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Space-efficient knot mosaics for prime knots with mosaic number 6

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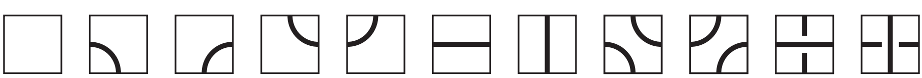
In 2008, Kauffman and Lomonaco introduced the concepts of a knot mosaic and the mosaic number of a knot or link K, the smallest integer n such that K can be represented on an n-mosaic. In 2018, the authors of this paper introduced and explored space-efficient knot mosaics and the tile number of K, the smallest number of nonblank tiles necessary to depict K on a knot mosaic. They determine bounds for the tile number in terms of the mosaic number. In this paper, we focus specifically on prime knots with mosaic number 6. We determine a complete list of these knots, provide a minimal, space-efficient knot mosaic for each of them, and determine the tile number (or minimal mosaic tile number) of each of them.

1. Introduction

Mosaic knot theory was first introduced in [Lomonaco and Kauffman 2008] and was later proven to be equivalent to tame knot theory in [Kuriya and Shehab 2014]. The idea of mosaic knot theory is to create a knot or link diagram on an $n \times n$ grid using *mosaic tiles* selected from the collection of 11 tiles shown below. The knot or link projection is represented by arcs, line segments, or crossings drawn on each tile. These tiles are identified, respectively, as T_0, T_1, T_2, ..., T_{10}. Tile T_0 is a blank tile, and we refer to the rest collectively as nonblank tiles.

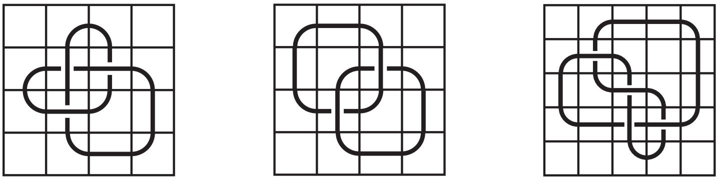


T_0 T_1 T_2 T_3 T_4 T_5 T_6 T_7 T_8 T_9 T_{10}

A *connection point* of a tile is a midpoint of a tile edge that is also the endpoint of a curve drawn on the tile. A tile is *suitably connected* if each of its connection points touches a connection point of an adjacent tile. An $n \times n$ *knot mosaic*, or *n-mosaic*, is an $n \times n$ matrix whose entries are suitably connected mosaic tiles. As is customary in the literature of knot mosaic theory, the term "knot mosaic" is used

MSC2010: primary 57M25; secondary 57M27.

Keywords: knots, knot mosaic, mosaic number, tile number, crossing number, space-efficient.



trefoil knot Hopf link figure-8 knot

Figure 1. Examples of knot mosaics.

for the mosaic, even when the resulting diagram on the mosaic depicts a link. See Figure 1 for some examples.

When listing prime knots with crossing number 10 or less, we will use the Alexander–Briggs notation, matching the table of knots in [Rolfsen 1976]. This notation names a knot according to its crossing number with a subscript to denote its order amongst all knots with that crossing number. For example, the 7_4 knot is the fourth knot with crossing number 7 in Rolfsen's table of knots. For knots with crossing number 11 or higher, we use the Dowker–Thistlethwaite name of the knot. This also names a knot according to its crossing number, with an "a" or "n" to distinguish the alternating and nonalternating knots and a subscript that denotes the lexicographical ordering of the minimal Dowker–Thistlethwaite notation for the knot. For example $11a_7$ is the seventh alternating knot with crossing number 11, and $11n_3$ is the third nonalternating knot with crossing number 11. For more details on these and other relevant information on traditional knot theory, we refer the reader to [Adams 1994].

The *mosaic number* of a knot or link K is the smallest integer n for which K can be represented as an n-mosaic. The mosaic number has previously been determined for every prime knot with crossing number 8 or less. For details, see [Lee, Ludwig, Paat, and Peiffer 2018]. In particular, it is known that the unknot has mosaic number 2, the trefoil knot has mosaic number 4, the 4_1, 5_1, 5_2, 6_1, 6_2, and 7_4 knots have mosaic number 5, and all other prime knots with crossing number 8 or less have mosaic number 6. In this paper, we determine the rest of the prime knots that have mosaic number 6, which includes prime knots with crossing numbers from 9 up to 13. This confirms, in the case where the mosaic number is $m = 6$, a result of [Howards and Kobin 2018], where they find that the crossing number is bounded above by $(m-2)^2 - 2$ if m is odd, and by $(m-2)^2 - (m-3)$ if m is even. We also determine that not all knots with crossing number 9 (or higher) have mosaic number 6.

Another number associated to a knot mosaic is the *tile number of a mosaic*, which is the number of nonblank tiles used to create the mosaic. From this we get an invariant called the *tile number* $t(K)$ *of a knot or link* K, which is the least number of nonblank tiles needed to construct K on a mosaic of any size. In [Heap and Knowles 2018], the authors explored the tile number of a knot or link and determined strict bounds for the tile number of a prime knot K in terms of the mosaic number $m \geq 4$. Specifically, if m is even, then $5m-8 \leq t(K) \leq m^2-4$. If m is odd, then $5m-8 \leq t(K) \leq m^2-8$. It follows immediately that the tile number of the trefoil knot must be 12, and the tile number of the prime knots mentioned above with mosaic number 5 must be 17. The authors also listed several prime knots with mosaic number 6 that have the smallest possible tile number $t(K) = 22$, which we summarize in Theorem 1. In this paper, we confirm that this list is complete. Knot mosaics in which the tile number is realized for each of these mosaics are given in [Heap and Knowles 2018] and also in the table of mosaics in the online supplement of this paper.

Theorem 1 [Heap and Knowles 2018]. *The following knots have the given tile numbers*:

(a) *Tile number* 4: *unknot.*

(b) *Tile number* 12: *trefoil knot.*

(c) *Tile number* 17: $4_1, 5_1, 5_2, 6_1, 6_2, 7_4$.

(d) *Tile number* 22: $6_3, 7_1, 7_2, 7_3, 7_5, 7_6, 7_7, 8_1, 8_2, 8_3, 8_4, 8_7, 8_8, 8_9, 8_{13}, 9_5, 9_{20}$.

Finally, in [Heap and Knowles 2018], the authors determine all of the possible layouts for any prime knot on an n-mosaic for $n \leq 6$. In this paper, we complete that work by determining which prime knots can be created from those layouts.

We also point out that throughout this paper we make significant use of the software package Knotscape [Thistlethwaite and Hoste 1999] to verify that a given knot mosaic represents a specific knot. Without this program, we would not have been able to complete the work.

2. Space-efficient knot mosaics

Two knot mosaic diagrams are of the *same knot type* (or *equivalent*) if we can change one to the other via a sequence of *mosaic planar isotopy moves* that are analogous to the planar isotopy moves for standard knot diagrams. An example of this is shown in Figure 2. A complete list of all of these moves is given and discussed in [Lomonaco and Kauffman 2008; Kuriya and Shehab 2014]. We will make significant use of these moves throughout this paper, as we attempt to reduce the tile number of mosaics in order to construct knot mosaics that use the least number of nonblank tiles.

