf{P}^{n-1}$, or "fat" points, which one can think of as points with multiplicity. Geramita-Maroscia-Roberts [**59**] gave a classification of the possible Hilbert functions of reduced points (in fact, of a reduced variety) in $\mathbf{P}^{n-1}$. One can ask the analogous questions in other situations, for example:

Open-Ended Problem 9.1. What are the possible Hilbert functions of sets of double points in $\mathbf{P}^{n-1}$? How about for sets of points in $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_r}$ (or even just $\mathbf{P}^1 \times \mathbf{P}^1$)?

For examples of progress on these questions, see work of Geramita-Migliore-Sabourin [**60**] for the former question and Guardo-Van Tuyl [**60**] for the latter. There are similar problems for Betti numbers as well.

Open-Ended Problem 9.2. Given a Hilbert function for a set of points in $\mathbf{P}^{n-1}$, what are the possible graded Betti numbers for sets of points with that Hilbert function?

There is an answer for $\mathbf{P}^2$, but little is known in general, cf. [**68**]. One could also ask the question for sets of fat points in $\mathbf{P}^{n-1}$. More fundamentally, there are many open problems about the possible minimal free resolutions for different configurations of fat points.

One can also isolate particular properties of sets of reduced points and ask about the Hilbert functions of sets with those characteristics. For example, a set of points possesses the Uniform Position Property (UPP) if any two subsets of the same cardinality have the same Hilbert function. The following question is open in $\mathbf{P}^3$ and higher [**91**].

Open-Ended Problem 9.3. What are the possible Hilbert functions of sets of points with the UPP?

The following conjecture was first stated as a problem in Geramita-Orecchia [**61**].

Ideal Generation Conjecture 9.4. If $X \subset \mathbf{P}^{n-1}$ is a generic set of q points, then the homogeneous coordinate ring of X has the maximal rank property. (We say that $R = S/I$ satisfies the *maximal rank property* if each map $R_p \otimes R_1 \longrightarrow R_{p+1}$ has maximal rank, for all $p \in \mathbf{N}$.)

Note that if S/I has the maximal rank property and its Hilbert function is known, then it is easy to determine the degrees of the elements in a minimal system of generators of the ideal I.

10. Problems from Algebraic Combinatorics.

The expository paper [**109**] discusses several open problems in Algebraic Combinatorics; see also [**111**]. We focus on problems related to Hilbert functions and resolutions.

Let Δ be a simplicial complex on vertex set $x_1, \dots, x_n$. Let $\mathcal{F}(\Delta)$ be the set of facets of Δ. It is said that Δ is *partitionable* if there exists a partition

$$\Delta = \cup_{i=1}^{r} [G_i : F_i] \,,$$

where $\mathcal{F}(\Delta) = \{ F_1, \dots, F_r \}$ and the closed interval $[G_i : F_i]$ is the set $\{ G_i \subseteq H \subseteq F_i \}$. For example, the simplicial complex Δ with facets $\mathcal{F}(\Delta) = \{ x_1x_2x_3, x_2x_3x_4, x_2x_4x_5, x_1x_2x_5 \}$ has a partition

$$\Delta = [\varnothing : x_1x_2x_3] \cup [x_4 : x_2x_3x_4] \cup [x_5 : x_2x_4x_5] \cup [x_1x_5 : x_1x_2x_5]\,.$$

A long standing and central conjecture in Combinatorics, cf. [**109**, Problem **6**], [**111**, Conjecture **2.7**] is:

Conjecture 10.1. If the Stanley-Reisner ring of a simplicial complex Δ is Cohen-Macaulay, then Δ is partitionable.

The conjecture clearly holds for shellable simplicial complexes, but is open for constructible ones. By Alexander duality [**34, 115**], a simplicial complex Δ is Cohen-Macaulay if and only if the Alexander dual ideal $I_{\Delta^\vee}$ has a linear resolution. Thus, Conjecture 10.1 leads to the conjecture that if a monomial ideal has a linear resolution, then it has a Stanley decomposition, cf. [**79, 73, 49**].

Recall that there exists a polynomial $h(t)$ such that the Hilbert series of S/I is $\dfrac{h(t)}{(1-t)^{\dim(S/I)}}$. The coefficients of this polynomial form the h-vector.

