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On weight-one solvable configurations of the Lights Out puzzle

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We show that the center-one configuration is always solvable in the Lights Out puzzle on a square grid with odd vertices.

1. Introduction

Let $\Gamma = (V, E)$ be a finite undirected simple graph, $n = \#V$ the number of vertices, and $\mathscr{F}$ the set of functions on V with values in $\mathbb{F}_2$, the field with two elements. We define the Laplacian $\Delta : \mathscr{F} \to \mathscr{F}$ by

$$(\Delta f)(v) := f(v) + \sum_{(v,w)\in E} f(w)$$

for $f \in \mathscr{F}$, $v \in V$. Let $\boldsymbol{e}_v$ denote the characteristic function of $v \in V$. Then $\{\boldsymbol{e}_v : v \in V\}$ is a basis of $\mathscr{F}$ as a vector space over $\mathbb{F}_2$, and by means of this basis we identify $\mathscr{F}$ with $\mathbb{F}_2^n$. Under this identification, Δ is a linear map represented by $I_n + \operatorname{adj}(\Gamma)$, where I_n denotes the identity matrix of degree n and $\operatorname{adj}(\Gamma)$ the adjacency matrix of Γ. Let the image and the kernel of Δ be denoted by $\mathscr{C}$ and $\mathscr{H}$, respectively. $\mathscr{C}$ is the set of solvable configurations of the Lights Out puzzle on Γ; see [Fleischer and Yu 2013; Goldwasser and Klostermeyer 1997; Goshima and Yamagishi 2010]. It is known that the all-one configuration is always solvable:

Theorem 1.1 [Sutner 1989]. *For any* Γ, *it holds that* $(1\ 1\ \cdots\ 1) \in \mathscr{C}$.

Since $\mathscr{C}$ is a linear subspace of $\mathbb{F}_2^n$, we may regard it as a binary linear code; see [Goldwasser and Klostermeyer 1997] for this point of view. The weight enumerator of $\mathscr{C}$ is defined by

$$W_{\mathscr{C}}(x, y) = \sum_{i=0}^{n} A_i x^{n-i} y^i,$$

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where A_i is the number of vectors in $\mathscr{C}$ which have Hamming weight i. By Sutner's theorem, we have $A_{n-i} = A_i$. If Δ is bijective, then $\mathscr{C} = \mathbb{F}_2^n$ and we have

$$A_i = \binom{n}{i}, \quad W_{\mathscr{C}}(x, y) = (x+y)^n.$$

In this paper, we are interested in A_1 of the classical $n \times n$ Lights Out puzzle. Our main result is Theorem 3.1, which states that the center-one configuration is always solvable when n is odd. Our proof is a neat application of Sutner's theorem and is not constructive. Theorem 3.1 implies in particular that the minimal distance of $\mathscr{C}$ is 1 when n is odd. For even n, it turns out that the minimal distance is at most 2.

We then look at the case $A_1 \le 1$ more closely, and make some conjectures based on numerical computations. We also make an attempt to "explain" the value of A_1.

2. Path and cycle graphs

Before proceeding to the main result, we consider the case of path and cycle graphs as first examples.

Let $\Gamma = \boldsymbol{P}_n$ be the path graph with n vertices. We have

$$\operatorname{adj}(\Gamma) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

under an obvious ordering of vertices. It is well known, see [Yamagishi 2015, Lemma 3.1], that the characteristic polynomial of $\operatorname{adj}(\Gamma)$ is $S_n(x)$, the n-th Chebyshev polynomial of the second kind, defined by

$$S_0(x) = 1, \quad S_1(x) = x, \quad S_n(x) = x S_{n-1}(x) - S_{n-2}(x) \quad (n \ge 2).$$

So we see that Δ is bijective if and only if $S_n(-1) \not\equiv 0 \pmod 2$ if and only if $n \not\equiv 2 \pmod 3$.