A knot mosaic is called *minimal* if it is a realization of the mosaic number of the knot depicted on it. That is, if a knot with mosaic number m is depicted on an m-mosaic, then it is a minimal knot mosaic. A knot mosaic is called *reduced*

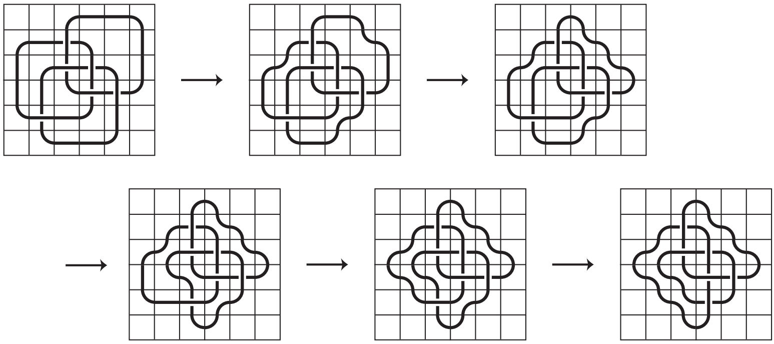


Figure 2. Example of mosaic planar isotopy moves.

if there are no unnecessary, reducible crossings in the knot mosaic diagram. See [Adams 1994] for more on reduced knot diagrams.

We have already defined the tile number of a mosaic and the tile number of a knot or link. A third type of tile number is the *minimal mosaic tile number* $t_M(K)$ *of a knot or link* K, which is the smallest number of nonblank tiles needed to construct K on a minimal mosaic. That is, it is the smallest possible tile number of all possible minimal mosaic diagrams for K. Much like the crossing number of a knot cannot always be realized on a minimal mosaic (such as the 6_1 knot), the tile number of a knot cannot always be realized on a minimal mosaic. Note that the tile number of a knot or link K is certainly less than or equal to the minimal mosaic tile number of K; that is, $t(K) \leq t_M(K)$. The fact that the tile number of a knot is not necessarily equal to the minimal mosaic tile number of the knot is confirmed later in Theorem 8. However, for prime knots, it is shown in [Heap and Knowles 2018] that $t_M(K) = t(K)$ when $t_M(K) \leq 27$.

A knot n-mosaic is *space-efficient* if it is reduced and the tile number is as small as possible on an n-mosaic without changing the knot type of the depicted knot, meaning that the tile number cannot be decreased through a sequence of mosaic planar isotopy moves. A knot mosaic is *minimally space-efficient* if it is minimal and space-efficient. The first four knot mosaics of the Borromean rings depicted in Figure 2 are not space-efficient because we can decrease the tile number through the depicted mosaic planar isotopy moves. In Figure 3, both mosaics are knot mosaic diagrams of the 5_1 knot. The first knot mosaic is not space-efficient, but the second knot mosaic is minimally space-efficient.

In addition to the original 11 tiles T_0–T_{10}, we will also make use of *nondeterministic tiles*, such as those in Figure 4, when there are multiple options for the tiles that can be placed in specific tile locations of a mosaic. For example, if a tile location must contain a crossing tile T_9 or T_{10} but we have not yet chosen which, we will use the nondeterministic crossing tile. Similarly, if we know that a tile

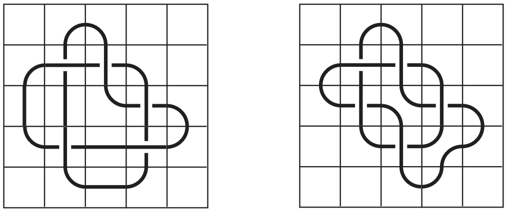


Figure 3. Space-inefficient and minimally space-efficient knot mosaics of 5_1.

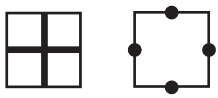
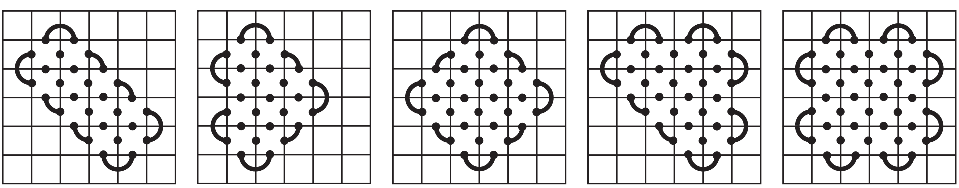


Figure 4. Nondeterministic crossing tile and a nondeterministic tile with four connection points.

location must have four connection points but we do not know if the tile is a double arc tile (T_7 or T_8) or a crossing tile (T_9 or T_{10}), we will indicate this with a tile that has four connection points.

In [Heap and Knowles 2018], the authors provide the possible tile numbers (and the layouts that result in these tile numbers) for all prime knots on a space-efficient 6-mosaic.

Theorem 2 [Heap and Knowles 2018]. *If we have a space-efficient* 6*-mosaic of a prime knot K for which either every column or every row is occupied, then the only possible values for the tile number of the mosaic are* 22, 24, 27, *and* 32. *Furthermore, any such mosaic of K is equivalent (up to symmetry) to one of the following mosaics:*



In order to determine all prime knots with mosaic number 6 and their minimal mosaic tile numbers, we need to determine which prime knots can be depicted on a knot mosaic with one of the layouts above. To help us with this, we make a few simple observations. All of these are easy to verify, and any rotation or reflection of these scenarios is also valid.

Consider the upper, right 3×3 corner of any space-efficient mosaic of a prime knot with mosaic number 6 and tile number 22, 27, or 32. (That is, we are

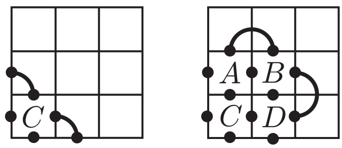


Figure 5. A partially filled block and a filled block, respectively.

considering every option except those with tile number 24.) It must be one of the two options in Figure 5. All other 3×3 corners are a rotation of one of these. We will refer to the first option as a *partially filled block* and the second option as a *filled block*.

Observation 1. In any space-efficient 6-mosaic of a prime knot, the tile in position C of a partially filled block is either a crossing tile or double arc T_7.

This is easy to see, as it must be a tile with four connection points, and the only space-efficient mosaics that results from using the double arc T_8 are composite knots or links with more than one component. In Figure 6, the first two examples are valid possibilities, but the third one is not.

Observation 2. In any space-efficient 6-mosaic of a prime knot, there must be at least two crossing tiles in a filled block.

If there are no crossing tiles in positions A, B, C, and D of the mosaic, then the mosaic is not space-efficient or it is a link with more than one component. Each one that is not a link reduces to one of the last two partially filled block options in Figure 6. If there is only one crossing tile and it is in position A, B, or D, then the mosaic is not space-efficient. For each option, if we fill the remaining tile positions with double arc tiles so that the block is suitably connected and we avoid the obvious inefficiencies we get the options shown in Figure 7. They are equivalent to each other via a simple mosaic planar isotopy move that rolls the crossing through each of these positions, and they all reduce to the first partially filled block in Figure 6. If there is only one crossing tile and it is in position C, then the mosaic is also not space-efficient and reduces to either of the first two options in Figure 6.

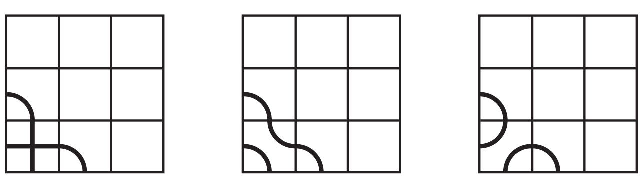


Figure 6. The first two examples are the only valid possibilities for a partially filled block.

