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The length spectrum of the sub-Riemannian three-sphere

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We determine the lengths of all closed sub-Riemannian geodesics on the three-sphere S^3. Our methods are elementary and allow us to avoid using explicit formulas for the sub-Riemannian geodesics.

1. Introduction

In the case of a compact Riemannian manifold (M, g) there is a relationship between closed geodesics, representing paths of free classical particles in periodic motion, and eigenfunctions of the Laplacian Δ, representing periodic free quantum "waves" (up to a phase factor). For this reason, the set of lengths of closed geodesics is called the *length spectrum*, in analogy to the spectrum of the Laplacian. There are in fact precise formulas relating lengths to eigenvalues; see for example the announcement [Guillemin and Weinstein 1976] for a readable discussion with references.

So far there is no such formula relating lengths and eigenvalues in the case of a compact sub-Riemannian (sR) manifold. We recall that an sR manifold is a manifold with a specified linear subbundle $\mathcal{H}$ (the "horizontal bundle") of its tangent bundle, along with a Riemannian metric on $\mathcal{H}$. Distances between points are then measured using curves that are constrained to have tangent vectors in $\mathcal{H}$ ("horizontal curves"). In fact, when $\mathcal{H}$ is the span of a set of bracket-generating vector fields, then the Chow–Rashevskii theorem says that any two points are connected by a horizontal curve, a result that even experts find surprising [Burago et al. 2001, p. 178]; thus given any two points there is a shortest horizontal curve connecting them; it is called an *sR geodesic*.

Sub-Riemannian geometry is of practical interest; for example, the problem of parallel parking a car, or, even worse, a car with a trailer, is a problem in sR geometry [Burago et al. 2001; Nelson 1967]. And there are further surprises from the purely mathematical point of view, one being Montgomery's proof of existence of singular sR geodesics, singular in the sense that they do not satisfy the geodesic equations

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(Hamilton's equations) [Montgomery 1994; 2002]. This and other relatively recent results in sR geometry inspired renewed interest in the sub-Laplacian: the operator naturally associated with the given (sub-)Riemannian metric on $\mathcal{H}$.

In this paper, with the goal of understanding a single example, we compute the sR length spectrum of the three-dimensional sphere S^3 with its standard sR structure; this is to be compared with the spectrum of the sub-Laplacian on S^3, known by Taylor [1986] and generalized to other connected, semisimple Lie groups by Domokos [2015]. We expect that a general theory relating the sR length spectrum to the spectrum of the sub-Laplacian would be amenable to the tools of microlocal analysis, as in the Riemannian setting; [Colin de Verdière et al. 2016] gives hope that this will be accomplished.

We focus on S^3 with its standard sR structure because it is perhaps the simplest compact manifold with an sR structure, and there are no singular sR geodesics on S^3; that is, all sR geodesics arise as projections of solutions of Hamilton's equations [Montgomery 2002]. Moreover, we wish to compare the sR setting to the Riemannian setting, in which the spheres S^n are of fundamental importance, as examples of manifolds all of whose geodesics are closed and have the same length T; in general this is equivalent to most of the spectrum of $\sqrt{-\Delta}$ being concentrated near an arithmetic progression $(2\pi/T)k+\beta,\ k=1,2,\dots,$ for some constant β [Duistermaat and Guillemin 1975]. As we will see, in the case of S^3 not all sR geodesics are closed, and not all have the same length:

Theorem. *The set of lengths of the closed sR geodesics on S^3 is*

$$\{2\pi\sqrt{n} : n \in \mathbb{N}\}.$$

Others have studied the sR geodesics on S^3 [Calin et al. 2009; Chang et al. 2009; Hurtado and Rosales 2008] (see also the survey article [D'Angelo and Tyson 2010]), but we compute their lengths and differ from the previous work in that we consistently use Hopf coordinates on S^3 and avoid using explicit formulas for the sR geodesics; we believe it clarifies the presentation to *not* use explicit formulas.

We introduce the sR structure and geodesic equations in Section 2 using Hopf coordinates, and in Section 3 we categorize the qualitatively different types of sR geodesics. In Section 4 we determine which sR geodesics are closed, and in Section 5 we compute their lengths, resulting in the theorem above. Finally, in Section 6 we compare the sR length spectrum to the previously known spectrum of the sub-Laplacian.

Remark. During peer review, it was pointed out that the above result is contained in [Chang et al. 2011] (see their Theorem 2). However, our proof is entirely new and has the advantage of being elementary after the introduction of Hamilton's equations (2) in our chosen coordinate system.

2. S^3 in Euclidean and Hopf coordinates

First we consider S^3 as a subset of $\mathbb{R}^4$:

$$S^3 = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 : x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\}.$$

On S^3 we have the orthonormal vector fields

$$\begin{aligned} V &:= -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial y_2}, \\ E_1 &:= -x_2 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial x_2} - y_1 \frac{\partial}{\partial y_2}, \\ E_2 &:= -y_2 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial y_2}, \end{aligned}$$

which satisfy the Lie bracket relations

$$[V, E_1] = -2E_2, \quad [E_2, V] = -2E_1, \quad [E_1, E_2] = -2V.$$

Thus $\mathcal{H}(S^3) = \operatorname{span}\{E_1, E_2\}$ is a bracket-generating tangent subbundle, and by the Chow–Rashevskii theorem any two points on S^3 are connected by an sR geodesic.