According to [**109**, Problem **1**], the question whether the g-Theorem holds for Gorenstein simplicial complexes is perhaps the main open problem in the subject of h-vectors. It states:

Problem 10.2. (Stanley) [**109**, Problem **1**] If I is a square-free monomial ideal such that the quotient S/I is Gorenstein with an h-vector $(h_0, \dots, h_q)$, then is it true that $h_0 \leq h_1 \leq \cdots \leq h_{\frac{q}{2}}$? (It might be reasonable to generalize the problem to non-monomial ideals.)

An overview of the cases when the g-Theorem holds is given in [**109**]. A possible goal is to characterize the Hilbert functions (equivalently, h-vectors) of Gorenstein simplicial complexes, cf. [**111**, Conjecture **6.2**]. Another goal was stated in Open-Ended Problem 4.5: to characterize the Hilbert functions of Gorenstein graded algebras; for example, see [**22**] for a result in this direction. Another conjecture in the Gorenstein case is:

Problem 10.3. (Charney-Davis-Stanley) [**102**], [**109**, Problem **4**] Let I be a quadratic square-free monomial ideal such that S/I is Gorenstein with h-vector $(h_0, \ldots, h_{2e})$. Is it true that

$$(-1)^e(h_0 - h_1 + h_2 - \cdots + h_{2e}) \geq 0?$$

It might be reasonable to generalize Problem 10.3 to all Koszul (non-monomial) ideals. (See Section 13 for definition of Koszulness.)

A sequence $c_0, \ldots, c_r$ of real numbers is called *unimodal* if for some $0 \leq s \leq r$ we have $c_0 \leq \cdots \leq c_{s-1} \leq c_s \geq c_{s+1} \geq \cdots \geq c_r$. The sequence is called *log-concave* if $c_i^2 \geq c_{i-1}c_{i+1}$ for all $1 \leq i \leq r-1$. A log-concave sequence of positive numbers is unimodal.

Conjecture 10.4. (Stanley) [**110**] If S/I is a Cohen-Macaulay integral domain, then its h-vector is unimodal.

See [**75**] for a result in this direction. The following problem is in the spirit of the above conjecture.

Open-Ended Problem 10.5.

(1) Find classes of graded ideals for which the sequence of Betti numbers is log-concave or just unimodal.

(2) The graded Betti numbers are *log-concave or just unimodal* if for every $j \geq 0$ the sequence of the Betti numbers in the j'th strand $\{b^S_{i,i+j}(S/I)\}_{i\geq 1}$ has the desired property. Find classes of graded ideals for which the graded Betti numbers are log-concave or just unimodal.

11. Betti numbers of infinite free resolutions.

We continue with open problems and conjectures on *infinite* free resolutions. Expository lectures in this area are given in [**3**]; see also [**5**, **6**]. In view of the examples in [**37**] which show that the beginning of an infinite free resolution can be unstructured, it is natural to focus on the asymptotic properties of the resolutions. In the rest of this paper, we assume that $R = S/I$ and $I \subseteq (x_1, \ldots, x_n)^2$.

The following problem seems to be the most basic open problem on infinite free resolutions.

Problem 11.1. (Avramov) [**4**] Is it true that the Betti numbers of every finitely generated graded R-module are eventually non-decreasing?

The following two special cases of 11.1 are of interest.

Problem 11.2. (Ramras) [**100**] Is it true that if the Betti numbers of a graded finitely generated R-module are bounded, then they are eventually constant?

Problem 11.3. Does there exist a graded finitely generated periodic R-module (that is, a module isomorphic to some of its syzygies) with non-constant Betti numbers?

In the rest of this section, M stands for a graded finitely generated R-module. The following question related to 11.1 is also open.

Problem 11.4. (Avramov) [**3**] Is the limit $\limsup_{n\to\infty} \dfrac{b^R_{i+1}(M)}{b^R_i(M)}$ always finite?

For a long time the rationality of the Poincarè series $\sum_{i\geq 0} b^R_i(M)t^i$ was a central problem in Commutative Algebra. After Anick's example of an irrational Poincarè series [**1**], the research can be continued in the following directions.