In the case $n \equiv 2 \pmod 3$, it is easy to see that $\mathscr{H}$ is one-dimensional, spanned by the vector

$$(1\ 1\ 0\ 1\ 1\ 0\ \cdots\ 0\ 1\ 1), \tag{2-1}$$

so that

$$W_{\mathscr{H}}(x, y) = x^n + x^{(n-2)/3} y^{(2n+2)/3}.$$

Since $\mathscr{C} = \mathscr{H}^{\perp}$, we have

$$W_{\mathscr{C}}(x, y) = \tfrac{1}{2}((x+y)^n + (x+y)^{(n-2)/3}(x-y)^{(2n+2)/3}) \tag{2-2}$$

by the MacWilliams identity [MacWilliams and Sloane 1977, p. 127]. In particular, expanding (2-2), we find that

$$A_1 = \tfrac{1}{3}(n-2), \quad A_2 = \tfrac{1}{18}(5n^2 - 5n + 8).$$

Note that A_1 and A_2 can be seen more quickly as follows. In the general setting, we have $\mathscr{C} = \mathscr{H}^{\perp}$ since $\mathrm{adj}(\Gamma)$ is a symmetric matrix. Suppose $\dim \mathscr{C} = k < n$, so that $\dim \mathscr{H} = n - k > 0$. Any basis of $\mathscr{H}$ gives a parity check matrix H (of size $(n-k) \times n$) of $\mathscr{C}$, and A_i is the number of unordered i-tuples of columns of H whose sum is the zero vector. In the case $\Gamma = \boldsymbol{P}_n$, $n \equiv 2 \pmod 3$, the vector (2-1) itself is a parity check matrix, and one easily sees that

$$A_1 = \tfrac{1}{3}(n-2), \quad A_2 = \binom{\frac{1}{3}(n-2)}{2} + \binom{\frac{1}{3}(2n+2)}{2}.$$

Next let $\Gamma = \boldsymbol{C}_n$ be the cycle graph with n vertices ($n \geq 3$). It is also well known, see [Yamagishi 2015, Lemma 3.1], that Δ is bijective if and only if $C_n(-1) \not\equiv 0 \pmod 2$ if and only if $n \not\equiv 0 \pmod 3$, where $C_n(x)$ is the n-th Chebyshev polynomial of the first kind, defined by

$$C_0(x) = 2, \quad C_1(x) = x, \quad C_n(x) = xC_{n-1}(x) - C_{n-2}(x) \quad (n \geq 2).$$

In the case $n \equiv 0 \pmod 3$, it is easy to see that $\mathscr{H}$ is two-dimensional, spanned by the row vectors of

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & \cdots & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & \cdots & 1 & 0 & 1 \end{pmatrix}, \tag{2-3}$$

so that

$$W_{\mathscr{H}}(x, y) = x^n + 3x^{n/3}y^{2n/3},$$
$$W_{\mathscr{C}}(x, y) = \tfrac{1}{4}((x+y)^n + 3(x+y)^{n/3}(x-y)^{2n/3}).$$

In particular, we obtain

$$A_1 = 0, \quad A_2 = \tfrac{1}{6}(n^2 - 3n).$$

As explained above, A_1 and A_2 can be seen directly from (2-3). This is clear for A_1. Since i-th and j-th columns add to zero if and only if $i \equiv j \pmod 3$, we see that $A_2 = \frac{1}{2}n\left(\frac{1}{3}n - 1\right)$. We also have an alternative proof for $A_1 = 0$ as follows. Suppose there is a vector in $\mathscr{C}$ with Hamming weight 1. Then any vector with Hamming weight 1 belongs to $\mathscr{C}$ since Δ commutes with "shifts". This implies $\mathscr{C} = \mathbb{F}_2^n$, which contradicts $n \equiv 0 \pmod 3$.

3. The main theorem

In the following, we let Γ be the Cartesian product $\boldsymbol{P}_n \times \boldsymbol{P}_n$, forgetting the previous meaning of n as the number of vertices. The corresponding objects $V, \mathscr{F}, \Delta, \mathscr{C}, \mathscr{H}$, and A_i will be denoted by V_n, $\mathscr{F}_n$, Δ_n, $\mathscr{C}_n$, $\mathscr{H}_n$, and $A_i(n)$, respectively. We use double indices for the vertices in a natural way:

$$V_n = \{v_{i,j} : 1 \le i, j \le n\},$$
$$v_{i,j} \text{ and } v_{k,l} \text{ are adjacent} \quad \Longleftrightarrow \quad |i-k|+|j-l| = 1.$$

Let $\boldsymbol{e}_{i,j}$ denote the characteristic function of $v_{i,j}$.

The main result of this paper is the following, which states that the center-one configuration is always solvable in the Lights Out puzzle on $\boldsymbol{P}_n \times \boldsymbol{P}_n$ when n is odd.