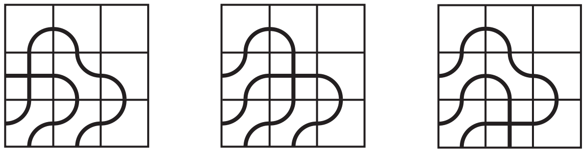
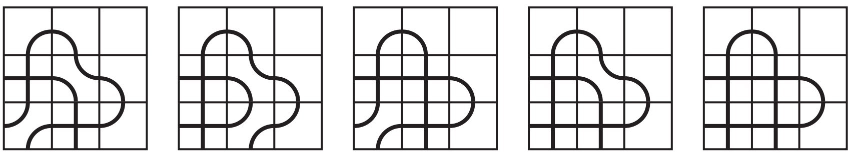


Figure 7. Suitably connected filled blocks with one crossing in position A, B, or D. None are space-efficient.

Observation 3. In a filled block in any space-efficient 6-mosaic of a prime knot, there are only two distinct possibilities for two crossing tiles, two distinct possibilities for three crossing tiles, and one possibility for four crossing tiles and they are shown below:



We will refer to the five filled blocks in Observation 3 together with the first two partially filled blocks in Figure 6 (and reflections and rotations of them) as *building blocks*. The observations provide a way for us to easily build all of the space-efficient 6-mosaics, as long as the tile number is 22, 27, or 32, but not 24.

Observation 4. In any space-efficient 6-mosaic of a prime knot, there is at most one of the filled block with four crossing tiles or the filled block with two crossings in positions A and C.

It is quite simple to verify that if there is more than one filled block with four crossings or more than one filled block with two crossings in positions A and C, the resulting mosaic must be a link with more than one component. If we use the indicated filled building block with two crossing tiles together with a filled block with four crossing tiles, the resulting mosaic will also be a link with more than one component. Several examples of these are pictured in Figure 8 with the second link component in each mosaic colored differently from the first link component.

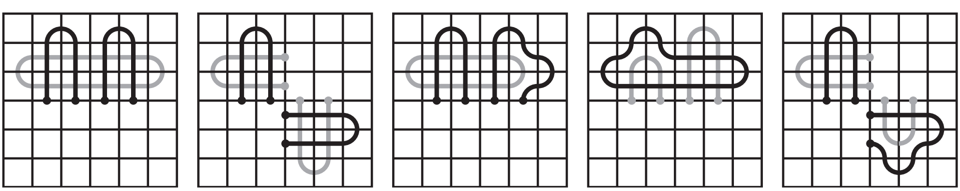


Figure 8. These layouts will always be multicomponent links.

3. All prime knots with mosaic number 6

We are now ready to determine the tile number of every prime knot with mosaic number 6. Theorem 2 says that the only possible tile numbers are 22, 24, 27, and 32. In order to determine which knots have these tile numbers, we simply compile a list of the prime knots that can fit within each of the layouts given in Theorem 2. Because we already know the tile number of every prime knot with crossing number 7 or less, we can restrict our search to knots with crossing number 8 or more. The process is simple, and the above observations help us tremendously. If the tile number is 22, 27, or 32, we use the building blocks. In the case of the mosaics with tile number 24, we look at all possible placements, up to symmetry, of eight or more crossing tiles within the mosaics and fill the remaining tile positions with double arc tiles so as to avoid composite knots and nonreduced knots. Once the mosaics are completed, we then eliminate any links, any duplicate layouts that are equivalent to others via obvious mosaic planar isotopy moves, and any mosaics for which the tile number can easily be reduced by a simple mosaic planar isotopy move. Finally, we use Knotscape to determine what knots are depicted in the mosaic by choosing the crossings so that they are alternating, as well as all possible nonalternating combinations. We provide minimally space-efficient knot mosaics for every prime knot with mosaic number less than or equal to 6 in the table of knots in the online supplement.

We have already listed several prime knots with tile number 22 in Theorem 1. This next theorem asserts that the list is complete.

Theorem 3. *The only prime knots K with tile number $t(K) = 22$ are*

(a) 6_3,

(b) 7_1, 7_2, 7_3, 7_5, 7_6, 7_7,

(c) 8_1, 8_2, 8_3, 8_4, 8_7, 8_8, 8_9, 8_{13},

(d) 9_5, *and* 9_{20}.

In order to obtain the minimally space-efficient knot mosaic for 7_3, we had to use eight crossings. None of the possible minimally space-efficient knot mosaics with 22 nonblank tiles and exactly seven crossings produced 7_3. The fewest number of nonblank tiles needed to represent 7_3 with only seven crossings is 24, and one such mosaic is given in Figure 9, along with a minimally space-efficient mosaic of 7_3 with eight crossings. In summary, on a minimally space-efficient knot mosaic, for the tile number (or minimal mosaic tile number) to be realized, it might not be possible for the crossing number to be realized. This is also the case with 8_1, 8_3, 8_7, 8_8, and 8_9, as nine crossing tiles are required to represent these knots on a mosaic with tile number 22.

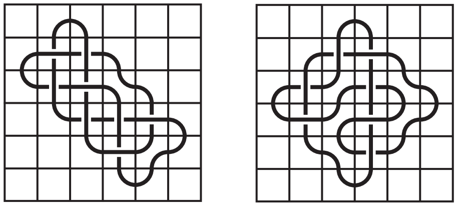
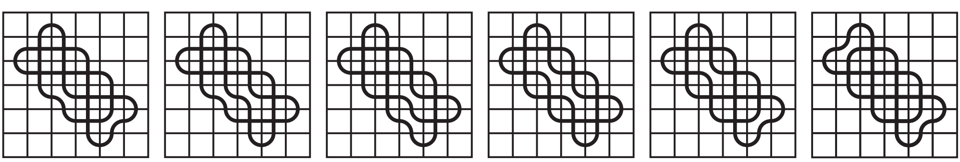


Figure 9. The 7_3 knot as a minimally space-efficient knot mosaic with eight crossing tiles and as a knot mosaic with seven crossing tiles.

Proof. We simply build the first two tile configurations (both with 22 nonblank tiles) in Theorem 2 using the 3×3 building blocks, eliminate any that do not satisfy the observations, choose specific crossing types, and see what we get. Whatever prime knots with eight or more crossings are missing are the ones we know cannot have tile number 22.

We begin with the first mosaic layout given in Theorem 2. Up to symmetry, there are only six possible configurations of this layout with eight crossings, and they are given in Figure 10. Notice that some of these are links that can be eliminated, including Figures 10(d) and (f). Furthermore, Figures 10(b) and (c) are equivalent to each other via a mosaic planar isotopy move that shifts one of the crossing tiles to a diagonally adjacent tile position. This leaves us with only three possible distinct configurations of eight crossings from this first layout, Figures 10(a), (b), and (e).

Now we do the same thing with the second mosaic layout given in Theorem 2 with 22 nonblank tiles. Up to symmetry, there are six possible configurations of this layout with eight crossings, and they are given in Figure 11. Again, Figures 11(d) and (f) are links, and Figures 11(b) and (c) are equivalent to each other. This leaves us again with only three possible configurations of eight crossings from this second layout, and they are Figures 11(a), (b), and (e). Moreover, each one of these is equivalent to the corresponding mosaics in Figure 10 via a few mosaic planar isotopy moves that shift the crossings in the lower-left building block into the lower-right building block of the mosaic.



(a) (b) (c) (d) (e) (f)

Figure 10. Possible placements of eight crossing tiles in the first layout with tile number 22.