Open-Ended Problem 11.5. Find classes of graded rings over which every finitely generated graded module has a rational Poincarè series.

Open-Ended Problem 11.6. If the Poincarè series of a finitely generated graded module is rational, then what can be said about its denominator and its roots?

A motivation for the above problem comes from the fact that the radius of convergence of the Poincarè series can provide a measure of the asymptotic behavior of the Betti numbers, cf. [**112**]. There are a number of questions on the asymptotic growth of the Betti numbers. It is known that the growth of the Betti numbers in a minimal free resolution over a quotient ring is at most exponential. The following problems are wide open, cf. [**3**].

Problem 11.7. (Avramov) What types of growth can the sequence of Betti numbers have? Are polynomial and exponential growth the only possibilities?

Polynomial growth can be measured by the *complexity*

$$\mathrm{cx}_R(M) = \inf\left\{ c \in \mathbf{N} \,\middle|\, \begin{array}{l} \text{there exists a polynomial } p(t) \text{ of degree } c-1, \\ \text{such that } b_i^R(M) \le p(i) \text{ for } i \ge 1 \end{array} \right\}.$$

Avramov raised the question whether complexity satisfies the analogue of the Auslander-Buchsbaum Formula:

Problem 11.8. (Avramov) [**3**] Suppose that $\mathrm{cx}_R(M) < \infty$. Is it true that $\mathrm{cx}_R(M) \le \mathrm{codepth}(R)$?

It is also unknown if finite complexity forces polynomial asymptotic behavior:

Problem 11.9. (Avramov) [**3**] Suppose that $c = \mathrm{cx}_R(M) < \infty$. Is it true that there exists a constant $a \in \mathbf{R}$ such that

$$\lim_{i\to\infty} \frac{b_i^R(M)}{i^{(c-1)}} = a\,?$$

The *curvature*

$$\mathrm{curv}_R(M) = \lim\ \sup_{i\to\infty} \frac{\ln(b_i^R(M))}{\ln(i)}$$

is another numerical invariant introduced by Avramov in order to measure growth.

Problem 11.10. (Avramov) [**3**] Does $\mathrm{curv}_R(M) = 1$ imply $\mathrm{cx}_R(M) < \infty$?

Problem 11.11. (Avramov) [**3**] Suppose that $\mathrm{curv}_R(M) > 1$. Is it true that there exists a constant $a \in \mathbf{R}$ such that

$$\lim_{i\to\infty} \frac{b_i^R(M)}{(\mathrm{curv}_R(M))^i} = a\,?$$

The sequence of Betti numbers $\{b_i^R(M)\}_{i\geq 0}$ has *strong polynomial growth* if there exist two polynomials $f(x)$ and $g(x)$ in $\mathbf{R}[t]$ of the same degree and with the same leading term, such that $f(i) \leq b_i^R(M) \leq g(i)$ for $i >> 0$. The sequence $\{b_i^R(M)\}_{i\geq 0}$ has *strong exponential growth* if there exist two numbers $\alpha, \beta \in \mathbf{R}$, such that $\alpha > 1$, $\beta > 1$, and $\alpha^i \leq b_i^R(M) \leq \beta^i$ for $i >> 0$. It is of interest to find modules with such types of growth.

Open-Ended Problem 11.12. Find classes of graded finitely generated modules with strong polynomial growth, or with strong exponential growth.

12. Complete intersections and exterior algebras.

There has been a lot of exciting progress on the structure of minimal free resolutions over complete intersections (cf. [**8, 12, 37**]) and over exterior algebras (cf. [**2, 40, 41, 42**]). Although we list only two specific problems, we believe that these areas are very fruitful and important, and that it is of high interest to continue studying such resolutions.