Theorem 3.1. *If $n = 2m+1$ ($m \ge 0$), then $\boldsymbol{e}_{m+1,m+1} \in \mathscr{C}_n$.*

Proof. The case $m = 0$ is trivial since Δ_1 is the identity map, so we suppose $m \ge 1$. We identify a function $f \in \mathscr{F}_n$ with the matrix $(a_{i,j})$ such that

$$f = \sum_{1 \le i,j \le n} a_{i,j}\boldsymbol{e}_{i,j} \quad (a_{i,j} \in \mathbb{F}_2).$$

Let $\mathbf{1}_{a,b}$ denote the $a \times b$ matrix whose entries are all 1, and $\mathbf{0}$ the zero matrix whose size will be clear from the context. Sutner's theorem states that $\mathbf{1}_{n,n} \in \mathscr{C}_n$. Applying Sutner's theorem to $\boldsymbol{P}_m \times \boldsymbol{P}_m$, we see that

$$f_1 := \begin{pmatrix} \mathbf{1}_{m,m} & \boldsymbol{x} & \mathbf{0} \\ \boldsymbol{y} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathscr{C}_n$$

for a suitable column vector $\boldsymbol{x}$ and a row vector $\boldsymbol{y}$. Since $\mathscr{C}_n$ is invariant under horizontal reflection, say α, and vertical reflection, say β, we find that

$$f_2 := \begin{pmatrix} \mathbf{1}_{m,m} & \mathbf{0} & \mathbf{1}_{m,m} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1}_{m,m} & \mathbf{0} & \mathbf{1}_{m,m} \end{pmatrix} = f_1 + \alpha(f_1) + \beta(f_1) + \alpha\beta(f_1) \in \mathscr{C}_n.$$

Similarly, we have

$$f_3 := (\mathbf{1}_{n,m} \;\; \boldsymbol{z} \;\; \mathbf{0}) \in \mathscr{C}_n$$

for a suitable column vector $\boldsymbol{z}$, so that

$$f_4 := (\mathbf{1}_{n,m} \;\; \mathbf{0} \;\; \mathbf{1}_{n,m}) = f_3 + \alpha(f_3) \in \mathscr{C}_n,$$

and likewise,

$$f_5 := \begin{pmatrix} \mathbf{1}_{m,n} \\ \mathbf{0} \\ \mathbf{1}_{m,n} \end{pmatrix} \in \mathscr{C}_n.$$

Therefore we have

$$\boldsymbol{e}_{m+1,m+1} = f_2 + f_4 + f_5 + \mathbf{1}_{n,n} \in \mathscr{C}_n$$

as desired. $\square$

Remark 3.2. Our proof is not constructive; in the context of Lights Out puzzle, we only know that $\boldsymbol{e}_{m+1,m+1}$ is solvable, but do not know any solution (an inverse image of $\boldsymbol{e}_{m+1,m+1}$ under Δ_n). It would be interesting to find out a unified description of a solution of $\boldsymbol{e}_{m+1,m+1}$.

Remark 3.3. The center-one configuration is the only universal solvable configuration of weight 1, since $A_1(n) = 1$ for some (infinitely many, under Conjecture 4.4 below) odd integers n.

Since $A_1(n)$ is the number of $\boldsymbol{e}_{i,j}$'s contained in $\mathscr{C}_n$, taking symmetry (i.e., invariance of $\mathscr{C}_n$ under the horizontal and vertical reflections) into account, we have:

Corollary 3.4. *$A_1(n) \equiv 1 \pmod 4$ if n is odd. $A_1(n) \equiv 0 \pmod 4$ if n is even.*

Let d_n denote the minimal distance of the linear code $\mathscr{C}_n$. By Theorem 3.1, we have $d_n = 1$ for odd n. We see that $d_n \leq 2$ in general by the following:

Lemma 3.5. *For $n \geq 4$, we have $\boldsymbol{e}_{1,4} + \boldsymbol{e}_{3,2} \in \mathscr{C}_n$.*

Proof. We have $\boldsymbol{e}_{1,4} + \boldsymbol{e}_{3,2} = \Delta_n(\boldsymbol{e}_{1,1} + \boldsymbol{e}_{1,2} + \boldsymbol{e}_{1,3} + \boldsymbol{e}_{2,2}) \in \mathscr{C}_n$. □

Note that $d_2 = 1$ since Δ_2 is bijective. Thus the determination of d_n is equivalent to answering the following:

Problem 3.6. Characterize (necessarily even) n such that $A_1(n) = 0$.

4. The case $A_1(n) \leq 1$

With the same notation as in the previous section, we consider the case $A_1(n) \leq 1$.

A first look at Table 1 leads to the following two conjectures.