The orbits of the flow generated by V are the circles of the Hopf fibration [Cannas da Silva 2008], so we find it convenient to use Hopf coordinates, see [Wikipedia 2015], on S^3:

$$\begin{aligned} x_1 &= \cos\theta_1 \sin\theta_0, \quad & y_1 &= \sin\theta_1 \sin\theta_0, \\ x_2 &= \cos\theta_2 \cos\theta_0, \quad & y_2 &= \sin\theta_2 \cos\theta_0 \end{aligned}$$

for $0 < \theta_0 < \frac{\pi}{2}$ and $0 < \theta_j < 2\pi$, $j = 1, 2$. We picture the $(\theta_0, \theta_1, \theta_2)$-space as "the Hopf cube" $\left(0, \frac{\pi}{2}\right) \times (0, 2\pi) \times (0, 2\pi)$. When we have occasion to exit the Hopf cube, we simply return to the definition of Hopf coordinates to make the correct interpretation:

(i) For the θ_1- and θ_2-coordinates the values 0 and 2π are identified.

(ii) When a point crosses the $\theta_0 = 0$ plane we have that θ_0 changes direction ("bounces") and (θ_1, θ_2) is identified with $(\theta_1 + \pi, \theta_2)$.

(iii) When a point crosses the $\theta_0 = \frac{\pi}{2}$-plane we have that θ_0 changes direction and (θ_1, θ_2) is identified with $(\theta_1, \theta_2 + \pi)$.

The (round) Riemannian metric in Hopf coordinates is

$$ds^2 = d\theta_0^2 + \sin^2\theta_0 \, d\theta_1^2 + \cos^2\theta_0 \, d\theta_2^2, \tag{1}$$

and the Laplacian is

$$\Delta = \frac{1}{\sin(2\theta_0)} \frac{\partial}{\partial \theta_0} \circ \sin(2\theta_0) \frac{\partial}{\partial \theta_0} + \csc^2\theta_0 \frac{\partial^2}{\partial \theta_1^2} + \sec^2\theta_0 \frac{\partial^2}{\partial \theta_2^2}.$$

We now write the sR structure in Hopf coordinates. We can introduce $r > 0$, to give coordinates to $\mathbb{R}^4$, allowing us to write the $\partial/\partial x_j,\ \partial/\partial y_j$ in terms of the $\partial/\partial\theta_j$, $\partial/\partial r$. Then restricting to functions on S^3 we get

$$\begin{aligned}
\frac{\partial}{\partial x_1} &= \cos\theta_1 \cos\theta_0 \frac{\partial}{\partial\theta_0} - \sin\theta_1 \csc\theta_0 \frac{\partial}{\partial\theta_1}, \\
\frac{\partial}{\partial y_1} &= \sin\theta_1 \cos\theta_0 \frac{\partial}{\partial\theta_0} + \cos\theta_1 \csc\theta_0 \frac{\partial}{\partial\theta_1}, \\
\frac{\partial}{\partial x_2} &= -\cos\theta_2 \sin\theta_0 \frac{\partial}{\partial\theta_0} - \sin\theta_2 \sec\theta_0 \frac{\partial}{\partial\theta_2}, \\
\frac{\partial}{\partial y_2} &= -\sin\theta_2 \sin\theta_0 \frac{\partial}{\partial\theta_0} + \cos\theta_2 \sec\theta_0 \frac{\partial}{\partial\theta_2}.
\end{aligned}$$

Our vector fields are then

$$\begin{aligned}
V &= \frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}, \\
E_1 &= -\cos(\theta_1+\theta_2) \frac{\partial}{\partial\theta_0} + \sin(\theta_1+\theta_2)\cot\theta_0 \frac{\partial}{\partial\theta_1} - \sin(\theta_1+\theta_2)\tan\theta_0 \frac{\partial}{\partial\theta_2}, \\
E_2 &= -\sin(\theta_1+\theta_2) \frac{\partial}{\partial\theta_0} - \cos(\theta_1+\theta_2)\cot\theta_0 \frac{\partial}{\partial\theta_1} + \cos(\theta_1+\theta_2)\tan\theta_0 \frac{\partial}{\partial\theta_2}.
\end{aligned}$$

The commutation relations hold, the same as before, and the vector fields are still orthonormal (of course, with respect to the Riemannian metric in Hopf coordinates).

The sR metric, written in Hopf coordinates, is

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta_0 \sin^2\theta_0 & -\cos^2\theta_0 \sin^2\theta_0 \\ 0 & -\cos^2\theta_0 \sin^2\theta_0 & \cos^2\theta_0 \sin^2\theta_0 \end{pmatrix}.$$

Indeed it is easy to check that E_1 and E_2 are orthonormal with respect to S, and V is in the kernel of S. Written as a two-tensor,

$$S = d\theta_0 \otimes d\theta_0 + \cos^2\theta_0 \sin^2\theta_0\, (d\theta_1 - d\theta_2) \otimes (d\theta_1 - d\theta_2).$$

The sR Laplacian, written in Hopf coordinates, is

$$\Delta_{\mathrm{sR}} = E_1^2 + E_2^2 = \frac{1}{\sin(2\theta_0)} \frac{\partial}{\partial\theta_0} \circ \sin(2\theta_0) \frac{\partial}{\partial\theta_0} + \left(\cot\theta_0 \frac{\partial}{\partial\theta_1} - \tan\theta_0 \frac{\partial}{\partial\theta_2}\right)^2.$$

