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On some obstructions of flag vector pairs (f_1, f_{04}) of 5-polytopes

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Motivated by the recent work of Sjöberg and Ziegler, who obtained a complete characterization of the pairs (f_0, f_{03}) of flag numbers for 4-polytopes, in this paper we give some new results about the possible flag vector pairs (f_1, f_{04}) of 5-polytopes.

1. Introduction

Let P be a d-dimensional convex polytope. For each $0 \le i \le d-1$, let $f_i(P)$ denote the number of i-dimensional faces of P. One fundamental combinatorial invariant of P is the *f-vector*

$$f(P) = (f_0(P), f_1(P), \dots, f_{d-1}(P)),$$

and characterizing all possible f-vectors of convex polytopes has been one of the central problems in convex geometry. For simplicity, throughout the paper a d-dimensional convex polytope will be called a *d-polytope*.

Let $\mathcal{F}^d$ denote the set of all f-vectors of d-polytopes, and let $\Pi_{i,j}(\mathcal{F}^d)$ denote the projection of $\mathcal{F}^d$ onto the coordinates f_i and f_j. Steinitz [1906] completely determined all possible f-vectors of 3-polytopes:

Theorem 1.1. *The set $\Pi_{0,1}(\mathcal{F}^3)$ of all f-vectors (f_0, f_1) of 3-polytopes is equal to*

$$\left\{(v, e) \;\middle|\; \tfrac{3}{2}v \le e \le 3v - 6\right\}.$$

In dimensions $d \ge 4$, any d-polytope P satisfies

$$\frac{d}{2} f_0(P) \le f_1(P) \le \binom{f_0(P)}{2}. \tag{1-1}$$

However, any complete determination of all possible f-vectors of d-polytopes for $d \ge 4$ is still elusive. As some partial results, for $d = 4$ the projections of the f-vector onto two of the four coordinates have been determined by Grünbaum

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[1967], Barnette and Reay [1973], and Barnette [1974] (see [Sjöberg and Ziegler 2018, Section 2] for more details).

Kusunoki and Murai [2019] characterized the first two entries of the f-vectors of 5-polytopes.

Theorem 1.2. *Let* $L = \left\{\left(v, \left[\frac{5}{2}v+1\right]\right) \,\middle|\, v \geq 7\right\}$, *and let*

$$G = \{(8, 20), (9, 25), (13, 35)\}.$$

Then we have

$$\Pi_{0,1}(\mathcal{F}^5) = \left\{(v, e) \,\middle|\, \frac{5}{2}v \leq e \leq \binom{v}{2}\right\} \setminus (L \cup G).$$

The same result has been independently proved by Pineda-Villavicencio, Ugon, and Yost [2018] (see also [Pineda-Villavicencio, Ugon, and Yost 2019]).

For a subset S of $\{0, 1, 2, \ldots, d-1\}$, let $f_S(P)$ denote the number of chains

$$F_1 \subset F_2 \subset \cdots \subset F_r$$

of faces F_i, $1 \leq i \leq r$, of P such that

$$S = \{\dim F_1, \dim F_2, \ldots, \dim F_r\}.$$

The *flag vector* of P is defined to be

$$(f_S(P))_{S \subset \{0,1,2,\ldots,d-1\}}.$$

For the sake of simplicity, from now on we use the notation $f_{i_1 i_2 \ldots i_k}(P)$ instead of $f_{\{i_1, i_2, \ldots, i_k\}}(P)$ for any subset $\{i_1, i_2, \ldots, i_k\}$ of $\{0, 1, 2, \ldots, d-1\}$.

In this paper, for any two subsets S_1 and S_2 of $\{0, 1, 2, \ldots, d-1\}$ a pair $(f_{S_1}(P), f_{S_2}(P))$, or simply (f_{S_1}, f_{S_2}), of flag numbers of P will be called a *flag vector pair*. More generally, for any k not necessarily mutually disjoint subsets $S_1, S_2, \ldots, S_k$ of $\{0, 1, 2, \ldots, d-1\}$, a k-tuple

$$(f_{S_1}(P), f_{S_2}(P), \ldots, f_{S_k}(P)),$$

or simply $(f_{S_1}, f_{S_2}, \ldots, f_{S_k})$, of flag numbers of P will be called a *flag vector k-tuple*.