Conjecture 4.1. *If $A_1(n) = 0$, then $n + 1 = 2^l \pm 1$ for some $l \geq 2$.*

Conjecture 4.2. *Let $n \geq 2$. We have $A_1(n) \leq 1$ if and only if $A_1(2n + 1) \leq 1$.*

The "if" part of Conjecture 4.2 follows from:

Proposition 4.3. *We have $A_i(n) \leq A_i(2n + 1)$ for $n \geq 1$ and $0 \leq i \leq n$.*

Proof. We define a map $\iota_n : \mathscr{F}_n \to \mathscr{F}_{2n+1}$ by

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_{1,1} & 0 & a_{1,2} & \cdots & a_{1,n} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_{2,1} & 0 & a_{2,2} & \cdots & a_{2,n} & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n,1} & 0 & a_{n,2} & \cdots & a_{n,n} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

which is an analog of $\iota^{\pm}_{m,n}$ used in [Goshima and Yamagishi 2010] for $\boldsymbol{C}_m \times \boldsymbol{C}_n$. One can then verify the identity $\iota_n \Delta_n = \Delta^2_{2n+1}\iota_n$, so it follows that $\iota_n(\mathscr{C}_n) \subset \mathscr{C}_{2n+1}$. Since ι_n preserves the Hamming weight, we have $A_i(n) \le A_i(2n+1)$ for $0 \le i \le n$. $\square$

n	$A_1(n)$	$\dim \mathscr{H}_n$	n	$A_1(n)$	$\dim \mathscr{H}_n$	n	$A_1(n)$	$\dim \mathscr{H}_n$	n	$A_1(n)$	$\dim \mathscr{H}_n$
1	1	0	41	701	2	81	6561	0	121	14641	0
2	4	0	42	1764	0	82	6724	0	122	14884	0
3	9	0	43	1849	0	83	1401	6	123	1	80
4	0	4	44	640	4	84	128	12	124	5376	4
5	5	2	45	2025	0	85	7225	0	125	1	50
6	36	0	46	2116	0	86	7396	0	126	0	56
7	49	0	47	9	30	87	7569	0	127	16129	0
8	64	0	48	2304	0	88	7744	0	128	0	56
9	1	8	49	401	8	89	829	10	129	1	56
10	100	0	50	196	8	90	8100	0	130	16900	0
11	9	6	51	2601	0	91	8281	0	131	1	86
12	144	0	52	2704	0	92	364	20	132	17424	0
13	169	0	53	1189	2	93	8649	0	133	17689	0
14	52	4	54	980	4	94	3060	4	134	6292	4
15	225	0	55	3025	0	95	9	62	135	1	64
16	0	8	56	3136	0	96	9216	0	136	18496	0
17	109	2	57	3249	0	97	9409	0	137	8189	2
18	324	0	58	3364	0	98	388	20	138	19044	0
19	1	16	59	53	22	99	801	16	139	1681	16
20	400	0	60	3600	0	100	10000	0	140	19600	0
21	441	0	61	1	40	101	197	18	141	19881	0
22	484	0	62	0	24	102	10404	0	142	20164	0
23	9	14	63	3969	0	103	10609	0	143	649	30
24	176	4	64	0	28	104	3760	4	144	7280	4
25	625	0	65	1	42	105	11025	0	145	21025	0
26	676	0	66	4356	0	106	11236	0	146	21316	0
27	729	0	67	1	32	107	2377	6	147	21609	0
28	784	0	68	4624	0	108	11664	0	148	21904	0
29	53	10	69	841	8	109	2201	8	149	2501	10
30	0	20	70	4900	0	110	12100	0	150	22500	0
31	961	0	71	361	14	111	12321	0	151	22801	0
32	0	20	72	5184	0	112	12544	0	152	2368	8
33	1	16	73	5329	0	113	5549	2	153	23409	0
34	372	4	74	1876	4	114	4532	4	154	240	24
35	217	6	75	5625	0	115	13225	0	155	5097	6
36	1296	0	76	5776	0	116	13456	0	156	24336	0
37	1369	0	77	2549	2	117	13689	0	157	24649	0
38	1444	0	78	6084	0	118	1380	8	158	24964	0
39	1	32	79	1	64	119	53	46	159	1	128
40	1600	0	80	6400	0	120	14400	0	160	25600	0

Table 1

Applying Conjecture 4.2 repeatedly and using Corollary 3.4, we easily arrive at the following:

Conjecture 4.4. *Let $n \geq 3$ be odd and let d be the maximal odd divisor of $n+1$. Then we have $A_1(n) = 1$ if and only if $d > 1$ and $A_1(d-1) = 0$.*