We can consider the sR metric as being the limit of certain penalty metrics, where the V-direction is penalized by a factor $\lambda > 1$. After simple linear algebra (multiplying the V-direction by λ, multiplying the other directions by 1, and *then* applying the Riemannian metric), the λ-penalty metric is given by the matrix

$$P_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\lambda^2-1)\sin^4\theta_0+\sin^2\theta_0 & (\lambda^2-1)\cos^2\theta_0\,\sin^2\theta_0 \\ 0 & (\lambda^2-1)\cos^2\theta_0\,\sin^2\theta_0 & (\lambda^2-1)\cos^4\theta_0+\cos^2\theta_0 \end{pmatrix}.$$

Indeed, one can check that in fact V, E_1, and E_2 are orthogonal with respect to this metric, that E_1 and E_2 have length 1, and that V has length λ. We can easily compute

$$\det P_\lambda = \lambda^2 \cos^2\theta_0 \sin^2\theta_0$$

and

$$P_\lambda^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cot^2\theta_0+\lambda^{-2} & \lambda^{-2}-1 \\ 0 & \lambda^{-2}-1 & \tan^2\theta_0+\lambda^{-2} \end{pmatrix}.$$

From this we find that the λ-penalty Laplacian on S^3 is

$$\Delta_\lambda = \frac{\partial^2}{\partial\theta_0^2} + 2\cot(2\theta_0)\,\frac{\partial}{\partial\theta_0} + \Big(\cot\theta_0\,\frac{\partial}{\partial\theta_1} - \tan\theta_0\,\frac{\partial}{\partial\theta_2}\Big)^2 + \lambda^{-2}\Big(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\Big)^2.$$

That is,

$$\Delta_\lambda = E_1^2 + E_2^2 + \lambda^{-2} V^2,$$

as might have been expected.

Montgomery discovered an example in which geodesics with respect to the λ-penalty metric converge (as $\lambda \to \infty$) to sR geodesics that do *not* solve the sR geodesic equations, in contrast to the Riemannian setting; that is, Montgomery [1994] discovered so-called *singular geodesics*. For the case of S^3 (and more generally, in the contact case), singular geodesics do not exist, so it suffices to study the geodesic equations, or, equivalently, Hamilton's equations [Montgomery 2002].

We denote the dual variable to θ_j by ξ_j. The sR Hamiltonian is then

$$H(\theta,\xi) = \tfrac{1}{2}\xi_0^2 + \tfrac{1}{2}(\cot\theta_0\,\xi_1 - \tan\theta_0\,\xi_2)^2.$$

Hamilton's equations, giving the sR geodesics, are then, for $j = 0, 1, 2$,

$$\dot\theta_j = \frac{\partial H}{\partial \xi_j}, \quad \dot\xi_j = -\frac{\partial H}{\partial\theta_j}.$$

Explicitly,

$$\begin{aligned} \dot\theta_0 &= \xi_0, & \dot\xi_0 &= \cot\theta_0\,\csc^2\theta_0\,\xi_1^2 - \tan\theta_0\,\sec^2\theta_0\,\xi_2^2, \\ \dot\theta_1 &= \cot^2\theta_0\,\xi_1 - \xi_2, & \dot\xi_1 &= 0, \\ \dot\theta_2 &= \tan^2\theta_0\,\xi_2 - \xi_1, & \dot\xi_2 &= 0. \end{aligned} \tag{2}$$

One obvious advantage of using Hopf coordinates is that ξ_1 and ξ_2 are constant along the flow; in addition, as always H is constant along the flow, so we already have three conserved quantities. Also, these equations have a clear symmetry; for example,

$$\cot\left(\tfrac{1}{2}\pi-\theta_0\right)\,\csc^2\left(\tfrac{1}{2}\pi-\theta_0\right)=\tan\theta_0\,\sec^2\theta_0.$$

The penalty Hamiltonian is

$$\begin{aligned} H_\lambda(\theta,\xi) &= H+\frac{1}{2\lambda^2}(\xi_1+\xi_2)^2 \\ &= \tfrac{1}{2}\xi_0^2+\tfrac{1}{2}(\cot\theta_0\,\xi_1-\tan\theta_0\,\xi_2)^2+\frac{1}{2\lambda^2}(\xi_1+\xi_2)^2. \end{aligned} \tag{3}$$

The corresponding penalty Hamiltonian equations, giving the penalty geodesics, are then

$$\begin{aligned} &\dot\theta_0=\xi_0, && \dot\xi_0=\cot\theta_0\,\csc^2\theta_0\,\xi_1^2-\tan\theta_0\,\sec^2\theta_0\,\xi_2^2, \\ &\dot\theta_1=\cot^2\theta_0\,\xi_1-\xi_2+\lambda^{-2}(\xi_1+\xi_2), && \dot\xi_1=0, \\ &\dot\theta_2=\tan^2\theta_0\,\xi_2-\xi_1+\lambda^{-2}(\xi_1+\xi_2), && \dot\xi_2=0. \end{aligned} \tag{4}$$

For the case of the Riemannian metric on S^3, that is, the case $\lambda=1$, the equations simplify, and we get

$$\dot\theta_1=\csc^2\theta_0\,\xi_1,\qquad \dot\theta_2=\sec^2\theta_0\,\xi_2.$$

When $\lambda=1$, the solutions of Hamilton's equations are great circles on S^3.