We denote by $\Pi_{S_1, S_2, \ldots, S_k}$ the projection of the flag vector $(f_S(P))_{S \subset \{0,1,2,\ldots,d-1\}}$ onto its coordinates $f_{S_1}, f_{S_2}, \ldots, f_{S_k}$. We call $(f_{S_1}, f_{S_2}, \ldots, f_{S_k})$ a *polytopal flag vector k-tuple* if

$$(f_{S_1}, f_{S_2}, \ldots, f_{S_k})$$

belongs to the image of the set of all flag vectors of d-dimensional polytopes under the projection map $\Pi_{S_1, S_2, \ldots, S_k}$, that is, if there is a d-polytope P such that

$$(f_{S_1}(P), f_{S_2}(P), \ldots, f_{S_k}(P)) = (f_{S_1}, f_{S_2}, \ldots, f_{S_k}).$$

Recently, Sjöberg and Ziegler [2018] obtained a complete characterization of the pairs (f_0, f_{03}) of flag numbers for 4-polytopes:

Theorem 1.3. *Let*

$$E = \begin{Bmatrix} (6,24), (6,25), (6,28), (7,28), (7,30), (7,31), (7,33), (7,34), (7,37), \\ (7,40), (8,33), (8,34), (8,37), (8,40), (9,37), (9,40), (10,40), (10,43) \end{Bmatrix}.$$

Then the set of all flag vector pairs (f_0, f_{03}) of 4-polytopes is equal to

$$\left\{ (f_0, f_{03}) \;\middle|\; \begin{array}{l} 20 \le 4f_0 \le f_{03} \le 2f_0(f_0-3), \\ f_{03} \ne 2f_0(f_0-3)-k,\ k \in \{1,2,3,5,6,9,13\} \end{array} \right\} \setminus E.$$

For the proof of Theorem 1.3, the classification of all combinatorial types of 4-polytopes with up to eight vertices by Altshuler and Steinberg [1984; 1985] played an important role.

Our primary aim of this paper is to provide some new results about the flag vector pairs (f_1, f_{04}) of 5-polytopes:

Theorem 1.4. *Let P be a 5-polytope. Then the flag vector pairs (f_1, f_{04}) of 5-polytopes satisfy the following inequalities*:

(1) *For a given flag number $f_{04}(P)$, we have*

$$\tfrac{5}{4}\Big(7+\sqrt{1+\tfrac{4}{5}f_{04}(P)}\Big) \le f_1(P) < \tfrac{1}{4} f_{04}(P)(f_{04}(P)-3). \tag{1-2}$$

(2) *For a given flag number $f_1(P)$, we have*

$$\tfrac{1}{2}\Big(3+\sqrt{9+16f_1(P)}\Big) < f_{04}(P) \le \tfrac{4}{5} f_1(P)^2 - 14 f_1(P) + 60. \tag{1-3}$$

Remark 1.5. (1) The lower (resp. upper) bound of the flag vector pairs (f_1, f_{04}) given in Theorem 1.4(1) (resp. (2)) are very sharp, since there is an explicit example, such as a 5-simplex with $(f_1, f_{04}) = (15, 30)$, which satisfies the equalities in (1-2) and (1-3).

(2) The upper (resp. lower) bound of the flag vector pairs (f_1, f_{04}) given in Theorem 1.4(1) (resp. (2)) might be improved further by using much sharper inequality instead of $\sum_{i=1}^{k} x_i^2 < \big(\sum_{i=1}^{k} x_i\big)^2$ for any positive $x_i > 0$ with $1 \le i \le k$ or by any other means (see Lemma 2.1 for more details). In this paper, we do not pursue this issue further, though.