Let I be generated by a homogeneous regular sequence $f_1, \dots, f_c$, and consider the complete intersection $R = S/I$. Given a complex of free R-modules $\mathbf{G}$, choose a sequence of homomorphisms of free S-modules $\tilde{\mathbf{G}}$, such that $\mathbf{G} = R \otimes_S \tilde{\mathbf{G}}$. Since $R \otimes \tilde{d} = 0$ (where $\tilde{d}$ is the differential in $\tilde{\mathbf{G}}$) there exist maps $\tilde{\tau}_i :\ \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ of degree -2, such that $\tilde{d}^2 = \sum_{i=1}^{c} f_i \tilde{\tau}_i$. Then for $1 \leq i \leq c$ define maps $\tau_i :\ \mathbf{G} \to \mathbf{G}$ of degree -2 by setting $\tau_i = R \otimes \tilde{\tau}_i$. These operators are called *Eisenbud operators* and were introduced by Eisenbud in [**37**]; cf. also [**10, 13**]. They are independent up to homotopy of the choice of the lifting $\tilde{d}$, they are homomorphisms of complexes, and they commute up to homotopy. Thus, $\mathrm{Tor}^R(M, k)$ and $\mathrm{Ext}_R(M, k)$ are graded modules over the polynomial ring in c variables. It would be helpful to have this property for the resolution itself, but the following conjecture is open.

Conjecture 12.1. (Eisenbud) [**37**] Let $\mathbf{G}$ be the minimal free resolution of a finitely generated module over a complete intersection. The Eisenbud operators on $\mathbf{G}$ can be chosen so that they commute asymptotically. (Here, "asymptotically" means that we ignore the beginning of the resolution and consider a high enough truncation.)

In [**47**] the following problem is stated in the case when R is a complete intersection, but the authors remark that this assumption might be unnecessary.

Problem 12.2. (Eisenbud-Huneke) [**47**] Let $\mathbf{G}$ be a graded minimal free resolution over R. Do there exist a number p and bases of the free modules in $\mathbf{G}$, such that for all $i \geq 0$ we have that each entry in the matrix of the differential d_i has degree less than p?

13. Koszul rings and Rate.

Throughout this section we assume that $R = S/I$ and $I \subseteq (x_1, \ldots, x_n)^2$. If regularity is infinite, then a meaningful numerical invariant is rate. Backelin introduced

$$\mathrm{rate}_R(k) = \sup\Big\{ \frac{p_i - 1}{i-1} \,\Big|\, i \geq 2 \Big\},$$
$$\text{where } p_i \;=\; \max\{\; j \mid b^R_{i,j}(k) \neq 0 \text{ or } j = i\},$$

called the *rate* of k over R (sometimes called the rate of R).

By [**50**], if I is a monomial ideal and q is the maximal degree of a monomial in its minimal system of monomial generators, then $\mathrm{rate}_{S/I}(k) = q - 1$. Furthermore, $\mathrm{rate}_{S/I}(k) \leq \mathrm{rate}_{S/in(I)}(k)$ for every initial ideal $in(I)$ of I. Therefore, $\mathrm{rate}_R(k) < \infty$. Thus, Open-Ended Problem 3.18 yields an upper bound on the rate.

Open-Ended Problem 13.1. Find classes of rings over which you can give a sharp bound on the rate of k.

Koszul rings play an important role in Commutative Algebra, Algebraic Geometry, and other fields. A ring R is called Koszul if the following equivalent conditions hold:

∘ if $i \neq j$ then the graded Betti number $b^R_{i,j}(k)$ of k over R vanishes

∘ the entries in the matrices of the differential (in the minimal free resolution of k) are linear forms.

∘ $\mathrm{rate}_R(k) = 1$.

Open-Ended Problem 13.2. Find classes of rings, which are Koszul.

Roos [**105**] constructed for each integer $j \geq 3$ a quotient ring W generated by 6 variables subject to 11 quadratic relations, with $b^W_{ij}(k) = 0$ for $i \neq i < j$ and $b^W_{j,j+1}(k) \neq 0$. Therefore, the Koszul property cannot be inferred from the knowledge of any finite number of Betti numbers of k. One of the most fruitful techniques in establishing the Koszul property is to obtain a quadratic Gröbner basis. Thus, Open-Ended Problem 13.2 leads to:

Open-Ended Problem 13.3. Find classes of ideals with quadratic Gröbner basis.

The following is the most interesting currently open conjecture on Koszul toric rings.

Problem 13.4. (Bϕgvad) Is the toric ring of a smooth projectively normal toric variety Koszul?

