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of the Lights Out puzzle

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We show that the center-one configuration is always solvable in the Lights Out puzzle on a square grid with odd vertices.

1. Introduction

Let $\Gamma = (V, E)$ be a finite undirected simple graph, $n = \#V$ the number of vertices, and \mathcal{F} the set of functions on V with values in \mathbb{F}_2 , the field with two elements. We define the Laplacian $\Delta : \mathcal{F} \rightarrow \mathcal{F}$ by

$$(\Delta f)(v) := f(v) + \sum_{(v,w) \in E} f(w)$$

for $f \in \mathcal{F}$, $v \in V$. Let e_v denote the characteristic function of $v \in V$. Then $\{e_v : v \in V\}$ is a basis of \mathcal{F} as a vector space over \mathbb{F}_2 , and by means of this basis we identify \mathcal{F} with \mathbb{F}_2^n . Under this identification, Δ is a linear map represented by $I_n + \text{adj}(\Gamma)$, where I_n denotes the identity matrix of degree n and $\text{adj}(\Gamma)$ the adjacency matrix of Γ . Let the image and the kernel of Δ be denoted by \mathcal{C} and \mathcal{H} , respectively. \mathcal{C} is the set of solvable configurations of the Lights Out puzzle on Γ ; see [Fleischer and Yu 2013; Goldwasser and Klostermeyer 1997; Goshima and Yamagishi 2010]. It is known that the all-one configuration is always solvable:

Theorem 1.1 [Sutner 1989]. *For any Γ , it holds that $(1 \ 1 \ \cdots \ 1) \in \mathcal{C}$.*

Since \mathcal{C} is a linear subspace of \mathbb{F}_2^n , we may regard it as a binary linear code; see [Goldwasser and Klostermeyer 1997] for this point of view. The weight enumerator of \mathcal{C} is defined by

$$W_{\mathcal{C}}(x, y) = \sum_{i=0}^n A_i x^{n-i} y^i,$$

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where A_i is the number of vectors in \mathcal{C} which have Hamming weight i . By Sutner’s theorem, we have $A_{n-i} = A_i$. If Δ is bijective, then $\mathcal{C} = \mathbb{F}_2^n$ and we have

$$A_i = \binom{n}{i}, \quad W_{\mathcal{C}}(x, y) = (x + y)^n.$$

In this paper, we are interested in A_1 of the classical $n \times n$ Lights Out puzzle. Our main result is [Theorem 3.1](#), which states that the center-one configuration is always solvable when n is odd. Our proof is a neat application of Sutner’s theorem and is not constructive. [Theorem 3.1](#) implies in particular that the minimal distance of \mathcal{C} is 1 when n is odd. For even n , it turns out that the minimal distance is at most 2.

We then look at the case $A_1 \leq 1$ more closely, and make some conjectures based on numerical computations. We also make an attempt to “explain” the value of A_1 .

2. Path and cycle graphs

Before proceeding to the main result, we consider the case of path and cycle graphs as first examples.

Let $\Gamma = P_n$ be the path graph with n vertices. We have

$$\text{adj}(\Gamma) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

under an obvious ordering of vertices. It is well known, see [\[Yamagishi 2015, Lemma 3.1\]](#), that the characteristic polynomial of $\text{adj}(\Gamma)$ is $S_n(x)$, the n -th Chebyshev polynomial of the second kind, defined by

$$S_0(x) = 1, \quad S_1(x) = x, \quad S_n(x) = x S_{n-1}(x) - S_{n-2}(x) \quad (n \geq 2).$$

So we see that Δ is bijective if and only if $S_n(-1) \not\equiv 0 \pmod{2}$ if and only if $n \not\equiv 2 \pmod{3}$.