(3) The question of whether or not all vector pairs (f_1, f_{04}) satisfying the inequalities (1-2) and (1-3) given in Theorem 1.4 are flag vector pairs of 5-polytopes is unknown, and the technique of this paper is insufficient to answer such a question.

This paper is organized as follows. In Section 2, we give a proof of Theorem 1.4 by a series of lemmas. In Section 3, we provide some concrete examples of 5-polytopes satisfying the inequalities given in Theorem 1.4 for the flag vector pairs (f_1, f_{04}) of 5-polytopes. In order to construct such examples, we make use of the well-known stacking and truncating operations.

2. Proof of Theorem 1.4

We begin with the following lemmas.

Lemma 2.1. *The flag vector pair* $(f_1(P), f_{04}(P))$ *of a* 5*-polytope* P *satisfies*

$$f_1(P) < \tfrac{1}{4} f_{04}(P)(f_{04}(P) - 3).$$

Proof. Let F be any facet of the 5-polytope P. Then it follows from [Sjöberg and Ziegler 2018, Theorem 2.1] that

$$f_3(F) \leq \tfrac{1}{2} f_0(F)(f_0(F) - 3).$$

Thus it is easy to obtain

$$\sum_{F \subset P} f_3(F) \leq \frac{1}{2} \sum_{F \subset P} f_0^2(F) - \frac{3}{2} \sum_{F \subset P} f_0(F). \tag{2-1}$$

Since

$$\sum_{i=1}^{k} x_i^2 < \left(\sum_{i=1}^{k} x_i\right)^2$$

for any positive x_i $(1 \leq i \leq k)$, it follows from (2-1) that

$$\begin{aligned} f_{34}(P) = \sum_{F \subset P} f_3(F) &< \frac{1}{2}\left(\sum_{F \subset P} f_0(F)\right)^2 - \frac{3}{2} \sum_{F \subset P} f_0(F) \\ &= \tfrac{1}{2} f_{04}(P)^2 - \tfrac{3}{2} f_{04}(P). \end{aligned} \tag{2-2}$$

By considering the dual polytope P^* of P, by (2-2) we can obtain

$$2 f_1(P^*) = f_{01}(P^*) < \tfrac{1}{2} f_{04}(P^*)(f_{04}(P^*) - 3).$$

Since P is an arbitrary polytope, so is its dual P^*. Therefore, we can obtain

$$f_1(P) < \tfrac{1}{4} f_{04}(P)(f_{04}(P) - 3).$$ □

Lemma 2.2. *The flag vector pair* $(f_0(P), f_{04}(P))$ *of a* 5*-polytope* P *satisfies*

$$5 f_0(P) \leq f_{04}(P) \leq 5(f_0(P) - 3)(f_0(P) - 4).$$

Proof. Note first that every vertex of a d-polytope meets at least d facets. Thus we have $5 f_0(P) \leq f_{04}(P)$, where equality holds if and only if P is a simple polytope.

On the other hand, it follows from [Sjöberg and Ziegler 2018, Lemma 2.6] (or [Billera and Björner 1997, Theorem 18.5.9]) that for any d-polytope Q with n vertices and for any subset $S \subset \{0, 1, 2, \ldots, d-1\}$ we have

$$f_S(Q) \leq f_S(C_d(n)),$$

where $C_d(n)$ denotes the d-dimensional cyclic polytope with $n = f_0(Q)$ vertices. Hence, we have

$$f_{04}(P) \le f_{04}(C_5(n)) = 5 f_4(C_5(n)). \tag{2-3}$$

Here, the second equality holds because $C_5(n)$ is simplicial, and the first inequality becomes an equality if and only if P is neighborly.