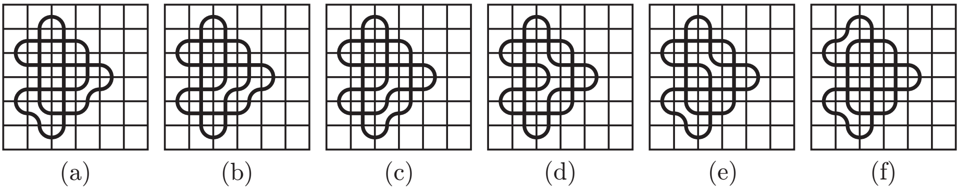


Figure 11. Possible placements of eight crossing tiles in the second layout with tile number 22.

This leaves us with only three distinct possible layouts for a minimally space-efficient 6×6 mosaic with eight crossings and tile number 22. If we choose crossings for the configuration in Figure 10(a) so that they are alternating, we get the 8_{13} knot. If we choose crossings for the configuration in Figure 10(b) so that they are alternating, we get the 8_4 knot. Finally, if we choose crossings for the configuration in Figure 10(e) so that they are alternating, we get the 8_2 knot. If we examine all possible nonalternating choices for each one, all of the resulting knots have crossing number 7 or less. (The minimally space-efficient knot mosaic for 7_3 must have eight crossing tiles and can be obtained by a choice of nonalternating crossings within any of the three distinct possible layouts in Figure 10.)

Now we go through the same process using nine crossing tiles. Up to symmetry, there are only four possible configurations of these layouts with nine crossings, and they are given in Figure 12. The mosaic in Figure 12(c) is equivalent to the mosaic in Figure 12(b) via a few mosaic planar isotopy moves that shift the crossings in the lower-left building block into the lower-right building block of the mosaic. This leaves us with only three possible configurations of nine crossing tiles.

If we choose crossings for the configuration in Figure 12(a) so that they are alternating, we get the 9_{20} knot. If we examine all possible nonalternating choices for the crossings, most of the resulting knots have crossing number 7 or less, but we do get some additions to our list of prime knots with tile number 22 and crossing number 8. In particular, we get 8_7, 8_8, and 8_9. (We also get 8_4, which was previously obtained with only eight crossings.) If we choose crossings for the configuration in Figure 12(b) so that they are alternating, we get the 9_5 knot. Again, if we examine

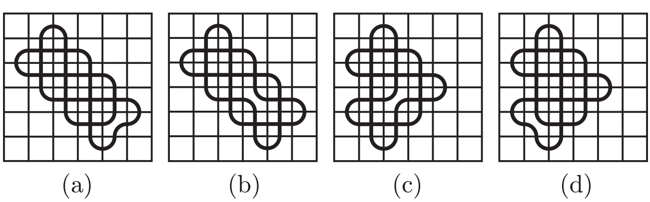


Figure 12. Possible placements of nine crossings with tile number 22.

the possible nonalternating choices for the crossings, we get two additional prime knots with tile number 22 and crossing number 8, and they are 8_1 and 8_3. Finally, if we choose crossings for the configuration in Figure 12(d), we get the exact same knots as we did for Figure 12(a).

By Observation 4, we cannot place more than nine crossing tiles on any mosaic with 22 nonblank tiles. We have now found every possible prime knot with tile number 22 and eight or more crossings, and they are exactly those listed in the theorem. All other prime knots with crossing number at least 8 must have tile number larger than 22. □

We now know precisely which prime knots have tile number 22 or less. Our next goal is to determine which prime knots have tile number 24.

Theorem 4. *The only prime knots K with tile number $t(K) = 24$ are*

(a) 8_5, 8_6, 8_{10}, 8_{11}, 8_{12}, 8_{14}, 8_{16}, 8_{17}, 8_{18}, 8_{19}, 8_{20}, 8_{21},

(b) 9_8, 9_{11}, 9_{12}, 9_{14}, 9_{17}, 9_{19}, 9_{21}, 9_{23}, 9_{26}, 9_{27}, 9_{31},

(c) 10_{41}, 10_{44}, 10_{85}, 10_{100}, 10_{116}, 10_{124}, 10_{125}, 10_{126}, 10_{127}, 10_{141}, 10_{143}, 10_{148}, 10_{155} *and* 10_{159}.

We will show that 8_6 must have nine crossing tiles to fit on a mosaic with tile number 24. None of the possible minimally space-efficient knot mosaics with exactly eight crossings produce these knots. Similarly, the minimally space-efficient mosaics for 9_{12}, 9_{19}, 9_{21}, and 9_{26} require 10 crossings.

Proof. We search for all of the prime knots that have tile number 24. In this particular case, the observations at the beginning of this section do not apply, meaning we cannot use the building blocks as we did in the proof of Theorem 3. We know from Theorem 2 that any prime knot with tile number 24 has a space-efficient mosaic, like the third layout there. We simply look at all possible placements of eight or more crossings within that layout, choose the type of each crossing, and keep track of the resulting prime knots.

First, we look at all possible placements, up to symmetry, of eight crossings within the mosaic and, we fill the remaining tile positions with double arc tiles so as to avoid composite knots and unnecessary loops. After eliminating any links and any duplicate layouts that are equivalent to others via simple mosaic planar isotopy moves, we get 17 possible layouts, which are shown in Figure 13. Not all of these will result in distinct knots, and in most cases it is not difficult to see that they will result in the same knot. However, we include all of them here because they differ by more than just simple symmetries or simple mosaic planar isotopy moves.

Choosing specific crossings so that the knots are alternating, we obtain only 14 distinct knots as shown in the following table:

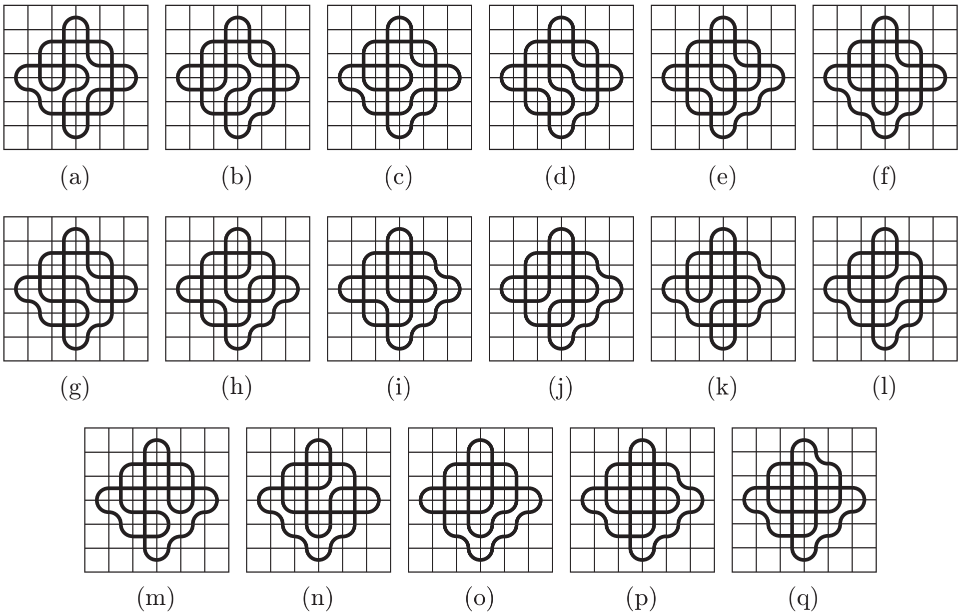


Figure 13. Only possible layouts, after elimination, with eight crossing tiles for a prime knot with tile number 24.