In particular, it is not known:

Problem 13.5. (Bϕgvad) Is the toric ideal of a smooth projectively normal toric variety generated by quadrics?

Problem 16.7 is a very challenging open question on Koszul rings and comes from the theory of Hyperplane Arrangements.

14. Generic Poincarè series.

This is a new area of research on infinite free resolutions. Avramov has recently raised the problem to search for meaningful conjectures and ideas on what generic behavior means for Poincarè series.

Let I be the ideal generated by the set $U = \{f_1, \dots, f_r\}$ of r generic forms of fixed degrees $a_1, \dots, a_r$. Let V be a matrix with homogeneous entries; for simplicity, we may assume that the entries are linear forms. The data U, V is parametrized by an algebraic variety. We consider the minimal free resolution of the cokernel of V over the quotient ring $R = S/I$. Thus, we consider the minimal free resolutions of cokernels of generic matrices over generic quotient rings.

There are several directions, which one might explore. One of them is to vary the generators of I.

Open-Ended Problem 14.1. (Avramov) Let I be generated by generic forms $f_1, \dots, f_r$ of degrees $a_1, \dots, a_r$. What can be said about the Poincarè series of k over R?

For example, Fröberg and Löwfall prove that if $a_1 = \dots = a_r = 2$ and $r \le n$ or $r \ge n/2 + n^2/4$ then generic quotient rings are Koszul, and Conca [Co] studies quadratic Gröbner basis. More generally one can ask:

Problem 14.2. (Avramov) Is the Poincarè series of k over R determined by the Hilbert series of R if I is generated by generic forms?

Another possibility is to fix the generators of I, but vary the module that we are resolving.

Open-Ended Problem 14.3. (Avramov) Let R be fixed (here we either want I to be generated by generic forms, or to have another assumption on I). What can be said about the Poincarè series of the generic finitely generated modules over R?

Yet another possibility is to fix some parameters or properties, and then search for generic behavior. An example of this kind is that if you take 7 generic quadrics in 4 variables and a module presented by a (2×2)-matrix with generic linear entries, then its Betti numbers are constant and equal to 2. Results over rings with $(x_1, \dots, x_n)^3 = 0$ are obtained in [**30, 11**]. Recall that the pair U, V is parametrized by an algebraic variety X. The following question was raised by Avramov, Iyengar, and Sega recently.

Problem 14.4. (Avramov, Iyengar, Sega) Let I be generated by generic quadrics. Here we vary both U and V. Denote by A the set of points in X for which the cokernel of V over R has constant Betti numbers. Does A contain a non-empty Zariski open set? Denote by B the set of points in X for which the cokernel of V over R has constant Betti numbers and a non-periodic minimal free resolution. Does B contain a non-empty Zariski open set?

Eisenbud raised (long time ago) a related question, where U is fixed.

Open-Ended Problem 14.5. Consider a fixed quotient ring $R = S/I$, where I is generated by (generic) quadrics (perhaps, we need to impose some restrictions on R). For example, consider the ring constructed in [**56**].

(1) Denote by B the set of points for which the cokernel of V over R has Betti numbers equal to 2 and a non-periodic minimal free resolution. Does B contain a non-empty Zariski open set? What can be said about B?

(2) Let T be a graded finitely generated R-module with Betti numbers equal to 2 and a non-periodic minimal free resolution. What can be said about the infinite set of points that are syzygies of T?

15. Monomial and Toric ideals.

The problems in this section concern both finite and infinite minimal free resolutions.

Some of the known concepts/constructions useful for studying finite resolutions of monomial ideals are: the Stanley-Reisner correspondence, simplicial and cellular resolutions, Alexander duality, the lcm-lattice, algebraic and combinatorial shifting, discrete Morse theory, the Scarf complex. The main goal in this area is:

Open-Ended Problem 15.1. Introduce new constructions and ideas on resolutions of monomial (or toric) ideals.

A monomial ideal M is p-Borel fixed if it is invariant under the action of the general linear group in characteristic p; such ideals are characterized by a combinatorial property on the multidegrees of their monomial generators. The interest in studying such ideals comes from the fact that the generic initial ideals are p-Borel fixed in characteristic p. The minimal free resolution of a 0-Borel fixed ideal is known; it is the Eliahou-Kervaire resolution.