In the case $n \equiv 2 \pmod{3}$, it is easy to see that \mathcal{H} is one-dimensional, spanned by the vector

$$(1 \ 1 \ 0 \ 1 \ 1 \ 0 \ \cdots \ 0 \ 1 \ 1), \tag{2-1}$$

so that

$$W_{\mathcal{H}}(x, y) = x^n + x^{(n-2)/3} y^{(2n+2)/3}.$$

Since $\mathcal{C} = \mathcal{H}^\perp$, we have

$$W_{\mathcal{C}}(x, y) = \frac{1}{2}((x + y)^n + (x + y)^{(n-2)/3} (x - y)^{(2n+2)/3}) \tag{2-2}$$

by the MacWilliams identity [MacWilliams and Sloane 1977, p. 127]. In particular, expanding (2-2), we find that

$$A_1 = \frac{1}{3}(n-2), \quad A_2 = \frac{1}{18}(5n^2 - 5n + 8).$$

Note that A_1 and A_2 can be seen more quickly as follows. In the general setting, we have $\mathcal{C} = \mathcal{H}^\perp$ since $\text{adj}(\Gamma)$ is a symmetric matrix. Suppose $\dim \mathcal{C} = k < n$, so that $\dim \mathcal{H} = n - k > 0$. Any basis of \mathcal{H} gives a parity check matrix H (of size $(n - k) \times n$) of \mathcal{C} , and A_i is the number of unordered i -tuples of columns of H whose sum is the zero vector. In the case $\Gamma = \mathbf{P}_n$, $n \equiv 2 \pmod{3}$, the vector (2-1) itself is a parity check matrix, and one easily sees that

$$A_1 = \frac{1}{3}(n-2), \quad A_2 = \binom{\frac{1}{3}(n-2)}{2} + \binom{\frac{1}{3}(2n+2)}{2}.$$

Next let $\Gamma = \mathbf{C}_n$ be the cycle graph with n vertices ($n \geq 3$). It is also well known, see [Yamagishi 2015, Lemma 3.1], that Δ is bijective if and only if $\mathbf{C}_n(-1) \not\equiv 0 \pmod{2}$ if and only if $n \not\equiv 0 \pmod{3}$, where $\mathbf{C}_n(x)$ is the n -th Chebyshev polynomial of the first kind, defined by

$$\mathbf{C}_0(x) = 2, \quad \mathbf{C}_1(x) = x, \quad \mathbf{C}_n(x) = x\mathbf{C}_{n-1}(x) - \mathbf{C}_{n-2}(x) \quad (n \geq 2).$$

In the case $n \equiv 0 \pmod{3}$, it is easy to see that \mathcal{H} is two-dimensional, spanned by the row vectors of

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & \cdots & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & \cdots & 1 & 0 & 1 \end{pmatrix}, \quad (2-3)$$

so that

$$\begin{aligned} W_{\mathcal{H}}(x, y) &= x^n + 3x^{n/3}y^{2n/3}, \\ W_{\mathcal{C}}(x, y) &= \frac{1}{4}((x+y)^n + 3(x+y)^{n/3}(x-y)^{2n/3}). \end{aligned}$$

In particular, we obtain

$$A_1 = 0, \quad A_2 = \frac{1}{6}(n^2 - 3n).$$

As explained above, A_1 and A_2 can be seen directly from (2-3). This is clear for A_1 . Since i -th and j -th columns add to zero if and only if $i \equiv j \pmod{3}$, we see that $A_2 = \frac{1}{2}n(\frac{1}{3}n - 1)$. We also have an alternative proof for $A_1 = 0$ as follows. Suppose there is a vector in \mathcal{C} with Hamming weight 1. Then any vector with Hamming weight 1 belongs to \mathcal{C} since Δ commutes with “shifts”. This implies $\mathcal{C} = \mathbb{F}_2^n$, which contradicts $n \equiv 0 \pmod{3}$.