Figure 13	knot	Figure 13	knot
(a)	8_1	(j)	8_{11}
(b), (c)	8_2	(k)	8_{12}
(d)	8_4	(l)	8_{13}
(e)	8_5	(m), (n)	8_{14}
(f), (g)	8_7	(o)	8_{16}
(h)	8_8	(p)	8_{17}
(i)	8_{10}	(q)	8_{18}

Not all of these have tile number 24. We already know 8_1, 8_2, 8_4, 8_7, 8_8, and 8_{13} have tile number 22. Each of the others have tile number 24. The nonalternating knots 8_{19}, 8_{20}, and 8_{21} are obtained by choosing nonalternating crossings in a few of these. Those pictured in the table of knots come from the layout in Figure 13(p). Mosaics for all of these are given in the table of knots in the online supplement. The only knots with crossing number 8 that we have not yet found are 8_6 and 8_{15}, and now we know that they cannot be represented with eight crossings and 24 nonblank tiles.

We now turn our attention to mosaics with nine crossings. Just as before, we look at all possible placements, up to symmetry, of nine crossings, eliminate any composite knots, unnecessary loops, links and any duplicate layouts that are equivalent to others via simple mosaic planar isotopy moves. In the end, we get

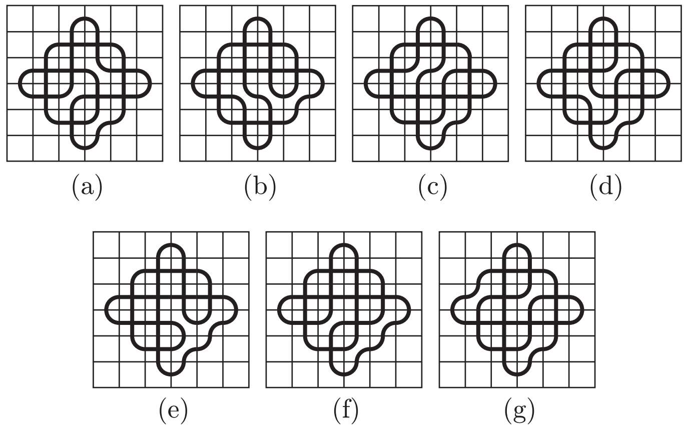
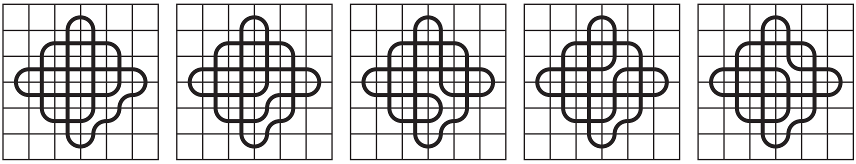


Figure 14. Only possible layouts, after elimination, with nine crossing tiles for a prime knot with tile number 24.

seven possible layouts, which are shown in Figure 14. Choosing specific crossings for each layout, in order, so that the knots are alternating, we obtain the seven knots 9_8, 9_{11}, 9_{14}, 9_{17}, 9_{23}, 9_{27}, and 9_{31}, all of which have tile number 24. If we look at all possible choices for nonalternating crossings, the only knot with tile number 24 that arises but did not show up with only eight crossing tiles is the 8_6 knot, whose knot mosaic in the table of knots comes from the layout in Figure 14(a). All other prime knots that arise using nonalternating crossings have been exhibited as a minimally space-efficient mosaic with fewer crossings or fewer nonblank tiles.

Now we do the same for 10 crossings. Again, we observe all possible placements of 10 crossings on the third mosaic in Theorem 2, and after eliminating any links and duplicate layouts up to reflection, rotation, or equivalencies via simple mosaic planar isotopy moves, we end up with five possible layouts, shown in Figure 15.

We begin with Figure 15(a). Choosing specific crossings so that the knot is alternating, we obtain the 10_{116} knot. If we look at all possible choices for nonalternating crossings, the only prime knots that we get with tile number 24 are the nonalternating knots 10_{124}, 10_{125}, 10_{141}, 10_{143}, 10_{155}, and 10_{159}. We do the same with Figure 15(b) and get the alternating knot 10_{100}. For the nonalternating choices, we get almost all of the same ones we just obtained, but we do not get any new additions to our list of knots. For Figure 15(c), with alternating crossings we get 10_{41}, and with nonalternating crossings we get 9_{19} and 9_{21} as the only new additions to our list. Neither of these came from considering only nine crossings. Now we observe the mosaic in Figure 15(d). By alternating the crossings, we obtain 10_{44}, and by using nonalternating crossings, the only new additions to our list are 9_{12} and 9_{26}. Finally, we end with Figure 15(e). Assigning alternating crossings, we get 10_{85}, and assigning nonalternating crossings, we get 10_{126}, 10_{127}, and 10_{148}.



(a) (b) (c) (d) (e)

Figure 15. Only possible layouts, after elimination, with 10 crossing tiles for a prime knot with tile number 24.

Finally, we can place 11 or 12 crossing tiles into the layout with 24 nonblank tiles, but the space-efficient results will always be a link with more than one component. Therefore, no minimally space-efficient prime knot mosaics arise from this consideration. We have considered every possible placement of crossing tiles on the third layout in Theorem 2 and have found every possible prime knot with tile number 24 and eight or more crossings, and they are exactly those listed in the theorem. Minimally space-efficient mosaics for all of these knots are given in the table of knots in the online supplement. All other prime knots with crossing number at least 8 must have tile number larger than 24. □

We now know precisely which prime knots have tile number less than or equal to 24, and we are ready to determine which prime knots with mosaic number 6 have tile number 27. We see our first occurrence of knots with crossing number larger than 10, and we use the Dowker–Thistlethwaite name of the knot.

Theorem 5. *The only prime knots K with mosaic number* 6, *tile number $t(K) = 27$, and minimal mosaic tile number $t_M(K) = 27$ are*

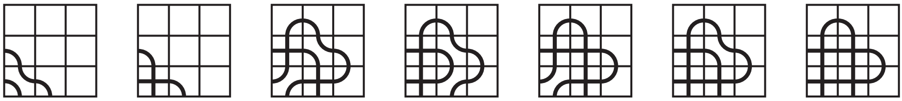
(a) 8_{15},

(b) 9_1, 9_2, 9_3, 9_4, 9_7, 9_9, 9_{13}, 9_{24}, 9_{28}, 9_{37}, 9_{46}, 9_{48},

(c) 10_1, 10_2, 10_3, 10_4, 10_{12}, 10_{22}, 10_{28}, 10_{34}, 10_{63}, 10_{65}, 10_{66}, 10_{75}, 10_{78}, 10_{140}, 10_{142}, 10_{144},

(d) $11a_{107}$, $11a_{140}$, *and* $11a_{343}$.

Notice that this theorem is only referring to prime knots with mosaic number 6. There are certainly prime knots with tile number 27 and mosaic number 7 that are not included in this theorem. Also, the requirement that the tile number equals the minimal mosaic tile number is necessary here. As far as we know now (and will verify below), there are knots with mosaic number 6 and tile number 27 which have minimal mosaic number 32. Some of these are listed in the next theorem. Finally, notice that up to this point we have determined the tile number for every prime knot with crossing number 8 or less.



(a) (b) (c) (d) (e) (f) (g)

Figure 16. The seven building blocks resulting from the observations at the beginning of this section.