On the other hand, by using the formula in [Buchstaber and Panov 2002, Lemma 1.34] we can directly calculate

$$\begin{aligned} f_4(C_5(n)) &= \sum_{q=0}^{2} \binom{q}{0}\binom{n+q-6}{q} + \sum_{p=0}^{2} \binom{5-p}{5-p}\binom{n+p-6}{p} \\ &= (n-3)(n-4). \end{aligned}$$

Hence, it follows from (2-3) that

$$f_{04}(P) \le 5 f_4(C_5(n)) = 5(f_0(P)-3)(f_0(P)-4). \qquad \square$$

Lemma 2.3. *The flag vector pair $(f_1(P), f_{04}(P))$ of a 5-polytope P satisfies*

$$f_1(P) \ge \tfrac{5}{4}\Bigl(7 + \sqrt{1 + \tfrac{4}{5} f_{04}(P)}\Bigr).$$

Proof. By Lemma 2.2, we have

$$f_0(P)^2 - 7 f_0(P) + 12 - \tfrac{1}{5} f_{04}(P) \ge 0.$$

Thus, since $f_0(P) \ge 6$, it is easy to obtain

$$f_0(P) \ge \tfrac{1}{2}\Bigl(7 + \sqrt{1 + \tfrac{4}{5} f_{04}(P)}\Bigr). \tag{2-4}$$

Recall now that $f_0(P) \le \frac{2}{5} f_1(P)$ by (1-1). Hence, it follows from (2-4) that

$$f_1(P) \ge \tfrac{5}{4}\Bigl(7 + \sqrt{1 + \tfrac{4}{5} f_{04}(P)}\Bigr),$$

as desired. $\square$

Theorem 1.4(1) is an immediate consequence of Lemmas 2.1 and 2.3.

Next, we want to prove Theorem 1.4(2). We begin with the *generalized Dehn–Sommerville equations*, given in the following theorem (see [Sjöberg and Ziegler 2018, Theorem 2.4] and [Bayer and Billera 1985, Theorem 2.1] for more details).

Theorem 2.4. *Let P be a d-polytope, and let S be a subset of* $\{0, 1, 2, \dots, d-1\}$. *If $\{i, k\}$ is a subset of $S \cup \{-1, d\}$ such that $i < k-1$ and such that there is no $j \in S$ for which $i < j < k$, then*

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S\cup\{j\}}(P) = f_S(P)\bigl(1 - (-1)^{k-i-1}\bigr).$$

Corollary 2.5. *The flag vector* 4*-tuple* $(f_{01}(P), f_{02}(P), f_{03}(P), f_{04}(P))$ *of a* 5*-polytope* P *satisfies*

$$f_{01}(P) - f_{02}(P) + f_{03}(P) - f_{04}(P) = 0. \tag{2-5}$$

Proof. Let $S = \{0\}$, $i = 0$, and $k = 5$. By applying Theorem 2.4 to these choices of S, i, and k, it is immediate to obtain (2-5). □

Lemma 2.6. *The flag vector* 3*-tuple* $(f_1(P), f_{02}(P), f_{04}(P))$ *of a* 5*-polytope* P *satisfies*

$$2f_1(P) - f_{02}(P) + f_{04}(P) \le 0.$$

Proof. As in the proof of Lemma 2.1, let F denote any facet of P. By [Sjöberg and Ziegler 2018, Theorem 2.2], we have

$$f_1(F) \ge 2f_0(F).$$

Thus, it is easy to obtain

$$f_{14}(P) = \sum_{F\subset P} f_1(F) \ge 2\sum_{F\subset P} f_0(F) = 2f_{04}(P). \tag{2-6}$$

By duality, it follows from (2-6) that

$$f_{03}(P) \ge 2f_{04}(P). \tag{2-7}$$

On the other hand, by Corollary 2.5 together with (2-6) we also have

$$\begin{aligned} f_{04}(P) &= f_{01}(P) - f_{02}(P) + f_{03}(P) \\ &\ge f_{01}(P) - f_{02}(P) + 2f_{04}(P). \end{aligned}$$

Since $2f_1(P) = f_{01}(P)$, finally we obtain

$$2f_1(P) - f_{02}(P) + f_{04}(P) \le 0,$$

as desired. □

Lemma 2.7. *The flag vector pair* $(f_0(P), f_{02}(P))$ *of a* 5*-polytope* P *satisfies*

$$f_{02}(P) \le 6(f_0(P)^2 - 6f_0(P) + 10).$$

Proof. As in the proof of Lemma 2.2, by applying the upper bound theorem stated in [Sjöberg and Ziegler 2018, Lemma 2.6] (see also [Billera and Björner 1997, Theorem 18.5.9]) we obtain

$$f_{02}(P) \le f_{02}(C_5(n)) = 3f_2(C_5(n)),$$

where $f_0(P) = n$ and the fact that $C_5(n)$ is a simplicial polytope was used in the last equality.