3. The main theorem

In the following, we let Γ be the Cartesian product $\mathbf{P}_n \times \mathbf{P}_n$, forgetting the previous meaning of n as the number of vertices. The corresponding objects V , \mathcal{F} , Δ , \mathcal{C} , \mathcal{H} ,

and A_i will be denoted by V_n , \mathcal{F}_n , Δ_n , \mathcal{C}_n , \mathcal{H}_n , and $A_i(n)$, respectively. We use double indices for the vertices in a natural way:

$$V_n = \{v_{i,j} : 1 \leq i, j \leq n\},$$

$$v_{i,j} \text{ and } v_{k,l} \text{ are adjacent} \iff |i - k| + |j - l| = 1.$$

Let $\mathbf{e}_{i,j}$ denote the characteristic function of $v_{i,j}$.

The main result of this paper is the following, which states that the center-one configuration is always solvable in the Lights Out puzzle on $\mathbf{P}_n \times \mathbf{P}_n$ when n is odd.

Theorem 3.1. *If $n = 2m + 1$ ($m \geq 0$), then $\mathbf{e}_{m+1,m+1} \in \mathcal{C}_n$.*

Proof. The case $m = 0$ is trivial since Δ_1 is the identity map, so we suppose $m \geq 1$. We identify a function $f \in \mathcal{F}_n$ with the matrix $(a_{i,j})$ such that

$$f = \sum_{1 \leq i, j \leq n} a_{i,j} \mathbf{e}_{i,j} \quad (a_{i,j} \in \mathbb{F}_2).$$

Let $\mathbf{1}_{a,b}$ denote the $a \times b$ matrix whose entries are all 1, and $\mathbf{0}$ the zero matrix whose size will be clear from the context. Sutner's theorem states that $\mathbf{1}_{n,n} \in \mathcal{C}_n$. Applying Sutner's theorem to $\mathbf{P}_m \times \mathbf{P}_m$, we see that

$$f_1 := \begin{pmatrix} \mathbf{1}_{m,m} & \mathbf{x} & \mathbf{0} \\ \mathbf{y} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathcal{C}_n$$

for a suitable column vector \mathbf{x} and a row vector \mathbf{y} . Since \mathcal{C}_n is invariant under horizontal reflection, say α , and vertical reflection, say β , we find that

$$f_2 := \begin{pmatrix} \mathbf{1}_{m,m} & \mathbf{0} & \mathbf{1}_{m,m} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1}_{m,m} & \mathbf{0} & \mathbf{1}_{m,m} \end{pmatrix} = f_1 + \alpha(f_1) + \beta(f_1) + \alpha\beta(f_1) \in \mathcal{C}_n.$$

Similarly, we have

$$f_3 := (\mathbf{1}_{n,m} \ \mathbf{z} \ \mathbf{0}) \in \mathcal{C}_n$$

for a suitable column vector \mathbf{z} , so that

$$f_4 := (\mathbf{1}_{n,m} \ \mathbf{0} \ \mathbf{1}_{n,m}) = f_3 + \alpha(f_3) \in \mathcal{C}_n,$$

and likewise,

$$f_5 := \begin{pmatrix} \mathbf{1}_{m,n} \\ \mathbf{0} \\ \mathbf{1}_{m,n} \end{pmatrix} \in \mathcal{C}_n.$$

Therefore we have

$$\mathbf{e}_{m+1,m+1} = f_2 + f_4 + f_5 + \mathbf{1}_{n,n} \in \mathcal{C}_n$$

as desired. □

Remark 3.2. Our proof is not constructive; in the context of Lights Out puzzle, we only know that $e_{m+1,m+1}$ is solvable, but do not know any solution (an inverse image of $e_{m+1,m+1}$ under Δ_n). It would be interesting to find out a unified description of a solution of $e_{m+1,m+1}$.

Remark 3.3. The center-one configuration is the only universal solvable configuration of weight 1, since $A_1(n) = 1$ for some (infinitely many, under [Conjecture 4.4](#) below) odd integers n .

Since $A_1(n)$ is the number of $e_{i,j}$'s contained in \mathcal{C}_n , taking symmetry (i.e., invariance of \mathcal{C}_n under the horizontal and vertical reflections) into account, we have:

Corollary 3.4. $A_1(n) \equiv 1 \pmod{4}$ if n is odd. $A_1(n) \equiv 0 \pmod{4}$ if n is even.