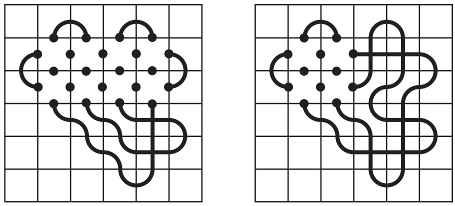
Again we claim that the minimally space-efficient mosaics for 9_3, 9_4, 9_{13}, 9_{37}, 9_{46}, and 9_{48} must have 10 crossing tiles. The minimally space-efficient mosaics for 9_7, 9_9, and 9_{24} must have 11 crossing tiles. None of the possible minimally space-efficient knot mosaics with exactly nine crossing tiles produce these knots. Similarly, the minimally space-efficient mosaics for 10_1, 10_3, 10_{12}, 10_{22}, 10_{34}, 10_{63}, 10_{65}, 10_{78}, 10_{140}, 10_{142}, and 10_{144} require 11 crossing tiles.

Proof. Similar to what we did in the proof of Theorem 3, we search for all of the prime knots that have mosaic number 6 and tile number 27, which have a space-efficient mosaic as depicted in the fourth layout of Theorem 2. We simply build this layout using the 3×3 building blocks that result from the observations at the beginning of this section, shown again in Figure 16. We then choose specific crossing types for each crossing tile and see what knots we get.

For bookkeeping purposes, we note that the knot 8_{15} has tile number 27, and this is the only knot with crossing number 8 for which we have not previously found the tile number. A minimally space-efficient mosaic for it is included in the table of knots in the online supplement. We now know the tile number for every prime knot with crossing number 8 or less, and from here we restrict our search to mosaics with nine or more crossing tiles.

Before we get started placing crossing tiles, we make a few more simple observations that apply to this particular case and help us reduce the number of possible configurations. Observe that if we place a partially filled building block with no crossing adjacent to the filled building block with two crossing tiles in Figure 16(c), the resulting mosaic will always reduce to a mosaic with tile number 22. The same result holds if the two blocks are not adjacent and one of the adjacent blocks is the filled building block with three crossings depicted in Figure 16(e). The mosaics in Figure 17 exhibit these scenarios. The same result also holds if the partially filled building block with one crossing is combined with two of the filled building blocks with two crossing tiles shown in Figure 16(c). Depending on the placement of these two filled blocks, the result will be equivalent to either Figure 17(a) or Figure 17(b) via a simple mosaic planar isotopy move that shifts the crossing in the partially filled block to another block.

First, we consider nine crossing tiles with the above observations in mind, together with the observations at the beginning of this section. Up to symmetry, there are



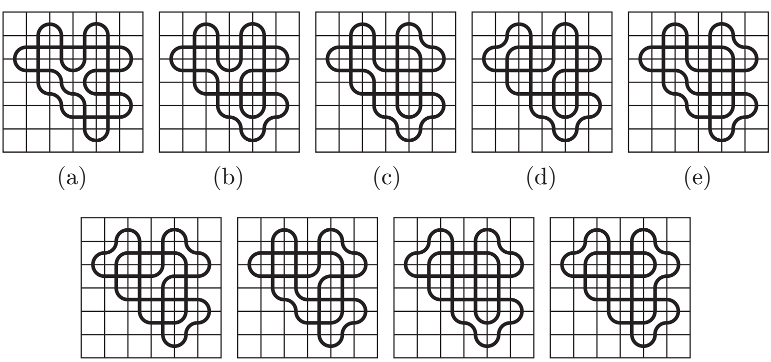
(a) (b)

Figure 17. These two mosaics are not minimally space-efficient.

only nine possible configurations of the building blocks after we eliminate the links, duplicate layouts that are equivalent to others via simple mosaic planar isotopy moves, and any mosaics for which the tile number can easily be reduced by a simple mosaic planar isotopy move. They are shown in Figure 18. Not all of these will result in distinct knots, and in several cases it is not difficult to see that they will result in the same knot. However, we include all of them here because they differ by more than just symmetries or a simple mosaic planar isotopy move.

Choosing specific crossings so that the knots are alternating, we obtain only seven distinct knots. The only ones with tile number 27 are Figure 18(a), which gives the 9_1 knot, Figure 18(b), which gives us 9_2, and Figures 18(h) and (i), which give us 9_{28}. Each of the remaining layouts give knots with tile number less than 27. In particular, Figures 18(c) and (d) are 9_8, Figures 18(e) and (f) are 9_{17}, and Figure 18(g) is 9_{20}. None of these configurations give nonalternating knots with crossing number 9.

Second, we do the same for 10 crossings. Again, we use the building blocks to build all possible configurations of the crossings, and up to symmetry, there are only



(a) (b) (c) (d) (e)

(f) (g) (h) (i)

Figure 18. Only possible layouts, after elimination, with nine crossing tiles for a prime knot with tile number 27.

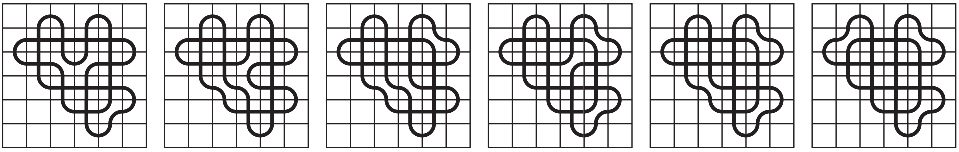


Figure 19. Only possible layouts, after elimination, with 10 crossing tiles for a prime knot with tile number 27.

six possibilities after eliminating any links and duplicate layouts that are equivalent via simple mosaic planar isotopy moves. These are shown in Figure 19.

Choosing specific crossings so that the knots are alternating, we obtain only five distinct knots, all of which have tile number 27. In particular, Figure 19(a) becomes the 10_2 knot, Figure 19(b) becomes 10_4, Figures 19(c) and (d) become 10_{28}, Figure 19(e) becomes 10_{66}, and Figure 19(f) becomes 10_{75}. Choosing nonalternating crossings, we also get some knots with crossing number 9, but we do not obtain any nonalternating knots with crossing number 10. We can get 9_3 from Figure 19(a), 9_4 from Figure 19(b), 9_{13} from Figure 19(c), and 9_{37}, 9_{46}, and 9_{48} from Figure 19(f). All other knots that are obtained by considering nonalternating crossings can be drawn with fewer crossings or a lower tile number.

Third, we consider the case where the mosaic has 11 crossing tiles. In this instance, we end up with the five possible layouts shown in Figure 20, and again, not all of these are distinct. Choosing alternating crossing in each layout results in three distinct knots with crossing number 11. Figures 20(a) and (b) become $11a_{107}$, Figures 20(c) and (d) become $11a_{140}$, and Figure 20(e) becomes $11a_{343}$. (Note that, for knots with crossing number greater than 10, we are using the Dowker–Thistlethwaite name of the knot.) Choosing nonalternating crossings in each of the layouts results in several knots with crossing number 9 or 10. In particular, we can obtain the knots 9_{24}, 10_{63}, 10_{65}, 10_{78}, 10_{140}, 10_{142}, and 10_{144} from Figure 20(a). We can obtain 9_7, 9_9, 10_{12}, 10_{22}, and 10_{34} from Figure 20(c). And we can obtain 10_1 and 10_3 from Figure 20(e). All of these are shown in the table of knots in the online supplement. All other knots that are obtained by considering nonalternating crossings can be drawn with fewer crossings or a lower tile number.