Problem 15.2. (Evans-Stillman) Describe the minimal free resolution and find a formula for the regularity (and the Betti numbers) of a p-Borel fixed ideal if $p > 0$. Note that the resolution is considered in characteristic 0.

There are a number of particular classes of monomial or toric ideals, for which one might expect to get interesting results on the structure of their minimal free resolutions. For example:

Open-Ended Problem 15.3. Find a formula (or at least sharp upper bounds) on the regularity of an edge monomial ideal in terms of the properties of the defining graph.

Open-Ended Problem 15.4. Find a formula (or at least sharp upper bounds) on the regularity or rate of an edge toric ideal, in terms of the properties of the defining graph.

Let **G** be the minimal free resolution of a monomial ideal over $R = S/I$. Assume that I is either a monomial or a toric ideal. It will be interesting to explore what can be said about the structure of **G**. For example:

Open-Ended Problem 15.5. Develop the theory of infinite cellular resolutions.

Open-Ended Problem 15.6. Find how to compute the Betti numbers of **G** using various simplicial complexes.

When I is a square-free monomial ideal and J is the maximal ideal, Problem 15.5 is solved by Berglund [**17**] and he also proved the conjecture by Charalambous-Reeves on the possible terms in the denominator of the Poincarè series.

Problem 15.7. Construct **G** in the case it is resolving a lex ideal. (For example, we can assume here that R is a Veronese or a Segre ring.)

Problem 15.8. Let $C = S/(x_1^{a_1}, \dots, x_n^{a_n})$, where $a_1 \leq a_2 \leq \cdots \leq a_n \leq \infty$, be a Clements-Lindström ring. Construct the minimal free resolution over C of a Borel ideal.

16. Problems on Subspace Arrangements.

The problems in this section concern both finite and infinite minimal free resolutions. For background on Subspace Arrangements, see [**94**]. A set $\mathcal{A}$ of subspaces in $\mathbf{C}^r$ is called a subspace arrangement.

Open-Ended Problem 16.1. Study problems that relate the properties of subspace arrangements and minimal free resolutions.

The above problem sounds vague, but this is a new area of research and Problem 16.1 is just inviting to explore in that direction. [**98**] provides a result of this type. Another similar problem is about the complement $\mathbf{C}^r \setminus A$, whose topology has been extensively studied in topological combinatorics. [**57**] provides a result in the spirit of the next problem.

Open-Ended Problem 16.2. Relate the cohomology algebra of the complement $\mathbf{C}^r \setminus A$ and Tor-algebras.

In the rest of this section, we assume that $\mathcal{A} = \bigcup_{i=1}^{n} H_i \subseteq \mathbf{C}^r$ is a central arrangement of n hyperplanes ("central" means that each of the hyperplanes contains the origin). The cohomology ring A of the complement $\mathbf{C}^r \setminus \mathcal{A}$ has a simple combinatorial description; it is a quotient of an exterior algebra by a combinatorially determined ideal. Namely, if E is the exterior algebra on n variables $e_1, \ldots, e_n$, then the Orlik-Solomon algebra is $A = E/J$ where J is generated by all elements

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{1 \le q \le p} (-1)^{q-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_q}} \wedge \cdots \wedge e_{i_p}$$
$$\text{for which } \operatorname{codim}(H_{i_1} \cap \cdots \cap H_{i_p}) < p\,;$$

such a set $\{i_1, \ldots, i_p\}$ is called *dependent*. The Orlik-Solomon algebra is similar in some ways to the Stanley–Reisner ring, but formulas for the graded Betti numbers are elusive.

Problem 16.3. Find a combinatorial description for the Betti numbers $\dim_{\mathbf{C}} \mathrm{Tor}_i^E(A, \mathbf{C})_j$ (in special cases).

First steps in this direction are taken in [**81**]. The Betti number $\dim_{\mathbf{C}} \mathrm{Tor}_i^E(A, \mathbf{C})_i$ is the i^{th} Chen rank [**107**]. The **Resonance Formula** in [**112**] is an intriguing conjecture for these numbers.