3. Categorizing sR geodesics

Our categorization of sR geodesics is based on a reduced problem. In Hamilton's equations (2), since ξ_1 and ξ_2 are constant along the flow, we can isolate the equations

$$\dot\theta_0=\xi_0,\qquad \dot\xi_0=\cot\theta_0\,\csc^2\theta_0\,\xi_1^2-\tan\theta_0\,\sec^2\theta_0\,\xi_2^2,$$

which are Hamilton's equations for the sR Hamiltonian H considered as a function of *two* variables

$$H(\theta_0,\xi_0)=\tfrac{1}{2}\xi_0^2+\tfrac{1}{2}(\cot\theta_0\,\xi_1-\tan\theta_0\,\xi_2)^2. \tag{5}$$

Equation (5) can be viewed as a *one-dimensional* energy equation: it is of the form

$$\text{energy}=\text{kinetic energy}+\text{potential energy},$$

with potential function

$$U=\tfrac{1}{2}(\cot\theta_0\,\xi_1-\tan\theta_0\,\xi_2)^2.$$

We now list the various disjoint cases:

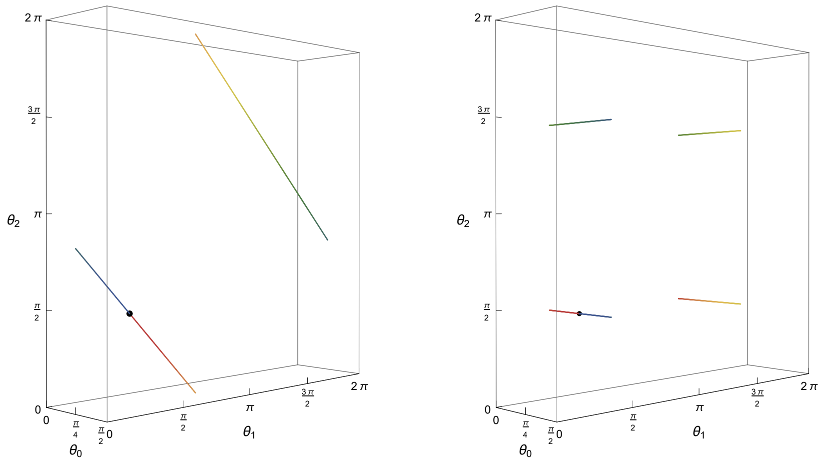


Figure 1. Case 1b (left) and Case 2 (right).

(1) A fixed point in the (θ_0, ξ_0) phase plane. From our original choice of coordinates we may assume that $\theta_0 \equiv \frac{\pi}{4}$, and then $\xi_1^2 = \xi_2^2$.

(a) $\xi_1 = \xi_2$. (This is precisely the case when $H = 0$.) Then from Hamilton's equations θ_0, θ_1, and θ_2 are constant; this gives a degenerate sR geodesic of length 0.

(b) $\xi_1 = -\xi_2 \neq 0$. Hamilton's equations then say that the speed on the Hopf cube is $\sqrt{2}\,|\xi_1 - \xi_2|$, and the length of the (simple) closed curve on the Hopf cube is $\sqrt{2}\,2\pi$, so the period is $2\pi/|\xi_1 - \xi_2|$. On S^3 the speed is $|\xi_1 - \xi_2|$, so the length of this closed sR geodesic is 2π. See Figure 1.

We categorize the remaining cases in terms of the potential function U.

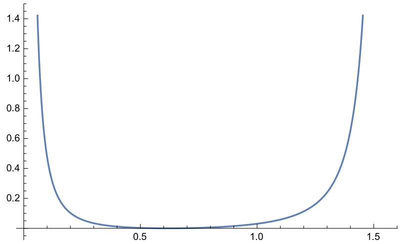
(2) The "free" case $U \equiv 0$. This happens precisely when $\xi_1 = \xi_2 = 0$. (We have already dispensed with the case when θ_0 is constant.) By Hamilton's equations, $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\xi}_0$ are also identically zero, while $\dot{\theta}_0 = \xi_0$. That is, we have a point with speed $|\xi_0|$ moving purely in the θ_0-direction; the length of this (simple) closed geodesic is 2π. (It is both a geodesic and an sR geodesic.) See Figure 1.

(3) $\xi_1 \neq 0$ and $\xi_2 \neq 0$. Then U is a potential well with a single nondegenerate minimum occurring when $\tan^4\theta_0 = \xi_1^2/\xi_2^2$. Typical potential functions are shown in Figure 2 for ξ_1 and ξ_2 with the same and opposite signs.

Since in this case θ_0 is *not* constant, its period is

$$\operatorname{period}(\theta_0) = 2\int_a^b \frac{d\theta_0}{\sqrt{2(H-U)}} = 2\int_a^b \frac{d\theta_0}{\sqrt{2H - (\cot\theta_0\,\xi_1 - \tan\theta_0\,\xi_2)^2}}.$$

Here a and b are the "turning points," where the kinetic energy is zero.



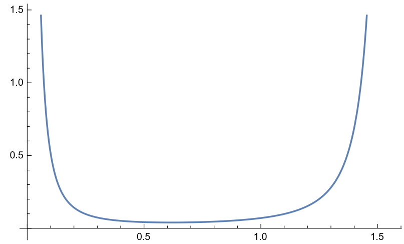


Figure 2. Potential functions with $\xi_1 = 0.1$, $\xi_2 = 0.2$ (left) and $\xi_1 = 0.1$, $\xi_2 = -0.2$ (right).