On the other hand, by using the formula of $f_2(C_5(n))$ given in [Buchstaber and Panov 2002, Lemma 1.34] it is straightforward to compute

$$\begin{aligned} f_2(C_5(n)) &= \sum_{q=0}^{2}\binom{q}{2}\binom{n+q-6}{q} + \sum_{p=0}^{2}\binom{5-p}{2}\binom{n+p-6}{p} \\ &= \binom{n-4}{2} + \binom{5}{2}\binom{n-6}{0} + \binom{4}{2}\binom{n-5}{1} + \binom{3}{2}\binom{n-4}{2} \\ &= 2(n^2-6n+10) = 2(f_0(P)^2 - 6f_0(P)+10). \end{aligned}$$

□

Lemma 2.8. *The flag vector pair $(f_1(P), f_{04}(P))$ of a 5-polytope P satisfies*

$$f_{04}(P) \le \tfrac{1}{25}(24 f_1(P)^2 - 410 f_1(P) + 1500).$$

Proof. By Lemma 2.6, it is easy to obtain

$$\begin{aligned} f_{04}(P) &\le -2f_1(P) + f_{02}(P) \\ &\le -2f_1(P) + 6(f_0(P)^2 - 6f_0(P) + 10) \\ &\le -2f_1(P) + 6\big(\tfrac{4}{25} f_1(P)^2 - \tfrac{12}{5} f_1(P) + 10\big) \\ &= \tfrac{1}{25}(24 f_1(P)^2 - 410 f_1(P) + 1500), \end{aligned}$$

where we used $f_0(P) \le \frac{2}{5} f_1(P)$ and $f_0(P) \ge 6$ in the third inequality. □

In fact, it turns out that for any values of $f_1(P) > 15$ the upper bound of $f_{04}(P)$ given in Lemma 2.8 can be improved further by using (1-2).

Lemma 2.9. *The flag vector pair $(f_1(P), f_{04}(P))$ of a 5-polytope satisfies*

$$f_{04}(P) \le \tfrac{4}{5} f_1(P)^2 - 14 f_1(P) + 60.$$

Proof. For the proof, note that by Lemma 2.3 we have

$$f_1(P) \ge \tfrac{5}{4}\big(7 + \sqrt{1 + \tfrac{4}{5} f_{04}(P)}\big).$$

Thus, it is easy to obtain

$$f_{04}(P) \le \tfrac{4}{5} f_1(P)^2 - 14 f_1(P) + 60.$$

□

For any 5-polytopes, $f_1(P) \ge 15$. Thus it is straightforward to show that

$$\tfrac{4}{5} f_1(P)^2 - 14 f_1(P) + 60 \le \tfrac{1}{25}(24 f_1(P)^2 - 410 f_1(P) + 1500),$$

where equality holds if and only if $f_1(P) = 15$.