Even more challengingly, we could ask for descriptions of the differentials in the resolutions. Investigating in special classes of rings (arrangements) is likely to yield results.

Problem 16.4. Construct the minimal free resolution of the Orlik-Solomon algebra A over the exterior algebra E (in special cases).

Problem 16.5. Construct a nicely structured non-minimal free resolution of the Orlik-Solomon algebra A over the exterior algebra E (in special cases).

We can also study the minimal free resolution of $\mathbf{C}$ over the quotient ring A.

Problem 16.6. Find a combinatorial description for the Betti numbers $\dim_{\mathbf{C}} \mathrm{Tor}_i^A(\mathbf{C}, \mathbf{C})_j$.

When A is Koszul, then a formula is known for the Betti numbers $\dim_{\mathbf{C}} \mathrm{Tor}_i^A(\mathbf{C}, \mathbf{C})_i$. These linear Betti numbers are of great interest in algebraic topology since they are related to the homotopy of the complement $\mathbf{C}^r \setminus \mathcal{A}$ by the formula

$$\prod\nolimits_{j=1}^{\infty} (1 - t^j)^{-\varphi_j} = \sum_{i \geq 0} \dim_{\mathbf{C}} \mathrm{Tor}_i^A(\mathbf{C}, \mathbf{C})_i t^i \,,$$

where φ_i is the i^{th} lower central series rank (LCS rank) of the fundamental group $\pi_1(\mathbf{C}^r \setminus \mathcal{A})$, cf. [**97**]. In the introduction to [**78**], Hirzebruch wrote: "The topology of the complement of an arrangement of lines in the projective plane is very interesting, the investigation of the fundamental group of the complement very difficult." The following problem is a very challenging open question in this direction.

Problem 16.7. [**52**, Problem **2.2**] Find a non-supersolvable central hyperplane arrangement for which the Lower Central Series Formula $\prod_{j=1}^{\infty}(1-t^j)^{\varphi_j} = \mathrm{Hilb}_A(-t)$ holds. Equivalently (by [**97**]), find a Koszul Orlik-Solomon algebra without a quadratic Gröbner basis.

See [**52**] for more problems in this direction. Problem 16.7 is similar in flavor to the example of the pinched Veronese ring, which was proved to be Koszul by Caviglia in [**23**].

A set $W \subseteq \{1, \dots, n\}$ is called a *circuit* if it is dependent and has minimal support among the dependent sets, and W is a *broken circuit* if there exists a hyperplane H_i such that $W \cup i$ is a circuit and $i > \max(W)$. We call the monomial $e_{i_1} \wedge \cdots \wedge e_{i_p}$ a circuit (or broken circuit) if W has that property. The *broken circuit ideal* T the monomial ideal in E generated by the broken circuits. Consider the lex order in the exterior algebra E with $e_1 > \cdots > e_n$. If $W = \{i_1 < \cdots < i_p\}$ is a circuit, then the initial term of $\partial(e_{i_1} \wedge \cdots \wedge e_{i_p})$ is the broken circuit $e_{i_1} \wedge \cdots \wedge e_{i_{p-1}}$. Therefore, the ideal T is contained in the initial ideal of J. By [**94**, Theorem **3.43**], it follows that T is the initial ideal. In view of the above questions, it is interesting to obtain information about E/T.

Problem 16.8. Consider problems 16.3–16.6 for E/T instead of A.

Another interesting object is the module $\Omega^1(\mathcal{A})$ of logarithmic one-forms with pole along the arrangement or (dually) the module $D(\mathcal{A})$ of derivations tangent to the arrangement.

17. Relevant Books.

We have compiled a list of books, that have appeared since 1985 and that contain material related to the topics in this paper. The list is probably not complete. The books are listed in the order they have appeared.

Bruns, Winfried; Gubeladze, Joseph: *Polytopes, Rings, and K-Theory*. Springer Monographs in Mathematics. Springer. to appear.

Jöllenbeck, Michael; Welker, Volkmar: *Resolution of the residue field via algebraic discrete Morse theory*. Memoirs of the AMS. AMS. to appear.

Greuel, Gert-Martin; Pfister, Gerhard; A.: *Singular introduction to commutative algebra*. Second, extended edition. With contributions by Olaf Bachmann, Christoph Lossen and Hans Schnemann. Springer, Berlin, 2008.