Let d_n denote the minimal distance of the linear code \mathcal{C}_n . By [Theorem 3.1](#), we have $d_n = 1$ for odd n . We see that $d_n \leq 2$ in general by the following:

Lemma 3.5. For $n \geq 4$, we have $e_{1,4} + e_{3,2} \in \mathcal{C}_n$.

Proof. We have $e_{1,4} + e_{3,2} = \Delta_n(e_{1,1} + e_{1,2} + e_{1,3} + e_{2,2}) \in \mathcal{C}_n$. □

Note that $d_2 = 1$ since Δ_2 is bijective. Thus the determination of d_n is equivalent to answering the following:

Problem 3.6. Characterize (necessarily even) n such that $A_1(n) = 0$.

4. The case $A_1(n) \leq 1$

With the same notation as in the previous section, we consider the case $A_1(n) \leq 1$.

A first look at [Table 1](#) leads to the following two conjectures.

Conjecture 4.1. If $A_1(n) = 0$, then $n + 1 = 2^l \pm 1$ for some $l \geq 2$.

Conjecture 4.2. Let $n \geq 2$. We have $A_1(n) \leq 1$ if and only if $A_1(2n + 1) \leq 1$.

The “if” part of [Conjecture 4.2](#) follows from:

Proposition 4.3. We have $A_i(n) \leq A_i(2n + 1)$ for $n \geq 1$ and $0 \leq i \leq n$.

Proof. We define a map $\iota_n : \mathcal{F}_n \rightarrow \mathcal{F}_{2n+1}$ by

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_{1,1} & 0 & a_{1,2} & \cdots & a_{1,n} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_{2,1} & 0 & a_{2,2} & \cdots & a_{2,n} & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n,1} & 0 & a_{n,2} & \cdots & a_{n,n} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

which is an analog of $\iota_{m,n}^\pm$ used in [Goshima and Yamagishi 2010] for $C_m \times C_n$. One can then verify the identity $\iota_n \Delta_n = \Delta_{2n+1}^2 \iota_n$, so it follows that $\iota_n(\mathcal{C}_n) \subset \mathcal{C}_{2n+1}$. Since ι_n preserves the Hamming weight, we have $A_i(n) \leq A_i(2n+1)$ for $0 \leq i \leq n$. \square

n	$A_1(n)$	$\dim \mathcal{H}_n$	n	$A_1(n)$	$\dim \mathcal{H}_n$	n	$A_1(n)$	$\dim \mathcal{H}_n$	n	$A_1(n)$	$\dim \mathcal{H}_n$
1	1	0	41	701	2	81	6561	0	121	14641	0
2	4	0	42	1764	0	82	6724	0	122	14884	0
3	9	0	43	1849	0	83	1401	6	123	1	80
4	0	4	44	640	4	84	128	12	124	5376	4
5	5	2	45	2025	0	85	7225	0	125	1	50
6	36	0	46	2116	0	86	7396	0	126	0	56
7	49	0	47	9	30	87	7569	0	127	16129	0
8	64	0	48	2304	0	88	7744	0	128	0	56
9	1	8	49	401	8	89	829	10	129	1	56
10	100	0	50	196	8	90	8100	0	130	16900	0
11	9	6	51	2601	0	91	8281	0	131	1	86
12	144	0	52	2704	0	92	364	20	132	17424	0
13	169	0	53	1189	2	93	8649	0	133	17689	0
14	52	4	54	980	4	94	3060	4	134	6292	4
15	225	0	55	3025	0	95	9	62	135	1	64
16	0	8	56	3136	0	96	9216	0	136	18496	0
17	109	2	57	3249	0	97	9409	0	137	8189	2
18	324	0	58	3364	0	98	388	20	138	19044	0
19	1	16	59	53	22	99	801	16	139	1681	16
20	400	0	60	3600	0	100	10000	0	140	19600	0
21	441	0	61	1	40	101	197	18	141	19881	0
22	484	0	62	0	24	102	10404	0	142	20164	0
23	9	14	63	3969	0	103	10609	0	143	649	30
24	176	4	64	0	28	104	3760	4	144	7280	4
25	625	0	65	1	42	105	11025	0	145	21025	0
26	676	0	66	4356	0	106	11236	0	146	21316	0
27	729	0	67	1	32	107	2377	6	147	21609	0
28	784	0	68	4624	0	108	11664	0	148	21904	0
29	53	10	69	841	8	109	2201	8	149	2501	10
30	0	20	70	4900	0	110	12100	0	150	22500	0
31	961	0	71	361	14	111	12321	0	151	22801	0
32	0	20	72	5184	0	112	12544	0	152	2368	8
33	1	16	73	5329	0	113	5549	2	153	23409	0
34	372	4	74	1876	4	114	4532	4	154	240	24
35	217	6	75	5625	0	115	13225	0	155	5097	6
36	1296	0	76	5776	0	116	13456	0	156	24336	0
37	1369	0	77	2549	2	117	13689	0	157	24649	0
38	1444	0	78	6084	0	118	1380	8	158	24964	0
39	1	32	79	1	64	119	53	46	159	1	128
40	1600	0	80	6400	0	120	14400	0	160	25600	0