Finally, we are in a position to give a proof of Theorem 1.4(2):

Theorem 2.10. *Given a flag number $f_1(P)$ of a 5-polytope P, $f_{04}(P)$ satisfies*

$$\tfrac{1}{2}(3 + \sqrt{9 + 16 f_1(P)}) < f_{04}(P) \le \tfrac{4}{5} f_1(P)^2 - 14 f_1(P) + 60.$$

Proof. By Lemma 2.9, it suffices to prove the first inequality. Indeed, recall from Lemma 2.1 that we have

$$4f_1(P) < f_{04}(P)(f_{04}(P)-3), \quad \text{i.e., } f_{04}(P)^2 - 3f_{04}(P) - 4f_1(P) > 0.$$

This immediately implies

$$f_{04}(P) > \tfrac{1}{2}(3+\sqrt{9+16f_1(P)}). \qquad \square$$

3. Some examples

The aim of this section is to provide some examples of 5-polytopes whose flag vector pairs (f_1, f_{04}) satisfy the inequalities (1-2) and (1-3) given in Theorem 1.4. In order to construct such examples, we use the well-known operations of stacking and truncating. In many instances, these operations turn out to be essential in finding new examples of polytopes for possible polytopal flag vector pairs.

To begin with, we have the following lemma.

Lemma 3.1. *Let P be a 5-polytope with at least one simple facet F, and let v be a point beyond F and beneath all other facets of P. Let Q be the 5-polytope obtained by stacking the vertex v over P; i.e., let Q be the convex hull of v and P. Then we have the identities*

$$f_0(Q) = f_0(P)+1,$$
$$f_1(Q) = f_1(P)+5,$$
$$f_{04}(Q) = f_{04}(P)+20.$$

Proof. By the way of the construction of Q, it suffices to show the last identity. To see it, note first that F is a 4-simplex with five vertices. If we apply the stacking operation to P with such a vertex v over F, then it is easy to see that the flag number f_{04} increases by $5\binom{5}{4}$ and decreases by 5. Thus the net change of f_{04} is equal to 20, and so we have

$$f_{04}(Q) = f_{04}(P)+20. \qquad \square$$

Let P be a d-polytope with a vertex v, and let H be a hyperplane intersecting the interior of P such that on one side of H the only vertex of P is v. Then we can obtain a new polytope Q by cutting off the side of H that contains v. This operation of obtaining a new polytope is called a *truncating at a vertex*.

The following lemma holds.

Lemma 3.2. *Let P be a 5-polytope with at least one simple vertex v, and let R be the 5-polytope obtained by truncating the vertex v from P. Then we have the identities*

$$
\begin{aligned}
f_0(R) &= f_0(P) + 4,\\
f_1(R) &= f_1(P) + 10,\\
f_{04}(R) &= f_{04}(P) + 20.
\end{aligned}
$$

Proof. By the way of the construction of R, once again it suffices to prove the last equality. Note first that by the truncating operation we have five new vertices, all of which are simple. Thus the flag number f_{04} increases by 5×5 and decreases by 5 coming from the old vertex v. This implies $f_{04}(R) = f_{04}(P) + 20$, as required. □

Note that the polytopes obtained through stacking over a simple vertex v and truncating at v all have a simple vertex and a simplex facet. Thus we can repeatedly stack vertices on simplex facets and truncate simple vertices.

With these understood, let P be a 5-polytope P with a 4-simplex facet and a simple vertex. By truncating simple vertices l times and stacking vertices on 4-simplex facets k times inductively, we can obtain a new 5-polytope Q with the flag vector pair

$$(f_1(Q), f_{04}(Q)) = (f_1(P) + 5k + 10l,\ f_{04}(P) + 20k + 20l), \quad k, l \geq 0. \tag{3-1}$$

Let $n = k + l$. Then it follows from (3-1) that

$$(f_1(Q), f_{04}(Q)) = (f_1(P) + 10n - 5k,\ f_{04}(P) + 20n), \quad n \geq 0,\ 0 \leq k \leq n. \tag{3-2}$$

As a special case, let P be a 5-simplex. Then the flag vector pair $(f_1(P), f_{04}(P))$ is equal to $(15, 30)$. Thus, by (3-2) we can obtain the flag vector pair

$$(f_1(Q), f_{04}(Q)) = (10n - 5k + 15,\ 20n + 30), \quad n \geq 0,\ 0 \leq k \leq n.$$

One may check directly that the flag vector pair $(f_1(Q), f_{04}(Q))$ satisfies the inequalities (1-2) and (1-3) given in Theorem 1.4.

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