Orlik, Peter; Welker, Volkmar: *Algebraic combinatorics*. Lectures from the Summer School held in Nordfjordeid, June 2003. Universitext. Springer, Berlin, 2007.

Iyengar, Srikanth B.; Leuschke, Graham J.; Leykin, Anton; Miller, Claudia; Miller, Ezra; Singh, Anurag K.; Walther, Uli: *Twenty-four hours of local cohomology*. Graduate Studies in Mathematics, 87. American Mathematical Society, Providence, RI, 2007.

Syzygies and Hilbert functions. Edited by Irena Peeva. Lecture Notes in Pure and Applied Mathematics, 254. Chapman & Hall/CRC, Boca Raton, FL, 2007.

Cox, David; Little, John; O'Shea, Donal: *Ideals, varieties, and algorithms*. An introduction to computational algebraic geometry and commutative algebra. Third edition. Undergraduate Texts in Mathematics. Springer, New York, 2007.

Boffi, Giandomenico; Buchsbaum, David A.: *Threading homology through algebra: selected patterns*. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, Oxford, 2006.

Huneke, Craig; Swanson, Irena: *Integral closure of ideals, rings, and modules*. London Mathematical Society Lecture Note Series, 336. Cambridge University Press, Cambridge, 2006.

Thomas, Rekha R.: *Lectures in geometric combinatorics*. Student Mathematical Library, 33. IAS/Park City Mathematical Subseries. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 2006.

Eisenbud, David: *The geometry of syzygies. A second course in commutative algebra and algebraic geometry*. Graduate Texts in Mathematics, 229. Springer-Verlag, New York, 2005.

Kreuzer, Martin; Robbiano, Lorenzo: *Computational commutative algebra. 2*. Springer-Verlag, Berlin, 2005.

Cox, David A.; Little, John; O'Shea, Donal: *Using algebraic geometry*. Second edition. Graduate Texts in Mathematics, 185. Springer, New York, 2005.

Miller, Ezra; Sturmfels, Bernd: *Combinatorial commutative algebra*. Graduate Texts in Mathematics, 227. Springer-Verlag, New York, 2005.

Lazarsfeld, Robert: *Positivity in algebraic geometry. II.* Positivity for vector bundles, and multiplier ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 49. Springer-Verlag, Berlin, 2004.

Lazarsfeld, Robert: *Positivity in algebraic geometry. I.* Classical setting: line bundles and linear series. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 48. Springer-Verlag, Berlin, 2004.

Trends in commutative algebra. Edited by Luchezar L. Avramov, Mark Green, Craig Huneke, Karen E. Smith and Bernd Sturmfels. Mathematical Sciences Research Institute Publications, 51. Cambridge University Press, Cambridge, 2004.

Schenck, Hal: *Computational algebraic geometry*. London Mathematical Society Student Texts, 58. Cambridge University Press, Cambridge, 2003.

Weyman, Jerzy: *Cohomology of vector bundles and syzygies*. Cambridge Tracts in Mathematics, 149. Cambridge University Press, Cambridge, 2003.

Computations in algebraic geometry with Macaulay 2. Edited by David Eisenbud, Daniel R. Grayson, Michael Stillman and Bernd Sturmfels. Algorithms and Computation in Mathematics, 8. Springer-Verlag, Berlin, 2002.

Computational commutative algebra and combinatorics. Proceedings of the 8th International Research Institute held at Osaka University, Osaka, July 21–30, 1999. Edited by Takayuki Hibi. Advanced Studies in Pure Mathematics, 33. Mathematical Society of Japan, Tokyo, 2002.

Villarreal, Rafael H.: *Monomial algebras.* Monographs and Textbooks in Pure and Applied Mathematics, 238. Marcel Dekker, Inc., New York, 2001.

Christensen, Lars Winther: *Gorenstein dimensions.* Lecture Notes in Mathematics, 1747. Springer-Verlag, Berlin, 2000.

Kreuzer, Martin; Robbiano, Lorenzo: *Computational commutative algebra. 1.* Springer Verlag, Berlin, 2000.

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