Table 1

Applying [Conjecture 4.2](#) repeatedly and using [Corollary 3.4](#), we easily arrive at the following:

Conjecture 4.4. *Let $n \geq 3$ be odd and let d be the maximal odd divisor of $n + 1$. Then we have $A_1(n) = 1$ if and only if $d > 1$ and $A_1(d - 1) = 0$.*

Proposition 4.5. *Conjectures 4.2 and 4.4 are equivalent.*

Proof. It suffices to show the implication [Conjecture 4.4](#) \Rightarrow [Conjecture 4.2](#). Let $n \geq 2$ and let d be the maximal odd divisor of $n + 1$ (and hence of $2n + 2$). By [Corollary 3.4](#), $A_1(2n + 1) \leq 1$ is equivalent to $A_1(2n + 1) = 1$, which, in turn, is equivalent to $d > 1$ and $A_1(d - 1) = 0$ by [Conjecture 4.4](#). If n is odd, then the same reasoning shows $A_1(n) \leq 1 \iff d > 1$ and $A_1(d - 1) = 0$, so we are done. If n is even, then $d = n + 1 > 1$ and we have $A_1(n) \leq 1 \iff A_1(d - 1) = 0$ by [Corollary 3.4](#). \square

Next we make an attempt to “explain” the value of $A_1(n)$. If the Laplacian Δ_n is bijective, then we have $\mathcal{C}_n = \mathbb{F}_2^{n^2}$ and hence $A_1(n) = n^2$. We comment here on the bijectivity of Δ_n . Sutner [2000] proved

$$\dim \mathcal{H}_n = \deg \gcd(S_n(x), S_n(x + 1)),$$

where S_n is the n -th Chebyshev polynomial of the second kind, regarded as a polynomial over \mathbb{F}_2 . Some sufficient conditions for the bijectivity of Δ_n follow from this identity and well-known properties of Chebyshev polynomials. For example, $n = 2^l - 1$ ($l \geq 1$) is sufficient [[Yamagishi 2015](#), Corollary 4.3]. Note that this confirms [Conjecture 4.4](#) for $n = 2^l - 1$, as $A_1(n) = n^2$ and $d = 1$. There seems to be no simple characterization of n for which Δ_n is bijective.

Now we consider the case where Δ_n is not bijective, i.e., $\dim \mathcal{H}_n > 0$. As in [Conjecture 4.4](#), the divisors d of $n + 1$ with $A_1(d - 1) = 0$ play an important role in the following two conjectures.