Finally, by Observation 4 we do not need to consider 12 or more crossing tiles in this layout, as no minimally space-efficient prime knot mosaics arise from this consideration. We have considered every possible placement of nine or more crossing tiles on the fourth layout in Theorem 2 and have found every possible prime knot with mosaic number 6 and tile number 27. They are exactly those listed in the theorem. All other prime knots with crossing number at least 9 and mosaic number 6 must have minimal mosaic tile number 32. □

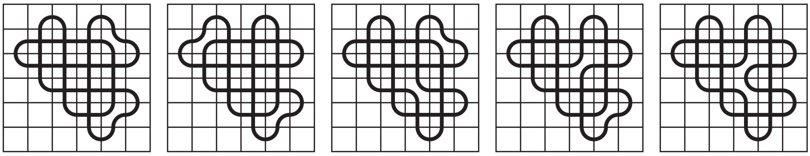


Figure 20. Only possible layouts, after elimination, with 11 crossing tiles for a prime knot with tile number 27.

Now we know the tile number for every prime knot with crossing number less than or equal to 8. Theorems 3, 4, and 5 tell us the tile number of some of the prime knots with crossing numbers 9, 10, and 11. Furthermore, we know that all other prime knots with mosaic number 6 must have minimal mosaic tile number 32 but not necessarily tile number 32. One problem that complicates the next step is that, as of the writing of this paper, we do knot know the mosaic number of all prime knots with crossing number 9 or more. That is, we do not know all prime knots with mosaic number 6. For this reason, we need to go through the same process as we did in the preceding proofs to determine which prime knots have mosaic number 6 and minimal mosaic tile number 32. By doing this, we will also be able to determine which prime knots have mosaic number greater than 6. The good news is that this is the final step in determining which prime knots have mosaic number 6 or less and determining the tile number or minimal mosaic tile numbers of all of these.

Theorem 6. *The only prime knots K with mosaic number* 6 *and minimal mosaic tile number $t_M(K) = 32$ are*

(a) 9_{10}, 9_{16}, 9_{35},

(b) 10_{11}, 10_{20}, 10_{21}, 10_{61}, 10_{62}, 10_{64}, 10_{74}, 10_{76}, 10_{77}, 10_{139},

(c) $11a_{43}$, $11a_{44}$, $11a_{46}$, $11a_{47}$, $11a_{58}$, $11a_{59}$, $11a_{106}$, $11a_{139}$, $11a_{165}$, $11a_{166}$, $11a_{179}$, $11a_{181}$, $11a_{246}$, $11a_{247}$, $11a_{339}$, $11a_{340}$, $11a_{341}$, $11a_{342}$, $11a_{364}$, $11a_{367}$,

(d) $11n_{71}$, $11n_{72}$, $11n_{73}$, $11n_{74}$, $11n_{75}$, $11n_{76}$, $11n_{77}$, $11n_{78}$,

(e) $12a_{119}$,$12a_{165}$, $12a_{169}$, $12a_{373}$, $12a_{376}$, $12a_{379}$, $12a_{380}$, $12a_{444}$,$12a_{503}$, $12a_{722}$, $12a_{803}$, $12a_{1148}$, $12a_{1149}$, $12a_{1166}$,

(f) $13a_{1230}$, $13a_{1236}$, $13a_{1461}$, $13a_{4573}$,

(g) $13n_{2399}$, $13n_{2400}$, $13n_{2401}$, $13n_{2402}$, *and* $13n_{2403}$.

Notice again our restriction to prime knots with mosaic number 6. Additionally, notice that this theorem only refers to the minimal mosaic tile number of the knot, not the tile number. Again, this is because we only know that these two numbers are equal when they are less than or equal to 27. Some of these knots may have (and actually do have) tile number less than 32.

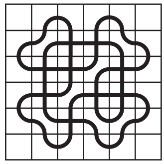


Figure 21. Only possible layout, after elimination, with nine crossing tiles for a prime knot with minimal mosaic tile number 32.

We claim that the minimally space-efficient mosaics for 9_{10}, 9_{16}, 10_{20}, 10_{21}, and 10_{77} need 11 crossing tiles. The minimally space-efficient mosaics for 9_{35}, 10_{11}, 10_{62}, 10_{64}, 10_{74}, 10_{139}, $11a_{106}$, $11a_{139}$, $11a_{166}$, $11a_{181}$, $11a_{341}$, $11a_{342}$, and $11a_{364}$ need 12 crossing tiles. And the minimally space-efficient mosaics for 10_{61}, 10_{76}, $11a_{44}$, $11a_{47}$, $11a_{58}$, $11n_{76}$, $11n_{77}$, $11n_{78}$, $11a_{165}$, $11a_{246}$, $11a_{339}$, $11a_{340}$, $12a_{119}$, $12a_{165}$, $12a_{169}$, $12a_{376}$, $12a_{379}$, $12a_{444}$, $12a_{803}$, $12a_{1148}$, and $12a_{1166}$ need 13 crossing tiles.

Proof. We simply go through the same process that we did in the previous proof. We search for all of the prime knots that have mosaic number 6 and minimal mosaic tile number 32. Whatever prime knots that do not show up in this process and that we have not previously determined the tile number for must have mosaic number greater than 6. We know from Theorem 2 that any prime knot with mosaic number 6 and minimal mosaic tile number 32 has a space-efficient mosaic with the fifth and final layout shown there.

As we have done several times previously, we use the building blocks to achieve all possible configurations, up to symmetry, of nine or more crossings within this mosaic. For this particular layout, we can only use the filled blocks, not the partially filled blocks. We can eliminate any layouts that do not meet the requirements of the observations, any multicomponent links, any duplicate layouts that are equivalent to others via simple mosaic planar isotopy moves, and any mosaics for which the tile number can easily be reduced by a simple mosaic planar isotopy move.

First, in the case of nine crossings, after we eliminate the unnecessary layouts we end up with only one possibility, and it is shown in Figure 21. However, once we choose specific crossings in an alternating fashion, it is the knot 9_8, which has tile number 24. Nothing new arises from considering nonalternating crossings either.

Second, we do the same for 10 crossings, and we end up with five possible layouts, shown in Figure 22. Choosing alternating crossings in each one, we again fail to get any prime knots with minimal mosaic tile number 32. Figure 22(a) is 10_1, Figure 22(b) and (c) are 10_{34}, and Figures 22(d) and (e) are 10_{78}. Nothing new arises from considering nonalternating crossings either.

Third, we consider the case where the mosaic has 11 crossing tiles. In this instance, we end up with the 10 possible layouts shown in Figure 23. With alternating

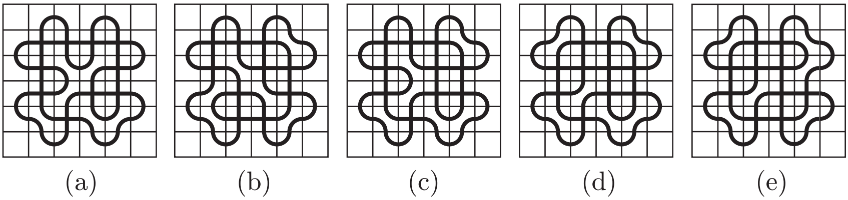


Figure 22. Only possible layouts, after elimination, with 10 crossing tiles for a prime knot with minimal mosaic tile number 32.

crossings, the first layout is $11a_{140}$, which we already know has tile number 27. The remaining layouts, given alternating crossings, lead to six distinct knots with minimal mosaic tile number 32, and with nonalternating crossings we get 10 additional knots that have minimal mosaic tile number 32. In particular, Figure 23(b) with alternating crossings is $11a_{43}$ and with nonalternating crossings can be made into $11n_{71}$, $11n_{72}$, $11n_{73}$, $11n_{74}$, and $11n_{75}$. Figures 23(c) and (d) are $11a_{46}$ when using alternating crossings and can be made into 9_{16} or 10_{77} with nonalternating crossings. Figures 23(e) and (f) are $11a_{59}$ when using alternating crossings and can be made into 10_{20} with nonalternating crossings. Figures 23(g) and (h) are $11a_{179}$ when using alternating crossings and can be made into 9_{10} or 10_{21} with nonalternating crossings. Figure 23(i) with alternating crossings is $11a_{247}$, and Figure 23(j) with alternating crossings is $11a_{367}$. Neither of these last two provide new knots to our list when considering nonalternating crossings.