Proposition 4.5. *Conjectures 4.2 and 4.4 are equivalent.*

Proof. It suffices to show the implication Conjecture 4.4 $\Rightarrow$ Conjecture 4.2. Let $n \geq 2$ and let d be the maximal odd divisor of $n+1$ (and hence of $2n+2$). By Corollary 3.4, $A_1(2n+1) \leq 1$ is equivalent to $A_1(2n+1) = 1$, which, in turn, is equivalent to $d > 1$ and $A_1(d-1) = 0$ by Conjecture 4.4. If n is odd, then the same reasoning shows $A_1(n) \leq 1 \Longleftrightarrow d > 1$ and $A_1(d-1) = 0$, so we are done. If n is even, then $d = n+1 > 1$ and we have $A_1(n) \leq 1 \Longleftrightarrow A_1(d-1) = 0$ by Corollary 3.4. □

Next we make an attempt to "explain" the value of $A_1(n)$. If the Laplacian Δ_n is bijective, then we have $\mathscr{C}_n = \mathbb{F}_2^{n^2}$ and hence $A_1(n) = n^2$. We comment here on the bijectivity of Δ_n. Sutner [2000] proved

$$\dim \mathscr{H}_n = \deg \gcd(S_n(x), S_n(x+1)),$$

where S_n is the n-th Chebyshev polynomial of the second kind, regarded as a polynomial over $\mathbb{F}_2$. Some sufficient conditions for the bijectivity of Δ_n follow from this identity and well-known properties of Chebyshev polynomials. For example, $n = 2^l - 1$ ($l \geq 1$) is sufficient [Yamagishi 2015, Corollary 4.3]. Note that this confirms Conjecture 4.4 for $n = 2^l - 1$, as $A_1(n) = n^2$ and $d = 1$. There seems to be no simple characterization of n for which Δ_n is bijective.

Now we consider the case where Δ_n is not bijective, i.e., $\dim \mathscr{H}_n > 0$. As in Conjecture 4.4, the divisors d of $n+1$ with $A_1(d-1) = 0$ play an important role in the following two conjectures.

Conjecture 4.6. *Let n be even. Then Δ_n is not bijective if and only if there exists a (necessarily odd) divisor $d > 1$ of $n+1$ such that $A_1(d-1) = 0$.*

Conjecture 4.7. *Suppose n is even and Δ_n is not bijective. Assume Conjecture 4.6, and let d_k ($1 \leq k \leq t$) be the divisors of $n+1$ such that $d_k > 1$ and $A_1(d_k - 1) = 0$. Then for $1 \leq i, j \leq n$, we have $\boldsymbol{e}_{i,j} \in \mathscr{C}_n$ if and only if*

$$i \equiv 0 \pmod{d_k} \quad \text{or} \quad j \equiv 0 \pmod{d_k} \tag{4-1}$$

for $k = 1, 2, \ldots, t$.

Example 4.8. If $A_1(n) = 0$, then we can take $d_1 = n+1$ and Conjecture 4.7 is trivially true. But this gives no explanation of why $A_1(n) = 0$. We exclude this case in the following examples.

Example 4.9. Suppose $t = 1$ and put $b = (n+1)/d_1$. The number of pairs (i, j) for which (4-1) with $k = 1$ fails is $(n-b+1)^2$, so we have $A_1(n) = n^2 - (n-b+1)^2$.

This applies for $n = 14, 24, 34, 44, 54, 74, 94, 104, 114, 124, 134, 144$ ($d_1 = 5$), $n = 50, 118, 152$ ($d_1 = 17$), $n = 92$ ($d_1 = 31$), and $n = 98$ ($d_1 = 33$).

Example 4.10. For $n = 84$, we have $t = 2$, $d_1 = 5$, $d_2 = 17$, and (4-1) for $k = 1, 2$ reads as $ij \equiv 0 \pmod{85}$. Thus we have $A_1(84) = 2(5-1)(17-1) = 128$. The same reasoning applies for $n = 154$: $t = 2$, $d_1 = 5$, $d_2 = 31$ and $A_1(154) = 2(5-1)(31-1) = 240$.

Finally, we note that an answer to Problem 3.6 would give, under Conjecture 4.4, a characterization of (necessarily odd) n with $A_1(n) = 1$, and, under Conjecture 4.6, a characterization of even n with nonbijective Δ_n.

We also point out that, in Table 1, there are four exceptions $n = 2, 6, 8, 14$ for the converse statement of Conjecture 4.1. Problem 3.6 would be settled if they are the only exceptions.

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