Fortunately it is possible to evaluate this integral using freshman calculus. Substituting

$$x = \cos^2\theta_0, \quad 0 < \theta_0 < \tfrac{\pi}{2},$$

we get

$$\operatorname{period}(\theta_0) = \int_{\cos^2 b}^{\cos^2 a} \frac{dx}{\sqrt{[-2H - (\xi_1+\xi_2)^2]x^2 + 2(H + \xi_1\xi_2 + \xi_2^2)x - \xi_2^2}}.$$

The limits of integration are exactly the points where the denominator vanishes (where the velocity is zero), and we recall that the Hamiltonian for Riemannian geodesics is $H_1 = H + \frac{1}{2}(\xi_1+\xi_2)^2$ (the case $\lambda = 1$), so we have

$$\operatorname{period}(\theta_0) = \frac{1}{\sqrt{2H_1}} \int_{\cos^2 b}^{\cos^2 a} \frac{dx}{\sqrt{(\cos^2 a - x)(x - \cos^2 b)}}.$$

This is an integral known to be solvable by elementary functions. Following [Woods 1934, p. 366],[1] we make the substitution defined by

$$z^2 + 1 = \frac{\cos^2 a - \cos^2 b}{x - \cos^2 b}$$

and finally get the answer

$$\operatorname{period}(\theta_0) = \sqrt{\frac{2}{H_1}} \int_0^\infty \frac{dz}{z^2+1} = \frac{\pi}{\sqrt{2H_1}}.$$

For future reference, we note that this is one-half the period of the (Riemannian) geodesic flow; after all, the speed of the geodesic flow is $\sqrt{2H_1}$, and we know the length of each geodesic, a great circle on S^3, to be 2π. (See Section 4.)

Examples are pictured in Figure 3, where ξ_1 and ξ_2 have the same and opposite signs.

[1]This is the book mentioned in [Feynman 1985] as giving him valuable tricks for integration.

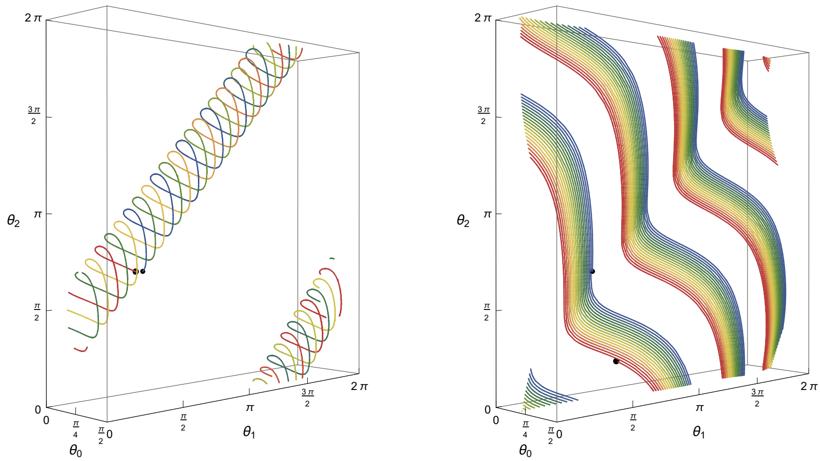


Figure 3. Case 3 when ξ_1 and ξ_2 have the same sign (left) and opposite signs (right).

(4) It remains to check the exceptional cases when $\{\xi_1 = 0 \text{ and } \xi_2 \neq 0\}$ and when $\{\xi_1 \neq 0 \text{ and } \xi_2 = 0\}$. For example, when $\xi_1 = 0$ the potential function is

$$U = \tfrac{1}{2}\tan^2\theta_0\,\xi_2^2, \qquad 0 < \theta_0 < \tfrac{\pi}{2}.$$

The force induced by this potential causes the point to exit the Hopf cube through the $\theta_0 = 0$ plane; rather we interpret it as bouncing off the plane, returning to the Hopf cube but with θ_1 shifted by π. (See Section 2.) With reasoning as in the previous case, we find that again $\operatorname{period}(\theta_0) = \pi/\sqrt{2H_1}$. The case when $\xi_1 \neq 0$ and $\xi_2 = 0$ follows by renaming the variables $\theta_0 \leftrightarrow \frac{\pi}{2} - \theta_0$ and $\xi_1 \leftrightarrow \xi_2$. In Figure 4 we

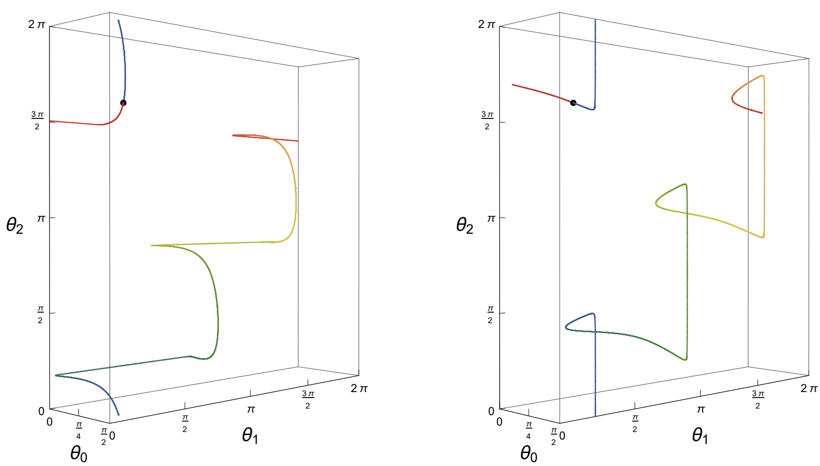


Figure 4. Case 4 with $0 < \xi_1 \ll \xi_2$ (left) and $0 < \xi_2 \ll \xi_1$ (right).

have the cases when $0 < \xi_1 \ll \xi_2$ and $0 < \xi_2 \ll \xi_1$, which illustrate how exiting the Hopf cube and re-entering after a π-shift appears as a limiting case.