Fourth, we consider the possibilities where the mosaic has 12 crossing tiles. In this case, we end up with the seven possible layouts shown in Figure 24. With alternating crossings, these layouts lead to five distinct knots with minimal mosaic

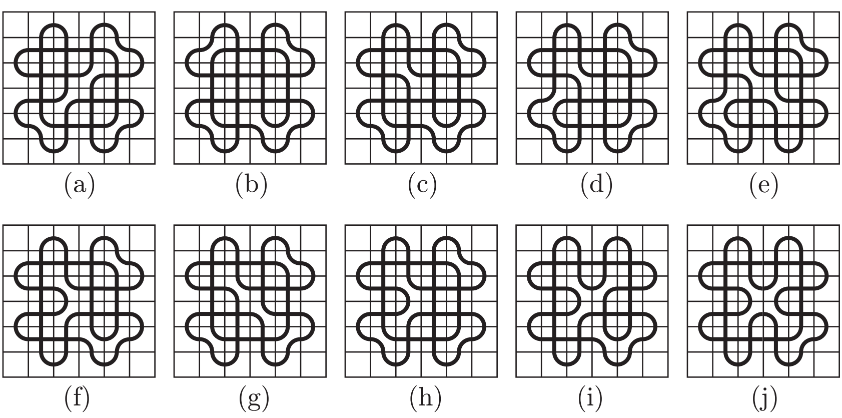
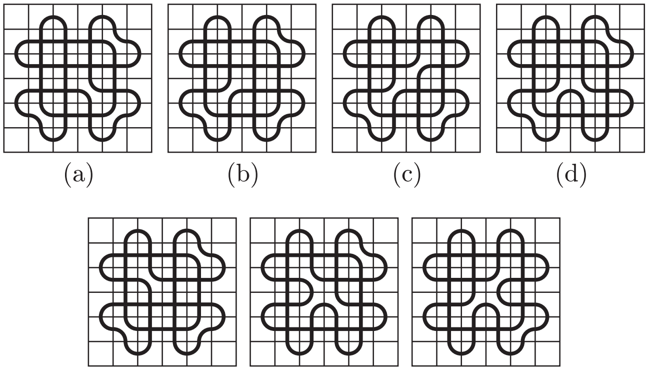


Figure 23. Only possible layouts, after elimination, with 11 crossing tiles for a prime knot with minimal mosaic tile number 32.



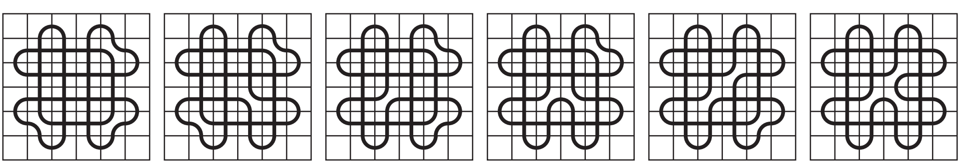
(a) (b) (c) (d)

(e) (f) (g)

Figure 24. Only possible layouts, after elimination, with 12 crossing tiles for a prime knot with minimal mosaic tile number 32.

tile number 32, and with nonalternating crossings we get 13 additional knots that have minimal mosaic tile number 32. In particular, Figures 24(a) and (b) with alternating crossings are $12a_{373}$ and with nonalternating crossings can be made into 10_{62}, 10_{64}, 10_{139}, $11a_{106}$, or $11a_{139}$. Figures 24(c) and (d) are $12a_{380}$ when using alternating crossings and can be made into 10_{11}, $11a_{166}$, or $11a_{341}$ with nonalternating crossings. Figure 24(e) is $12a_{503}$ when using alternating crossings and can be made into 9_{35}, 10_{74}, or $11a_{181}$ with nonalternating crossings. Figure 24(f) is $12a_{722}$ when using alternating crossings and can be made into $11a_{364}$ with nonalternating crossings. Figure 24(g) with alternating crossings is $12a_{1149}$ and with nonalternating crossings can be $11a_{342}$.

Fifth, we consider what happens when we place 13 crossing tiles on the mosaic. In this instance, we end up with the six possible layouts shown in Figure 25. With alternating crossings, the layouts lead to four distinct knots with minimal mosaic tile number 32, and with nonalternating crossings we get 26 additional knots that have minimal mosaic tile number 32. In particular, Figure 25(a) with alternating crossings is $13a_{1230}$ and with nonalternating crossings can be made into $11a_{44}$, $11a_{47}$, $11n_{76}$, $11n_{77}$, $11n_{78}$, $12a_{119}$, $13n_{2399}$, $13n_{2400}$, $13n_{2401}$, $13n_{2402}$, or $13n_{2403}$.



(a) (b) (c) (d) (e) (f)

Figure 25. Only possible layouts, after elimination, with 13 crossing tiles for a prime knot with minimal mosaic tile number 32.

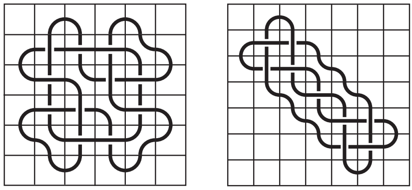


Figure 26. The 9_{10} knot represented as a minimally space-efficient 6-mosaic with minimal mosaic tile number 32 and as a space-efficient 7-mosaic with tile number 27.

Figures 25(b) and (c) are $13a_{1236}$ when using alternating crossings and can be made into 10_{61}, 10_{76}, $11a_{58}$, $11a_{165}$, $11a_{340}$, $12a_{165}$, $12a_{376}$, or $12a_{444}$ with nonalternating crossings. Figures 25(d) and (e) are $13a_{1461}$ when using alternating crossings and can be made into $11a_{246}$, $11a_{339}$, $12a_{169}$, $12a_{379}$, or $12a_{1148}$ with nonalternating crossings. Figure 25(f) is $13a_{4573}$ when using alternating crossings and can be made into $12a_{803}$ or $12a_{1166}$ with nonalternating crossings.

Finally, by Observation 4, we do not need to consider 14 or more crossing tiles in this layout. We have considered every possible placement of nine or more crossing tiles on the final layout of Theorem 2 and have found every possible prime knot with mosaic number 6 and minimal mosaic tile number 32. □

Because of the work we have completed, we now know every prime knot with mosaic number 6 or less. We also know the tile number or minimal mosaic tile number of each of these prime knots. In the table of knots in online supplement, we provide minimally space-efficient knot mosaics for all of these. These preceding theorems lead us to the following interesting consequences.

Corollary 7. *The prime knots with crossing number at least* 9 *not listed in Theorems 3, 4, 5, or 6 have mosaic number* 7 *or higher.*

Theorem 8. *The tile number of a knot is not necessarily equal to the minimal mosaic tile number of a knot.*

Proof. According to Theorem 6, the minimal mosaic tile number for 9_{10} is 32. However, on a 7-mosaic, this knot can be represented using only 27 nonblank tiles, as depicted in Figure 26. Also note that, as a 7-mosaic, this knot could be represented with only nine crossings, whereas 11 crossings were required to represent it as a 6-mosaic. □

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