Conjecture 4.6. *Let n be even. Then Δ_n is not bijective if and only if there exists a (necessarily odd) divisor $d > 1$ of $n + 1$ such that $A_1(d - 1) = 0$.*

Conjecture 4.7. *Suppose n is even and Δ_n is not bijective. Assume [Conjecture 4.6](#), and let d_k ($1 \leq k \leq t$) be the divisors of $n + 1$ such that $d_k > 1$ and $A_1(d_k - 1) = 0$. Then for $1 \leq i, j \leq n$, we have $e_{i,j} \in \mathcal{C}_n$ if and only if*

$$i \equiv 0 \pmod{d_k} \quad \text{or} \quad j \equiv 0 \pmod{d_k} \tag{4-1}$$

for $k = 1, 2, \dots, t$.

Example 4.8. If $A_1(n) = 0$, then we can take $d_1 = n + 1$ and [Conjecture 4.7](#) is trivially true. But this gives no explanation of why $A_1(n) = 0$. We exclude this case in the following examples.

Example 4.9. Suppose $t = 1$ and put $b = (n + 1)/d_1$. The number of pairs (i, j) for which (4-1) with $k = 1$ fails is $(n - b + 1)^2$, so we have $A_1(n) = n^2 - (n - b + 1)^2$.

This applies for $n = 14, 24, 34, 44, 54, 74, 94, 104, 114, 124, 134, 144$ ($d_1 = 5$), $n = 50, 118, 152$ ($d_1 = 17$), $n = 92$ ($d_1 = 31$), and $n = 98$ ($d_1 = 33$).

Example 4.10. For $n = 84$, we have $t = 2$, $d_1 = 5$, $d_2 = 17$, and (4-1) for $k = 1, 2$ reads as $ij \equiv 0 \pmod{85}$. Thus we have $A_1(84) = 2(5 - 1)(17 - 1) = 128$. The same reasoning applies for $n = 154$: $t = 2$, $d_1 = 5$, $d_2 = 31$ and $A_1(154) = 2(5 - 1)(31 - 1) = 240$.

Finally, we note that an answer to [Problem 3.6](#) would give, under [Conjecture 4.4](#), a characterization of (necessarily odd) n with $A_1(n) = 1$, and, under [Conjecture 4.6](#), a characterization of even n with nonbijective Δ_n .

We also point out that, in [Table 1](#), there are four exceptions $n = 2, 6, 8, 14$ for the converse statement of [Conjecture 4.1](#). [Problem 3.6](#) would be settled if they are the only exceptions.

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no. 4

Euler's formula for the zeta function at the positive even integers	541
SAMYUKTA KRISHNAMURTHY AND MICAH B. MILINOVICH	
Descents and des-Wilf equivalence of permutations avoiding certain nonclassical patterns	549
CADEN BIELAWA, ROBERT DAVIS, DANIEL GREESON AND QINHAN ZHOU	
The classification of involutions and symmetric spaces of modular groups	565
MARC BESSON AND JENNIFER SCHAEFER	
When is $a^n + 1$ the sum of two squares?	585
GREG DRESDEN, KYLIE HESS, SAIMON ISLAM, JEREMY ROUSE, AARON SCHMITT, EMILY STAMM, TERRIN WARREN AND PAN YUE	
Irreducible character restrictions to maximal subgroups of low-rank classical groups of types B and C	607
KEMPTON ALBEE, MIKE BARNES, AARON PARKER, ERIC ROON AND A. A. SCHAEFFER FRY	
Prime labelings of infinite graphs	633
MATTHEW KENIGSBERG AND OSCAR LEVIN	
Positional strategies in games of best choice	647
AARON FOWLKES AND BRANT JONES	
Graphs with at most two trees in a forest-building process	659
STEVE BUTLER, MISA HAMANAKA AND MARIE HARDT	
Log-concavity of Hölder means and an application to geometric inequalities	671
AUREL I. STAN AND SERGIO D. ZAPETA-TZUL	
Applying prospect theory to multiattribute problems with independence assumptions	687
JACK STANLEY AND FRANK P. A. COOLEN	
On weight-one solvable configurations of the Lights Out puzzle	713
YUKI HAYATA AND MASAKAZU YAMAGISHI	