Finally, we note that all sR geodesics are simple curves; that is, they do not self-intersect except trivially for closed curves. In Cases 1 and 2 above it is obvious. In Cases 3 and 4 we only need to wait until ξ_0 is zero, corresponding to the θ_0-particle having zero kinetic energy in the potential well U. When $(\theta_0, \theta_1, \theta_2)$ returns to that value, clearly ξ_0 is zero again, ξ_1, ξ_2 are the same as always, and the $\dot{\theta}_j$ and $\dot{\xi}_j$ return to their values; thus the curve only self-intersects in the case of a closed curve, at the end of a period.

4. Determining which sR geodesics are closed

In this section we identify the closed sR geodesics on S^3; we only need to consider Cases 3 and 4, and we may assume that the initial value of ξ_0 is zero. (See the comment at the end of Section 3.) Hurtado and Rosales [2008] found a necessary and sufficient condition in terms of geodesic curvature (see also [D'Angelo and Tyson 2010]):

Theorem [Hurtado and Rosales 2008]. *Let $\gamma : \mathbb{R} \to S^3$ be a complete sR geodesic of curvature λ. Then γ is a closed curve diffeomorphic to a circle if and only if $\lambda/\sqrt{1+\lambda^2}$ is a rational number. Otherwise γ is diffeomorphic to $\mathbb{R}$ and is dense in some group translate of a Clifford torus.*

Their proof relies on closed-form expressions of the sR geodesics. Here we give a condition which does not rely on closed-form expressions.

From the λ-penalty Hamilton's equations (4), we see that the sR Hamiltonian vector field for the Hamiltonian H is the difference of the Hamiltonian vector fields for the Hamiltonians H_1 and $H_V = \frac{1}{2}(\xi_1+\xi_2)^2$. Moreover, the vector fields Lie-commute (it is easy to see that the Poisson bracket of H_1 and H_V is zero), so the Hamiltonian flows for H_1 and H_V commute. We can thus consider the H-flow as an H_1-flow followed by an H_V-flow.

The Hamiltonian for the Riemannian geodesics may be written as

$$H_1(\theta,\xi) = \tfrac{1}{2}\xi_0^2 + \tfrac{1}{2}(\csc^2\theta_0\,\xi_1^2 + \sec^2\theta_0\,\xi_2^2),$$

(the penalty Hamiltonian (3) with $\lambda = 1$), so the first of Hamilton's equations, giving the velocities, are then

$$\dot{\theta}_0 = \xi_0, \quad \dot{\theta}_1 = \csc^2\theta_0\,\xi_1, \quad \dot{\theta}_2 = \sec^2\theta_0\,\xi_2.$$

We see that the speed, measured using the Riemannian metric (1), is $\sqrt{2H_1}$, which is constant. Moreover, the length of the Riemannian geodesic is 2π, being a great circle, so that the period of the closed orbit is $2\pi/\sqrt{2H_1} = 2 \times \text{period}(\theta_0)$.

On the other hand, the Hamiltonian H_V has Hamiltonian equations

$$\dot{\theta}_1 = \xi_1 + \xi_2, \quad \dot{\theta}_2 = \xi_1 + \xi_2, \quad \dot{\xi}_1 = \dot{\xi}_2 = 0.$$

Thus the speed (with respect to the Euclidean metric on the Hopf cube) is $\sqrt{2}|\xi_1+\xi_2|$. The length of the orbit (a circle fiber of the Hopf fibration) is $\sqrt{2}\cdot 2\pi$, so the period of the H_V-flow is $2\pi/|\xi_1+\xi_2|$. (It might seem strange that we find the speed and length with respect to the *Euclidean* metric on the Hopf cube, but the Euclidean metric is sufficient to compute the period of the H_V-flow.)

For a combination of an H_1-flow and an H_V-flow to result in a closed curve, we need the H_1-flow to return θ_0 to its original value (since the H_V-flow has no $\partial/\partial\theta_0$ component). Thus the time elapsed must be an integer multiple of $\text{period}(\theta_0) = \pi/\sqrt{2H_1}$. If the integer is odd, the H_1-flow takes the point to its antipodal point, and we would need a half-period of the H_V-flow to return to the starting point. If the integer is even, the H_1-flow takes the point back to itself, and we could only allow full periods of the H_V-flow. To summarize, a necessary and sufficient condition for a closed sR geodesic is

$$\text{time elapsed} = p \times \frac{\pi}{|\xi_1+\xi_2|} = q \times \frac{\pi}{\sqrt{2H_1}},$$

where $p, q \in \{1, 2, 3, \dots\}$ are either both odd or both even. In particular,

$$\frac{p}{q} = \frac{|\xi_1+\xi_2|}{\sqrt{2H_1}} = \sqrt{1-\frac{H}{H_1}} \in \mathbb{Q}\cap(0,1), \tag{6}$$

The quantity p/q is conserved along the flow and is positively homogeneous of degree zero in the ξ-variables. The condition (6) is also sufficient to have a closed sR geodesic. If it holds, then we have

$$H\text{-period} = p \times \frac{\pi}{|\xi_1+\xi_2|} = q \times \frac{\pi}{\sqrt{2H_1}}$$

for the least such integers $0 < p < q$ that are either both odd or both even.

When plotting sR geodesics in Cases 3 and 4, we can fix any $r \in \mathbb{Q}\cap(0,1)$ and rewrite the closure condition (6) as

$$\xi_0^2 = \frac{(\xi_1+\xi_2)^2}{r^2} - \csc^2\theta_0\,\xi_1^2 - \sec^2\theta_0\,\xi_2^2.$$

We can always find initial conditions satisfying this. Indeed, in Case 3 we can take any nonzero ξ_1 and ξ_2 and then take θ_0 to maximize the right-hand side: $\tan^2\theta_0 = |\xi_1/\xi_2|$. If ξ_1 and ξ_2 have the same sign, the right-hand side is always positive. If ξ_1 and ξ_2 have opposite signs, we need

$$\left|\frac{\xi_1+\xi_2}{\xi_1-\xi_2}\right| > r,$$

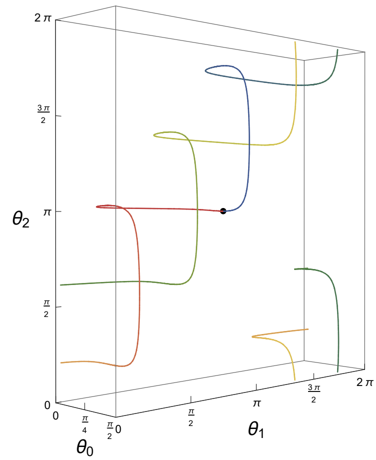


Figure 5. An example with $r = \frac{1}{5}$.

which is only valid for certain ξ_1 and ξ_2. Case 4 is similar. Then we can solve for ξ_0, use those numbers as the initial conditions in Hamilton's equations, and then plot the closed sR geodesic. Taking, for example, $r = \frac{1}{5}$, $\xi_1 = 0.6$, and $\xi_2 = 0.7$ we get the sR geodesic in Figure 5.

5. The sR length spectrum

To calculate the lengths of the closed sR geodesics we again only need to consider Cases 3 and 4 (the cases where θ_0 oscillates). We found in the previous section that an sR geodesic is closed when the period of the H_1-flow and the period of the H_V-flow are commensurable. Then we have

$$\text{period of } H\text{-flow} = p \times \frac{\pi}{|\xi_1 + \xi_2|} = q \times \frac{\pi}{\sqrt{2H_1}} \tag{7}$$

for the least such integers $0 < p < q$ where p, q are either both odd or both even. Since we know the speed of the sR geodesic is a constant $\sqrt{2H}$, we have that the length is

$$\text{length} = \text{period} \times \text{speed} = \frac{\pi q}{\sqrt{2H_1}} \times \sqrt{2H} = \pi q \sqrt{\frac{H}{H_1}} = \pi \sqrt{q^2 - p^2} \tag{8}$$

for the least integers $0 < p < q$ satisfying (7) where p, q are either both odd or both even.

We have another formulation of length that explains the repeating patterns seen in the figures. We know that the distance traveled in one θ_0-period is

$$\text{period}(\theta_0) \times \text{speed} = \pi \sqrt{\frac{H}{H_1}}.$$

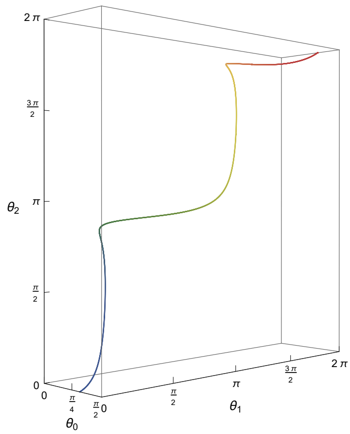


Figure 6. A Riemannian geodesic in Hopf coordinates.

Thus the length of a closed sR geodesic is

$$\text{length} = \pi \times (\text{number of } \theta_0\text{-oscillations}) \times \sqrt{\frac{H}{H_1}}.$$

Comparing with (8), we find that

$$\text{number of } \theta_0\text{-oscillations} = q.$$

Moreover, we see from Hamilton's equations that the curve segments traced out by θ_0-oscillations are congruent to each other. A similar argument shows that Riemannian geodesics in Hopf coordinates consist of two θ_0-oscillations, as illustrated in Figure 6.

To summarize, we have found that if an sR geodesic is closed then the initial conditions must satisfy

$$\sqrt{1 - \frac{H}{H_1}} = \frac{|\xi_1 + \xi_2|}{\sqrt{2H_1}} \in \mathbb{Q} \cap (0, 1)$$

and that the length of the closed sR geodesic is

$$\text{length} = \pi\sqrt{q^2 - p^2}$$

for the least integers $0 < p < q$ satisfying (7) where p, q are either both odd or both even.

In fact, every such number is attained as a length; we simply follow the procedure:

(i) Choose any $p/q \in \mathbb{Q} \cap (0, 1)$, with $\gcd(p, q) = 1$.

(ii) As seen at the end of Section 4, we can choose initial conditions so that

$$\frac{p}{q} = \sqrt{1 - \frac{H}{H_1}} = \frac{|\xi_1 + \xi_2|}{\sqrt{2H_1}}.$$

Thus
$$p \times \frac{\pi}{|\xi_1 + \xi_2|} = q \times \frac{\pi}{\sqrt{2H_1}}.$$
(iii) If p and q are both odd, then the sR geodesic with those initial conditions has length $\pi\sqrt{q^2 - p^2}$. If one of $\{p, q\}$ is odd and the other is even, the sR geodesic with those initial conditions has length $2\pi\sqrt{q^2 - p^2}$.

Thus the length spectrum consists of 2π and the numbers
$$\pi\sqrt{q^2 - p^2},$$
where $0 < p < q$ are odd integers with $\gcd(p, q) = 1$, and
$$2\pi\sqrt{q^2 - p^2},$$
where $0 < p < q$ are integers, one odd and the other even, with $\gcd(p, q) = 1$.

We now give an alternative characterization of these numbers. It is simpler to work with squares of lengths divided by π^2. Then we wish to characterize the set S of numbers consisting of 4 and
$$\epsilon(q^2 - p^2),$$
where $0 < p < q$ are integers with $\gcd(p, q) = 1$ and
$$\epsilon = \begin{cases} 1 & \text{if } p \text{ and } q \text{ are both odd,} \\ 4 & \text{if one of } p, q \text{ is odd and the other is even.} \end{cases}$$

In the $\epsilon = 1$ case we take the examples $p = 2k - 1$ and $q = 2k + 1$, $k \in \mathbb{N}$, to get
$$q^2 - p^2 = 4(2k), \quad k \in \mathbb{N}.$$
In the $\epsilon = 4$ case, we take the examples $p = k$ and $q = k + 1$, $k \in \mathbb{N}$, to get
$$4(q^2 - p^2) = 4(2k + 1), \quad k \in \mathbb{N}.$$
This shows that $4\mathbb{N} \subset S$. Now suppose that $n \in S$ and $4 \nmid n$. Then clearly n can only be in the $\epsilon = 1$ case, so there would be odd integers $0 < p < q$ with $\gcd(p, q) = 1$ such that $n = q^2 - p^2$. This is easily seen to be impossible. Thus in fact $4\mathbb{N} = S$.

We note that if $n \in S$ and $8 \mid n$, then n cannot be in the $\epsilon = 4$ case, and that if $n \in S$ and $n = 4(2k + 1)$, $k \in \mathbb{N}$, then n cannot be in the $\epsilon = 1$ case. Both of these statements easily follow from parity arguments.

Converting back to the language of lengths, we find that the set of lengths of the closed sR geodesics is
$$\{2\pi\sqrt{n} : n \in \mathbb{N}\}.$$
By the previous paragraph, odd n correspond to “full periods” of the H_V-flow and geodesic flow (the $\epsilon = 4$ case), and even n correspond to “half periods” of both the H_V-flow and geodesic flow (the $\epsilon = 1$ case).

6. The spectrum of the sub-Laplacian

The sub-Laplacian $-\Delta_{\mathrm{sR}}$ has a compact resolvent, and hence has a pure discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$, with $\lambda_n \to +\infty$ as $n \to +\infty$, and a complete orthonormal set of eigenfunctions. (See, for example, the recent paper [Colin de Verdière et al. 2016].) In fact, in the case of S^3, the eigenfunctions of the sub-Laplacian are the same as the eigenfunctions of the Laplacian. We recall that $\Delta_\lambda = E_1^2 + E_2^2 + \lambda^{-2}V^2$ is the λ-penalty Laplacian, with $\lambda = 1$ giving the Riemannian Laplacian on the sphere Δ_{S^3}, and $\lambda = \infty$ giving the sub-Laplacian on the sphere. In Hopf coordinates we have

$$\Delta_{\mathrm{sR}} = E_1^2 + E_2^2 = \frac{1}{\sin(2\theta_0)}\frac{\partial}{\partial\theta_0} \circ \sin(2\theta_0)\frac{\partial}{\partial\theta_0} + \left(\cot\theta_0 \frac{\partial}{\partial\theta_1} - \tan\theta_0 \frac{\partial}{\partial\theta_2}\right)^2.$$

It is easy to see that $V = \partial/\partial\theta_1 + \partial/\partial\theta_2$ commutes with Δ_{sR}; hence Δ_{sR} commutes with Δ_{S^3}. Thus Δ_{sR} and Δ_{S^3} have a *common* complete orthonormal set of eigenfunctions [Dirac 1947; von Neumann 1955]; the eigenfunctions of Δ_{sR} are simply the spherical harmonics.

Particularly noteworthy is $(x_1 + iy_1)^k = \sin^k\theta_0\, e^{ik\theta_1}$. It is a "Gaussian beam": a family of eigenfunctions of both Δ_{S^3} and Δ_{sR} that concentrates along a great circle. Zelditch [2016, pp. 185–186] singles out this example in the Riemannian setting. It would be interesting to see if it is possible to construct, localized to each sR geodesic, a quasimode or Gaussian beam in the spirit of [Ralston 1976; 1977].

Taylor [1986] used the Peter–Weyl theorem to find the eigenvalues of Δ_{sR}; Domokos [2015] generalized, using subelliptic Peter–Weyl and Plancherel theorems on compact, connected, semisimple Lie groups. To summarize, the eigenvalues of $-\Delta_{S^3}$ are $m(m+2)$ for $m \in \{0, 1, 2, \ldots\}$, and the eigenvalues of $-\Delta_{\mathrm{sR}}$ are (for the same m; the operators have the same complete orthonormal set of eigenfunctions)

$$4mj - 4j^2 + 2m, \quad j \in \{0, 1, 2, \ldots, m\}.$$

For reference, the eigenvalues of the λ-penalty Laplacian

$$-\Delta_\lambda = -\Delta_{\mathrm{sR}} - \lambda^{-2}V^2$$

are

$$(1-\lambda^{-2})4j(m-j) + m(2+\lambda^{-2}m),$$

for $m \in \{0, 1, 2, \ldots\}$ and $j \in \{0, 1, 2, \ldots, m\}$.

At this point we will not conjecture a general formula relating the sR length spectrum of a bracket-generating compact sR manifold (which for S^3 is $\{2\pi\sqrt{n} : n \in \mathbb{N}\}$) to the set of eigenvalues of the sub-Laplacian counted with or without multiplicities (which for S^3 is $\{2m : m = 0, 1, 2, \ldots\}$).

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